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Conditional independence structure and its closure: Inferential rules and algorithms

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ABSTRACT

In this paper, we deal with conditional independence models closed with respect to graphoid properties. Such models come from different uncertainty measures, in particular in a probabilistic setting. We study some inferential rules and describe methods and algorithms to compute efficiently the closure of a set of conditional independence statements.

1. Introduction

Conditional independence structures arise in different frameworks, in particular, in probability and in multivariate statistics. For example, graphical models [9,10,14,16,15,22,24,27,29,36] have been deeply developed as a tool for representing conditional independence relations or associations among the relevant variables and to simplify computation. Usually, probabilistic conditional independence structures are based on the classical notion of conditional independence. It is well known that the classical definition of stochastic independence leads to some counter-intuitive situations (see for example [4,6,30]) when some events with probability 0 are involved and when logical links among the variables are present. So other definitions of independence have been introduced in literature to encompass such situations. In particular, cs-independence introduced in [4] within the framework of coherent conditional probability [19,6] avoids the usual inconsistency related to logical dependence. The relationship between cs-independence and classical independence is described in [30] by considering graphoid properties.

It is well known [14] that for any probability measure P, the associated independence model $\mathscr{M}$, under the classical definition, is a semi-graphoid (i.e. it satisfies symmetry, decomposition, weak union, contraction) and if P is strictly positive, then $\mathscr{M}$ is a graphoid (also intersection property holds). On the other hand, cs-independence induces a structure not necessarily closed under symmetry, but its reinforcement (requiring symmetry) induces independence models closed under graphoid properties [30,33].

Among the peculiarities of coherent conditional probability framework, we recall that it allows to deal with partial assessments with (possible) conditioning events of zero probability, which represent a very crucial feature not only from a merely

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theoretical point of view, but they are met in many real problems, for example in medical diagnosis, statistical mechanics, physics, etc. (see, e.g. [5,21,20]). In medical diagnosis, for example, given probability assessments relative to some symptoms conditionally to some of the diseases, the following problems arise: (i) is this partial likelihood coherent? (ii) given an assessment on diseases is the assignment (the latter one together with the likelihood) global coherent? (iii) given a set of conditional independence relations on the diseases and on symptoms, are they compatible with the coherent assessment? If the answers are all yes, then we may try to "update" (coherently) the priors into the posteriors and discover the further induced conditional independence relations. Then, starting from a partial assessment and a set J of conditional independence statements given by an expert field, we can check whether the assessment is coherent [6,19] and whether it is compatible with the set J of independence statement [34].

The significance of independence models and graphoid structure is not limited to probabilistic models: in fact many independence models arising from different uncertainty measures are tested on the basis of graphoid properties (see e.g. [1,7,11,8,12,13,15,17,18,23,25–27,32,35]).

The aim of this paper is to consider a set J of conditional independence statements, compatible with a (coherent) conditional probability assessment, and to build in an efficient way the closure through graphoid properties. The obtained results are valid not just in the coherent setting, but also for classical independence models closed with respect to graphoid property.

This topic by considering semi-graphoid properties has already been faced successfully by Studený [27,28]. Since the computation of the closure is infeasible due to its size, which is exponentially larger than the size of J, our aim is, like in [27,28], to find a suitable subset of the closure which represents the same independence structure. This set should be as small as possible and from it all the relations in the closure should be easily deducible.

In other words, this small set of independence statements, which is called "fast closure", can be considered a basis for the closure.

The computation of the fast closure is relevant for the complexity of selection (based essentially on statistical tests) of a model on the basis of data for building, for example, the relevant Bayesian network. This is one of the motive of our effort.

We describe two algorithms to compute the reduced set. The first, called FC2, uses a generalization of the contraction and intersection (see also [28]). The second one, called FC1, is based on a unique inference rule introduced in this paper.

An empirical evaluation of the performance of FC2 and FC1 is provided by comparing computation times and number of iterations, as well as a comparison between the needed time to compute the fast closure and the time for computing the complete closure (the size of both closures is compared).

The paper is organized as follows: in Section 2 some preliminaries concepts about graphoids, closure and implications for independence relations are recalled. In Section 3, we describe the generalized inference rules, the fast closure and the algorithm FC2; while in Section 4 a system based on a unique inference rule and its corresponding algorithm FC1 are introduced. In Section 5, we describe and comment some experimental results. The last section is devoted to the conclusions.

2. Graphoid

Throughout the paper the symbol $\widetilde{S} = \{Y_1, \ldots, Y_n\}$ denotes a finite not empty set of variables. Given a subset $I \subseteq S = \{1, \ldots, n\}$ of indices, we denote by Y_I the vector $(Y_i : i \in I)$ of random variables, and given an uncertainty measure φ, a conditional independence statement $Y_A \perp\!\!\!\perp Y_B | Y_C$ (compatible with φ), where A, B, C are disjoint subsets of S, will be denoted simply also as an ordered triple (A, B, C). A conditional independence model, related to an uncertainty measure φ, is a subset of all ordered triples (A, B, C) of disjoint subset of S, such that A and B are not empty. In particular, we refer to probabilistic independence models. The properties of such models depend obviously on the independence notion taken into account. The classical definition of stochastic independence of two events

$$P(E \wedge H) = P(E)P(H) \tag{1}$$

gives rise to counter-intuitive situations when one of the events has probability 0 or 1. For instance, an event E with $P(E) = 0$ (or $P(E) = 1$) is stochastically independent on itself, while it is natural (due to the intuitive meaning of independence) to require for any event to be dependent on itself. Among other classical formulations, we recall

$$P(E|H) = P(E|H^c), \tag{2}$$

that is equivalent to (1) for events such that the probability of H is different from 0 and 1, in fact in these "extreme" cases the relevant conditional probabilities may even lack meaning according to the Kolmogorovian definition of conditional probability. Anyway, also by considering the stronger formulation (2) in the more general framework of de Finetti some critical situations continue to exist [30,32], for this reason other stronger independence notions have been introduced (see, e.g., cs-independence [4,6]). The particularity of de Finetti's approach is also to manage partial assessments by checking whether a partial assessment P on a set of conditional events is coherent (i.e. there exists a conditional probability in the sense of de Finetti [19] that extend the partial assessment P, see also [4]) and whether a given set of conditional independence statements is compatible with the coherent partial assessment P [34] (i.e. there exists a conditional probability among the extensions of P, inducing the given set of independence statements).

We recall that a conditional independence model (i.e. the set of conditional independence statements) arising from the classical independence notion is closed under semi-graphoid properties. Moreover, if the probability is strictly positive, the associated conditional independence model is also closed under graphoid properties [14]. For the properties of the conditional independence models arising from cs-independence, see [30], in particular we recall that these models are not necessarily closed with respect to symmetry [31] but, by reinforcing cs-independence (in a way requiring symmetry) the associated models are closed with respect to graphoid properties [30].

Let $S^{(3)}$ be the set of triples (A, B, C) of disjoint sets of S such that A and B are not empty. We recall that a graphoid is a couple $(S, \mathscr{I})$, where $\mathscr{I}$ is a ternary relation on the set S, which satisfies the following properties:

G1 if $(A, B, C) \in \mathscr{I}$, then $(B, A, C) \in \mathscr{I}$ (Symmetry);
G2 if $(A, B, C) \in \mathscr{I}$, then $(A, B', C) \in \mathscr{I}$ for any nonempty subset B' of B (Decomposition);
G3 if $(A, B_1 \cup B_2, C) \in \mathscr{I}$ with B_1 and B_2 disjoint, then $(A, B_1, C \cup B_2) \in \mathscr{I}$ (Weak Union);
G4 if $(A, B, C \cup D) \in \mathscr{I}$ and $(A, C, D) \in \mathscr{I}$, then $(A, B \cup C, D) \in \mathscr{I}$ (Contraction);
G5 if $(A, B, C \cup D) \in \mathscr{I}$ and $(A, C, B \cup D) \in \mathscr{I}$, then $(A, B \cup C, D) \in \mathscr{I}$ (Intersection).

A semi-graphoid is a couple $(S, \mathscr{I})$ satisfying only the properties G1–G4. The symmetric version of rules G2 and G3 will be denoted by

G2s if $(A, B, C) \in \mathscr{I}$, then $(A', B, C) \in \mathscr{I}$ for any nonempty subset A' of A;
G3s if $(A_1 \cup A_2, B, C) \in \mathscr{I}$, then $(A_1, B, C \cup A_2) \in \mathscr{I}$.

Let $\theta, \theta' \in S^{(3)}$, we denote by

$$\theta \vdash_R \theta'$$

the fact that θ' is obtained by applying once the property R to θ, where in this context R can be G1, G2 or G3.

Moreover, let $\theta_1, \theta_2, \theta \in S^{(3)}$;

$$\theta_1, \theta_2 \vdash_R \theta$$

denotes that θ is obtained by applying once R to the pair θ_1, θ_2 of triples. In this case R can be either G4 or G5.

Now, we start from a set $J \subset S^{(3)}$ of triples, compatible with a coherent conditional probability, and we are interested to establish whether a triple $\theta \in S^{(3)}$ can be derived from J, in symbols

$$J \vdash^* \theta.$$

This means that θ can be obtained by applying a finite number of times the rules G1–G5 starting from the set of triples J. This problem is called "implication problem" and has been already studied, for instance, in [37].

A strictly related problem is to compute the closure of a set J, defined as

$$\bar{J} = \{\theta \in S^{(3)} : J \vdash^* \theta\}.$$

It is clear that the implication problem can be easily solved once the closure of J has been computed. But the computation of the closure is infeasible because its size is exponentially larger than the size of J. Then, in the following sections we describe how it is possible to compute a smaller set of triples having the same information as the closure. This problem has been already faced in [28], with particular attention to semi-graphoid structures.

3. Generalized inference rules

In order to compute efficiently the closure of a set of conditional independence statements we introduce in Section 3.1 a notion of generalized inclusion, that is related to the notion of dominance given in [27]. Moreover in Section 3.2 we study some properties of intersection and contraction that lead to suitable inferential rules. In Section 3.3, we provide a procedure to compute a "small" set that can be considered a sort of basis for the closure, with respect to graphoid, of a given set of conditional independence statements.

3.1. Generalized inclusion

Let us focus our attention to the first three graphoid rules. Given a triple $\theta_2 \in S^{(3)}$ it is possible to compute all the triples θ_1 which can be obtained from θ_2 with a finite number of applications of G1, G2 and G3. We will say that any such triple θ_1 is *generalized-included* in θ_2 (briefly g-included), in symbol $\theta_1 \sqsubseteq \theta_2$.

In order to simplify the notation in the following, given a triple $\theta_i = (A_i, B_i, C_i), X_i$ stands for $(A_i \cup B_i \cup C_i)$.

Proposition 1. *Given $\theta_1 = (A_1, B_1, C_1)$ and $\theta_2 = (A_2, B_2, C_2)$, then $\theta_1 \sqsubseteq \theta_2$ if and only if the following conditions hold*

(i) $C_2 \subseteq C_1 \subseteq X_2$;
(ii) *either $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$ or $A_1 \subseteq B_2$ and $B_1 \subseteq A_2$.*

Proof. We prove first that if (i) and (ii) hold, then $\theta_1 \sqsubseteq \theta_2$.

Suppose that $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, θ_1 is obtained from θ_2 by the following steps: take $B'_2 = B_2 \setminus C_1$ and $C'_2 = C_2 \cup (C_1 \cap B_2)$, so $B_1 \subseteq B'_2$ and $(A_2, B_2, C_2) \vdash_{G3} (A_2, B'_2, C'_2)$; $(A_2, B'_2, C'_2) \vdash_{G2} (A_2, B_1, C'_2)$; furthermore $(A_2, B_1, C'_2) \vdash_{G1} (B_1, A_2, C'_2)$.

Now, let $A'_2 = A_2 \setminus C_1$, so $C_1 = C'_2 \cup (C_1 \cap A_2), A_1 \subseteq A'_2$ and $(B_1, A_2, C'_2) \vdash_{G3} (B_1, A'_2, C_1)$; $(B_1, A'_2, C_1) \vdash_{G2} (B_1, A_1, C_1)$; moreover $(B_1, A_1, C_1) \vdash_{G1} (A_1, B_1, C_1)$.

The case $A_1 \subseteq B_2$ and $B_1 \subseteq A_2$ follows from the previous case by applying as first step the symmetry.

Now, we need to prove that if $\theta_1 \sqsubseteq \theta_2$, then (i) and (ii) hold. If $\theta_1 \sqsubseteq \theta_2$, then there exist $\theta'_i, R_i, i = 1, \dots, n$, such that $\theta'_1 = \theta_2, \theta'_{n+1} = \theta_1$, $\theta'_i \vdash_{R_i} \theta'_{i+1}$, with $R_i \in \{G1, G2, G3\}$. We will prove by induction on i that $\theta'_i \sqsubseteq \theta_2$. For $i = 1$, it is trivial. If it is true for i (i.e. $\theta'_i \sqsubseteq \theta_2$), suppose that $A'_i \subseteq A_2$ and $B'_i \subseteq B_2$, we have the following three cases

(1) $\theta'_i \vdash_{G1} \theta'_{i+1}$ with $A'_{i+1} = B'_i \subseteq B_2, B'_{i+1} = A'_i \subseteq A_2, C'_{i+1} = C'_i$;
(2) $\theta'_i \vdash_{G2} \theta'_{i+1}$ with $A'_{i+1} = A'_i \subseteq A_2, B'_{i+1} \subseteq B'_i \subseteq B_2, C'_{i+1} = C'_i$;
(3) $\theta'_i \vdash_{G3} \theta'_{i+1}$ with $A'_{i+1} = A'_i \subseteq A_2, B'_{i+1} \subseteq B'_i \subseteq B_2, C'_{i+1} = C'_i \cup (B'_i \setminus B'_{i+1}), C_2 \subseteq C'_i \subseteq C'_{i+1}$. Furthermore, $B'_i \setminus B'_{i+1} \subseteq B_2$ and $C'_i \subseteq X_2$ imply $C'_{i+1} \subseteq X_2$.

The case $A'_i \subseteq B_2$ and $B'_i \subseteq A_2$ follows analogously to the previous one. □

Generalized inclusion is strictly related to the following *partial order relation* $\sqsubseteq_a$ on $S^{(3)}$, defined in [27] and called dominance: the triple $\theta = (A, B, C)$ is said to dominate $\theta' = (A', B', C')$ (in symbol $\theta' \sqsubseteq_a \theta$) if θ' can be derived from θ by means of decomposition, weak union and their symmetric properties (i.e. G2, G3, G2s and G3s).

Proposition 2. *Given $\theta_1 = (A_1, B_1, C_1)$ and $\theta_2 = (A_2, B_2, C_2), \theta_1 \sqsubseteq_a \theta_2$ if and only if*

(i) $C_2 \subseteq C_1 \subseteq X_2$;
(ii) $A_1 \subseteq A_2$ *and* $B_1 \subseteq B_2$.

Proof. The proof goes along the same lines of that one of Proposition 1. □

Given $\theta = (A, B, C)$, denote with $\theta^T = (B, A, C)$ the transpose of θ. The relation between $\sqsubseteq$ and $\sqsubseteq_a$ is simple: $\theta' \sqsubseteq \theta$ if and only if either $\theta' \sqsubseteq_a \theta$ or $\theta' \sqsubseteq_a \theta^T$.

As the next result shows, the g-inclusion verifies almost all the properties of a partial order relation on $S^{(3)}$, in fact the anti-symmetric property is verified in a weaker form called "weak anti-symmetry":

$$(AS)^* \theta_1 \sqsubseteq \theta_2 \quad \text{and} \quad \theta_2 \sqsubseteq \theta_1 \text{ implies either } \theta_1 = \theta_2 \text{ or } \theta_1 = \theta_2^T.$$

Proposition 3. *The g-inclusion satisfies reflexive, transitive and the weak anti-symmetric $(AS)^*$ properties.*

Proof. Reflexivity is trivial. To prove transitivity, let $\theta_1 \sqsubseteq \theta_2$ and $\theta_2 \sqsubseteq \theta_3$, we have the following four cases:

(1) if $\theta_1 \sqsubseteq_a \theta_2$ and $\theta_2 \sqsubseteq_a \theta_3$, then $\theta_1 \sqsubseteq_a \theta_3$ from transitivity of dominance (since $\sqsubseteq_a$ is an order relation);
(2) if $\theta_1 \sqsubseteq_a \theta_2^T$ and $\theta_2 \sqsubseteq_a \theta_3^T$ (being $\theta_2 \sqsubseteq_a \theta_3^T$equivalent to $\theta_2^T \sqsubseteq_a \theta_3$), then $\theta_1 \sqsubseteq_a \theta_3$ again from transitivity of dominance;
(3) if $\theta_1 \sqsubseteq_a \theta_2^T$ and $\theta_2 \sqsubseteq_a \theta_3$ (so $\theta_1^T \sqsubseteq_a \theta_2$), then $\theta_1^T \sqsubseteq_a \theta_3$;
(4) if $\theta_1 \sqsubseteq_a \theta_2^T$ and $\theta_2 \sqsubseteq_a \theta_3^T$, then $\theta_1 \sqsubseteq_a \theta_3$.

Now, suppose $\theta_1 \sqsubseteq \theta_2$ and $\theta_2 \sqsubseteq \theta_1$, we have four cases:

(1) if $\theta_1 \sqsubseteq_a \theta_2$ and $\theta_2 \sqsubseteq_a \theta_1$, then $\theta_1 = \theta_2$ since $\sqsubseteq_a$ is anti-symmetric.
(2) If $\theta_1 \sqsubseteq_a \theta_2^T$ and $\theta_2 \sqsubseteq_a \theta_1^T$, then (being $\theta_2 \sqsubseteq_a \theta_1^T$ equivalent to $\theta_2^T \sqsubseteq_a \theta_1$) $\theta_1 = \theta_2^T$.
(3) If $\theta_1 \sqsubseteq_a \theta_2$ and $\theta_2 \sqsubseteq_a \theta_1^T$, then $A_1 \subseteq A_2, A_2 \subseteq B_1, B_1 \subseteq B_2$ and $B_2 \subseteq A_1$, so by transitivity $A_1 \subseteq A_2 \subseteq B_1$ which is impossible since A_1 and B_1 are disjoint.
(4) If $\theta_1 \sqsubseteq_a \theta_2^T$ and $\theta_2 \sqsubseteq_a \theta_1$ we get a contradiction as in the previous step. □

Now, we extend the definition of g-inclusion between triples to the case of sets of triples and we show its properties.

Definition 4. Let H, J be subsets of $S^{(3)}$. J is a covering of H (in symbol $H \sqsubseteq J$) if and only if for any triple $\theta \in H$ there exists a triple $\theta' \in J$ such that $\theta \sqsubseteq \theta'$.

We show that the g-inclusion between sets of triples verifies some properties of g-inclusion between triples.

Proposition 5. *The g-inclusion between subsets of $S^{(3)}$ satisfies reflexivity and transitivity.*

Proof. Reflexivity is trivial. To prove transitivity suppose $H \sqsubseteq K$ and $K \sqsubseteq J$, with $H, K, J \subseteq S^{(3)}$. Then, for any $\theta \in H$ there exists $\theta' \in K$ such that $\theta \sqsubseteq \theta'$. For $\theta' \in K$, since $K \sqsubseteq J$, there exists $\theta'' \in J$ such that $\theta' \sqsubseteq \theta''$. From Proposition 3, $\theta \sqsubseteq \theta''$. □

The following example shows that the g-inclusion between sets of triples does not satisfy the anti-symmetry neither in its weak form.

Example 1. Given $S = \{1, 2, 3, 4\}$, consider the triples $\theta = (\{1\}, \{2\}, \{3\}), \theta' = (\{1, 4\}, \{2\}, \{3\}) \in S^{(3)}$ and the subsets $H = \{\theta, \theta'\}$ and $J = \{\theta'\}$ of $S^{(3)}$. It is easy to check that $H \sqsubseteq J$ and $J \sqsubseteq H$, but there is $\theta \in H$ such that $\theta \notin J$ and $\theta^T \notin J$.

However, in Proposition 16 we will show for which particular sets the anti-symmetry holds.

3.2. Generalization of G4 and G5

In order to study the inferential rules, we first prove a sort of monotonicity property for G4 and G5.

Proposition 6. *Let $\theta_1, \theta_2, \theta_3, \theta_4$ be triples such that $\theta_1 \sqsubseteq \theta_3, \theta_2 \sqsubseteq \theta_4$ and it is possible to apply the contraction rule to both pairs (θ_1, θ_2) and (θ_3, θ_4). If $\theta_1, \theta_2 \vdash_{G4} \theta$ and $\theta_3, \theta_4 \vdash_{G4} \theta'$, then $\theta \sqsubseteq \theta'$.*

Proof. Suppose that $\theta_1 \sqsubseteq_a \theta_3$ and $\theta_2 \sqsubseteq_a \theta_4$, so $A_1 \subseteq A_3; B_1 \subseteq B_3; C_3 \subseteq C_1 \subseteq X_3; A_2 \subseteq A_4; B_2 \subseteq B_4; C_4 \subseteq C_2 \subseteq X_4$.

If there exists a triple $\theta = (A, B, C)$ such that $\theta_1, \theta_2 \vdash_{G4} \theta$, then $A = A_1 = A_2, C_1 = (B_2 \cup C_2), (B_1 \cap B_2) = \phi, B = (B_1 \cup B_2)$ and $C = C_2$.

If there exists a triple $\theta' = (A', B', C')$ such that $\theta_3, \theta_4 \vdash_{G4} \theta'$, then $A' = A_3 = A_4, C_3 = (B_4 \cup C_4), (B_3 \cap B_4) = \phi, B' = (B_3 \cup B_4)$ and $C' = C_4$.

Then, it is simple to observe that $\theta \sqsubseteq_a \theta'$ being $A = A_1 \subseteq A_3 = A'; B = B_1 \cup B_2 \subseteq B_3 \cup B_4 = B'; C' = C_4 \subseteq C_2 = C; C = C_2 \subseteq X_4 \subseteq X_4 \cup B_3 = A_4 \cup (B_3 \cup B_4) \cup C_4 = (A' \cup B' \cup C')$.

Now, if $\theta_1 \sqsubseteq_a \theta_3^T$ and $\theta_2 \sqsubseteq_a \theta_4$ (analogously the cases $\theta_1 \sqsubseteq_a \theta_3$ and $\theta_2 \sqsubseteq_a \theta_4^T$, or $\theta_1^T \sqsubseteq_a \theta_3^T$ and $\theta_2^T \sqsubseteq_a \theta_4^T$) it is not possible to apply contraction either between θ_1 and θ_2 or between θ_3 and θ_4. □

Proposition 7. *Let $\theta_1, \theta_2, \theta_3, \theta_4$ be triples such that $\theta_1 \sqsubseteq \theta_3, \theta_2 \sqsubseteq \theta_4$ and it is possible to apply the intersection rule to both the pairs (θ_1, θ_2) and (θ_3, θ_4). If $\theta_1, \theta_2 \vdash_{G5} \theta$ and $\theta_3, \theta_4 \vdash_{G5} \theta'$, then $\theta \sqsubseteq \theta'$.*

Proof. Suppose that $\theta_1 \sqsubseteq_a \theta_3$ and $\theta_2 \sqsubseteq_a \theta_4$, so $A_1 \subseteq A_3; B_1 \subseteq B_3; C_3 \subseteq C_1 \subseteq X_3; A_2 \subseteq A_4; B_2 \subseteq B_4; C_4 \subseteq C_2 \subseteq X_4$.

If there exists a triple $\theta = (A, B, C)$ such that $\theta_1, \theta_2 \vdash_{G5} \theta$, then $A = A_1 = A_2, B_1 \cup C_1 = B_2 \cup C_2, C = C_1 \cap C_2, B = B_1 \cup B_2$.

If there exists a triple $\theta' = (A', B', C')$ such that $\theta_3, \theta_4 \vdash_{G5} \theta'$, then $A' = A_3 = A_4, B_3 \cup C_3 = B_4 \cup C_4, C' = C_3 \cap C_4, B' = B_3 \cup B_4$.

Then, it is simple to observe that $\theta \sqsubseteq_a \theta'$ being $A = A_1 \subseteq A_3 = A'; B = B_1 \cup B_2 \subseteq B_3 \cup B_4 = B'; C' = C_3 \cap C_4 \subseteq C_1 \cap C_2 = C; C = C_1 \cap C_2 \subseteq X_3 \cap X_4 = A_3 \cup (B_3 \cap B_4) \cup (B_3 \cap C_4) \cup (C_3 \cap B_4) \cup (C_3 \cap C_4) = (A' \cup B' \cup C')$.

Now, if $\theta_1 \sqsubseteq_a \theta_3^T$ and $\theta_2 \sqsubseteq_a \theta_4$ (analogously the case $\theta_1 \sqsubseteq_a \theta_3$ and $\theta_2 \sqsubseteq_a \theta_4^T$, or $\theta_1^T \sqsubseteq_a \theta_3^T$ and $\theta_2^T \sqsubseteq_a \theta_4^T$) it is not possible to apply intersection either between θ_1 and θ_2 or between θ_3 and θ_4. □

3.3. Closure through two generalized rules

Now, our target, as that in [27], is to find a fast method to compute a reduced (with respect to g-inclusion $\sqsubseteq$) set J^* included in $\bar{J}$ and having the same information of $\bar{J}$; this means that for any triple $\theta \in \bar{J}$ there exists a triple $\theta' \in J^*$ such that $\theta \sqsubseteq \theta'$.

Therefore, the computation of J^* provides a solution to the implication problem for J. The strategy to compute J^* is to use a generalized version of the remaining graphoid rules G4, G5 and their symmetric ones (see also [28]).

Given $\theta_1, \theta_2 \in S^{(3)}$, let

$$W_C(\theta_1, \theta_2) = \{\tau : \theta'_1, \theta'_2 \vdash_{G4} \tau, \text{ with } \theta'_1 \sqsubseteq_a \theta_1, \theta'_2 \sqsubseteq_a \theta_2\}.$$

Proposition 8. *Let $\theta_1 = (A_1, B_1, C_1), \theta_2 = (A_2, B_2, C_2)$ be a pair of triples belonging to $S^{(3)}$, then*

(1) $W_C(\theta_1, \theta_2)$ is not empty if and only if all the following five conditions hold:
  - *(a) $(A_1 \cap A_2) \neq \phi$;*
  - *(b) $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$;*
  - *(c) $(B_1 \setminus C_2) \neq \phi$;*
  - *(d) $B_2 \cap X_1 \neq \phi$;*
  - *(e) $|(B_1 \setminus C_2) \cup (B_2 \cap X_1)| \geqslant 2$.*

(2) If $W_C(\theta_1, \theta_2)$ is not empty then

$$gc(\theta_1, \theta_2) = (A_1 \cap A_2, (B_1 \setminus C_2) \cup (B_2 \cap X_1), C_2 \cup (A_2 \cap C_1))$$

is in $W_C(\theta_1, \theta_2)$ and dominates any triple belonging to $W_C(\theta_1, \theta_2)$.

Proof. If $W_C(\theta_1,\theta_2) \neq \phi$ then for any $\tau = (A,B,C) \in W_C(\theta_1,\theta_2)$ there exist $\theta'_1 = (A'_1,B'_1,C'_1) \sqsubseteq_a \theta_1$ and $\theta'_2 = (A'_2,B'_2,C'_2) \sqsubseteq_a \theta_2$ such that $\theta'_1,\theta'_2 \vdash_{G4} \tau$. Therefore, the following conditions hold:

- $A'_1 \subseteq A_1, A'_2 \subseteq A_2$, $A'_1 = A'_2$, then $A_1 \cap A_2 \neq \phi$.
- $C'_1 = B'_2 \cup C'_2, C_1 \subseteq C'_1 \subseteq X_1, C_2 \subseteq C'_2 \subseteq X_2$ and $\phi \neq B'_2 \subseteq B_2$. This implies $C_1 \subseteq C'_1 = B'_2 \cup C'_2 \subseteq X_2$, so $C_1 \subseteq X_2$. From $C'_2 \subseteq C'_1$ it follows $C_2 \subseteq X_1$.
- $B'_2 \subseteq C'_1 \subseteq X_1$, $B'_2 \subseteq B_2$, so $B'_2 \subseteq B_2 \cap X_1$ and then $B_2 \cap X_1 \neq \phi$.
- $B'_1 \cap C'_1 = \phi$, $C'_1 = B'_2 \cup C'_2, B'_1 \cap C'_2 = \phi, \phi \neq B'_1 \subseteq B_1$ and $C_2 \subseteq C'_2$, then it follows $B'_1 \subseteq B_1 \setminus C_2$ and hence $B_1 \setminus C_2 \neq \phi$.
- Moreover, from $B'_1 \cap B'_2 = \phi$, $B'_1 \subseteq B_1 \setminus C_2$ and $B'_2 \subseteq X_1$ it follows $|(B_1 \setminus C_2) \cup (B_2 \cap X_1)| \geqslant 2$. In fact, $B'_1 \neq \phi$ and $B'_2 \neq \phi$ so $(B_1 \setminus C_2) \cup (B_2 \cap X_1)$ contains at least two elements (otherwise there are no two disjoint subsets).

Suppose that the conditions *(a)–(e)* hold, it is possible to find two disjoint nonempty sets B^1 and B^2 such that $B^1 \subseteq B_1 \setminus C_2, B^2 \subseteq B_2 \cap X_1$ and $B^1 \cup B^2 = (B_1 \setminus C_2) \cup (B_2 \cap X_1)$. Let $C^2 = C_2 \cup (C_1 \cap A_2)$, the triples $\theta_a = (A_1 \cap A_2, B^1, B^2 \cup C^2)$ and $\theta_b = (A_1 \cap A_2, B^2, C^2)$ are such $\theta_a \sqsubseteq_a \theta_1, \theta_b \sqsubseteq_a \theta_2$ and $\theta_a, \theta_b \vdash_{G4} gc(\theta_1,\theta_2) = (A_{gc}, B_{gc}, C_{gc})$. This result implies that $W_C(\theta_1,\theta_2)$ is not empty and $gc(\theta_1,\theta_2) \in W_C(\theta_1,\theta_2)$.

Now, it is simple to prove that $\tau \sqsubseteq_a gc(\theta_1,\theta_2)$. In fact it is straightforward to show that $A \subseteq A_{gc}$ and $B \subseteq B_{gc}$. Since $C_2 \subseteq C'_2 = C$ and $(A_2 \cap C_1) \subseteq C'_2$ then $C_{gc} = C_2 \cup (A_2 \cap C_1) \subseteq C$. On the other hand, since $C'_2 \subseteq C'_1 \subseteq X_1$ and $C'_2 \subseteq X_2$ then $C \subseteq (X_1 \cap X_2)$ which is a subset of $A_{gc} \cup B_{gc} \cup C_{gc}$. $\square$

When $W_C(\theta_1,\theta_2)$ is empty, we set $gc(\theta_1,\theta_2) = \perp$. The function $gc(\cdot,\cdot)$ has already been introduced in [28] in an essentially equivalent form. The conditions *(a)–(e)*, which assure that $W_C(\theta_1,\theta_2)$ is not empty, are however stronger than those given in [28]: in fact, we are looking for the triple dominating all the triples obtained through G4 by θ_1 and θ_2 or by some their dominated triples. This is clarified in the next example.

Example 2. Consider the triples $\theta_1 = (\{1,4\},\{2\},\{3\})$ and $\theta_2 = (\{1,3\},\{2\},\{4\})$. The condition (e) fails, since $(B_1 \setminus C_2) = (B_2 \cap X_1)$ and it contains just the element 2. Then, in this case $W_C(\theta_1,\theta_2) = \phi$, however it could be noted that by applying G3 to one of the two triples we get $\theta = (\{1\},\{2\},\{3,4\}) \sqsubseteq_a \theta_i$ (for $i = 1,2$) and so θ adds no further information.

We denote with $GC(\theta_1,\theta_2)$ the set formed by the possible (i.e. belonging to $S^{(3)}$) triples among $gc(\theta_1,\theta_2), gc(\theta_1,\theta_2^T), gc(\theta_1^T,\theta_2)$ and $gc(\theta_1^T,\theta_2^T)$. Obviously, $GC(\theta_1,\theta_2)$ is in general different from $GC(\theta_2,\theta_1)$.

Remark 1. Given a pair $\theta_1 = (A_1,B_1,C_1)$ and $\theta_2 = (A_2,B_2,C_2)$ such that $\theta_1,\theta_2 \vdash_{G4} \tau = (A,B,C)$, then $\tau = gc(\theta_1,\theta_2)$. In fact, $A = A_1 = A_2 = A_1 \cap A_2$, $B = B_1 \cup B_2, C = C_2, C_1 = B_2 \cup C_2$. From $A_2 \cap B_2 = A_2 \cap C_2 = \phi$ it follows that $A_2 \cap C_1 = \phi$, so $C = C_2 \cup (A_2 \cap C_1)$. Furthermore, from $B_2 \subseteq C_1 \subseteq X_1$ it follows that $B_2 = B_2 \cap X_1$ and from $B_1 \cap C_2 \subseteq B \cap C = \phi$ it follows that $B_1 = B_1 \setminus C_2$, so $B = (B_2 \cap X_1) \cup (B_1 \setminus C_2)$.

Now, we provide a result similar to Proposition 8 by considering the set

$$W_I(\theta_1,\theta_2) = \{\tau : \theta'_1,\theta'_2 \vdash_{G5} \tau, \text{with} \theta'_1 \sqsubseteq_a \theta_1, \theta'_2 \sqsubseteq_a \theta_2\}$$

Proposition 9. *Let $\theta_1 = (A_1,B_1,C_1), \theta_2 = (A_2,B_2,C_2)$ be a pair of triples belonging to $S^{(3)}$, then*

(1) $W_I(\theta_1,\theta_2)$ is not empty if and only if all the following five conditions hold:
  - *(a) $A_1 \cap A_2 \neq \phi$;*
  - *(b) $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$;*
  - *(c) $B_1 \cap X_2 \neq \phi$;*
  - *(d) $B_2 \cap X_1 \neq \phi$;*
  - *(e) $|(B_1 \cap X_2) \cup (B_2 \cap X_1)| \geqslant 2$.*

(2) If $W_I(\theta_1,\theta_2)$ is not empty, then

$$gi(\theta_1,\theta_2) = (A_1 \cap A_2, (B_1 \cap X_2) \cup (B_2 \cap X_1), (C_1 \cap A_2) \cup (C_2 \cap A_1) \cup (C_2 \cap C_1))$$

is in $W_I(\theta_1,\theta_2)$ and dominates any triple belonging to $W_I(\theta_1,\theta_2)$.

Proof. If $W_I(\theta_1,\theta_2) \neq \phi$ then for any $\tau = (A,B,C) \in W_I(\theta_1,\theta_2)$ there exist $\theta'_1 = (A'_1,B'_1,C'_1) \sqsubseteq_a \theta_1$ and $\theta'_2 = (A'_2,B'_2,C'_2) \sqsubseteq_a \theta_2$ such that $\theta'_1,\theta'_2 \vdash_{G5} \tau$. Then, the following conditions hold:

- From $A'_1 \subseteq A_1, A'_2 \subseteq A_2, A'_1 = A'_2$, it follows $(A_1 \cap A_2) \neq \phi$.
- From $B'_1 \cup C'_1 = B'_2 \cup C'_2$ it follows that $C'_1 \subseteq B'_2 \cup C'_2$. Therefore, $C_1 \subseteq C'_1 \subseteq B'_2 \cup C'_2 \subseteq X_2$.From $B'_1 \cup C'_1 = B'_2 \cup C'_2$ it also follows that $C'_2 \subseteq B'_1 \cup C'_1$. Therefore, $C_2 \subseteq C'_2 \subseteq B'_1 \cup C'_1 \subseteq X_1$.
- From $B'_1 \subseteq C'_2 \subseteq X_2$ and $B'_1 \subseteq B_1$, it follows $B'_1 \subseteq B_1 \cap X_2 \neq \phi$.
- From $B'_2 \subseteq C'_1 \subseteq X_1$ and $B'_2 \subseteq B_2$, it follows $B'_2 \subseteq B_2 \cap X_1 \neq \phi$.
- Moreover, from $B'_1 \cap B'_2 = \phi$, $B'_1 \subseteq B_1, B'_2 \subseteq B_2, B_1 \cap X_2 \neq \phi$ and $B_2 \cap X_1 \neq \phi$, it follows $|(B_1 \cap X_2) \cup (B_2 \cap X_1)| \geqslant 2$.

Suppose the conditions *(a)–(e)* hold, it is possible to find two disjoint nonempty set B^1 and B^2 such that $B^1 \subseteq B_1 \cap X_2, B^2 \subseteq B_2 \cap X_1$ and $B^1 \cup B^2 = (B_1 \cap X_2) \cup (B_2 \cap X_1)$. Let $C^1 = (C_1 \setminus B_2) \cup (C_2 \setminus B_1) \cup B^2$ and $C^2 = (C_1 \setminus B_2) \cup (C_2 \setminus B_1) \cup B^1$, then the triples $\theta_a = (A_1 \cap A_2, B^1, C^1)$ and $\theta_b = (A_1 \cap A_2, B^2, C^2)$ are such $\theta_a \sqsubseteq_a \theta_1, \theta_b \sqsubseteq_a \theta_2$ and $\theta_a, \theta_b \vdash_{G5} gi(\theta_1, \theta_2) = (A_{gi}, B_{gi}, C_{gi})$. This result implies that $W_I(\theta_1, \theta_2)$ is not empty and $gi(\theta_1, \theta_2) \in W_I(\theta_1, \theta_2)$.

Now, it simple to prove that $\tau \sqsubseteq_a gi(\theta_1, \theta_2)$. In fact it is straightforward to show that $A \subseteq A_{gi}$ and $B \subseteq B_{gi}$. For showing that $C_{gi} \subseteq C$ note that $C_1 \subseteq C'_1, C_2 \subseteq C'_2, A_2 \cap C_1 \subseteq C'_2, A_1 \cap C_2 \subseteq C_1$.

On the other hand, since $C \subseteq C'_1 \subseteq X_1, C \subseteq C'_2 \subseteq X_2$ then $C \subseteq (X_1 \cap X_2) = (A_{gi} \cup B_{gi} \cup C_{gi})$. $\square$

Again, when $W_I(\theta_1, \theta_2)$ is empty, we set $gi(\theta_1, \theta_2) = \perp$. Also the function $gi(\cdot, \cdot)$ has already been introduced in [28] in an essentially equivalent form.

Given two triples θ_1, θ_2, Proposition 9 gives rise to the dominant triple generated through G5 by θ_1, θ_2 or by some dominated triples, respectively, by θ_1 and θ_2. The set $GI(\theta_1, \theta_2)$ is formed by the possible (i.e. belonging to $S^{(3)}$) triples among $gi(\theta_1, \theta_2), gi(\theta_1, \theta_2^T), gi(\theta_1^T, \theta_2)$ and $gi(\theta_1^T, \theta_2^T)$. Then, $GI(\theta_1, \theta_2) = GI(\theta_2, \theta_1)$. Also in this case, whether $\theta_1, \theta_2 \vdash_{G5} \tau$ then $\tau = gi(\theta_1, \theta_2)$.

The previous sets *GC* and *GI* are used to introduce two new inference rules

$G4^*$ "generalized contraction": from θ_1, θ_2 deduce any triple $\tau \in GC(\theta_1, \theta_2)$;
$G5^*$ "generalized intersection": from θ_1, θ_2 deduce any triple $\tau \in GI(\theta_1, \theta_2)$;

which, as explained above, generalize the two classical inference rules.

It is possible to compute the closure of a set J of triples in $S^{(3)}$, with respect to the generalized contraction $G4^*$ and generalized intersection $G5^*$, that is

$$J^* = \{\tau : J \vdash_G^* \tau\}, \tag{3}$$

where $J \vdash_G^* \tau$ means that τ is obtained by applying a finite number of times the rules $G4^*$ and $G5^*$.

We show the relationship between the two closures J^* and $\bar{J}$.

First we prove that if a triple can be deduced through $G4^*$ or $G5^*$, then it can be deduced by means of G1–G5.

Proposition 10. *Let J be a subset of $S^{(3)}$ and denote by J^* and $\bar{J}$ the closure, respectively, with respect to the generalized rules $G4^*$–$G5^*$ and the graphoid properties G1–G5. Then $J^* \subseteq \bar{J}$.*

Proof. The proof can be done by structural induction: we need to show that $\theta_1, \theta_2 \vdash_{G4^*} \tau$ implies $\{\theta_1, \theta_2\} \vdash^* \tau$ and $\theta_1, \theta_2 \vdash_{G5^*} \tau$ implies $\{\theta_1, \theta_2\} \vdash^* \tau$. From Proposition 8, if $W_C(\theta_1, \theta_2) \neq \phi$, then $gc(\theta_1, \theta_2) \in W_C(\theta_1, \theta_2)$ and dominates any triple belonging to $W_C(\theta_1, \theta_2)$. Hence, if $\theta_1, \theta_2 \vdash_{G4^*} \tau$, then $\tau = gc(\theta_1, \theta_2)$ and $\tau \in W_C(\theta_1, \theta_2)$. Being $\tau \in W_C(\theta_1, \theta_2)$, τ is obtained with G4 by some θ'_1 and θ'_2 with $\theta'_1 \sqsubseteq_a \theta_1, \theta'_2 \sqsubseteq_a \theta_2$. Finally, by definition of dominance $\sqsubseteq_a$, θ'_1 and θ'_2 are obtained through G2, G3, G2s, G3s, so $\{\theta_1, \theta_2\} \vdash^* \theta'_i$ for $i = 1, 2$. Obviously, this is true also for τ.

If $\theta_1, \theta_2 \vdash_{G5^*} \tau$, we get the same conclusion by Proposition 9. $\square$

Now, we prove that any triple obtained through G1–G5 is *g*-included in a triple deduced from $G4^*$ and $G5^*$.

Proposition 11. *Let J be a subset of $S^{(3)}$ and denote by J^* and $\bar{J}$ the closure, respectively, with respect to the generalized rules $G4^*$–$G5^*$ and the graphoid properties G1–G5. Then $\bar{J} \sqsubseteq J^*$.*

Proof. The proof is by induction. We can obtain, starting from $J_0 = J$,

$$\bar{J} = \bigcup_{i=0}^{\infty} J_i,$$

where $J_i = J_{i-1} \cup \{\tau : \tau \text{ is obtained by applying any graphoid property to } J_{i-1}\}$.

Since J is finite this iterative process ends when $J_k = J_{k+1}$, $k \in \mathbb{N}$ and $J_k = \bar{J}$. We show that $J_i \sqsubseteq J^*$. For $i = 0$ it is trivial. Suppose that $J_i \sqsubseteq J^*$ and let $\tau \in J_{i+1} \setminus J_i$.

If τ is obtained by means of G1, G2, G3 from $\theta \in J_i$, then $\tau \sqsubseteq \theta$ and, since $\theta \in J_i$, by hypothesis $\exists \bar{\theta} \in J^*$ such that $\theta \sqsubseteq \bar{\theta}$, so by transitivity $\tau \sqsubseteq \bar{\theta}$.

If $\theta_1, \theta_2 \vdash_{G4} \tau$ with $\theta_1, \theta_2 \in J_i$, then, by hypothesis, there exist $\bar{\theta}_1, \bar{\theta}_2 \in J^*$ such that $\theta_1 \sqsubseteq \bar{\theta}_1$ and $\theta_2 \sqsubseteq \bar{\theta}_2$, $\tau \in W_C(\bar{\theta}_1, \bar{\theta}_2)$ and, from Proposition 8, $\tau \sqsubseteq gc(\bar{\theta}_1, \bar{\theta}_2) \in J^*$.

The proof of the case $\theta_1, \theta_2 \vdash_{G5} \tau$ goes in the same line of the previous one and it is based on Proposition 9. $\square$

Note that J^* is a subset of $\bar{J}$, so even if J^* has the same information of $\bar{J}$, is smaller than $\bar{J}$. Actually, J^* contains some "redundant" triples, that means that are *g*-included in the other ones.

3.4. Algorithm with two generalized rules

Starting from a set $J \subseteq S^{(3)}$, in order to reduce as much as possible the cardinality of $\bar{J}$ without losing information, we define the "maximal" (with respect to *g*-inclusion) triple set

$$J_{/\sqsubseteq} = \{\tau \in J : \nexists \bar{\tau} \in J \text{ with } \bar{\tau} \neq \tau, \tau^T \text{ such that } \tau \sqsubseteq \bar{\tau}\}.$$

Obviously, $J_{/\sqsubseteq} \subseteq J$. Now we prove that there is no loss of information by using $J_{/\sqsubseteq}$ instead of J.

Lemma 12. *Let S be a finite set and $J \subseteq S^{(3)}$. Then, $J \sqsubseteq J_{/\sqsubseteq}$.*

Proof. Let $\theta \in J$, if $\nexists \bar{\theta} \in J$ such that $\theta \sqsubseteq \bar{\theta}, \theta \neq \bar{\theta}$ and $\theta \neq \bar{\theta}^T$, then $\theta \in J_{/\sqsubseteq}$. Otherwise, i.e. $\theta \in J \setminus J_{/\sqsubseteq}$, since J is finite, any chain

$$\theta_1 \sqsubseteq \theta_2 \sqsubseteq \cdots \sqsubseteq \theta_n \sqsubseteq \cdots$$

with $\theta_i \in J$ and $i \geqslant 1$, must have a maximal element θ_n, which necessarily belongs to $J_{/\sqsubseteq}$. □

Then, given a set J of triples in $S^{(3)}$, we compute J^* and then we cut redundant triples by taking its "maximal" triples, i.e. $J^*_{/\sqsubseteq}$. We call the set $J^*_{/\sqsubseteq}$ "fast closure" and we denote it, for simplicity, with J_*.
The proof of the following relationships is trivial.

Proposition 13. *Given a subset J of $S^{(3)}$, then $J_* \subseteq \bar{J}$ and $\bar{J} \sqsubseteq J_*$.*

It is interesting to show that $\bar{J}_{/\sqsubseteq}$ and J_* essentially coincide, as shown in the following result:

Proposition 14. *Given a subset J of $S^{(3)}$, then $\bar{J}_{/\sqsubseteq} \sqsubseteq J_*$ and $J_* \sqsubseteq \bar{J}_{/\sqsubseteq}$.*

Proof. By Proposition 13 it follows that $\bar{J}_{/\sqsubseteq} \subseteq \bar{J} \sqsubseteq J_*; J_* = J^*_{/\sqsubseteq} \subseteq J^* \subseteq \bar{J} \sqsubseteq \bar{J}_{/\sqsubseteq}$. □

Now, we propose a procedure to compute the closure and we show some properties, but before we introduce the notion of "maximal" set, that will be useful for proving completeness and correctness of the procedure.

Definition 15. A subset J of $S^{(3)}$ is said maximal if $J = J_{/\sqsubseteq}$.

The next result shows that g-inclusion on maximal subsets of $S^{(3)}$ satisfies weak anti-symmetric property.

Proposition 16. *Let J, H be two maximal sets of $S^{(3)}$, then $H \sqsubseteq J$ and $J \sqsubseteq H$ if and only if for any $\theta \in H$ either $\theta \in J$ or $\theta^T \in J$ and for any $\tau \in J$ either $\tau \in H$ or $\tau^T \in H$.*

Proof. Consider the case $H \sqsubseteq J$ and $J \sqsubseteq H$. From $H \sqsubseteq J$ for any $\theta \in H$ there exists $\tau \in J$ such that $\theta \sqsubseteq \tau$. From $J \sqsubseteq H$ there exists $\sigma \in H$ such that $\tau \sqsubseteq \sigma$. By transitivity we have $\theta \sqsubseteq \sigma$, but, being H maximal set, $\theta = \sigma$, so $\tau \sqsubseteq \theta$ and $\theta \sqsubseteq \tau$ imply that $\theta = \sigma$ or $\theta = \sigma^T$.
The vice versa, it is trivial by Definition 4. □

The following algorithm computes J_* by avoiding to build before J^*,[1]

Algorithm 1 Fast closure by $G4^*$ and $G5^*$

function FC2 J
  $J_0 \leftarrow J$
  $N_0 \leftarrow J$
  $k \leftarrow 0$
  repeat
    $k \leftarrow k + 1$
    $N_k := \bigcup_{\theta_1 \in J_{k-1}, \theta_2 \in N_{k-1}} \{GC(\theta_1, \theta_2) \cup GC(\theta_2, \theta_1) \cup GI(\theta_1, \theta_2)\}$
    $J_k \leftarrow FindMaximal\ (J_{k-1} \cup N_k)$
  until $J_k = J_{k-1}$
  return J_k
end function

where FindMaximal is the function which computes $J_{/\sqsubseteq}$ for a given set $J \subseteq S^{(3)}$.
The completeness and correctness of the above procedure is provided in the following result.

Theorem 17. *For any subset J of $S^{(3)}$, then*

(1) $FC2(J) \sqsubseteq J_*$;
(2) $J_* \sqsubseteq FC2(J)$.

[1] A program which computes the closure with respect to semi-graphoids (through $G4^*$, G1 and by using dominance relation), based on the results in the papers of Milan Studený [27,28], has been implemented by Kumicak at Charles University (2004).

Proof. First we give a proof of $J^* \sqsubseteq \mathrm{FC2}(J)$, since $J_* \sqsubseteq J^*$, (2) holds. To compute J^* we will use the following recursive schema $J'_0 = J$

$$J'_k = J'_{k-1} \cup \{\tau : \tau = GC(\theta_1, \theta_2) \text{ or } \tau = GC(\theta_2, \theta_1) \text{ or } \tau = GI(\theta_1, \theta_2) \text{ with } \theta_1, \theta_2 \in J'_{k-1}\}.$$

Let also $N'_0 = J$ and $N'_k = J'_k \setminus J'_{k-1}$.

It is possible to prove that, for any $h \in \mathbb{N}, N'_h \sqsubseteq N_h$ and $J'_h \sqsubseteq J_h$ by induction on h. For $h = 0$ it is trivial. If it is true for h, consider $\tau' \in N'_{h+1}$, for example $\tau' = gc(\theta'_1, \theta'_2)$ with $\theta'_1, \theta'_2 \in J'_h$. It is easy to see that either $\theta'_1 \in N'_h$ or $\theta'_2 \in N'_h$, otherwise τ' would be already present in J'_h.

By inductive hypothesis, there exists $\theta_1 \in J_h, \theta_2 \in N_h$ such that $\theta'_1 \sqsubseteq \theta_1$ and $\theta'_2 \sqsubseteq \theta_2$. Therefore, $\tau \in GC(\theta_1, \theta_2) \subseteq N_{h+1}$ with $\tau' \sqsubseteq \tau$.

Along the same line it is possible to prove the other cases.

Now, for any $\tau' \in J'_{h+1}$ one has the following cases:

- $\tau' \in N'_{h+1}$, then there exists $\tau \in N_{h+1}$ such that $\tau' \sqsubseteq \tau$. Now, if $\tau \in J_{h+1}$ then the claim is true. Otherwise, there exists $\bar{\tau} \in J_{h+1}$ such that $\tau \sqsubseteq \bar{\tau}$, so by transitivity $\tau' \sqsubseteq \bar{\tau}$.
- $\tau' \notin N_{h+1}$, then $\tau' \in J'_h$ and by hypothesis there exists $\tau \in J_h$ such that $\tau' \sqsubseteq \tau$.

Since J is finite, then there exists $n \in \mathbb{N}$ such that $J'_n = J'_{n-1} = J^*$ and $J_n = J_{n-1} = FC2(J)$, then $J^* \sqsubseteq \mathrm{FC2}(J)$.

Furthermore, by observing that $J_n \subseteq \bar{J}$ by Proposition 10, $\bar{J} \sqsubseteq J^*$ by Proposition 11 and $J^* \sqsubseteq J_*$ by Lemma 12 it follows that $\mathrm{FC2}(J) \sqsubseteq J_*$. □

Note that FC2 is based on g-inclusion (instead of dominance), so it does not require to apply G1, but just $G4^*$ and $G5^*$.

It is easy to see that the function FC2, so called because it uses two inference rules, terminates after a finite number of steps for each possible set $J \subseteq S^{(3)}$, because of the finiteness of $S^{(3)}$.

4. A unique inference rule

In Section 3, we have described a procedure to compute efficiently the closure of a set of conditional independence statements. This procedure arises essentially from the generalized contraction and intersection rules. Now, in order to improve such procedure, we look for a unique inferential rule with the aim of simplifying the procedure. By taking into account Propositions 8 and 9, that provide necessary and sufficient conditions for applying generalized contraction and intersection, the notion of almost complete pair of triples is introduced in order to characterize the dominant triples arising from generalized rules.

Definition 18. Let $\theta_1 = (E_{(1,1)}, E_{(1,2)}, E_{(1,3)}), \theta_2 = (E_{(2,1)}, E_{(2,2)}, E_{(2,3)})$ be in to $S^{(3)}$, then (θ_1, θ_2) is said almost complete pair of triples if

- $(E_{(1,i)} \cap E_{(2,j)}) \geqslant 1$ for $i, j = 1, 2, 3$;
- $(E_{(1,i)} \setminus X_2) \neq \phi, (E_{(2,i)} \setminus X_1) \neq \phi$ for $i = 1, 2$.

From Propositions 8 and 9, condition (b) (i.e., $E_{(1,3)} \subseteq X_2$ and $E_{(2,3)} \subseteq X_1$) is necessary for applying $G4^*$ and $G5^*$ to $\theta_1 = (E_{(1,1)}, E_{(1,2)}, E_{(1,3)})$ and $\theta_2 = (E_{(2,1)}, E_{(2,2)}, E_{(2,3)})$. Such condition can be reformulated, according to the above notation, as $E_{(i,3)} \subseteq X_{3-i}$ with $i = 1, 2$. Note that Definition 18 does not require the above condition.

First, we introduce for $\theta_1 = (E_{(1,1)}, E_{(1,2)}, E_{(1,3)})$ and $\theta_2 = (E_{(2,1)}, E_{(2,2)}, E_{(2,3)})$ such that $E_{(i,3)} \subseteq X_{3-i}$ $(i = 1, 2)$, the following functions.

Definition 19. If $(E_{(i,j)} \cap E_{(3-i,k)}) \neq \phi$, let

$$\hat{\theta}_{(i,j,k)}(\theta_1, \theta_2) = \left(E_{(i,j)} \cap E_{(3-i,k)}, E_{(i,3-j)} \cup (E_{(3-i,3-k)} \cap X_i), C\right)$$

with $C = (E_{(1,3)} \cap E_{(2,3)}) \cup (E_{(i,3)} \cap E_{(3-i,k)}) \cup (E_{(3-i,3)} \cap E_{(i,j)})$, otherwise $\hat{\theta}_{(i,j,k)}(\theta_1, \theta_2) = \perp$.

Definition 20. If for any $i \in \{1, 2\}$ $(E_{(1,1)} \cap E_{(2,i)}) \neq \phi, (E_{(1,2)} \cap E_{(2,3-i)}) \neq \phi$ and at least one of these sets $((E_{(1,1)} \cap E_{(2,3-i)})$ or $(E_{(1,2)} \cap E_{(2,i)}))$ is not empty let $v(\theta_1, \theta_2)$ be

$$\left((E_{(1,1)} \cap E_{(2,2)}) \cup (E_{(1,2)} \cap E_{(2,1)}), (E_{(1,1)} \cap E_{(2,1)}) \cup (E_{(1,2)} \cap E_{(2,2)}), E_{(1,3)} \cup E_{(2,3)}\right)$$

otherwise $v(\theta_1, \theta_2) = \perp$.

For example, given $\theta_1 = (E_{(1,1)}, E_{(1,2)}, E_{(1,3)}), \theta_2 = (E_{(2,1)}, E_{(2,2)}, E_{(2,3)})$, we have that $\hat{\theta}_{(1,1,1)}(\theta_1, \theta_2)$ is $(E_{(1,1)} \cap E_{(2,1)}, E_{(1,2)} \cup (E_{(2,2)} \cap X_1), (E_{(1,3)} \cap E_{(2,3)}) \cup (E_{(1,3)} \cap E_{(2,1)}) \cup (E_{(2,3)} \cap E_{(1,1)}))$.

Given θ_1, θ_2, we denote by $K(\theta_1, \theta_2)$ the set

$$K(\theta_1, \theta_2) = \{\theta_1, \theta_2, v(\theta_1, \theta_2), \hat{\theta}_{(i,j,k)}(\theta_1, \theta_2) : i, j, k \in \{1, 2\}\}. \tag{4}$$

Some properties of the set $K(\theta_1,\theta_2)$ are shown in the following.

Theorem 21. *Let θ_1,θ_2 be an almost complete pair of triples of $S^{(3)}$ such that $E_{(i,3)} \subseteq X_{3-i}$ for $i = 1,2$. Then, $K(\theta_1,\theta_2)$ is closed with respect to $G4^*$ and $G5^*$ and*

$$\{\theta_1,\theta_2\}_* \sqsubseteq K(\theta_1,\theta_2).$$

Proof. First of all we prove the closure of $K(\theta_1,\theta_2)$ with respect to $G4^*$ and $G5^*$. Since (θ_1,θ_2) is an almost complete pair, it is easy to see that $K(\theta_1,\theta_2)$ is composed by 11 elements, because all the $\hat{\theta}_{(i,j,k)}$'s, as well as v, are not $\perp$. With simple (but tedious) computations it is possible to see that applying in all the possible ways $G4^*$ and $G5^*$ to $\hat{\theta}_{(1,1,1)}$ and the elements of $K(\theta_1,\theta_2)$ we obtain the results shown in the Table 1. The condition 1 of Definition 18 assures that it is possible to apply the generalized rules as much as possible, and then the number of null entries in Table 1 is minimized.

For each entry in the table, it is possible to see that there are only three possibilities:

(1) the entry is $\perp$, because the rule cannot be applied;
(2) the entry corresponds to an element of $K(\theta_1,\theta_2)$;
(3) the entry is g-included into an element of $K(\theta_1,\theta_2)$.

Thus the application of generalized rules to $\hat{\theta}_{(1,1,1)}$ does not produce any further non redundant triple.

The results of the application of generalized rules to the others $\hat{\theta}_{(i,j,k)}$ are similar to those shown in Table 1 and are not reported here.

At the same way, Tables 2 and 3 show that the application of generalized rules to v and, respectively, to θ_1 (as well as θ_2, whose corresponding results table is not reported) produces only triples g-included in $K(\theta_1,\theta_2)$ (sometimes member of $K(\theta_1,\theta_2)$).

Now, we prove that $K(\theta_1,\theta_2) \sqsubseteq \{\theta_1,\theta_2\}_*$. Since θ_1,θ_2 form an almost complete pair of triples, it is easy to check that conditions *(a)–(e)* of Propositions 8 and 9 hold for any pair $\theta_i^{(j)}, \theta_{3-i}^{(k)}$, with $i,j,k \in \{1,2\}$ and where

$$\theta_i^{(j)} = \begin{cases} \theta_i & \text{for } j = 1, \\ \theta_i^T & \text{for } j = 2. \end{cases}$$

Therefore, $gc(\theta_i^{(j)}, \theta_{3-i}^{(k)}) \neq \perp$ and $gi(\theta_i^{(j)}, \theta_{3-i}^{(k)}) \neq \perp$. On the other hand, it is easy to check that conditions *(a)–(e)* of Proposition 8 are satisfied by the pair $gc(\theta_i^{(j)}, \theta_{3-i}^{(k)}), gi(\theta_i^{(j)}, \theta_{3-i}^{(k)})$ and

$$\hat{\theta}_{(i,j,k)}(\theta_i^{(j)}, \theta_{3-i}^{(k)}) = gc(gc(\theta_i^{(j)}, \theta_{3-i}^{(k)}), gi(\theta_i^{(j)}, \theta_{3-i}^{(k)})).$$

Finally, the conditions *(a)–(e)* of Proposition 9 hold for the pair of triples $(\hat{\theta}^T_{(i,j,k)}(\theta_1,\theta_2), \hat{\theta}^T_{(l,3-j,3-k)}(\theta_1,\theta_2))$ and it is easy to check that

$$v(\theta_1,\theta_2) = gi(\hat{\theta}^T_{(i,j,k)}(\theta_1,\theta_2), \hat{\theta}^T_{(l,3-j,3-k)}(\theta_1,\theta_2)).$$

Since all the elements of $K(\theta_1,\theta_2)$ are obtained through $G4^*$ and $G5^*$, then its elements must be g-included into $\{\theta_1,\theta_2\}_*$. □

Actually, Theorem 21 implies that for an almost complete pair θ_1,θ_2 of triples the set $K(\theta_1,\theta_2)$ "coincides" with $\{\theta_1,\theta_2\}_*$, being also a maximal set.

Table 1
$\hat{\theta}_{(1,1,1)}$ versus all the elements in $K(\theta_1,\theta_2)$.

	$\hat{\theta}_{(1,1,1)}$	$\hat{\theta}_{(1,1,2)}$	$\hat{\theta}_{(1,2,1)}$	$\hat{\theta}_{(1,2,2)}$	$\hat{\theta}_{(2,1,1)}$	$\hat{\theta}_{(2,1,2)}$	$\hat{\theta}_{(2,2,1)}$	$\hat{\theta}_{(2,2,2)}$	θ_1	θ_2	v
$gc(\cdot,\hat{\theta}_{(1,1,1)})$	$=\hat{\theta}_{(1,1,1)}$	$=\perp$	$=\perp$	$=\perp$	$=\hat{\theta}_{(2,1,1)}$	$=\perp$	$=\perp$	$=\perp$	$=\hat{\theta}_{(1,1,1)}$	$=\hat{\theta}_{(2,1,1)}$	$=\perp$
$gc(\cdot^T,\hat{\theta}_{(1,1,1)})$	$=\perp$	$=\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a \hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$
$gc(\cdot,\hat{\theta}^T_{(1,1,1)})$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,2)}$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(2,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(2,2,1)}$	$\sqsubseteq_a\hat{\theta}_{(2,2,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(2,1,2)}$	$=v$
$gc(\cdot^T,\hat{\theta}^T_{(1,1,1)})$	$=\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\theta_1$	$\sqsubseteq_a\theta_2$	$=v$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\theta_2$	$\sqsubseteq_a\theta_1$	$=v$	$\sqsubseteq_a\theta_1$	$\sqsubseteq_a\theta_2$	$\sqsubseteq_a\hat{\theta}_{(1,2,2)}$
$gc(\hat{\theta}_{(1,1,1)},\cdot)$	$=\hat{\theta}_{(1,1,1)}$	$=\perp$	$=\perp$	$=\perp$	$=\hat{\theta}_{(1,1,1)}$	$=\perp$	$=\perp$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$
$gc(\hat{\theta}_{(1,1,1)},\cdot^T)$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$
$gc(\hat{\theta}^T_{(1,1,1)},\cdot)$	$=\perp$	$=\hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,2)}$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,2,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,1)}$	$=v$
$gc(\hat{\theta}^T_{(1,1,1)},\cdot^T)$	$=\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\theta_1$	$\sqsubseteq_a\theta_2$	$=v$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\theta_2$	$\sqsubseteq_a\theta_1$	$=v$	$\sqsubseteq_a\theta_1$	$\sqsubseteq_a\theta_2$	$\sqsubseteq_a\hat{\theta}_{(1,2,2)}$
$gi(\cdot,\hat{\theta}_{(1,1,1)})$	$=\hat{\theta}_{(1,1,1)}$	$=\perp$	$=\perp$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$	$=\perp$	$=\perp$	$=\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$
$gi(\cdot^T,\hat{\theta}_{(1,1,1)})$	$=\perp$	$=\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$=\perp$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$
$gi(\cdot,\hat{\theta}^T_{(1,1,1)})$	$=\perp$	$=\hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,2)}$	$=\perp$	$\sqsubseteq_a\hat{\theta}_{(1,2,1)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a\hat{\theta}_{(1,2,1)}$	$=v$
$gi(\cdot^T,\hat{\theta}^T_{(1,1,1)})$	$=\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\theta_1$	$\sqsubseteq_a\theta_2$	$=v$	$\sqsubseteq_a\hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a\theta_2$	$\sqsubseteq_a\theta_1$	$=v$	$\sqsubseteq_a\theta_1$	$\sqsubseteq_a\theta_2$	$\sqsubseteq_a\hat{\theta}_{(1,2,2)}$

Table 2
v versus θ_1, θ_2, v.

	θ_1	θ_2	v
$gc(\cdot, v)$	$\sqsubseteq_a \hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a \hat{\theta}_{(2,1,2)}$	$= v$
$gc(\cdot^T, v)$	$\sqsubseteq_a \hat{\theta}_{(1,2,1)}$	$\sqsubseteq_a \hat{\theta}_{(2,2,1)}$	$= \bot$
$gc(\cdot, v^T)$	$\sqsubseteq_a \hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a \hat{\theta}_{(2,1,1)}$	$= \bot$
$gc(\cdot^T, v^T)$	$\sqsubseteq_a \hat{\theta}_{(1,2,2)}$	$\sqsubseteq_a \hat{\theta}_{(2,2,2)}$	$= v$
$gc(v, \cdot)$	$\sqsubseteq_a \hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a \hat{\theta}_{(1,2,1)}$	$= v$
$gc(v, \cdot^T)$	$\sqsubseteq_a \hat{\theta}_{(1,2,1)}$	$\sqsubseteq_a \hat{\theta}_{(1,1,2)}$	$= \bot$
$gc(v^T, \cdot)$	$\sqsubseteq_a \hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a \hat{\theta}_{(1,1,1)}$	$= \bot$
$gc(v^T, \cdot^T)$	$\sqsubseteq_a \hat{\theta}_{(1,2,2)}$	$\sqsubseteq_a \hat{\theta}_{(1,2,2)}$	$= v$
$gi(\cdot, v)$	$\sqsubseteq_a \hat{\theta}_{(1,1,2)}$	$\sqsubseteq_a \hat{\theta}_{(1,2,1)}$	$= v$
$gi(\cdot^T, v)$	$\sqsubseteq_a \hat{\theta}_{(1,2,1)}$	$\sqsubseteq_a \hat{\theta}_{(1,1,2)}$	$= \bot$
$gi(\cdot, v^T)$	$\sqsubseteq_a \hat{\theta}_{(1,1,1)}$	$\sqsubseteq_a \hat{\theta}_{(1,1,1)}$	$= \bot$
$gi(\cdot^T, v^T)$	$\sqsubseteq_a \hat{\theta}_{(1,2,2)}$	$\sqsubseteq_a \hat{\theta}_{(1,2,2)}$	$= v$

Table 3
θ_1 versus θ_1, θ_2.

	θ_1	θ_2
$gc(\cdot, \theta_1)$	$\sqsubseteq_a \theta_1$	$\sqsubseteq_a \hat{\theta}_{(2,1,1)}$
$gc(\cdot^T, \theta_1)$	$= \bot$	$\sqsubseteq_a \hat{\theta}_{(2,2,1)}$
$gc(\cdot, \theta_1^T)$	$= \bot$	$\sqsubseteq_a \hat{\theta}_{(2,1,2)}$
$gc(\cdot^T, \theta_1^T)$	$\sqsubseteq_a \theta_1$	$\sqsubseteq_a \hat{\theta}_{(2,2,2)}$
$gc(\theta_1, \cdot)$	$\sqsubseteq_a \theta_1$	$\sqsubseteq_a \hat{\theta}_{(1,1,1)}$
$gc(\theta_1, \cdot^T)$	$= \bot$	$\sqsubseteq_a \hat{\theta}_{(1,1,2)}$
$gc(\theta_1^T, \cdot)$	$= \bot$	$\sqsubseteq_a \hat{\theta}_{(1,2,1)}$
$gc(\theta_1^T, \cdot^T)$	$\sqsubseteq_a \theta_1$	$\sqsubseteq_a \hat{\theta}_{(1,2,2)}$
$gi(\cdot, \theta_1)$	$\sqsubseteq_a \theta_1$	$\sqsubseteq_a \hat{\theta}_{(1,1,1)}$
$gi(\cdot^T, \theta_1)$	$= \bot$	$\sqsubseteq_a \hat{\theta}_{(1,1,2)}$
$gi(\cdot, \theta_1^T)$	$= \bot$	$\sqsubseteq_a \hat{\theta}_{(1,2,1)}$
$gi(\cdot^T, \theta_1^T)$	$\sqsubseteq_a \theta_1$	$\sqsubseteq_a \hat{\theta}_{(1,2,2)}$

Now the aim is to extend the previous result to any pair of triples θ_1, θ_2, using for the set $K(\theta_1, \theta_2)$ the same definition as for almost complete pair, without including the undefined triples. To achieve this goal we need to give some preliminary results based on the following notion of projection of a triple.

Definition 22. Given $\theta = (A, B, C)$ and $Y \subseteq (A \cup B \cup C)$, if $(A \cap Y) \neq \phi$ and $(B \cap Y) \neq \phi$, then

$$\pi_Y(\theta) = (A \cap Y, B \cap Y, C \cap Y)$$

is said the projection of θ on Y.

Now, it is straightforward to prove that

Lemma 23. *Given $\theta_1 = (A_1, B_1, C_1), \theta_2 = (A_2, B_2, C_2)$ and $Y \subseteq X_1$ with $(A_1 \cap Y) \neq \phi, (B_1 \cap Y) \neq \phi$. If $\theta_1 \sqsubseteq_a \theta_2$, then $\pi_Y(\theta_1) \sqsubseteq_a \pi_Y(\theta_2)$.*

In the following we show how the generalized rules work with respect to projections.

Lemma 24. *Given $\theta_1 = (A_1, B_1, C_1), \theta_2 = (A_2, B_2, C_2)$ with $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$, let Y be a subset of $X_1 \cap X_2$ such that the projections $\pi_Y(\theta_1)$ and $\pi_Y(\theta_2)$ are defined. If $gc(\pi_Y(\theta_1), \pi_Y(\theta_2)) \neq \bot$, then $gc(\theta_1, \theta_2) \neq \bot$, moreover*

$$gc(\pi_Y(\theta_1), \pi_Y(\theta_2)) = \pi_Y(gc(\theta_1, \theta_2)).$$

Proof. Since $\theta'_i = \pi_Y(\theta_i)$ is defined, for $i = 1, 2$, let $\theta'_i = (A'_i, B'_i, C'_i)$ and $gc(\theta'_1, \theta'_2) = \theta' = (A', B', C')$. We need to prove that if conditions *(a)–(e)* of Proposition 8 hold for θ'_1 and θ'_2, then they are also verified for θ_1 and θ_2.

From $A'_1 \cap A'_2 = (A_1 \cap A_2) \cap Y \neq \phi$, it follows $(A_1 \cap A_2) \neq \phi$.

From $B'_2 \cap X'_1 = (B_2 \cap X_1) \cap Y \neq \phi$, it follows $(B_2 \cap X_1) \neq \phi$. From $B'_1 \setminus C'_2 = (B_1 \cap Y) \setminus (C_2 \cap Y) = (B_2 \setminus C_2) \cap Y \neq \phi$, so $(B_1 \setminus C_2) \neq \phi$. Moreover, from Theorem 21, it follows that $B' = (B'_1 \setminus C'_2) \cup (B'_2 \cap X'_1) = [(B_1 \setminus C_2) \cup (B_2 \cap X_1)] \cap Y$ with $|B'| \geqslant 2$, so $|(B_1 \setminus C_2) \cup (B_2 \cap X_2)| \geqslant 2$. Therefore, it is possible to apply $G4^*$ to θ_1, θ_2, obtaining $gc(\theta_1, \theta_2) = \theta = (A, B, C)$. Since $(A_1 \cap A_2) \cap Y = A'_1 \cap A'_2 = A', [(B_1 \setminus C_2) \cup (B_2 \cap X_1)] \cap Y = B', [C_2 \cup (C_1 \cap A_2)] \cap Y = C'$, one has $\pi_Y(\theta) = \theta'$. $\square$

Lemma 25. *Given $\theta_1 = (A_1, B_1, C_1), \theta_2 = (A_2, B_2, C_2)$ with $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$, let Y be a subset of $X_1 \cap X_2$ such that the projections $\pi_Y(\theta_1)$ and $\pi_Y(\theta_2)$ are defined. If $gi(\pi_Y(\theta_1), \pi_Y(\theta_2)) \neq \bot$, then $gi(\theta_1, \theta_2) \neq \bot$, and moreover*

$$gi(\pi_Y(\theta_1), \pi_Y(\theta_2)) = \pi_Y(gi(\theta_1, \theta_2)).$$

Proof. It follows the same line of that Lemma 24. □

Lemma 26. *Given $\theta_1 = (A_1, B_1, C_1), \theta_2 = (A_2, B_2, C_2)$ with $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$, let Y be a subset of $X_1 \cap X_2$ such that the projections $\pi_Y(\theta_1)$ and $\pi_Y(\theta_2)$ are defined.*

If $\hat{\theta}_{(i,j,k)}(\pi_Y(\theta_1), \pi_Y(\theta_2)) \neq \perp$, then $\hat{\theta}_{(i,j,k)}(\theta_1, \theta_2) \neq \perp$ and

$$\hat{\theta}_{(i,j,k)}(\pi_Y(\theta_1), \pi_Y(\theta_2)) = \pi_Y(\hat{\theta}_{(i,j,k)}(\theta_1, \theta_2)).$$

If $\nu(\pi_Y(\theta_1), \pi_Y(\theta_2)) \neq \perp$, then $\nu(\theta_1, \theta_2) \neq \perp$ and

$$\nu(\pi_Y(\theta_1), \pi_Y(\theta_2)) = \pi_Y(\nu(\theta_1, \theta_2)).$$

Proof. The proof is a straightforward consequence from the definition of the function $\hat{\theta}_{(i,j,k)}(\cdot, \cdot)$ and $\nu(\cdot, \cdot)$ and the properties of π_Y. □

Now, we show that for any pair θ_1, θ_2 of triples, the set $K(\theta_1, \theta_1)$ is closed. For this aim, note that given $\theta_1 = (A_1, B_1, C_1)$ and $\theta_2 = (A_2, B_2, C_2)$, we can find an almost complete pair of triples $\tilde{\theta}_1, \tilde{\theta}_2$, with $\tilde{\theta}_i = (E_{(i,1)}, E_{(i,2)}, E_{(i,3)})$, and a suitable set Y, such that for $i = 1, 2$

$$E_{(i,3)} \subseteq X_{3-i} \quad \text{and} \quad \theta_i = \pi_Y(\tilde{\theta}_i). \tag{5}$$

Note that it is sufficient that $A_i \subseteq E_{(i,1)}, B_i \subseteq E_{(i,2)}, C_i \subseteq E_{(i,3)}$ and any component of $\tilde{\theta}_i$ contains other elements in a way that $E_{(1,i)} \cap E_{(2,j)} \neq \phi$, for $i, j = 1, 2, 3$, and $(E_{(k,i)} \setminus X_{3-k}) \neq \phi$, for $i, k = 1, 2, 3$.

Theorem 27. *Let θ_1, θ_2 be such that $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$. Then $K(\theta_1, \theta_2)$ is closed with respect to $G4^*$ and $G5^*$.*

Proof. Given θ_1 and θ_2, we can build (not necessarily uniquely) an almost complete pair $\tilde{\theta}_1, \tilde{\theta}_2$ of triples such that (5) hold for a suitable set Y.

Note that, by Lemma 26, if $\hat{\theta}_{(i,j,k)}(\theta_1, \theta_2) \neq \perp$, then $\hat{\theta}_{(i,j,k)}(\tilde{\theta}_1, \tilde{\theta}_2) \neq \perp$ and $\hat{\theta}_{(i,j,k)}(\theta_1, \theta_2) = \pi_Y(\hat{\theta}_{(i,j,k)}(\tilde{\theta}_1, \tilde{\theta}_2))$.

Again by Lemma 26, if $\nu(\theta_1, \theta_2) \neq \perp$, then $\nu(\tilde{\theta}_1, \tilde{\theta}_2) \neq \perp$ and $\nu(\theta_1, \theta_2) = \pi_Y(\nu(\tilde{\theta}_1, \tilde{\theta}_2))$.

Let $\theta'_1, \theta'_2 \in K(\theta_1, \theta_2)$. If $\theta' \in GC(\theta'_1, \theta'_2)$ (i.e. it is different from $\perp$) then there exist τ_1, τ_2 in $K(\tilde{\theta}_1, \tilde{\theta}_2)$, with $\theta'_1 = \pi_Y(\tau_1)$ and $\theta'_2 = \pi_Y(\tau_2)$, such that (by Lemma 24) there exists $\tau \in GC(\tau_1, \tau_2)$ with $\theta' = \pi_Y(\tau)$. If $\theta' \in GI(\theta'_1, \theta'_2)$, by Lemma 25, it is possible to arrive at the same conclusion. Since $K(\tilde{\theta}_1, \tilde{\theta}_2)$ is closed under $G4^*$ and $G5^*$, by Theorem 21, there is $\sigma \in K(\tilde{\theta}_1, \tilde{\theta}_2)$ such that $\tau \sqsubseteq \sigma$. Moreover $\theta' = \pi_Y(\tau) \sqsubseteq \pi_Y(\sigma)$ by Lemma 23.

If σ coincides with $\tilde{\theta}_1$ or $\tilde{\theta}_2$ the conclusion is trivial. Otherwise, $\sigma = \hat{\theta}_{(i,j,k)}(\tilde{\theta}_1, \tilde{\theta}_2)$ or $\sigma = \nu(\tilde{\theta}_1, \tilde{\theta}_2)$. Since $\pi_Y(\tau) \sqsubseteq \pi_Y(\sigma)$, one has $\pi_Y(\sigma) \neq \perp$. By Lemma 26, $\pi_Y(\sigma) = \sigma'$, for $\sigma' \in K(\theta_1, \theta_2)$, then $\theta' \sqsubseteq \sigma'$ and $K(\theta_1, \theta_2)$ is closed with respect to $G4^*$ and $G5^*$. □

Also the vice versa holds: the elements of $K(\theta_1, \theta_2)$ are g-included in $\{\theta_1, \theta_2\}^*$.

Theorem 28. *Let θ_1, θ_2 be such that $C_1 \subseteq X_2$ and $C_2 \subseteq X_1$. Then,*

$$K(\theta_1, \theta_2) \sqsubseteq \{\theta_1, \theta_2\}^*.$$

Proof. Let $\theta_i = (A_i, B_i, C_i)$ for $i = 1, 2$. From (4) the elements of $K(\theta_1, \theta_2)$ are $\theta_1, \theta_2, \hat{\theta}_{(i,j,k)}(\theta_1, \theta_2)$, with $i, j, k = 1, 2$, and $\nu(\theta_1, \theta_2)$. In the following we denote $\hat{\theta}_{(i,j,k)}(\theta_1, \theta_2)$ and $\nu(\theta_1, \theta_2)$ with $\hat{\theta}_{(i,j,k)}$ and ν, respectively.

Let us start to prove that

$$\hat{\theta}_{(1,1,1)} = (A_1 \cap A_2, B_1 \cup (B_2 \cap X_1), (C_1 \cap C_2) \cup (A_1 \cap C_2) \cup (A_2 \cap C_1))$$

is g-included in $\{\theta_1, \theta_2\}^*$. Firstly, note that the triple $\hat{\theta}_{(1,1,1)}$ is defined if and only if $A_1 \cap A_2 \neq \phi$ by Definition 19.

Now, we need to show that there is $\theta \in \{\theta_1, \theta_2\}^*$ such that $\hat{\theta}_{(1,1,1)} \sqsubseteq \theta$.

If $B_2 \cap X_1$ is empty, $\hat{\theta}_{(1,1,1)}$ reduces to $(A_1 \cap A_2, B_1, C_1 \cup (A_1 \cap C_2))$, and therefore $\hat{\theta}_{(1,1,1)} \sqsubseteq \theta_1$.

Otherwise, if $B_2 \cap X_1$ is nonempty, we can have four cases. Let us denote by $\sigma = gi(\theta_1, \theta_2)$ and by $\tau = gc(\theta_1, \theta_2)$.

(1) If σ and τ are not defined, since $\hat{\theta}_{(1,1,1)}$ is defined, by Propositions 8 and 9, the only possible situation[2] is

$$|(B_1 \setminus C_2) \cup (B_2 \cap X_1)| \leqslant 1 \quad \text{and} \quad |(B_1 \cap X_2) \cup (B_2 \cap X_1)| \leqslant 1$$

and from $B_2 \cap X_1 \neq \phi$ it follows that $(B_1 \setminus C_2) \cup (B_2 \cap X_1) = B_1 \cap B_2, (B_1 \cap X_2) \cup (B_2 \cap X_1) = B_1 \cap B_2$ and $|B_1 \cap B_2| = 1$. Therefore, we know that $B_1 \subset B_1 \cup (B_2 \cap X_1)$ and $C_1 \subseteq (C_1 \cap C_2) \cup (A_2 \cap C_1) \cup (A_1 \cap C_2)$, so $\hat{\theta}_{(1,1,1)} \sqsubseteq \theta_1$.

(2) If $\sigma = \perp$ but $\tau \neq \perp$, then either $B_1 \cap X_2 = \phi$ or $|(B_1 \cap X_2) \cup (B_2 \cap X_1)| \leqslant 1$. In both cases $\hat{\theta}_{(1,1,1)} = \tau$.

[2] The other combinations lead to contradiction.

(3) If $\sigma \neq \perp$ but $\tau = \perp$, then either $B_1 \setminus C_2 = \phi$ or $|(B_1 \setminus C_2) \cup (B_2 \cap X_1)| \leqslant 1$. In both cases $\hat{\theta}_{(1,1,1)} = \sigma$.
(4) If $\sigma \neq \perp$ and $\tau \neq \perp$ are defined, then it follows easily that $gc(\tau, \sigma)$ is defined and coincides with $\hat{\theta}_{(1,1,1)}$. In fact, it is easy to show that $\tau = (A_3, B_3, C_3)$ and $\sigma = (A_4, B_4, C_4)$ satisfy the conditions *(a)–(e)* of Proposition 8.

By a few simple steps it is possible to compute $gc(\tau, \sigma)$ and to observe that it coincides with $\hat{\theta}_{(1,1,1)}$.

In general, it is possible to compute $\hat{\theta}_{(i,j,k)}$ from $\theta_i^{(j)}$ and $\theta_{3-i}^{(k)}$ following the same steps used to determinate $\hat{\theta}_{(1,1,1)}$. The triple v can be obtained in four possible ways. For example, if $A_1 \cap A_2 \neq \phi, B_1 \cap B_2 \neq \phi$ and $A_1 \cap B_2 \neq \phi$, then $\hat{\theta}_{(1,1,1)}, \hat{\theta}_{(1,2,2)}$ and $gi(\hat{\theta}^T_{(1,1,1)}, \hat{\theta}^T_{(1,2,2)})$ are all defined and $v = gi(\hat{\theta}^T_{(1,1,1)}, \hat{\theta}^T_{(1,2,2)})$. In fact, it is very easy to see that $\hat{\theta}^T_{(1,1,1)}$ and $\hat{\theta}^T_{(1,2,2)}$ satisfy the conditions *(a)–(e)* of Proposition 9. Finally, it is simple to observe that $gi(\hat{\theta}^T_{(1,1,1)}, \hat{\theta}^T_{(1,2,2)}) = v$.

The proof of the other three cases goes along the same line. □

We show that $K(\theta_1, \theta_2)$ is closed under $G4^*$ and $G5^*$ for any pair of triples θ_1, θ_2.

Theorem 29. *For any pair of triples θ_1, θ_2, we have that*

$$\{\theta_1, \theta_2\}^* \sqsubseteq K(\theta_1, \theta_2) \quad \text{and} \quad K(\theta_1, \theta_2) \sqsubseteq \{\theta_1, \theta_2\}^*.$$

Proof. Since $\{\theta_1, \theta_2\} \subseteq K(\theta_1, \theta_2)$, it follows that $\{\theta_1, \theta_2\} \sqsubseteq K(\theta_1, \theta_2)$. Since $K(\theta_1, \theta_2)$ is closed with respect to $G4^*$ and $G5^*$, the conclusion $\{\theta_1, \theta_2\}^* \sqsubseteq K(\theta_1, \theta_2)$ follows.

The relation has been already proved in Theorems 21 and 28. □

Note that in general $K(\theta_1, \theta_2)$ may not coincide with $\{\theta_1, \theta_2\}_*$ because it could contain some redundant triple. However, it is easy to see that $K(\theta_1, \theta_2)_{/\sqsubseteq} \sqsubseteq \{\theta_1, \theta_2\}_*$ and $\{\theta_1, \theta_2\}_* \sqsubseteq K(\theta_1, \theta_2)_{/\sqsubseteq}$ from Definition 15 and Proposition 16, since both sets are maximal. Therefore, the set $K(\theta_1, \theta_2)$ allows to compute $\{\theta_1, \theta_2\}_*$ without the algorithm FC2: in fact, it is possible to build up such a set and apply to it the function FindMaximal. All this computation requires a constant number of steps with respect to the size of θ_1, θ_2.

By using $\{\theta_1, \theta_2\}_*$, it is possible to provide a new inference rule

U: from θ_1, θ_2 deduce any triple $\tau \in \{\theta_1, \theta_2\}_*$.

In the next section the main properties of this rule are studied.

4.1. Correctness and completeness of U

With the aim to prove the correctness and completeness of the inference rule U we denote with J^+ the set of triples obtained by applying a finite number of times the rule U. First let us show that U is correct.

Proposition 30. *Given a set J of triples in $S^{(3)}$, then $J^+ \subseteq \bar{J}$.*

Proof. Note that for $\theta_1, \theta_2 \in \bar{J}, \{\theta_1, \theta_2\}_* \subseteq \overline{\{\theta_1, \theta_2\}} \subseteq \bar{J}$.

Let $\tau \in J^+$. Then, it is possible to find a derivation $\theta_1, \theta_2 \vdash_U \tau_1, \ldots, \theta_{2n-1}, \theta_{2n} \vdash_U \tau_n$, in which $\tau_n = \tau$ and for $i = 1, \ldots, 2n$ either $\theta_i \in J$ or $\theta_i = \tau_j$ for $j < \lfloor \frac{i+1}{2} \rfloor$. Now, we show by induction that each $\tau_i \in \bar{J}$ for $i = 1, \ldots, n$.

Since $\theta_1, \theta_2 \in J$, then $\tau_1 \in \bar{J}$. Suppose that $\tau_1, \ldots, \tau_{k-1} \in \bar{J}$, then $\theta_{2k-1}, \theta_{2k} \in \bar{J}$ and so also $\tau_k \in \bar{J}$. In fact, $\{\theta_{2k-1}, \theta_{2k}\}_* \subseteq \overline{\{\theta_{2k-1}, \theta_{2k}\}}$ by Proposition 13. Since $\bar{J}$ is closed with respect to graphoid properties $\overline{\{\theta_{2k-1}, \theta_{2k}\}} \subseteq \bar{J}$ and so $\{\theta_{2k-1}, \theta_{2k}\}_* \subseteq \bar{J}$. □

We give now some preliminary lemmas useful for proving the completeness of rule U.

Lemma 31. *If $\theta'_1, \theta'_2 \vdash_{G4^*} \tau'$ and $\theta'_i \sqsubseteq \theta_i$ for $i = 1, 2$, then $\theta_1, \theta_2 \vdash_{G4^*} \tau$ with $\tau' \sqsubseteq \tau$.*

Proof. If τ' is obtained by $G4^*$ from θ'_1 and θ'_2, then, by definition, there exist $\theta''_1 \sqsubseteq \theta'_1$ and $\theta''_2 \sqsubseteq \theta'_2$ such that $\theta''_1, \theta''_2 \vdash_{G4} \tau'$. Since $\theta''_i \sqsubseteq \theta_i$, for $i = 1, 2$, it follows that $\tau' \in W_C(\theta_1, \theta_2) \cup W_C(\theta_1, \theta_2^T) \cup W_C(\theta_1^T, \theta_2) \cup W_C(\theta_1^T, \theta_2^T)$ and so there exists $\tau \in GC(\theta_1, \theta_2)$ such that $\tau' \sqsubseteq \tau$, by Proposition 8. □

Lemma 32. *If $\theta'_1, \theta'_2 \vdash_{G5^*} \tau'$ and $\theta'_i \sqsubseteq \theta_i$ for $i = 1, 2$, then $\theta_1, \theta_2 \vdash_{G5^*} \tau$ with $\tau' \sqsubseteq \tau$.*

Proof. The proof goes along the same lines of Lemma 31. □

Lemma 33. *If $\theta'_1 \sqsubseteq \theta_1$ and $\theta'_2 \sqsubseteq \theta_2$, then $\{\theta'_1, \theta'_2\}^* \sqsubseteq \{\theta_1, \theta_2\}^*$.*

Proof. Let $\tau' \in \{\theta'_1, \theta'_2\}^*$. Then τ' is obtained by applying $G4^*$ and $G5^*$ to θ'_1 and θ'_2. Therefore from θ_1 and θ_2 it is possible to get a triple τ with $\tau' \sqsubseteq \tau$, by Lemmas 31 and 32. □

In the following result the monotonicity of the inference rule U is shown.

Corollary 34. *If $\theta'_1, \theta'_2 \vdash_U \tau'$ and $\theta'_i \sqsubseteq \theta_i$ for $i = 1, 2$, then $\theta_1, \theta_2 \vdash_U \tau$ with $\tau' \sqsubseteq \tau$.*

Proof. It is enough to observe that $\{\theta'_1,\theta'_2\}_* \subseteq \{\theta'_1,\theta'_2\}^* \sqsubseteq \{\theta_1,\theta_2\}^* \sqsubseteq \{\theta_1,\theta_2\}_*$. $\square$

Now we can prove the completeness of U.

Proposition 35. *Let J be a nonempty subset of $S^{(3)}$. Then $\bar{J} \sqsubseteq J^+$.*

Proof. We need to show that $J^* \sqsubseteq J^+$. The conclusion follows from the completeness of J^* with respect to $G4^*$ and $G5^*$.

Let $J^* = \bigcup_{h=0}^{\infty} J^{(h)}$ with $J^{(0)} = J$

$$J^{(h)} = J^{(h-1)} \cup \{\theta : \theta_1,\theta_2 \vdash_{G4^*} \theta \text{ or } \theta_1,\theta_2 \vdash_{G5^*} \theta \text{ with } \theta_1,\theta_2 \in J^{(h-1)}\}.$$

We show by induction that $J^{(k)} \sqsubseteq J^+$ for any k. For $k = 0$ it is trivial. Suppose that it holds for $k > 0$ and let $\tau \in J^{(k+1)} \setminus J^{(k)}$ with, for example, $\theta_1,\theta_2 \vdash_{G4^*} \tau$ for $\theta_1,\theta_2 \in J^{(h)}$. By inductive hypothesis there exist $\tau_1,\tau_2 \in J^+$ with $\theta_i \sqsubseteq \tau_i$. Then, by Lemma 33, there exists $\bar{\tau} \in \{\theta_1,\theta_2\}^*$ with $\tau \sqsubseteq \bar{\tau}$. $\square$

4.2. Algorithm with one generalized rule

We provide an algorithm, alternative to FC2, and some theoretical justifications showing its better performance with respect to FC2.

Algorithm 2 Fast closure by U

function FC1 J
  $J_0 \leftarrow J$
  $N_0 \leftarrow J$
  $k \leftarrow 0$
  repeat
    $k \leftarrow k+1$
    $N_k := \bigcup_{\theta_1 \in J_{k-1},\theta_2 \in N_{k-1}} \{\theta_1,\theta_2\}_*$
    $J_k \rightarrow FindMaximal(J_{k-1} \cup N_k)$
  until $J_k = J_{k-1}$
   return J_k
end function

Theorem 36. *Let J be a nonempty subset of $S^{(3)}$, then*

(1) $FC1(J) \sqsubseteq J_*$;
(2) $J_* \sqsubseteq FC1(J)$.

Proof. The proof of condition (2) goes into the same line of that of the Theorem 17 and uses the same sequences of sets J'_k and N'_k for $k = 0,1,2,\ldots$. To show that $J_* \sqsubseteq$ FC1(J) we prove by induction that $J'_k \sqsubseteq J_k$ and $N'_k \sqsubseteq N_k$ for any k. For $k = 0$ it is trivial. Suppose that it holds for $k = h-1$ and let, for example, $\tau' = gc(\theta'_1,\theta'_2)$ with $\theta'_1 \in J_{h-1}$ and $\theta'_2 \in N_{h-1}$. By inductive hypothesis there are $\theta_1 \in J_{h-1}, \theta_2 \in N_{h-1}$ such that $\theta'_1 \sqsubseteq \theta_1$, $\theta'_2 \sqsubseteq \theta_2, \tau' \in \overline{\{\theta'_1,\theta'_2\}} \sqsubseteq \{\theta'_1,\theta'_2\}_* \sqsubseteq \{\theta_1,\theta_2\}_* \sqsubseteq N_h$. Analogously, $J'_h \sqsubseteq J_h$.

Let N be such that $J'_N = J'_{N-1} = J^*$ and $J_N = J_{N-1} =$ FC1(J), then it follows that $J_* \sqsubseteq J^* = J'_N \sqsubseteq J_N =$ FC1(J).

For the condition (1), note that FC1(J) $\sqsubseteq J_N \subseteq \bar{J} \sqsubseteq J^* \sqsubseteq J_*$. $\square$

Note that FC1 is more efficient of FC2 since it builds the closure of a pair of two triples in just one step (with one application of the rule U), while FC2 can require two steps (with several applications of $G4^*$ and $G5^*$) to get the same result. For this reason it is easy to see that when FC1 computes the fast closure of a set J of n triples, with $n > 2$, each triple belonging to J_* will be generated in a number of steps which is always not greater than the number of steps required by FC2. Therefore, the number of iterations of the loop in FC1 is not greater than the number of iterations needed by the analogous loop in FC2.

Moreover, FC1 can be optimized by observing that if θ'_1 and θ'_2 belong to $\{\theta_1,\theta_2\}_*$, then $\{\theta'_1,\theta'_2\}_*$ is g-included to $\{\theta_1,\theta_2\}_*$. The validity of this observation follows easily since $\{\theta'_1,\theta'_2\}_* \sqsubseteq \{\theta'_1,\theta'_2\}^* \sqsubseteq \{\theta_1,\theta_2\}^* \sqsubseteq \{\theta_1,\theta_2\}_*$.

Therefore, it is not necessary to apply the inference rule U to a pair of triples θ'_1 and θ'_2, generated by U from the same two triples θ_1 and θ_2, since from θ'_1 and θ'_2 we would obtain only redundant triples, which would be discarded by the function FindMaximal.

Note that for the same reasons, we do not need to apply the rule U between a triple θ and another one θ' generated from θ (by combining θ with another triple θ''): in fact if $\theta' \in \{\theta,\theta''\}_*$, then $\{\theta,\theta'\} \subseteq \{\theta,\theta''\}^*$ and so $\{\theta,\theta'\}_* \sqsubseteq \{\theta,\theta''\}_*$, which implies that no maximal triple can be obtained.

Therefore, the use of the inference rule U in FC1 can be enhanced by keeping track of the "parents" of each triple and by neglecting the pairs which satisfies the two previously described situations ("sibling" triples and "father–child"). For this reason the number of triples generated by FC1 will be less than the number of triples generated by FC2, since in the latter a similar improvement is not possible.

Summing up these two considerations (number of iterations to generate the closure and improvements) about the computational differences between FC1 and FC2, we can expect that FC1 is faster than FC2 and generates a less number of triples. The experimental results shown in the next section will confirm this intuition.

Another possible improvement for the implementation of FC1 would be to avoid (as shown by Theorem 28) to generate $\hat{\theta}_{(1,1,1)}$ when either $B_2 \cap X_1 = \phi$ or $|(B_2 \cap X_1) \cup (B_1 \cap X_2)| \leqslant 1$ and $|(B_1 \setminus C_2) \cup (B_2 \cap X_1)| \leqslant 1$ since in these cases $\hat{\theta}_{(1,1,1)} \sqsubseteq \theta_1$.

However, note that even after this last optimization, $K(\theta_1, \theta_2)$ could be not maximal, therefore it is necessary to apply again FindMaximal on $K(\theta_1, \theta_2)$.

In our implementation, we use the first two optimizations, but we consider $K(\theta_1, \theta_2)$ instead of $\{\theta_1, \theta_2\}_*$, because in any case in each cycle of FC1 a call to function FindMaximal is however performed.

5. Experimental results

In this section, we describe some experimental results obtained with an implementation of the algorithms FC2 and FC1, as well as an implementation of an algorithm to compute the complete closure (with respect to G1–G5). The main purposes of these experiments is to give strong empirical justifications to some intuitive ideas, as well as to provide a numerical counterpart to the theoretical results shown in the previous sections.

The first answer we expect from the experiments is a comparison between the two different algorithms implemented for the computation of the fast closure. We provided some heuristic motivations to support that FC1 should have a better behaviour than FC2.

The other question is which is the quantitative difference in size and in computation time of the fast closure with respect to the complete closure. It is simple to see that each triple θ in the fast closure corresponds to several triples in the complete closure, i.e. all the triple g-included in θ.

The experiments were run on an AMD Dual Core Opteron at 1.8 GHz with 2 GByte main memory. We applied a cut-off of 5,000,000 triples that can be stored (to avoid problems with memory) and a time-out of 3600 s. Some preliminary results, with different experimental parameters, have already been given in [3,2].

In the first set of experiments we compare the two algorithms described in the previous sections. We generate 200 random sets of triples having nv variables and nr triples, for $nr = 10, 15, 20, 25$ and $nv = \lfloor 0.5 \cdot nr \rfloor, nr, \lfloor 1.5 \cdot nr \rfloor$. We compute the fast closure with FC2 and FC1 (see Table 4).

As we expected, FC1 is clearly faster than FC2, needs a smaller number of iterations for computing the closure and generates a sensitively lesser number of triples. Furthermore, the number of instances resolved by FC1 is slightly larger than those resolved by FC2, see the last column of Table 4, and then any instance solved by FC2 is solved by FC1.

The third column contains the average computation times in seconds, the fourth column contains the average number of iterations needed by each algorithm to find the closure, the fifth column contains the average number of the overall generated triples. The previous average values are computed over instances solved by FC2. The last column contains the number of instances in which each algorithm has been able to compute the closure within the memory and time limits.

In the second set of experiments we compare the computation time needed for finding the complete closure and its size with respect to the time and size of the fast closure. The complete closure is obtained by using an algorithm similar to FC1 and FC2, which uses all the inference rules G1–G5, without calling FindMaximal. Furthermore, we did not apply for it any cut-off with respect to number of triples.

Table 4
Fast closure: FC2 versus FC1.

nr	nv	Time		Size	Iterations		Generated		Solved	
		FC2	FC1		FC2	FC1	FC2	FC1	FC2	FC1
10	5	0.00	0.00	10.84	4.17	4.00	691.33	201.97	200	200
10	10	2.15	1.06	95.94	7.31	6.43	63561.67	27523.80	200	200
10	15	60.14	30.77	215.88	7.78	6.25	429100.75	197149.92	197	198
15	7	0.21	0.09	46.85	5.86	5.51	15382.56	5840.90	200	200
15	15	638.44	335.52	862.23	12.04	9.84	2916763.34	1421116.05	116	126
15	22	68.71	33.34	282.33	8.87	6.42	359679.62	162101.98	156	161
20	10	142.28	79.19	433.84	8.10	7.41	1396140.47	652607.94	200	200
20	20	310.16	143.46	670.94	12.76	9.84	1280049.88	558191.29	49	55
20	30	44.91	18.97	202.70	7.70	5.34	225146.12	92839.77	181	187
25	12	1254.51	599.83	1054.48	9.21	8.31	7011081.22	3025176.67	67	99
25	25	171.89	81.14	554.83	12.17	8.74	697131.20	311854.02	65	70
25	37	12.93	5.94	112.97	5.66	3.77	88332.23	38183.21	194	195

Table 5
Fast closure with FC1.

nr	*nv*	Time	Size	Iter.	Gen.
4	4	0	3.95	2.75	12.1
4	6	0	5.85	2.95	29.2
7	7	2E−03	18.65	4.95	559.25
7	10	1.8E−02	32.05	4.7	1756.15
10	10	0.6755	86.9	5.95	18415
10	15	42.7225	320.45	6.7	335910.5

Table 6
Complete closure.

nr	*nv*	Time	Size	Iter.	Gen.	Res.
4	4	0	64	7	57	20
4	6	0.05	527	8.9	899	20
7	7	1.75	3282	13.15	9526	20
7	10	248	28,808	13.89	147,249	19
10	10	603	50,760	16.67	268,381	15
10	15	3513	159,164	14	683,991	1

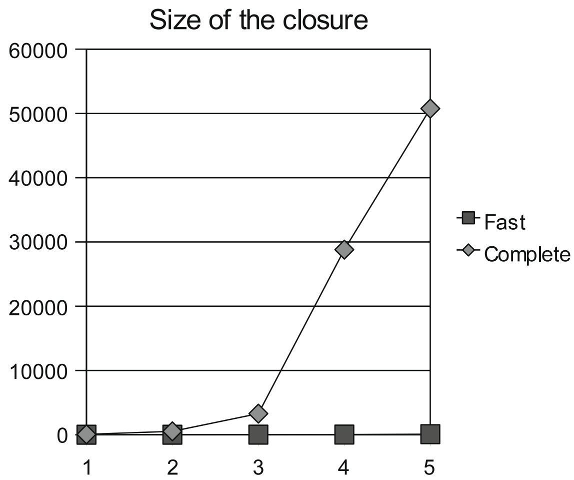

Fig. 1. Sizes of the closure.

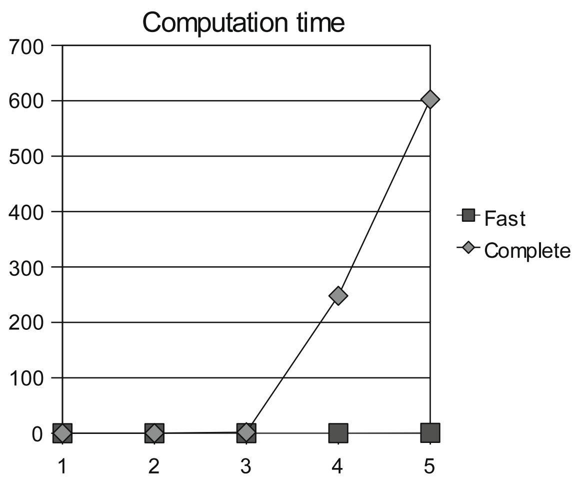

Fig. 2. Computation times.

The fast closure is obtained by FC1, since it is faster than FC2. Since we expect that the complete closure is much larger than its fast version, we have run new experiments with smaller instances, instead of using the previous one. In particular, we generate 20 sets of nr triples and nv variables, for $nr = 4, 7, 10$ and $nv = nr, \lfloor 1.5 \cdot nr \rfloor$.

In Table 5 the results for the fast closure are reported, with the average values calculated with respect to the solved instances by FC1, the average computation time is negligible, except that in the last row, where we obtain results similar in magnitude order, as those displayed in Table 4. The algorithm FC1 has been able to build the closure for each instance.

In Table 6 we report the results obtained in the computation of the complete closure. The last column contains the number of instances for which the algorithm has been able to compute the complete closure within an hour of computation. Note that with $nr = 10$ and $nv = 15$ we could solve only one instance, which almost reached the time limit, while the fast closure of this instance has only 27 triples and has been found in a negligible amount of time. The values in the last column are used to compute the average values showed in Table 5.

The comparison of the size between fast and complete closure is impressive, as it is possible to see in the graph of Fig. 1 (the last rows of both tables have been ignored).

Clearly also the computation times for computing the complete closure are much higher than the time needed to compute the fast closure, as displayed in the Fig. 2.

6. Conclusions

We show some properties of graphoid structures arising from conditional independence models, with the aim to compute efficiently the closure of a set J of conditional independence statements. In particular, we provide a method which is able to compute the "fast" closure of a set of triples using graphoid rules and it is able to compute the closure in medium size instances.

A straightforward extension of this work is to adapt this framework for computing the closure by using semi-graphoid axioms and compare it with that proposed in [28].

From the theoretical point of view, it could be worth to study whether there exist other groups of inference rules, other than $G4^*$ and $G5^*$, by which it is possible to compute the fast closure.

A further point of investigation to enhance the performance of our implementation is to look for suitable data structures for representing sets of triples. Now, the sets of triples are represented by sequential unordered lists, in which the insertions are performed at the end of the list, thus making simpler the step in which the N_k's are computed. To test wether a given triple is implied by the set, a linear search has to be performed. Moreover, the function FindMaximal takes a quadratic number of steps. Therefore, it is desirable to look for a data structure in which the implication and FindMaximal procedures can be solved in a faster way.

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