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Nonuniqueness of Best Rational Chebyshev Approximations on the Unit Disk

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In the past it has been unknown whether complex rational best Chebyshev approximations (BAs) on the unit disk need be unique. This paper answers this and related questions by exhibiting examples in which: (a) the BA is not unique, (b) the number of distinct BAs is arbitrarily large, (c) the BA to a real analytic function f (i.e., $f(\bar{z}) = \overline{f(z)}$) among rational functions with real coefficients is not unique, and (d) the complex BAs to such a function are better than any approximation with real coefficients. Except in case (d), our constructions hold for approximation of arbitrary type (m, n) with $n \geqslant 1$. Finally, by the same methods we also establish the new result that if a function is approximated on a small disk about 0 of radius ε (or on an interval of length ε), then as $\varepsilon \to 0$, the BA need not in general approach the corresponding Padé approximant in a sense considered by J. L. Walsh.

1. INTRODUCTION

Let S denote the unit circle $\{z: |z| = 1\}$, Δ the closed unit disk $\{z : |z| \leqslant 1\}$, and $\|\cdot\|$ the supremum norm $\|\phi\| = \sup_{z \in \Delta} |\phi(z)|$. Let $A = A(\Delta)$ be the set of functions continuous on Δ and analytic in the interior, and for arbitrary integers $m, n \geqslant 0$, let $R_{mn} \subseteq A$ be the subspace of rational functions of type (m, n) with complex coefficients and no poles in Δ. For simplicity we

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will also write $P_m = R_{m0}$. Given $f \in A$, a *best approximation* (BA) to f on Δ in R_{mn} is a function $r^* \in R_{mn}$ that satisfies

$$\|f - r^*\| = \inf_{r \in R_{mn}} \|f - r\|.$$

The *existence* of rational BAs on the disk, and more generally on an arbitrary compact subset of $\mathbb{C}$ with no isolated points, was established by Walsh in 1931 [17]. The question of *uniqueness* has been less fully understood. In 1934 Walsh showed that on at least some complex domains BAs are not unique, by exhibiting an example in which the domain of approximation is a crescent-shaped Jordan region or arc that is symmetric with respect to the unit circle [17, 18]. For many years this was apparently the only known instance of nonuniqueness in complex rational Chebyshev approximation. A more natural domain was added to the collection when Gončar [6] and Lungu [10] and Saff and Varga [12, 13, 16] found that complex rational BAs to a real function on a real interval can be nonunique for all $m \geqslant 0$, $n \geqslant 1$. But the question of whether approximations on the disk must be unique has remained open [16].

Certain related matters have also remained unresolved, including two questions mentioned by Ellacott in [4]. Let us say that f is a *real analytic* function if $f(\bar{z}) = \overline{f(z)}$, that is, if its Maclaurin series has all real coefficients, and let A^{r} and R^{r}_{mn} be the subsets of real analytic functions in A and R_{mn}, respectively. Ellacott asks: Are BAs to real analytic functions unique, if one restricts attention to real analytic approximations? Can a real analytic function on the disk, in contrast to the situation on the interval, always be approximated as well in R^{r}_{mn} as in R_{mn}?

In this paper we show that the answers to all of the above questions are negative. Thus rational Chebyshev approximation on the disk, despite some expectations to the contrary, is apparently fairly typical among nonlinear approximation problems, where nonuniqueness is the rule. For other examples, best complex rational least-squares approximations on both the circle and the interval are nonunique [5, 9], and so are real Chebyshev approximations by sums of exponentials to continuous real functions on an interval, if confluent exponents are permitted [1]. (It is interesting that in the latter case, there is a definite limit to the number of distinct BAs of given degree that a function can possess [2]; we will see that this is not the case here.) On the other hand, real rational BAs to a continuous real function on a real interval are well known to be unique, and they are characterized by an equioscillation condition due to Achieser [11].

All of our proofs consist of elementary symmetry arguments. But to make sure that the underlying idea is not obscured by details, we will now consider the special case of type (0, 1) approximation before turning to general (m, n) in the next section.

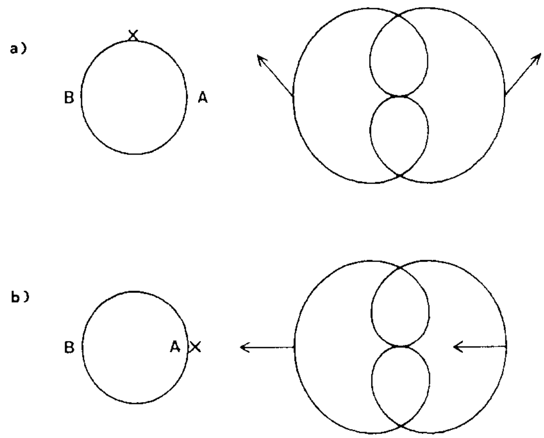

FIG. 1. (a) $f(z)=z+z^3$, $r(z)=1/(z-1.1i)$; (b) $f(z)=z+z^3$, $r(z)=1/(z-1.1)$.

Consider the function $f(z)=z+z^3$. As illustrated in Fig. 1, f maps S onto an oblong loop oriented along the real axis that attains maximum modulus at the points $A=1$ and $B=-1$. Now for $r\in R_{01}$ to be a better approximation to f than 0, r must have positive real part at A and negative real part at B. Conversely, if r is any such function, then obviously $\|f-\varepsilon r\|<\|f\|$ for small enough $\varepsilon>0$.

From these considerations it follows that 0 *is not a BA to f in R_{01}*. The demonstration of this is that the function $r(z)=1/(z-1.1i)$, as illustrated in Fig. 1a (the cross indicates the pole, the arrows indicate $r/|r|$ at $f(A)$ and $f(B)$), has the required real parts at A and B. On the other hand, 0 *is a BA to f in R^{r}_{01}*. For when only real coefficients are permitted, the denominator of r and hence r itself must have uniform sign on $[-1,1]$, in particular at A and B, so a correction of the required form is impossible. This is suggested in Fig. 1b.

We have shown: *there exists a function f in A^{r} that can be better approximated in R_{01} than in R^{r}_{01}*. This answers one of the questions posed by Ellacott. The same argument applies to approximation in R_{0n} for any $n\geqslant 1$.

Moreover, symmetry implies that if $r^*(z)$ is a BA to f with complex coefficients, then $\overline{r^*(\bar{z})}$ is another one. Thus *best approximations in R_{01} need not be unique.*

Now rotate the figure by 90°, and define $f(z)=z-z^3$. This function attains maximum modulus at $A=i$ and $B=-i$, as illustrated in Fig. 2. As before the function f can be approximated better by a function in R^{r}_{01} with a pole near -1 or 1 than by 0, which is the best approximation among functions with no poles. Therefore any BA r^* must have a pole, which is necessarily asymmetrically situated with respect to the imaginary axis. This

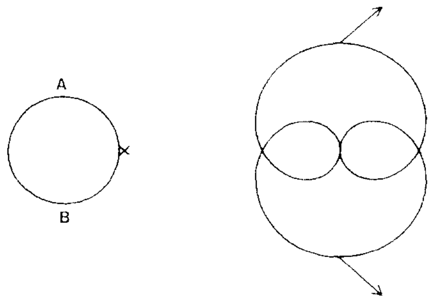

FIG. 2. $f(z) = z - z^3$, $r(z) = -1/(z - 1.1)$.

implies that $-r^*(-z)$ is another distinct BA, and we have shown: *best approximations of real-analytic functions in R_{01}^r need not be unique.*

The organization of the remainder of this paper is as follows. In Section 2 we establish results (a) (Theorem 1) and (b) (Theorem 2) mentioned in the abstract for general (m, n). In Section 3 we treat result (c) (Theorem 3). Finally, in Section 4, we turn to the question of best approximation on small disks and intervals. A variation of our symmetry arguments shows there that *r^* need not approach the Padé approximant r^{p} as the size of the disk or interval decreases to* 0 (Theorem 4). This conclusion is counter to what one might expect on the basis of a theorem of Chui *et al.* [3] which states that r^* does approach r^{p} if attention is restricted to real coefficients. We show further that both nonuniqueness and the $r^* \to r^{\mathrm{p}}$ question are closely connected to a normality condition appearing often in Padé approximation that requires a Hankel matrix of Maclaurin coefficients to be nonsingular.

2. NONUNIQUENESS IN R_{mn}

If K is a positive integer, let ω_K denote the primitive Kth root of unity

$$\omega_K = e^{2\pi i/K}.$$

We will say that a function ϕ is *K-symmetric* if $\phi(\omega_K z) \equiv \phi(z)$. Equivalently, ϕ is K-symmetric if its Maclaurin series takes the form

$$\phi(z) = a_0 + a_1 z^K + a_2 z^{2K} + \cdots. \tag{1}$$

We will also say that a set $M \subseteq \mathbb{C}$ (e.g., the set of poles of ϕ) is *K-symmetric* if $\omega_K M = M$.

In all of the arguments of this section, f_K denotes any function with the following three properties:

(a) $f_K \in A$,

(b) $zf_K(z)$ is K-symmetric (i.e., $f_K(z) = a_1 z^{K-1} + a_2 z^{2K-1} + \cdots$),

(c) f_K attains maximum modulus at exactly K points of S.

These points of extreme modulus will necessarily be just the Kth roots of unity times some constant $e^{i\tau}$, and let us denote them by $\{\zeta_k\}$,

$$\zeta_k = e^{i\tau}\omega_K^k, \qquad 0 \leqslant k \leqslant K-1.$$

By (b) and (c) we also have, for some nonzero $c \in \mathbb{C}$,

$$f(\zeta_k) = \bar{c}\zeta_k^{-1}, \qquad 0 \leqslant k \leqslant K-1.$$

As a concrete example, throughout this section one can take f_K to be

$$f_K(z) = z^{K-1} + z^{2K-1},$$

in which case the constants are $e^{i\tau} = 1$ and $c = 2$.

We begin with some easy lemmas. For an arbitrary function $r \in R_{mn}$, there is no simple test to determine whether or not r is a BA to f_K: the local Kolmogorov condition is necessary but not sufficient for this, while the global Kolmogorov condition (or Meinardus–Schwedt condition) is sufficient but not necessary [8]. However, in the special case $r = 0$, the two conditions coalesce and one has the following:

LEMMA 1. *Let R denote R_{mn} or a subset of it (such as R_{mn}^{r}) that is closed under multiplication by real scalars. The zero function is a BA to f_K in R if and only if there exists no $r \in R$ satisfying*

$$\mathrm{Re}[czr(z)] > 0 \qquad \textit{for} \quad z = \zeta_k,\ 0 \leqslant k \leqslant K-1. \tag{2}$$

Proof. Equation (2) is a specialization to the present context of the condition

$$\mathrm{Re}[\overline{f(z)}\, r(z)] > 0 \qquad \forall\, z \in S \text{ s.t. } |f(z)| = \|f\|, \tag{3}$$

which can be established by the usual derivation of the Kolmogorov criterion for linear approximation [11, Theorem 18]. In brief, if $\|f_K - r\| < \|f_K\|$ for some $r \in R$, then obviously r must satisfy (3). Conversely, if r satisfies (3), it is easy to show $\|f_K - \varepsilon r\| < \|f_K\|$ for all sufficiently small $\varepsilon > 0$. ∎

The next lemma states that the K-rotations of a BA to f_K are also BAs, up to a multiplicative constant. This observation has nothing to do with the fact that $\|\ \|$ is the Chebyshev norm, and for an application of the same idea in a least-squares approximation context, see [5, p. 54].

LEMMA 2. *If r^* is a BA to f_K, then so is the function $\hat{r}^*$ defined by $\hat{r}^*(z) = \omega_K r^*(\omega_K z)$.*

Proof. Since $zf_K(z)$ is K-symmetric we have $zf_K(z) = \omega_K zf_K(\omega_K z)$, hence $f_K(z) = \omega_K f_K(\omega_K z)$, from which we compute $\|f_K - \hat{r}^*\| = \|f_K(z) - \omega_K r^*(\omega_K z)\| = \|\omega_K f_K(\omega_K z) - \omega_K r^*(\omega_K z)\| = \|f_K - r^*\|$. ∎

The third lemma has more substance, and is essentially half of the nonuniqueness argument.

LEMMA 3. *If $n \geqslant 1$, then 0 is not a BA to f_K in R_{mn}.*

Proof. Let $\sigma \in S$ be any number with $\sigma \neq \zeta_k \; \forall k$. We claim that for all sufficiently small $\varepsilon > 0$, the function $r \in R_{01} \subseteq R_{mn}$ defined by

$$r(z) = c^{-1}/(z - (1+\varepsilon)\sigma) \tag{4}$$

satisfies (2) of Lemma 1. To establish this, it is enough to take $\varepsilon = 0$, because the pole at $\sigma(1+\varepsilon)$ remains bounded away from each ζ_k as $\varepsilon \to 0$. Thus if s denotes the Moebius transformation $s(z) = z/(z-\sigma)$, it will suffice to show that s maps S into $\operatorname{Re} z > 0$.

In fact, s maps S onto the line $\operatorname{Re} z = \frac{1}{2}$. To see this, note that s maps the straight line through $\sigma, 0, -\sigma$ onto $\mathbb{R}$. Therefore it must map S, which intersects that straight line at σ and $-\sigma$ with right angles, onto a generalized circle orthogonal to $\mathbb{R}$ at $z = \frac{1}{2}$ that passes through ∞, namely, the line $\operatorname{Re} z = \frac{1}{2}$. ∎

Our final lemma provides the other half of the argument.

LEMMA 4. *Suppose $m, n \geqslant 0$ and $K \geqslant m+2$. Then 0 is a BA to f_K among functions in R_{mn} whose set of poles is K-symmetric.*

Proof. Following Lemma 1, consider a function r in this subset of R_{mn}, which we can write $r(z) = p(z)/q(z)$ with $p \in P_m$ and $q \in P_n$, where p and q have no common factors. The K-symmetry of the poles of r implies that q is a K-symmetric function, and, in particular, $q(\zeta_k)$ has the same value for all k, which we can take to be 1 by dividing both p and q by this number.

Equation (2) of Lemma 1 thus reduces to the Kolmogorov criterion for the numerator p: does a polynomial $p \in P_m$ exist for which

$$\operatorname{Re}[czp(z)] > 0 \qquad \text{for } z = \zeta_k, 0 \leqslant k \leqslant K-1? \tag{5}$$

Since the polynomial $czp(z)$ has degree at most $m+1 < K$, its value at $z = 0$, namely, 0, is given by a discrete mean value over the points ζ_k,

$$0 = \frac{1}{K} \sum_{k=0}^{K-1} c\zeta_k\, p(\zeta_k).$$

By taking the real part we obtain

$$0 = \frac{1}{K} \sum_{k=0}^{K-1} \operatorname{Re}[c\zeta_k\, p(\zeta_k)],$$

which contradicts (5). This implies that the polynomial in question cannot exist, and the lemma is proved. ∎

Our first two main results are now straightforward consequences of the lemmas. The first theorem shows that for any $m \geqslant 0$ and $n \geqslant 1$, there exist functions whose BAs in R_{mn} are not unique.

THEOREM 1. *Suppose $m \geqslant 0$, $n \geqslant 1$, $K \geqslant m+2$. Then the BA to f_K in R_{mn} is nonunique.*

Proof. By Lemma 4, 0 is a BA to f_K among functions in R_{mn} with K-symmetric pole sets. On the other hand, by Lemma 3, it is not a BA in all of R_{mn}. This implies that any r^* of f_k must have a pole distribution that is not K-symmetric. Therefore the function $\hat{r}^*$ defined by $\hat{r}^*(z) = \omega_K r^*(\omega_K z)$ is distinct from r^*. On the other hand, by Lemma 2, $\hat{r}^*$ is also a BA to f_K. ∎

In the special case $m = 0$, we can take $K = 2$, and the above argument shows that the BA of type $(0, n)$ to any odd function $f \in A$ is nonunique, unless f attains maximum modulus at more than two points on S. Thus nonuniqueness in rational approximation on the disk is by no means confined to pathological examples.

In Theorem 1 there is no condition relating K and n. By making K large enough, we obtain examples for any $m \geqslant 0$ and $n \geqslant 1$ in which the number of BAs is arbitrarily large.

THEOREM 2. *Suppose $m \geqslant 0$ and $n \geqslant 1$, and let $K \geqslant m+2$ be an integer with no divisors j in the range $2 \leqslant j \leqslant n$. Then f_K has at least K distinct BAs in R_{mn}.*

Proof. We have seen in the last proof that a BA r_0^* must have a pole set that is not K-symmetric, which means, in particular, it must have at least one pole. On the other hand, since $r_0^* \in R_{mn}$, it can have at most n of them. Let ν be the number of poles of r_0^*. The hypothesis implies that ν and K are relatively prime, and it is obvious hat this implies that all of the funcions r_j^* defined by

$$r_j^*(z) = \omega_K^j r_0^*(\omega_K^j z), \qquad 0 \leqslant j \leqslant K-1,$$

must have distinct pole sets. By Lemma 2, these are all BAs to f_K in R_{mn}. ∎

This proof shows in fact not only that f_K has at least K distinct BAs, but that the number of them is an integral multiple of K. Note that it does not, however, exhibit a function that has an infinite number of BAs, although Ruttan has shown that such a situation can occur in complex rational approximation on an interval [16].

By essentially the same argument as the one above, it is not hard to construct functions that have nonunique BAs for many (m, n). For example, any function $f \in A$ of the form

$$f(z) = a_1 z + a_3 z^3 + a_7 z^7 + a_{15} z^{15} + \ldots, \qquad a_k > 0,$$

has nonunique BAs of all orders (m, n) with $n \geqslant 1$ and $m \neq 1, 3, 7, 15, \ldots$. We do not know whether there exist functions whose BAs of *all* orders with $n \geqslant 1$ are nonunique.

3. Nonuniqueness in R^{r}_{mn}

As in Section 1, let A^{r} and R^{r}_{mn} denote the subsets of real analytic functions in A and R_{mn}, respectively. In this section we are concerned with BAs to $f \in A^{\mathrm{r}}$ out of R^{r}_{mn}, which we will again denote by r^*. Existence of at least one such BA is guaranteed by the theory of Walsh [17].

Let f^{r}_K denote any function that satisfies

(a′) $f^{\mathrm{r}}_K \in A^{\mathrm{r}}$,

(b) $zf^{\mathrm{r}}_K(z)$ is K-symmetric,

(c) f^{r}_K attains maximum modulus at exactly K points $\{\zeta_k\}$ of S,

(d) $1 \notin \{\zeta_k\}$.

Note the presence of the new condition (d). In the theorem below K is even, so (d) implies also $-1 \notin \{\zeta_k\}$. Since $f^{\mathrm{r}}_K \in A^{\mathrm{r}}$, the points of extreme modulus of f^{r}_K on S will be the "skew Kth roots of unity" $\zeta_k = \omega_K^{k+1/2}$, $0 \leqslant k \leqslant K-1$. The fact $f^{\mathrm{r}}_K \in A^{\mathrm{r}}$ also implies that the constant c of Section 2 is real. We now show that for any $m \geqslant 0$ and $n \geqslant 1$, there exist functions whose BAs in R^{r}_{mn} are not unique.

Theorem 3. *Suppose $m \geqslant 0$, $n \geqslant 1$, $K \geqslant m+2$, and moreover $K = 2^J$ for some positive integer J. Then the BA to f^{r}_K in R^{r}_{mn} is not unique.*

Proof. First we observe that 0 is not a BA to f^{r}_K in R^{r}_{mn}, for in the proof of Lemma 3 we have already constructed a better approximation, namely, the function $r \in R^{\mathrm{r}}_{mn}$ given by (4) with $\sigma = 1$ or $\sigma = -1$.

On the other hand, Lemma 4 shows that 0 is a BA to f^{r}_K among functions in R^{r}_{mn} with K-symmetric pole sets. A fortiori, 0 is a BA among functions $r \in R^{\mathrm{r}}_{mn}$ with the property that $zr(z)$ is K-symmetric. Our proof will proceed by showing that the assumption that r^* is unique implies that $zr^*(z)$ is K-symmetric after all, a contradiction. We argue by induction, showing that $zr^*(z)$ is 2^j-symmetric successively for $j = 0, 1, \ldots, J$.

Case $j = 0$. Trivial, because any function is 1-symmetric.

Case $j+1$, $0 \leqslant j \leqslant J-1$. Let us write $\mu = 2^j$ for abbreviation. If $zr^*(z)$ is μ-symmetric, the Maclaurin series of r^* has the form

$$r^*(z) = a_1 z^{\mu-1} + a_2 z^{2\mu-1} + a_3 z^{3\mu-1} + \ldots, \qquad a_i \in \mathbb{R}.$$

Now $(\omega_K^{K/2\mu})^\mu = -1$, and therefore by applying Lemma 2 to r^*, $K/2\mu$ times in succession, we obtain a new function $\hat{r}^* \in R_{mn}^{\mathrm{r}}$,

$$\hat{r}^*(z) = -a_1 z^{\mu-1} + a_2 z^{2\mu-1} - a_3 z^{3\mu-1} + \ldots$$

which must also be a BA to f_K^{r}. If r^* is unique, then $\hat{r}^*$ must be the same as r^*, so the coefficients of odd index in this expansion are 0, and we have

$$r^*(z) = a_2 z^{2\mu-1} + a_4 z^{4\mu-1} + \ldots .$$

Thus $zr^*(z)$ is 2μ-symmetric, which completes the induction step. ∎

As in Section 2, there is some flexibility in this proof; we could for example take K to be any number containing some power of 2 larger than n as a factor, and then count poles. However, it appears that no argument of this type will establish the existence of more than two BAs to a given function. Thus the question of whether a large number of distinct real analytic BAs can occur must remain open.

On the other hand, it is easy to see that result (d) of the abstract, which we proved in Section 1 for type $(0, n)$, $n \geqslant 1$, holds for type $(m, 1)$, $m \geqslant 0$, also: *If f satisfies* (a′), (b), (c) *for an even integer $K > 2m$, and if $1 \in \{\zeta_k\}$, then f can be better approximated in R_{m1} than in R_{m1}^{r}.* For again 0 is not a BA from R_{m1} (by Lemma 3), but 0 is best in R_{m1}^{r}, as can be seen as follows: Assume $r \in R_{m1}^{\mathrm{r}}$ is better and has a finite pole $z_0 > 0$, say. Then we can write $r(z) = p(z)/q(z)$ with $q(z) = z_0^{K/2} - z^{K/2}$ and $p \in P_{m-K/2-1}$. Note that $q(\zeta_k) > 0$, $0 \leqslant k \leqslant K-1$, so $\mathrm{Re}[zp(z)]$ has constant sign on $\{\zeta_k\}$, which, as in the proof of Lemma 4, contradicts the discrete mean value theorem. By the same argument, 0 is best in R_{m0}^{r}. Hence, 0 is best in R_{m1}^{r}.

4. Padé and Best Approximation on Small Disks and Intervals

If $m, n \geqslant 0$ are given and f is analytic at the origin, the Padé approximant of type (m, n) to f is the unique rational function r^{p} of type (m, n), analytic at the origin, whose Maclaurin series matches that of f to as many terms as possible. Thus r^{p} is in a sense the optimal approximation to f at the point $z = 0$. A natural question posed by Walsh [18] is: if for each $\varepsilon > 0$, r_ε^* is a BA to f on $\varepsilon\Delta$, must one have $r_\varepsilon^* \to r^{\mathrm{p}}$ as $\varepsilon \to 0$? By $r_\varepsilon^* \to r_{\mathrm{p}}$ we mean that r_ε^* approaches r^{p} uniformly on any compact subset of $\mathbb{C}$ that contains no poles of r^{p}.

In 1964 Walsh answered this question in the affirmative for the restricted set of functions that satisfy the following normality condition [19]. Let f have the Maclaurin series $f(z) = a_0 + a_1 z + \ldots$, and define $a_k = 0$ for $k < 0$.

Assumption B. The $n \times n$ Hankel matrix

$$H = \begin{pmatrix} a_{m-n+1} & & a_m \\ & \iddots & \\ a_m & & a_{m+n-1} \end{pmatrix}$$

is nonsingular.

Assumption B appears frequently in the theory of Padé approximation, for it is readily seen that the coefficients of r^{p} satisfy a linear system of equations involving the matrix H.

Walsh's result asserts: *if f satisfies Assumption B, then $r_\varepsilon^* \to r^{\mathrm{p}}$ as $\varepsilon \to 0$.* But Walsh did not determine whether Assumption B is actually needed for this conclusion to be valid. It is easy to imagine that it might not be, for in 1974 Chui *et al.* published a result to the effect that in real approximation on a small interval $[0, \varepsilon]$, $r_\varepsilon^* \to r^{\mathrm{p}}$ for *any* f [3]. (We will return to their result below.)

Nevertheless, a variation of our symmetry arguments shows that Assumption B is essential after all. Following Section 2, let f_K be any function that satisfies the conditions

(a) $f_K \in A$,

(b) $zf_K(z)$ is K-symmetric,

(c′) $f_K^{(K-1)}(0) \neq 0$, $f_K^{(2K-1)}(0) \neq 0$.

THEOREM 4. *Suppose $m \geqslant 0$, $n \geqslant 1$, $K \geqslant 2m+3$, $f = f_K$. Then $r^{\mathrm{p}} \equiv 0$, but $r_\varepsilon^* \not\to 0$ as $\varepsilon \to 0$.*

Proof. Observe first that since $K - 1 > m$, the Maclaurin coefficients of f_K satisfy $a_k = 0$ for $k \leqslant m$, which implies hat the matrix H is singular. Thus f_K does not satisfy Assumption B, so the claim does not contradict Walsh's result. The fact $r^{\mathrm{p}} \equiv 0$ is also a consequence of $a_k = 0$ for $k \leqslant m$.

To prove that r_ε^* does not approach the zero function as $\varepsilon \to 0$, it is enough to show that not all poles of r_ε^* converge to ∞ as $\varepsilon \to 0$: that is, there exists $\rho > 0$ such that for all sufficiently small ε, r_ε^* has a pole in $\rho\Delta$. Let the given problem be rescaled from $\varepsilon\Delta$ to Δ by defining for each $\varepsilon > 0$ $F_\varepsilon(z) = f_K(\varepsilon z)$, $R_\varepsilon^*(z) = r_\varepsilon^*(\varepsilon z)$. We will in fact prove the stronger result that R_ε^* has a pole in $\rho\Delta$ for some ρ, which amounts to showing that at least one pole of r_ε^* converges linearly to the origin as $\varepsilon \to 0$. The proof as usual has two halves:

(i) 0 is not a BA to F_ε in R_{mn} (for all sufficiently small ε):

(ii) 0 is a BA to F_ε among functions in R_{mn} with no poles in $\rho\Delta$ (for some suitable fixed ρ).

Proof of (i). The function F_ε satisfies conditions (a) and (b) of Section 2 trivially for any $\varepsilon \in [0, 1]$. From (c′) it can be seen that for all sufficiently small positive ε, it also satisfies condition (c). Therefore Lemma 3 applies.

Proof of (ii). If 0 is not a BA to F_ε in the class mentioned, then by Lemma 1 there exists $r \in R_{mn}$ that satisfies (2) but has no poles in $\rho\Delta$. Let r be written $r(z) = p(z)/q(z)$ in lowest terms, with $q(0) = 1$. If ρ is large, q must be approximately 1 on Δ. In particular, for any $\theta > 0$ we can pick ρ large enough so that necessarily

$$\max_{z \in S} |\arg q(z)| \leqslant \theta,$$

which means that (2) implies

$$|\arg[czp(z)]| < \pi/2 + \theta \qquad \text{for} \quad z = \zeta_k, 0 \leqslant k \leqslant K-1. \tag{6}$$

For each k, define $\sigma_k, \tau_k \in \mathbb{R}$ by $c\zeta_k p(\zeta_k) = \sigma_k + i\tau_k$, and in addition, define $\sigma_k^+ = \max\{0, \sigma_k\}$ and $\sigma_k^- = \min\{0, \sigma_k\}$. Let $\sigma, \tau, \sigma^-, \sigma^+$ be the corresponding K-vectors. In this notation (6) amounts to the condition

$$|\sigma_k^-| \leqslant \tan\theta\, |\tau_k|, \qquad 0 \leqslant k \leqslant K-1.$$

By summing over k, we obtain

$$\|\sigma^-\|_1 \leqslant \tan\theta\, \|\tau\|_1$$

(vector 1-norms). At the same time, since $K > m+1$, our usual mean-value argument from Lemma 4 implies $\sum_{k=0}^{K-1} \sigma_n = 0$, hence $\|\sigma^-\|_1 = \frac{1}{2}\|\sigma\|_1$, and therefore we have

$$\|\sigma\|_1 \leqslant 2\tan\theta\, \|\tau\|_1. \tag{7}$$

On the other hand, σ and τ cannot differ too greatly in norm. The real and imaginary parts of $czp(z)$ on S are conjugate trigonometric polynomials in

$\arg z$ of degree $m+1$, with constant terms 0, and therefore they have equal L_2 norms on S. Since $K \geqslant 2m+3 = 2(m+1)+1$, they are moreover the unique trigonometric polynomials of degree $m+1$ that interpolate $\{\sigma_k\}$ and $\{\tau_k\}$ at the points $\{\arg \zeta_k\}$, and the equality of L_2 norms on S carries over to equality of l_2 norms on $\{\zeta_k\}$:

$$\|\tau\|_2 = \|\sigma\|_2. \tag{8}$$

(In fact $\tau = W_K \sigma = -W_K^2 \tau$, where W_K is the so-called *Wittich matrix*, with $\|W_K\|_2 = 1$.) By discrete Hölder inequalities, we have in general

$$\|\tau\|_1 \leqslant \sqrt{K}\, \|\tau\|_2, \qquad \|\sigma\|_2 \leqslant \|\sigma\|_1,$$

and so (8) implies

$$\|\tau\|_1 \leqslant \sqrt{K}\, \|\sigma\|_1. \tag{9}$$

It is now clear that (7) and (9) will be inconsistent, contradicting the assumption that r exists, provided ρ is taken large enough so that θ is small enough to ensure $2\sqrt{K} \tan\theta < 1$. This proves (ii). ∎

Before closing, we will make some remarks on related matters.

Approximation on small intervals. How does Theorem 5 relate to the result of Chui *et al.* mentioned above? Suppose $f(x)$ is a complex function of class $C^{m+n+1}[0,1]$, and let r_ε^* be a BA to f in R_{mn} on $[0,\varepsilon]$. First, Walsh showed in the early 1970s that his result of 1964 extends to this problem too: if f satisfies Assumption B, then $r_\varepsilon^* \to r^{\mathrm{P}}$ as $\varepsilon \to 0$ [20]. The purpose of the paper of Chui *et al.* [3] was to extend Walsh's result by removing the hypothesis of Assumption B. However, although their proof does not require Assumption B, it assumes that f is real and that r_ε^* denotes its (unique) BA with *real* coefficients. In contrast, by an argument much like that above, one can readily show for at least some (m,n) that $r_\varepsilon^* \to r^{\mathrm{P}}$ can fail to hold if r_ε^* is a BA with complex coefficients, even when f is real. For example, take $f(x) = x$ on $[-\varepsilon, \varepsilon]$ and $(m,n) = (0,1)$; then the argument of Fig. 1 shows that r_ε^* has a pole for every ε, but since f is linear, the problem is scale-invariant, so this pole will approach 0 linearly with ε.

Uniqueness and Assumption B. In all of our examples in which r^* is nonunique, f has failed to satisfy Assumption B. Can it be that Assumption B is enough to ensure uniqueness? To see that the answer is no, take $(m,n) = (0,1)$ and consider the function

$$f(z) = \eta/(z-2) + z + z^3.$$

For any $\eta > 0$, f satisfies Assumption B, but for all sufficiently small η, a variation of the argument of Fig. 1 shows that any BA has non-real coefficients and hence is nonunique.

On the other hand, to the best of our knowledge it is possible that for any function satisfying Assumption B, the BA on $\varepsilon\Delta$ is unique for all sufficiently small ε.

Approximate uniqueness and approximate real analyticity. Suppose r^{p} has exactly n finite poles and, in addition, the Maclaurin series of f and r^{p} agree through degree $m+n$ but no further. This condition, which implies Assumption B, is called *Assumption A* in [14] and [15], and a number of asymptotic results are proved there by the *CF method* regarding approximation of such functions on small disks $\varepsilon\Delta$. One of these is that BAs are "approximately unique": any two BAs r_1^*, r_2^* satisfy $r_1^* - r_2^* = O(\varepsilon^{m+n+2})$ uniformly on compact sets containing no poles of r^{p} [14, Section 6]. In contrast, the construction of Theorem 4 here shows that in the absence of Assumption B, $r_1^* - r_2^*$ need not approach zero with ε at all. Alternatively, Assumption A also implies $\|r_1^* - r_2^*\|_{\varepsilon\Delta} = O(e^{2m+2n+3})$, while in the absence of Assumption B it appears that this must be weakened to $O(\varepsilon^{2m+3})$.

For $f \in A^{\mathrm{r}}$, analogous estimates follow from [14] for how close r_ε^* must be to R_{mn}^{r}, in particular to the *CF approximant* $r_\varepsilon^{\mathrm{cf}} \in R_{mn}^{\mathrm{r}}$: $\|r_\varepsilon^* - r_\varepsilon^{\mathrm{cf}}\| = O(\varepsilon^{m+n+2})$ on compact sets with no poles of r^{p}, and $\|r_\varepsilon^* - r_\varepsilon^{\mathrm{cf}}\|_{\varepsilon\Delta} = O(\varepsilon^{2m+2n+3})$. Again it seems that without Assumption B, these reduce to ε^0 and ε^{2m+3}, respectively.

In summary, although degeneracy of the Padé approximant is not necessary for nonuniqueness and associated phenomena in complex rational approximation, it is evidently a related factor.

Notes added in proof. (i) *Block structure in the Walsh table.* It is well known that if the best real approximations $\{r_{mn}^*\}$ to a continuous real function f on a real interval are arranged in a so-called *Walsh table* indexed by m and n, then degenerate situations in which a single rational function is best for several degrees (m, n) always occur precisely in square blocks, except where $r_{mn}^* \equiv 0$ [11]. (An analogous block structure appears in the Padé table.) From the example at the end of Section 2 one may however conclude that in a complex approximation such a block structure need not occur, at least in the top row ($n = 0$) of the Walsh table. Indeed, our arguments show that for each $k \geqslant 1$, the best approximations to f of types $(2^k - 1, 0), \ldots, (2^{k+1} - 2, 0)$ are all equal to the polynomial section of f of degree $2^k - 1$, while for any $n \geqslant 1$ the best approximations of types $(2^k, n), \ldots, (2^{k+1} - 2, n)$ are better.

(ii) *Padé and best approximation.* Several additional results have been obtained concerning the Padé and best approximation questions discussed in Section 4; see [21]. In particular, further examples show that $r^* \nrightarrow r^{\mathrm{p}}$ can occur even for *real* approximations of real functions, both on $[0, \varepsilon]$ and on $[-\varepsilon, \varepsilon]$; the same is also true for the analogously restricted best approximation on small disks $\varepsilon\Delta$. Thus the result of [3] quoted above is *false*.

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