Restrictions on the structure of subgroup lattices of finite alternating and symmetric groups

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\textbf{Abstract}

Let $G$ be a finite alternating or symmetric group. We describe an infinite class of finite lattices, none of which is isomorphic to any interval $[H, G]$ in the subgroup lattice of $G$.

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\textbf{1. Introduction}

We exhibit restrictions on the structure of the lattice $O_G(H)$ of subgroups of $G$ containing $H \leq G$ when $G$ is a finite alternating or symmetric group. This is part of a program in which we aim to exhibit such restrictions on $O_G(H)$ for an arbitrary finite group $G$.

For a finite lattice $L$, let $L'$ be the poset obtained by removing the minimum and maximum elements of $L$. For a positive integer $n$, let $\Delta(n)$ be the lattice of all subsets of $[n] := \{1, \ldots, n\}$, ordered by inclusion. For positive integers $m_1, \ldots, m_t$, let $D\Delta(m_1, \ldots, m_t)$ be the lattice such that

\begin{itemize}
  \item $D\Delta(m_1, \ldots, m_t)'$ is the disjoint union of posets $C_1, \ldots, C_t$,
  \item $C_i \cong \Delta(m_i)'$ for all $i$, and
  \item if $i \neq j$ then no element of $C_i$ is comparable to any element of $C_j$.
\end{itemize}

The lattice $\Delta(3, 3)$ is pictured below.

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We have conjectured that if \( t > 1 \) and each \( m_i > 2 \), then there do not exist finite groups \( H \leq G \) such that \( \mathcal{O}_G(H) \cong D\Delta(m_1, \ldots, m_t) \). Indeed, \( D\Delta(m_1, \ldots, m_t) \) is a CD-lattice, as defined in [As3]. Also, this conjecture is a special case of [Sh1, Conjecture A]. The problem of determining whether every finite lattice is isomorphic with some \( \mathcal{O}_G(H) \) with \( G \) finite arose originally in universal algebra (see for example [PâPu]). Significant, but not dispositive, progress on another approach to settling this problem appears in [BaLu].

Fix integers \( t > 1 \) and \( m_1 \geq m_2 \geq \cdots \geq m_t \geq 3 \). Set
\[
L(m_1, \ldots, m_t) := \{(H,G) : H \leq G, \ G \text{ finite}, \mathcal{O}_G(H) \cong D\Delta(m_1, \ldots, m_t)\}.
\]
In [As3], the first author of this paper shows that if \( L(m_1, \ldots, m_t) \neq \emptyset \) then one of the following cases must hold.

(Simp) There is some \((H,G) \in L(m_1, \ldots, m_t)\) with \( G \) almost simple.

(Sig) There is some \( \tau = (G,H,I) \) such that \( G \) is an almost simple group, \( I \triangleleft H \leq G \) with \( H/I \) almost simple, \( G \) is generated by \( H \) and the signalizers of \( H \), and \( L(m_1, \ldots, m_t) \) is isomorphic to the lower signalizer lattice \( \Xi(\tau) \), as defined in [As4].

As mentioned above, we have the following result.

**Theorem 1.1.** There is no \((H,G) \in L(m_1, \ldots, m_t)\) such that \( G \) is an alternating or symmetric group.

Here is an outline of the proof of Theorem 1.1. We consider separately the cases where \( H \) is intransitive, imprimitive or primitive on the natural \( G \)-set \( \Omega \). The case where \( H \) is primitive and \(|\Omega|\) is not prime is eliminated in [As2,As1]. All primitive groups of prime degree are known, and this allows us to handle quickly the remaining cases where \( H \) is primitive, see Proposition 6.3. Eliminating the case where \( H \) is intransitive is relatively straightforward, see Proposition 6.4. The most challenging case is that where \( H \) is imprimitive. A key point here is that if \( H < K < G \) then \( \mathcal{O}_G(K) \cong \Delta(n) \) for some \( n \). In particular, there are some \( K \in \mathcal{O}_G(H) \) such that \( \mathcal{O}_G(K) = (K,M_1,M_2,G) \) where \( M_1 \) is an imprimitive maximal subgroup of \( G \) and \( M_2 \) is an imprimitive or primitive maximal subgroup of \( G \). All possibilities for \( K \) are described in Theorem 5.2, which depends on several preparatory results in Sections 3 and 4, and the elimination of the case where \( H \) is imprimitive follows fairly directly. Some of the preparatory results just mentioned are also of use in examining the signalizer lattice problem.

**2. Notation and basic definitions**

In this section, we introduce notation and terminology (other than that given in the introduction) that will be used in the following sections.

Throughout this paper, \( \Omega \) is a finite set of size \( n \), and \( S \) and \( A \) are, respectively, the symmetric and alternating groups on \( \Omega \). For \( \Gamma \subseteq \Omega \) and \( H \leq S, H_\Gamma \) is the pointwise stabilizer of \( \Gamma \) in \( H \). We often write \( H_\omega \) for \( H_{\{\omega\}} \). The setwise stabilizer of \( \Gamma \) in \( H \) will be denoted by \( N_H(\Gamma) \). The complement of \( \Gamma \) in \( \Omega \) will sometimes be denoted by \( \Gamma' \). If a group \( G \) acts on a set \( X \) and \( Y \subseteq X \) then \( Y^G \) will denote the orbit of \( Y \) under the induced action of \( G \) on the power set of \( X \). Also, if \( U \subseteq X \) is \( G \)-invariant then \( G^U \) will denote the image of \( G \) in the symmetric group on \( U \) determined by the action of \( G \) on \( U \). For \( g \in S \), we define \( M(g) := \{\omega \in \Omega : \omega g \neq \omega\} \).
If a group $H$ acts on another group $K$, then $\mathcal{I}_K(H)$ is the lattice of subgroups of $K$ fixed (setwise) by $H$. We use the following standard notation from group theory. For a group $G$ and a prime $p$, $m_p(G)$ is the largest rank of an elementary abelian $p$-subgroup of $G$, $O_p(G)$ is the largest normal $p$-subgroup of $G$ and $O_p^*(G)$ is the smallest normal subgroup of $G$ having index a power of $p$.

Let $L$ be a finite lattice. As mentioned above, $L'$ will denote the poset obtained from $L$ by removing its unique minimum element and its unique maximum element. The set of coatoms of $L$ (that is, maximal elements of $L'$) will be denoted by $L^*$. For groups $H \leq G$, we will write $\mathcal{M}(H)$ for $O_{\mathcal{C}}(H)^*$, so $\mathcal{M}(H)$ is the set of maximal subgroups of $G$ that contain $H$. Given a group $Y$ acting on a poset $P$, $P(Y)$ will denote the poset of $Y$-invariant elements of $P$. We sometimes write $P^*(Y)$ for $P(Y)^*$. For any elements $x, y$ of any poset $P$, we write $[x, y]$ and $(x, y)$, respectively, for $\{z \in P : x \leq z \leq y\}$ and $\{z \in P : x < z < y\}$.

The set $\mathcal{P}$ of all partitions of $\Omega$ is a lattice, with the reverse refinement order. That is, say $\pi = \{P_1, \ldots, P_k\} \in \mathcal{P}$, so $\mathcal{P}$ is the disjoint union of the nonempty subsets $P_i$. If $\rho = \{R_1, \ldots, R_l\} \in \mathcal{P}$ then $\pi \succeq \rho$ if each $P_i$ is contained in some $R_j$. For $H \leq S$, $H_{\pi}$ consists of those $h \in H$ that fix each $P_i$ setwise. The stabilizer of $\pi$ in $H$, that is, the set of all $h \in H$ that map each $P_i$ to some $P_j$, will be denoted by $N_H(\pi)$. The partition $\pi$ is an equipartition or regular partition if $|P_i| = |P_j|$ for all $i, j \in [k]$. If $\pi$ consists of $l$ sets, each of size $m$, we will sometimes call $\pi$ a regular $(m, l)$-partition. Note that if $Y \leq S$ is transitive then every element of $\mathcal{P}(Y)$ is an equipartition. The minimum and maximum elements of $\mathcal{P}$ will be denoted by $\emptyset$ and $\infty$, respectively.

If a group $H$ is the direct product of pairwise isomorphic subgroups $L_1, \ldots, L_k$, a full diagonal subgroup of $H$ is a subgroup $D$ such that the restriction of the standard projection $\pi_1 : H \to L_1$ to $D$ is an isomorphism for all $i$.

Let $H$ be a subgroup of $S$ that acts primitively on $\Omega$, let $\omega \in \Omega$ and let $D$ be the generalized Fitting subgroup of $H$. We call $H$

- **affine** if $H$ has a normal elementary abelian subgroup that acts regularly on $\Omega$;
- **doubled** if $H$ has two distinct minimal normal subgroups;
- **complemented** if $D$ is unique minimal normal subgroup of $H$, is nonabelian, and acts regularly on $\Omega$;
- **diagonal** if $D = L_1 \times \cdots \times L_k$, where
  - $k > 1$,
  - the $L_i$ are pairwise isomorphic nonabelian simple groups permuted transitively by $H$, and
  - there is some maximal (with respect to the order on partitions described above) $H$-invariant partition $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ of $\{L_1, \ldots, L_k\}$ such that $D_\omega$ is the direct product of subgroups $E_1, \ldots, E_l$, with each $E_i$ a full diagonal subgroup of $\prod_{j \in \sigma_i} L_j$.

An affine primitive subgroup of $G \leq [A, S]$ that is maximal in $G$ is called the stabilizer of an affine structure. If $|\Omega| = m^k$, the stabilizer of a regular $(m, k)$-product structure in $S$ is a wreath product of $S_m$ by $S_k$ embedded in $S$ through a bijection from $[m]^k$ to $\Omega$. An element of this wreath product is of the form $(\sigma, t)$, with $\sigma \in S_k$ and $t = (\tau_1, \ldots, \tau_k) \in S_m^k$. For $v = (v_1, \ldots, v_k) \in [m]^k$, we have $v(\sigma, t) = (v_{1\sigma^{-1}}, \tau_1, \ldots, v_{k\sigma^{-1}}, \tau_k)$. The stabilizer of a regular $(m, k)$-product structure in $A$ is the intersection of such a wreath product with $A$. A primitive group $H \leq S$ stabilizes a product structure if $H$ is contained in the stabilizer of a regular product structure.

3. Lemmas on permutation groups

Given a partition $\Gamma$ of $\Omega$, set $K_+^+(\Gamma) = \langle A_{\Omega - \gamma} : \gamma \in \Gamma \rangle$.

3.1. Suppose $G$ is a primitive subgroup of $S$.

(1) If $G$ contains a transposition or a 3-cycle, then $A \leq G$.

(2) If $G$ contains an involution $t$ such that $|M(t)| = 4$ then one of the following conditions holds.

(a) $A \leq G$.

(b) We have $n = 5$ and $O_5(G) \neq 1$. 

(c) We have \( n = 6 \) and \( F^*(G) \cong L_2(5) \).
(d) We have \( n = 7 \) and \( G \cong L_3(2) \).
(e) We have \( n = 8 \) and \( G \cong AGL_3(2) \) is the stabilizer of an affine structure.

**Proof.** Claim (1) is a result of C. Jordan (see for example [As6, Exercise 5.6.2] or [DiMo, Theorem 3.3A]), as is the fact that if primitive \( G \) contains an involution \( t \) with \( |M(t)| = 4 \) then either \( n \leq 8 \) or \( A \leq G \) (see for example [DiMo, Example 3.3.1]). All primitive permutation groups of degree at most eight are known, and (2) follows quickly after examining these groups. (Note that if \( n = 8 \) and \( G \) is affine, we have \( G = H \cdot V \), where \( V \cong \mathbb{Z}_2^3 \) acts regularly and \( H \) acts irreducibly on \( V \). It follows from \( t \in G \) that \( |H| \) is even, and from this and the irreducibility of \( H \) we get \( H = GL(V) \).) \( \square \)

### 3.2. Assume \( Y \) is a transitive subgroup of \( S, \alpha \subseteq \Omega \), and \( A_{\Omega - \alpha} \leq X \leq S_{\Omega - \alpha} \) with \( 1 \neq X \leq W \leq Y \). Then

(1) If \( Y \) is primitive on \( \Omega \) then \( A \leq W \).
(2) If \( Y \) is imprimitive on \( \Omega \), then \( \mathcal{P}'(Y) \) has a greatest member \( \Gamma = \Gamma_Y \). Further \( \alpha \subseteq \gamma \leq \Gamma, X \leq S_\Gamma, \) and \( K_+(\Gamma) \leq W \).

**Proof.** As \( 1 \neq X \) and \( A_{\Omega - \alpha} \leq X \leq S_{\Omega - \alpha} \), \( X = \langle \chi \rangle \), where \( \chi \) is the set of transpositions or 3-cycles in \( X \). Thus (1) holds if \( Y \) is primitive on \( \Omega \) by 3.1, so we may assume \( Y \) is imprimitive on \( \Omega \).

Let \( \Gamma = \gamma^Y \in \mathcal{P}^*(Y) \) with \( \alpha \cap \gamma \neq \emptyset \). Let \( x \in \chi \), set \( m = |x| \), and suppose \( \beta^x \neq \beta \in \Gamma \). Then \( m = 2 \) or \( 3 \) and \( \{ \beta^i : 1 \leq i \leq m \} \) is an orbit of \( \langle x \rangle \) on \( \Gamma \) of length \( m \). Thus \( |M(x)| = m \) and \( \beta \), contradicting \( \beta^x = \beta^x \).

We have shown \( x \in S_\Gamma \). Therefore \( X = \langle x \rangle \subseteq S_\Gamma \). Then as \( X \) is transitive on \( \alpha, \alpha \subseteq \gamma \). By maximality of \( \Gamma, N_\gamma(\gamma) \) is primitive on \( \gamma \), so by 3.1, \( N_\gamma(Y) \) contains the alternating group on \( \gamma \). Thus \( A_{\Gamma - Y} \leq \langle X^{N_\gamma(\gamma)} \rangle \leq N_\gamma(Y) \), so as \( Y \) is transitive on \( \Gamma, K_+(\Gamma) \leq W \).

Set \( \chi' = A_{\Gamma - \gamma} \) and let \( \gamma' \in \Gamma' \in \mathcal{P}^*(Y) \) with \( \gamma \cap \gamma' \neq \emptyset \). The tuple \( \langle \gamma', \chi', \Gamma' \rangle \) satisfies the hypotheses of \( \alpha, X, \Gamma \), so by the previous paragraph, \( \gamma \subseteq \gamma' \). By symmetry, \( \gamma' \subseteq \gamma \), so \( \Gamma' = (\gamma')^Y = \gamma^Y = \Gamma \), completing the proof of (2). \( \square \)

### 3.3. Suppose \( H \leq S \) and let \( \alpha \) be an orbit of \( H \) on \( \Omega \). Assume \( W \in I_5(H) \) and \( A_{\Omega - \alpha} \leq X \leq S_{\Omega - \alpha} \) with \( 1 \neq X \leq W \). Let \( \Sigma \) be the orbit of \( W \) on \( \Omega \) containing \( \alpha \). Then \( H \) acts on \( \Sigma \) and either:

(1) \( A_{\Omega - \Sigma} \leq W \), or
(2) \( H = WH \) is imprimitive on \( \Sigma \) and \( \mathcal{P}'(Y^\Sigma) \) has a greatest member \( \Gamma = \Gamma_Y \). Further \( \alpha \subseteq \gamma \leq \Gamma \) and \( A_{\Omega - \gamma} \leq W \).

**Proof.** As \( H \) acts on \( W \) and \( \alpha, H \) acts on \( \Sigma \). We apply 3.2 with \( \alpha, W_\Sigma, Y_\Sigma, X_\Sigma, \Sigma \) in the respective roles of \( \alpha, W, Y, X, \Omega \), and conclude that either \( A_{\Sigma} \leq W_\Sigma \), or \( \mathcal{P}^*(Y_\Sigma) = \langle \Gamma \rangle \) with \( \alpha \subseteq \gamma \leq \Gamma \) and \( A_{\Sigma - \gamma} \leq W_\Sigma \). In the former case \( A_{\Omega - \Sigma} = \langle X^W \rangle \leq W \), while in the latter case, \( A_{\Omega - \gamma} = \langle X^{N_\gamma(Y)} \rangle \leq W \), completing the proof. \( \square \)

### 3.4. Let \( \Gamma, \Sigma \in \mathcal{P}(\Omega) \).

(1) \( S_\Gamma \cap S_\Sigma \leq S_{\Gamma \vee \Sigma} \).
(2) \( S_{\Gamma \wedge \Sigma} \leq S_{\Gamma \wedge \Sigma} \).

**Proof.** Let \( K = S_\Gamma \) and \( J = S_\Sigma \). For \( \alpha \in \Gamma \) and \( \beta \in \Sigma \) with \( \gamma = \alpha \cap \beta \neq \emptyset \), \( K \cap J \) acts on \( \alpha \) and \( \beta \), and hence on \( \gamma \). Thus (1) holds.

Similarly \( \delta \in \Delta = \Gamma \wedge \Sigma \) is the union of blocks of \( \Gamma \), each of which is fixed by \( K \), so \( K \) acts on \( \delta \). Thus \( K \leq S_\Delta \), so (2) holds. \( \square \)

### 3.5. Assume \( H \) is a transitive subgroup of \( S, \Gamma \in \mathcal{P}(H) \), \( \gamma \leq \Gamma \), and \( \Omega = H_{\Omega - \gamma} \) is transitive on \( \gamma \). Assume \( G \in O_5(H) \) is primitive on \( \Omega \). Then \( G \) is 2-transitive on \( \Omega \) and either \( G \) is almost simple or the following hold:
(1) $n = 2^a + 1$ for some positive integer $a$, and $G$ is the stabilizer in $S$ of an affine structure on $\Omega$. Set $D = F^a(G)$ and let $\omega \in \Omega - \gamma$.

(2) $|\Gamma| = 2$.

(3) $Q \cong E_{2\omega}$ is the full group of transvections in $G_{\omega} \cong GL(D)$ with axis $B = C_D(Q)$.

(4) $N_G(\Gamma) = N_G(B) = N_G((\mathbb{Q}, Q^d))$ for $d \in D - B$, $\{Q, Q^d\} = Q^H$, and $B$ is a full diagonal subgroup of $Q \times Q^d$.

**Proof.** By Jordan's Theorem (cf. 15.17.1 in [As6]), $G$ is 2-transitive on $\Omega$. Thus $G$ is almost simple or affine, and we may assume the latter. Then $n = |D| = p^e$ for some prime $p$ and positive integer $e$.

Let $X = N_G(D)$. Then $X = DX_\omega$, with $X_\omega$ acting faithfully as $GL(D)$ on $D$ (see for example [DiMo, pp. 54–55]). Further $G = DG_\omega$ with $Q \leq G_\omega \leq X_\omega$. Next $|\gamma|$ divides $n = p^e$, so $|\gamma| = p^a$ for some $0 < a < e$. Now

\[ p^a(p^e - a - 1) = p^e - p^a = n - |\gamma| = |\text{Fix}(Q)| = |B| = p^b \]

for some nonnegative integer $b$. Thus as $a < e$, we conclude that $p = 2$ and $e = a + 1$. Thus

\[ |\Gamma| = n/|\gamma| = 2^{a+1}/2^a = 2, \]

establishing (2), and $|B| = 2^a$ so that $B$ is a hyperplane of $D$. Then as $Q$ is transitive on $\gamma$, $Q$ is the full group of transvections in $GL(D)$ with axis $B$. As $G$ is primitive on $\Omega$, $G_\omega$ irreducible on $D$, so as $G_\omega$ contains the full group $Q$ of transvections with axis $B$, $G_\omega = GL(D)$. Therefore $G = X$, completing the proof of (1) and (3).

Next $\Gamma = \{\gamma, \alpha\}$, where $\alpha = \Omega - \gamma = \text{Fix}(Q) = \omega B$. Thus $N_G(Q) \leq N_G(\Gamma)$. Now the set $\mathcal{O}$ of orbits of $B$ on $\Omega$ is of order 2, so $\mathcal{O} = \Gamma$. Of course $N_G(B)$ acts on $\mathcal{O}$, so $N_G(B) \leq N_G(\Gamma)$, and in particular $D \leq N_G(\Gamma)$. Thus $N_G(\Gamma) = DN_G(D)$ and $N_G(\Gamma)$ acts on $\alpha = \omega B$, and hence on $B$. Thus $N_G(B) = N_G(\Gamma)$. Further $N_G(D) = N_G(Q)$, so $|N_G(B): N_G(Q)| = 2$, so $Q_{N_G(B)} = Q^H$, as $|Q^H| = 2$ by transitivity of $H$ on $\Gamma$. Thus $N_G(B) = N_G((\mathbb{Q}, Q^d))$ for $d \in D - B$.

Finally $Q^d \leq Q_{\mathcal{D}}(Q) = Q B$ and $Q^d \cap D = Q \cap D = 1$, so $B$ is a full diagonal subgroup of $Q \times Q^d$, completing the proof. $\square$

3.6. Say $\Omega$ is the disjoint union of $B$ and $C$ with $|B| > |C| > 1$. Let $Y$ be the setwise stabilizer of $B$ (and $C$) in $G$. Let $L$ be a maximal subgroup of $Y$ with $[Y : L] \neq 2$.

(1) If $G = S$ then there exist $E \in \{B, C\}$ and a maximal subgroup $T$ of $S_E$ such that $L = S_T \times T$.

(2) If $G = \Theta$ then one of the following conditions holds.

(a) We have $|B| = 4$, $|C| = 3$, and there is a surjective homomorphism $\phi : S_C \to S_B$ with $L = \{(x, \phi(x)) : x \in S_C\}$.

(b) There exist $E \in \{B, C\}$, $T \leq A_E$ and $y \in L \setminus (A_B \times A_C)$ such that

- $T(y)$ is a maximal subgroup of $A_E(y)$, and
- $L = (A_T \times T)(y)$.

**Proof.** Let $M = S_B \times S_C$, so $Y = G \cap M$. For $E \in \{B, C\}$ let $\pi_E : M \to S_E$ be the natural projection, so kernel($\pi_E$) = $S_{E}$.

Fix $E \in \{B, C\}$. Since $L$ is maximal in $Y$, either $A_E \leq L$ or $Y = A_E L$. If $Y = A_E L$ then $\pi_E(L) = \pi_E(Y) = S_E$, and every subgroup of $S_E$ normalized by $L$ is normal in $S_E$. Therefore, if $L$ contains neither $A_B$ nor $A_C$, then either $S_B \cap L = S_C \cap L = 1$ or there is some $E \in \{B, C\}$ with $|E| = 4$ and $|L \cap S_E| = 4$. In the first case, we get

\[ S_C = \pi_C(L) \cong L \cong \pi_B(L) = S_B, \]

contradicting the fact that $|B| > |C|$. In the second case, we have $|E| \neq 4$, so $L \cap S_E = 1$ and $L \cong \pi_E(L) \cong S_4$. Moreover, by a well known result about subgroups of direct products, there is a surjection
\[ \phi : \pi_E(L) \to \pi_E(L)/(S_E \cap L) \]

with kernel \( S_E \cap L \). It follows that \( B = E \) and \( |C| = 3 \). Moreover, \( L \leq A_7 \) and \( |L| = 24 \). Thus condition (a) of (2) holds.

We may now assume that \( A_E \leq L \) for some \( E \in \{ B, C \} \). Say \( G = S \), so \( Y = M \). If \( S_E \not\leq L \) then, by the maximality of \( L \) in \( M \), we have \( L \cap E = M \) and \( \pi_E(L) = \pi_E(M) \). This gives \( [M : L] = 2 \), a contradiction. Therefore \( S_E \subseteq L \) and (1) follows.

Say \( G = A \). Since \( A_E \leq L \), we have \( L \cap (A_B \times A_C) = A_E(L \cap A_F) \). Since \( [Y : L] \neq 2 \), we cannot have \( A_T \leq L \). Since \( L \) is maximal in \( Y \), there is some \( y \in L \setminus (A_B \times A_C) \). Let \( T = L \cap A_F \). Then \( L = A_E T(y) \), and if \( T(y) \leq X \leq A_T(y) \) then \( L \leq A_E X \leq Y \). Therefore, condition (b) of (2) holds. \( \square \)

The next result appears in [LPS].

3.7. Let \( G = A \) or \( S \), let \( \emptyset \neq B \subset \Omega \) and let \( \pi \) be a \((k, l)\)-regular partition of \( \Omega \) with \( 1 < k < n \).

\( \begin{align*}
(1) & \text{ } N_G(B) \text{ is maximal in } G \text{ if and only if } |B| \neq |\Omega|/2. \\
(2) & \text{ } N_G(\pi) \text{ is maximal in } G \text{ unless } G = A, k = 2 \text{ and } l = 4, \text{ in which case } N_G(\pi) \text{ is contained in the stabilizer of an affine structure.}
\end{align*} \)

4. Permutations with many fixed points

Let \( S \) be the set of permutations of cycle type \( 2^2 \) on \( \Omega \). Let \( R \) be the set of permutations \( r \in S \) of prime order such that \( |\text{Fix}(r)| \geq n/2 \).

4.1. Suppose \( s = (a, b)(c, d) \in S \), and \( \Gamma \in \mathcal{P}'(s) \) is regular. Then either:

\( \begin{align*}
(1) & \text{ } s \in S_{\Gamma}, \text{ or } \\
(2) & \text{ } \text{The blocks of } \Gamma \text{ are of size } 2 \text{ and } |a, c) \text{ or } |a, d) \in \Gamma.
\end{align*} \)

Proof. Straightforward. \( \square \)

The next result is essentially due to Guralnick and Magaard.

4.2. Say \( s \in R \) and \( H \) is a primitive overgroup of \( s \) in \( S \).

\( \begin{align*}
(1) & \text{ One of the following conditions holds.} \\
(a) & \text{ } H \text{ is almost simple, and one of the following conditions holds.} \\
& \text{ (i) We have } n = \binom{m}{k} \text{ with } 1 \leq k \leq \frac{m}{4} \text{ and } F^*(H) \text{ acts as the alternating group on } m \text{ points does on } k-\text{sets.} \\
& \text{ (ii) } F^*(H) \text{ is an orthogonal group over } \mathbb{F}_2. \\
(b) & \text{ We have } n = 2^a, s \text{ is an involution with } |\text{Fix}(s)| = n/2, H \text{ is affine, and } s \text{ centralizes a hyperplane of } F^*(H). \\
(c) & \text{ } H \text{ stabilizes a regular } (m, r)-\text{product structure } F \text{ on } \Omega = \Gamma_1 \times \cdots \times \Gamma_r, \text{ and } s = s_1 \cdots s_r \text{ is in the kernel } K_1 \times \cdots \times K_r \text{ of } N_S(F) \text{ on } \{ \Gamma_1, \ldots, \Gamma_r \}, \text{ with } s_i \in K_i, \text{ where } K_i \text{ acts faithfully as } \text{Sym}(\Gamma_i) \text{ on the } i\text{th factor } \Gamma_i \text{ and trivially on } \Gamma_j \text{ for } j \neq i. \text{ Further}
\end{align*} \)

\[ \frac{|\text{Fix}(s)|}{n} = \prod_{i=1}^{r} f(s_i), \]

where \( f(s_i) = |\text{Fix}_{\Gamma_i}(s_i)|/m. \text{ Thus } 1 \neq f(s_i) \geq 1/2 \text{ for some } 1 \leq i \leq k. \)

(2) If \( H \) is a doubly transitive maximal subgroup of \( G \in \{ A, S \} \) then one of the following conditions holds.

\( \begin{align*}
(a) & \text{ } H = A \text{ and } G = S. \\
(b) & \text{ We have } n = 2^a, a \geq 3, H \text{ is the stabilizer of an affine structure, and } G = A.
\end{align*} \)
Proof. Claim (2) follows from Claim (1) after comparing the groups described in (1) with the list of doubly transitive permutation groups (found for example in [DiMo, Section 7.7]). Now we prove (1). It follows from [GuMa, Theorem 1] that either one of (a) or (b) holds or $G$ stabilizes some regular $(m,r)$-product structure on $\Omega$. Assume the last case holds. We have $m > 2$, since if $\Gamma$ is a $(2,r)$-product structure then the stabilizer of a point in the action of $N = N_5(\Gamma)$ on $\Omega$ is isomorphic to $S_r$ and not maximal in $N$. Let $\sigma$ be the permutation of $\{T_1, \ldots, T_r\}$ induced by $s$ and let $c$ be the number of orbits of $\sigma$. Then $|\text{Fix}(s)| \leq m^c$. Since $m > 2$, we see that $\sigma$ is the identity permutation. If $s$ acts as $s_i$ on $T_i$ for each $i \in [r]$ then

$$\prod_{i=1}^{r} |\text{Fix}_{T_i}(s_i)|, \quad \Box$$

Hypothesis 4.3. $n = 2m > 4$, $\Gamma$ is a regular $(2,m)$-partition of $\Omega$, $M = N_5(\Gamma)$, and $T = A_\Gamma$. Let $\mathcal{H}(T)$ be the set of primitive overgroups $G$ of $T$ in $S$ that do not contain $A$.

4.4. Assume Hypothesis 4.3. Then

1. $T \cong E_{2m-1}$, $\Gamma$ is the set of orbits of $T$ on $\Omega$, and $N_5(T) = M$.
2. $\Gamma = (T \cap S)$. 
3. If $\Sigma \in \mathcal{P}(T)$ then $\Sigma \leq \Gamma$.
4. $T$ is weakly closed in $M$ with respect to $S$.

Proof. Parts (1) and (2) are straightforward. Assume the hypothesis of (3). If $T \cap S \subsetneq S_\Sigma$ then $T \subseteq S_\Sigma$ by (2). Then by (1), $\Sigma \subseteq \Gamma$, so that (3) holds. Thus we may assume $s = (a, b)(c, d) \in T - S_\Sigma$. Therefore by 4.1, we may assume $\sigma = (a, c) \in \Sigma$. However as $n \geq 6$ there is $t = (a, b)(e, f) \in T$ with $e, f \notin \{c, d\}$. Then $t$ fixes $c$ so $t$ acts on $\sigma$, contradicting $b = a \notin \sigma$. This completes the proof of (3).

Suppose $g \in S$ with $T^g \subseteq M$. By (3), $\Gamma \leq T^g$, so $\Gamma = T^g$ and hence $T = T^g$, so (4) holds. \Box

Theorem 4.5. Assume Hypothesis 4.3. Then one of the following holds:

1. $\mathcal{H}(T) = \emptyset$.
2. $n = 8$, $|\mathcal{H}(T)| = 2$, $M$ is transitive on $\mathcal{H}(T)$, and each member of $\mathcal{H}(T)$ is the stabilizer of an affine structure on $\Omega$.
3. $n = 6$ and $\mathcal{H}(T) = \{X, N_5(X), X^a, N_5(X^a)\}$ for $a \in M - N_5(X)$, with $X = N_4(X) \cong L_2(5)$ and $N_5(X) \cong PGL_2(5)$.

Proof. Note that $T$ contains an involution $t$ such that $|\text{Fix}(t)| = 4$. By 3.1(2), it remains to show that (2) holds when $n = 8$ and (3) holds when $n = 6$.

Assume first that $n = 8$. Let us show first that $\mathcal{H}(T) \neq \emptyset$. Let $G$ be the stabilizer in $S$ of any affine structure on $\Omega$, let $V = F(G)$, let $0 \neq v \in V$ and let $H$ be a complement to $V$ in $G$. Then $C_H(v) \cong S_4$ stabilizes the partition $\Sigma(v)$ determined by the orbits of $v$, and acts as $S_3$ on $\Sigma(v)$. Let $K$ be the kernel of this action. Then $(K, v)$ is conjugate to $T$.

Now assume $G \in \mathcal{H}(T)$, so $G$ is the stabilizer of an affine structure by 3.1(2). There are $|S : M| = 105$ conjugates of $T$ and $|S : G| = 30$ conjugates of $G$ in $S$. Let $V = F(G)$. The image $\tilde{T}$ of $T$ in $G/V \cong L_3(2)$ is elementary abelian, so $|\tilde{T}| = 4$ and $T \cap V \neq 1$. On the other hand, for each nonidentity $v \in V$, there is exactly one $(2,4)$-regular $\Sigma \in \mathcal{P}$ such that $v \in A_\Sigma$, namely $\Sigma(v)$. Thus $G$ contains exactly seven $S$-conjugates of $T$, so $T$ is contained in exactly two $S$-conjugates of $G$. Since $M$ does not normalize $G$, we see that $M$ acts transitively on $\mathcal{H}(T)$.

Now assume $n = 6$. Again, we have $\mathcal{H}(T) \neq \emptyset$. Indeed, since $L_2(5) \cong A_5$ acts 2-transitively on the set of six 1-spaces from its natural 2-space, we have that the stabilizer of two points in this action has order two. It follows that the image of any Klein 4-group from $L_2(5)$ in this action is conjugate to $T$ in $S$. Now let $L_2(5) \cong G \in \mathcal{H}(T)$ and let $N = N_5(G) \cong PGL_2(5)$, so every primitive member of
\( \mathcal{M}(T) \) is conjugate to \( N \) in \( S \). There are \( [S:M]=15 \) conjugates of \( T \) and \( [S:N]=6 \) conjugates of \( N \) in \( S \). Now \( T \in \text{Syl}_2(G) \) and \( G = N \cap A \). It follows that \( N \) contains exactly five \( S \)-conjugates of \( T \). Therefore \( T \) is contained in exactly two \( S \)-conjugates of \( N \). Since \( M \) does not normalize \( N, M \) acts transitively on the set of these two conjugates, and it follows that (3) holds. \( \square \)

**Corollary 4.6.** Assume Hypothesis 4.3 and \( G \) is a transitive overgroup of \( T \) in \( S \). Then one of the following holds:

1. \( A \leq G \).
2. \( G \) is imprimitive on \( \Omega \) and for each \( \Sigma \in \mathcal{P}'(G) \), \( \Sigma \leq \Gamma \).
3. \( n = 8 \) and \( G \) is the stabilizer of an affine structure on \( \Omega \).
4. \( n = 6 \) and \( G \cong L_2(5) \) or \( \text{PGL}_2(5) \).

**Proof.** If \( G \) is imprimitive on \( \Omega \) then (2) holds by 4.4.3, so we may assume \( G \) is primitive. Then the corollary follows from Theorem 4.5. \( \square \)

5. A lemma on diamonds

Throughout this section we assume the following hypothesis:

**Hypothesis 5.1.** \( G = S \) or \( A, \Sigma \in \mathcal{P}' \) is regular, \( G_1 = N_G(\Sigma) \), \( G_2 \) is a maximal subgroup of \( G \) distinct from \( A \), and \( H = G_1 \cap G_2 \). Assume \( \mathcal{M}(H) = \{G_1, G_2\} \), \( H \) is maximal in \( G_1 \) and \( G_2 \), and \( H \) is transitive on \( \Omega \).

In this section we prove:

**Theorem 5.2.** Assume Hypothesis 5.1. Then one of the following holds:

1. For \( i = 1, 2 \) there exist \( \Sigma_i \in \mathcal{P}' \) such that \( G_i = N_G(\Sigma_i) \), and \( \Sigma_i \leq \Sigma_{3-i} \) for some \( i \in \{1, 2\} \). Further \( n \geq 8 \) and if \( n = 8 \) then \( G = S \).
2. \( G = A, n = 2^{2+i+1} \) for some integer \( i > 1 \), \( G_2 \) is affine, \( V = F^*(G_2) \leq H \), \( V_\Sigma \) is a hyperplane of \( V \), the elements of \( \Sigma \) are the two orbits of \( V_\Sigma \) on \( \Omega \), and \( H = N_G(V_\Sigma) \).
3. \( G = A, n \equiv 0 \mod 4, n > 8 \), and for \( i = 1, 2 \) there exists \( \Sigma_i \in \mathcal{P}' \) such that
   a. \( G_i = N_G(\Sigma_i) \).
   b. \( \Sigma_1 \) and \( \Sigma_2 \) are lattice complements in \( \mathcal{P} \), and
   c. one of \( \Sigma_1, \Sigma_2 \) is \( (2, n/2) \)-regular and the other is \( (n/2, 2) \)-regular.

We prove Theorem 5.2 in a series of reductions.

5.3.

1. \( n \geq 6 \).
2. If \( n = 8 \) and \( |\Sigma| = 4 \), then \( N_A(\Sigma) \) is not maximal in \( A \), so \( G \neq A \).

**Proof.** Suppose (1) fails. As \( \mathcal{P}' \neq \emptyset \), \( n \) is not prime, so \( n = 4 \). If \( G = A \) then \( G_1 = O_2(G) \), so \( |H| = 2 \) as \( H \) is maximal in \( G_1 \). But now \( H \) is not transitive on \( \Omega \), contrary to Hypothesis 5.1. Therefore \( G = S \) and \( G_1 \in \text{Syl}_2(G) \), so \( |H| = 4 \) by maximality of \( H \) in \( G_1 \). If \( H = O_2(G_1) \) then \( |\mathcal{M}(H)| = 4 \), while if \( H \neq O_2(G_1) \) then \( \mathcal{M}(H) = \{G_1\} \). In either case Hypothesis 5.1 is not satisfied, so (1) is established.

Claim (2) is part of 3.7(2). \( \square \)

5.4. If \( G_2 \) is imprimitive then Theorem 5.2 holds.
**Proof.** Assume $G_2$ is imprimitive on $\Omega$. Then as $G_2$ is maximal in $G$, $G_2 = N_G(\Sigma_2)$ for some $\Sigma_2 \in \mathcal{P}'$. Set $\Sigma_1 = \Sigma$, $n_i = |\Sigma_i|$, $l_i = n/n_i$, $\Sigma_0 = \Sigma_1 \land \Sigma_2$, and $\Sigma_3 = \Sigma_1 \lor \Sigma_2$. Then $\Sigma_i \in \mathcal{P}(H)$ for $i = 0, 3$, so as $\mathcal{M}(H) = \{G_1, G_2\}$, either

(i) $\Sigma_0 = 0$ and $\Sigma_3 = \infty$, or

(ii) $\Sigma_1 < \Sigma_{3-i}$ for some $i \in \{1, 2\}$.

Suppose (ii) holds. Then $n$ is not a prime, or of the form $pq$ with $p, q$ prime. Thus $n \geq 8$, and if $n = 8$ then $G = S$ by 5.3(2). Therefore conclusion (1) of Theorem 5.2 holds in this case, so we may assume (i) holds.

Let $V_i = G_{\Sigma_i}$ and $D_1 = F^*(G_i)$. Then $V_0 = V_1 \cap V_2 = 1$ by 3.4.1 and the assumption that $\Sigma_3 = \infty$.

By 5.3(1), $n > 5$. Suppose $D_1 \leq G_2$. Let $\alpha \in \Sigma$. If $l_1 \notin \{2, 4\}$ or if $l_1 = 2$ and $G = S$ then we have

1. $F^*(G_{\Omega-\alpha}) \leq D_1 \triangleleft H$ and
2. $1 \neq A_{\Omega} \leq F^*(G_{\Omega}) \leq S_{\Omega}.$

Thus by 3.2(2), $\mathcal{P}'(H)$ has a maximum element, contradicting our assumption that (i) holds. If $l_1 = 2$ and $G = A$ then $D_1 = G_{\Sigma}$ and it follows from 4.4(3) that $\Sigma_2 \leq \Sigma_1$, again contradicting (i).

Finally if $l_1 = 4$, consider any block $\{a, b, c, d\} \in \Sigma$. We have $(ab)(cd) \in D_1 \leq G_2$, so by 4.1, one of $\{a, b, c, d\}$, $\{a, c\}$ or $\{a, d\}$ is a block of $\Sigma_2$. It follows that either $\Sigma_1 \leq \Sigma_2$ or $\Sigma_2 \leq \Sigma_1$. Again, this contradicts (i).

Thus, for $i \in \{1, 2\}$, $D_i \notin \Sigma_{3-i}$, so $D_i \not\triangleleft H$. Therefore as $H$ is maximal in $G_i$, $G_i = D_iH$. Thus $V_i = D_iU_i$, where $U_i = V_i \cap H$. Let $j = 3 - i$. Now $U_j \cap V_i \leq V_j \cap V_i = 1$, so

$$U_j \cong U_j^{\Sigma_j} \triangleleft H^{\Sigma_j} = G_i^{\Sigma_i}.$$ 

However $G_i^{\Sigma_i} \cong S_{n_i}$, whereas $V_j = D_jU_j$, so

$$U_j/(U_j \cap D_j) \cong U_jD_j/D_j = V_j/D_j \cong$$

$$\begin{align*}
\mathbb{Z}_{n_j}^{n_j} & \quad l_j \notin \{2, 4\} \text{ and } G = S, \\
\mathbb{Z}_{2}^{n_j} & \quad l_j \notin \{2, 4\} \text{ and } G = A, \\
\mathbb{S}_{3}^{n_j} & \quad l_j = 4 \text{ and } G = S, \\
\mathbb{Z}_{3}^{n_j} & \quad l_j = 4 \text{ and } G = A, \\
1 & \quad l_j = 2.
\end{align*}$$

We conclude that either:

(a) $G = A$, $n_j = 2$, and $U_j^{\Sigma_j} \cong S_{n_j}$, or

(b) $l_j = 2$.

Since $\Sigma_1 \lor \Sigma_2 = \infty$, each block of $\Sigma_i$ intersects each block of $\Sigma_j$ in at most one element. Thus if some $n_j = 2$ then $l_j = 2$ and each block of $\Sigma_i$ intersects each block of $\Sigma_j$ in exactly one element. If neither $n_j = 2$ then each $l_j = 2$. In any case, consider the bipartite graph $C$ with one “red” vertex for each block of $\Sigma_1$ and one “blue” vertex for each block of $\Sigma_2$, and with an edge connecting a red vertex to a blue vertex if the corresponding blocks have nonempty intersection. The edges of $C$ are in bijection with $\Omega$, and $H$ is contained in the subgroup of $\text{Aut}(C)$ that preserves the given vertex coloring.

If $n \equiv 0 \mod 4$ then every element of $\text{Aut}(C) \cong S_2 \times S_{n/2}$ induces an even permutation.
of the edges of $C$. Since $n > 4$ by 5.3(1), we have $n \geq 8$. Now $n > 8$ by 5.3(2), and condition (3) of Theorem 5.2 holds. If each $l_j = 2$ then every vertex of $C$ has degree two. Now $C$ is connected, since $\Sigma_1 \cap \Sigma_2 = 0$. Thus $C$ is an $n$-cycle and $H \leq D_{2n}$. Since $H$ preserves color classes and is transitive on $\Omega$, $H$ must act regularly on the edges of $C$, and this contradicts the fact that $|\mathcal{P}'(H)| = 2$. □

By 5.4, we may assume that $G_2$ is primitive on $\Omega$. Set $D = F^*(G_1)$, $m = |\Sigma|$, pick $\sigma \in \Sigma$, and set $k = |\sigma|$. Thus $n = mk$.

5.5.

(1) If $D \leq H$ then conclusion (2) of Theorem 5.2 holds. Moreover, if $k = 2$ then $A_\Sigma \not\leq H$.

(2) If conclusion (2) of Theorem 5.2 does not hold, then $G_1 = HD$.

(3) If $n = 6$ then $k \neq 2$.

Proof. Let $X = X(\sigma) = D_{\Sigma \sigma}$. If $k \neq 2$ or 4 then by 3.2.1, $X \not\leq G_2$, so (1) holds.

Say $k = 4$ and $D \leq H$. Since $D$ contains an involution $t$ with $|\mathcal{M}(t)| = 4$, so does $G_2$. By Theorem 3.1(2), we have $n = 8$ and $G_2$ is the stabilizer of an affine structure on $\Omega$. Thus $G_2 = LV$, where $L \cong L_2(2)$ is a point stabilizer and $V \cong E_{23}$ acts regularly on $\Omega$. We identify $\Omega$ with $V$. Since $L$ is simple and the nonidentity elements of $V$ have cycle type $2^4$, we have $G_2 \leq A$, so $G = A$. Since $H \leq G_1$, we see that $H$ contains no element of order seven, so $HV/V \not\leq G_2/V$. Since $H$ is maximal in $G_2$, we must have $V \leq H$. We have $\Sigma = \{\sigma_1, \sigma_2\}$, and since $V \leq N_G(\Sigma)$ is regular on $\Omega$, $U = V_\Sigma$ is a hyperplane in $V$, and $\Sigma$ is the set of orbits of $U$ on $\Omega$. Since $V \leq H \leq N_G(\Sigma)$, we have $U \leq H$, so as $H$ is maximal in $G_2$, we have $H = N_{G_2}(U)$ and conclusion (2) of Theorem 5.2 holds.

Finally suppose $k = 2$ and $A_\Sigma \leq H$. By 5.3.1, $n > 4$ so by Theorem 4.5, either $n = 8$ and $G_2$ is the stabilizer of an affine structure on $\Omega$, or $n = 6$ and $G_2$ is $L_2(5)$ or $PGL_2(5)$. The first case does not hold by 5.3.2. In the second, $H \cong A_4$ or $S_4$ is of index 2 in $G_1$, so $G_2$ is a subset of $\mathcal{M}(H)$ of order 2, contradicting $\mathcal{M}(H) = \{G_1, G_2\}$. Thus the proof of (1) is complete.

Observe that (1) and maximality of $H$ in $G_1$ implies (2). Suppose $n = 6$ and $k = 2$. By (1) and (2), $O_3(H) \neq 1$. Now $O_3(H)$ stabilizes a unique regular $(3, 2)$-partition $\Gamma$, and $N_G(\Gamma) \leq \mathcal{M}(H)$, contradicting $\mathcal{M}(H) = \{G_1, G_2\}$. Thus (3) holds. □

5.6.

(1) $H^\Sigma$ and $H^\sigma$ are primitive.

(2) We may assume that $k \leq 3$.

Proof. As $\mathcal{M}(H) = \{G_1, G_2\}$ and $G_2$ is primitive on $\Omega$, $\mathcal{P}'(H) = \{\Sigma\}$. This implies (1).

Suppose $k > 4$. Then $\Sigma = \{\sigma_i: 1 \leq i \leq m\}$ and $D = D_1 \times \cdots \times D_m$, where $D_i = A_{\Sigma - \sigma_i} \cong A_k$. Now $H_i = H \cap D_i \leq N_H(\sigma_i)$, so if $H_i \neq 1$ then $H_i$ is transitive on $\sigma_i$, as $H^\sigma_i$ is primitive by (1). Hence $G_2$ is doubly transitive on $\Omega$ by 3.5. Since any nontrivial element of $H_i$ fixes at least $\frac{n}{l \sigma_i}$ points, it follows from 4.2(2) that $G_2$ is the stabilizer of an affine structure and $G = A$. Indeed conclusion (2) of Theorem 5.2 holds by 3.5.

Thus we may assume that $H_1 = 1$. Recall $H$ is maximal in $G_1$, so $G_1$ is primitive on $G_1/H$. Then as $H_1 = 1$, $G_1$ is complemented, doubled, or diagonal on $G_1/H$ (see [As2, 2.2]). As $H$ is transitive on $\Omega$, $H$ is transitive on $\{D_i: 1 \leq i \leq m\}$, so $G_1$ is not doubled. Let $U = H \cap G_\Sigma$. By 5.5.2, $G_\Sigma = DU$, and as $m > 1$, $U \neq D$. If $G_1$ is complemented, then $U$ is a complement to $D$ in $G_\Sigma$, so $U \cong Z_2$ with $l \in \{m, m - 1\}$. In particular, $Aut_U(D_1) \cong Z_2$ is an abelian normal subgroup of $Aut_H(D_1)$. Therefore, $Aut_U(D_1)$ cannot contain $inn(D_1)$. It follows that $N_H(D_1)$ normalizes some nontrivial proper subgroup $Y$ of $D_1$. Indeed, this is clear if $Aut_D(D_1) = Aut_U(D_1)$, and otherwise $N_H(D_1)$ fixes the preimage in $D_1$ of $Aut_H(D_1) \cap inn(D_1)$. Now $H \leq (H, Y) \leq G_1$, contradicting Hypothesis 5.1. Thus $G_1$ is not complemented, so $G_1$ is diagonal. Indeed as $H^\Sigma$ is primitive, $H \cap D$ is a full diagonal subgroup of $D$. Thus by [AsSc, 1.4], we have

$$X = N_S(\Sigma) \cap N_S(H \cap D) = T \times X_{\Sigma}$$
with $X_{\Sigma} \cong S_k$ and $T \cong S_m$. However $U/(U \cap D) \cong E_{2r}$, with $a = m$ or $a = m - 1$ for $G = S$ or $G = A$, respectively, while as $U \leq X_{\Sigma}$, we have $|U: H \cap D| \leq 2$. It follows that $G = A$, $m = 2$, and $a = 1$. Then as $H$ is transitive on $\Omega$, we have $|H| = 2 |U| = |T|$, so $H = T$. This is a contradiction as $|T: T \cap A| = 2$.

Therefore $k \leq 4$. If $k = 4$ then as $G_{\Sigma} = UD$, $U \leq G_2$ contains a 3-cycle, contrary to 3.2.1. □

5.7. $k = 2$.

Proof. Assume otherwise. Then by 5.6.2, $k = 3$. Further by 5.5.2, $G_1 = HD$, so $G_{\Sigma} = UD$, where $U = H \cap D$, and $H^D = G^D \cong S_m$. Now $UD/D \cong E_{2}\Sigma$, $a = m$ or $m - 1$, and $H$ is transitive on $\Sigma$, so $H$ is irreducible on $D$. Then as $D \nsubseteq H$ by 5.5.1, $H \cap D = 1$, so $H$ is a complement to $D$ in $G_1$, with $H^D = U \cong E_{2}\Sigma$. In particular $U \not\leq N_H(\sigma)$ and $U$ fixes a unique point of $\sigma$, whereas by 5.6.1, $N_H(\sigma)$ is transitive on $\sigma$, a contradiction. This contradiction completes the proof of the lemma. □

5.8. Let $z$ be the involution in $S$ such that $\Sigma$ is the set of orbits of $Z = \langle z \rangle$. Then

1. $z \in G_1$.
2. $m = n/2$ and $G = S$ if and only if $m$ is odd.
3. $H = Z \times K$, where $K \cong S_m$ is a complement to $D$ in $G_1$, which has two orbits on $\Omega$ of length $m$, and contains an involution of type $2^2$.
4. $n > 8$.

Proof. By 5.7, $k = 2$, so $m = n/k = n/2$. By 5.5.2, $G_1 = HD$, so $H^D = G^D \cong S_m$. By 5.5.1, $T = A_{\Sigma} \nsubseteq H$. Then as $H$ is indecomposable on $T$ and irreducible on $T/(T \cap Z)$, $H \cap T \leq Z$. Further if $Z \leq G_1$ but $Z \nsubseteq H$, then $G_1 = HZ$ by maximality of $H$, contradicting $H \cap T \leq Z$. Thus $Z \leq G_1$ if and only if $Z \leq H$.

Let $G_0 = N_S(\Sigma)$ and $G_0 = G_0/Z$. There is a complement $K_0$ to $D_0 = S_\Sigma$ in $G_0$ such that $K_0$ has two orbits of length $m$ on $\Omega$. In particular if $t_0 \in K_0$ with $t_0^\Sigma$ a transposition then $t$ is of type $2^2$, so $K_0 \leq A \leq G$. Thus $G_1 = G_0 \cap G = (D_0 \cap G)K_0 = D_0 K_0$, so $K_0$ is a complement to $D$ in $G_1$.

Let $L_0 = O^2(K_0)$, and suppose $m > 4$. Then from the 1-cohomology of $A_m$ on its natural $F_2$-module $\overline{T}$ (see for example [As6, Exercise 6.3]), $D_0$ is transitive on the complements to $\overline{D}$ in $\overline{L}_0 \overline{D}$, and $ZK_0 = NG_0(L_0)$, so $O^2(H) \leq L_0^D$, and hence $H \leq (ZK_0)^d$ for some $d \in D_0$. In particular if $H \cap D_0 = Z$ then $H = Z \times K_0^d$.

By 5.5.3, $n > 6$. Suppose $m$ is even. Then $z \in A_1$, so $z \in G_1$ and hence $Z \leq H$ by paragraph one. If $G = S$ then $T < D$, so as $G_1 = TH$, there exists $d \in H \cap D - T$. But then $T = [H, d] \leq H$, a contradiction. Therefore $G = A$, so (1) and (2) hold in this case. By 5.3.2, $n \neq 8$, completing the proof of (4). Then (3) follows from paragraph three.

So assume $m$ is odd. Then $z$ is odd, so $Z \leq G_1$ if and only if $G = S$, in which case (1) and (2) hold, and (3) follows from paragraph three. Thus we may assume $G = A$, so $D = T$ and $H$ is a complement to $D$ in $G_1$ by paragraph one. By paragraph three, we may take $H = L_0(t)$, where $t \in ZK_0$ with $t^\Sigma$ a transposition. Then as $t$ is even, $t \in K_0$ is of type $2^2$ by paragraph two, so $H = K_0$. But then $H$ is not transitive on $\Sigma$, contrary to Hypothesis 5.1. □

Theorem 5.2 now follows. Indeed, by 3.1, parts (3) and (4) of 5.8 are contradictory.

6. The proof of Theorem 1.1

In this section, we present our proof of Theorem 1.1. Thus we may assume that $(H, G) \in L(m_1, \ldots, m_r)$. We begin with a useful result that follows directly from the definitions.

6.1. Let $H \leq K \leq L \leq G$. If $H \neq K$ or $L \neq G$ then the interval $[K, L]$ in $O_G(H)$ is isomorphic with $A(m)$ for some $m$.

The next lemma will be used at several places and might be useful in further study of our main conjecture.
6.2. Let $Y$ be any finite group and let $X \leq Y$. If $O_Y(X) \cong D_\Delta(m_1, \ldots, m_t)$ with $t > 1$ and each $m_i > 2$, then for each $K \in O_Y(X)$ we have $N_Y(K) = K$.

**Proof.** We proceed by downward induction on $O_Y(X)$, with the base case $K = Y$ holding trivially. Assume $K < Y$. If $K < N_Y(K)$, there is some $L \leq N_Y(K)$ such that $K$ is maximal in $L$. If $L = Y$, so $K$ is maximal and normal in $Y$, then for every maximal subgroup $M \not= K$, we have $K \cap M$ maximal in $M$. However, there is some such $M$ with $K \cap M = H$, contradicting our assumption that each $m_i > 2$. Now assume $L \not= Y$. Pick $T \leq Y$ such that $L$ is a maximal subgroup of $T$. Note that, because each $m_i > 2$, if $K = T$ then $T \not= Y$. By 6.1, there is some $U$ such that $[K, T] = [K, U, T]$. By inductive hypothesis, $U$ is not normal in $T$. But now, for any $g \in L \setminus K$, the group $U^g$ lies in $[K, T]$ and is distinct from $L$ and $U$. □

We are now in a position to prove Theorem 1.1 through a series of reductions.

6.3. $H$ is not primitive.

**Proof.** In [As1, Theorem D], a list of all possibilities for the isomorphism type of $O_C(H)$ is given under the assumptions that $|\Omega|$ is not prime, $G = A$ or $S$, $H < G$ is primitive on $\Omega$ and for every maximal subgroup $M$ of $G$ containing $H$, there is some maximal $L$ with $L \cap M = H$. The lattices $D\Lambda(m_1, \ldots, m_t)$ satisfy the last given condition whenever $t > 1$ and do not appear on the given list. It remains to examine the case where $|\Omega| = p$ is prime. In this case (see for example [Gu]), for each primitive $K \leq G$, one of the following holds.

(a) $P \leq K \leq N_C(P)$ for some Sylow $p$-subgroup $P \leq G$.

(b) $F^*(K) \cong L_d(q)$ for some prime $d$ and some prime power $q$, $p = \frac{d^2 - 1}{q - 1}$ and the action of $K$ on $\Omega$ is equivalent to its action on $1$-spaces or $(d - 1)$-spaces from $\mathbb{F}_q^d$.

(c) We have $p = 11$ and $K \cong M_{11}$.

(d) We have $p = 11$, $K \cong L_2(11)$ and the stabilizer of a point in $K$ is isomorphic to $A_5$.

(e) We have $p = 23$ and $K \cong M_{23}$.

If one of cases (b)–(e) holds then $K \leq A$ (see for example [Sh2, Corollary 3.2]). Moreover, since $N_C(P)/P$ is cyclic for every Sylow $p$-subgroup $P$ of $S$, we see that if $K \in O_C(H)$ satisfies (a) then by 6.2 we have $K = H = N_C(P)$. It now follows that some $K \in \{H, G\}$ satisfies one of (b)–(e) (otherwise $O_C(H) = [H, G]$) which forces in turn that $G = A$ (otherwise we have $A \in (H, G)$, violating 6.2).

Say $K \in (H, A)$ satisfies one of (b)–(e). There is no $L \in (H, G)$ such that $F^*(K) < F^*(L)$ (see [LPS]), so $N_C(F^*(K))$ is the unique maximal subgroup of $G$ containing $K$. By 6.1, $K = N_C(F^*(K))$. It now follows that if $H = G_0 < \cdots < G_m = A$ is a chain in $O_C(H)$ with $m > 2$ then $p = 11$, $m = 3$, $H = N_C(P)$ for some Sylow $p$-subgroup $P$ of $G$, $G_1 \cong L_2(11)$ and $G_2 \cong M_{11}$. Therefore, every maximal element of $(H, G)$ is isomorphic with $M_{11}$. For fixed maximal $M \in (H, G)$, the number of $G$-conjugates of $M$ containing $H$ is the number of $M$-conjugacy classes containing $G$-conjugates of $H$, by 6.2 (and easy double counting). Since $H = N_C(P) = N_M(P)$ as above, there is one such $M$-conjugacy class. Since $M_{11}$ has a unique conjugacy class of subgroups of index 11, every subgroup of $S_{11}$ isomorphic with $M_{11}$ is conjugate to $M$ in $S_{11}$. The conjugacy class of $M$ splits into two $A_{11}$ classes, from which it follows that $(H, G)$ contains exactly two maximal elements, contradicting our assumptions on $O_C(H)$. □

6.4. $H$ is transitive.

**Proof.** Assume for contradiction that $H$ has $t > 1$ orbits on $\Omega$. Say $t \geq 3$. Let $\pi = \{P_1, P_2, P_3\}$ be a partition of $\Omega$ such that each orbit of $H$ is contained in some $P_i$. Set $K = G \cap \prod_{i=1}^3 S_{\Omega \setminus P_i}$, so $K \in O_C(H)$. If two of the $P_i$ have the same size then $K < N_C(K)$ and if some $|P_i| = \frac{p}{2}$ then $L := G \cap (S_{P_1} \times S_{\Omega \setminus P_1}) \in O_C(H)$ with $L < N_C(L)$. By 6.2, neither of these conditions can hold. Thus we may assume that $|P_1| > |P_2| > |P_3|$. Then $|P_1| \geq 3$, so $K$ contains a 3-cycle. It follows from 3.1(1) and 6.2.
that no member of $\mathcal{M}(K)$ is primitive. Also, $|P_1| > n/3$. Assume for contradiction that there is some regular $\Sigma \in \mathcal{P}^*(K)$. Since $K_{O_1(P_1)}$ contains the alternating group on $P_1$, some part of $\Sigma$ contains $P_1$ by 3.2(2). Thus $|\Sigma| = 2$. Since $K$ fixes $P_1$ setwise, we have $K \leq G_\Sigma$, so each part of $\Sigma$ is the union of some of the $P_i$. This forces $|P_1| = n/2$, giving the desired contradiction. We see now that

$$\mathcal{M}(K) = \{N_G(P_i) : i \in [3]\}.$$ 

But now $K$ is the intersection of any pair of members of $\mathcal{M}(K)$, contradicting 6.1.

Say $t = 2$, so $H$ has orbits $B, C$. Let $Y$ be the unique maximal intransitive subgroup of $G$ containing $H$. Assume first that $|C| = 1$, so $Y$ is a point stabilizer in the natural action. Note that $H$ stabilizes no equipartition of $\Omega$, so any maximal subgroup in $O_C(\Omega)$ other than $Y$ is primitive. Let $K$ be such a maximal subgroup that does not lie in the same connected component of $O_C(H)'$ as $Y$. Then $K \cap Y = H$ is not contained in $K$, contradicting the primitivity of $K$.

Now assume that $|B|, |C| > 1$. By 6.2, $|B| \neq |C|$. So, we assume that $|C| < |B|$. By 3.7, $Y$ is maximal in $G$. Let $C$ by the connected component of $O_C(H)'$ containing $Y$, and let $M_1, M_2$ be maximal elements of $C$, neither of which is $Y$. Set $Y_1 = Y \cap M_1, Y_2 = Y \cap M_2$, $Z = M_1 \cap M_2$. By Lemma 6.1, $Z$ is not contained in $Y$. Therefore, $Z$ is transitive.

Let us assume first that $G = S$. By Lemmas 6.2 and 3.6, each $Y_i$ contains a transposition. By 3.1, each $M_i$ is imprimitive. Assume $M_i$ preserves the equipartition $\pi_i$ of $\Omega$, for $i \in \{1, 2\}$. Then $|B| > \frac{3}{2}$, neither $M_i$ can contain $AC$. Thus by 3.6 each $Y_i = S_B \times T_i$ with $T_i$ transitive on $B$. It follows that $C \in \pi_i$ for each $i$. However, two distinct equipartitions with a common part cannot be simultaneously preserved by the transitive group $Z$.

Now assume $G = A$. By 6.2 and 3.6, each $Y_i$ satisfies one of conditions (a), (b) from 3.6(2). In fact, neither $Y_i$ satisfies (a). Indeed, if (a) is satisfied then $H$ is contained in some $X$ that is isomorphic to a maximal subgroup of $S_b$, and the orbits of $X$ are $B$ and $C$. Now $X$ is isomorphic to one of $A_4, D_8$ or $S_3$. In the first case, we have a contradiction to Lemma 6.2, as $X \leq Y_i$. In the second case, $X$ has no orbit of size three and in the third case $X$ has no orbit of size four. Therefore, each $Y_i$ satisfies (b).

If $|C| > 2$ then each $Y_i$ contains a 3-cycle and therefore each $M_i$ is not primitive by 3.1. We can now argue as we did above – since neither $M_i$ can contain $AC$, we see that each $M_i$ stabilizes a partition that has $C$ as a part, contradicting the fact the $Z$ is transitive.

Finally, assume that $|C| = 2$. Say some regular $(k, l)$-partition $\rho$ of $B$ is $H$-invariant. Let $N = N_Y(\rho)$, so $H \subseteq N$. We claim that $N$ is maximal in $Y$. Indeed, let $\pi_B, \pi_C$ be, respectively, the natural projections of $N_2(B) = N_2(C)$ onto $S_2$ and $S_2$. Then $\pi_2(N) = N_2(C)(\rho)$, which is maximal in $S_2$ by 3.7, and $\pi_2(N) = S_2$. Moreover, $N$ has index 2 in $\pi_2(N) \times \pi_2(N)$, which is not contained in $A$. It follows that if $N \leq M \leq Y$ then $\pi_2(M) = S_2$. Since the only nontrivial quotient of $S_2$ that is isomorphic to a quotient of $S_B$ is $Z_2$, we have $A_4 \times A_2 \leq M \leq N_2(B) = Y$. Since $N \nleq A_4 \times A_2$, we have $M = Y$ as claimed.

By Lemma 6.1, we have $N = K \cap Y$ for some maximal $K \in O_C(H)$. If $k > 2$ then $N$ contains a 3-cycle and it follows from 3.1 that $K$ is not primitive. On the other hand, if $K$ stabilizes a partition $\tau$ of $\Omega$ then, as $N$ contains $A_2^{\Omega \setminus T}$ for each part $T$ of $\rho$, if $k > 2$ by 3.2(2) every $Q \leq \rho$ is contained in some $P \in \tau$ and every $P \in \tau$ such that $P \cap \tau = \emptyset$ has size a multiple of $k$. The remaining $P \in \tau$ do not have such size, giving a contradiction. Thus $k = 2$ and if $\phi = \rho \cup \{C\}$ then $H \leq N_C(\phi)$. Moreover, since $Y \cap N_C(\phi)$ is maximal in $Y$, we see that $N_C(\phi)$ lies in the same connected component of $O_C(H)'$ as $Y$ and that $H < Y \cap N_C(\phi)$.

Let $K \in O_C(H)'$ lie in a different connected component than $Y$. Then $N_K(C) = K \cap Y = H$. If we can show that each such $K$ is 2-homogeneous then we are done, as then each such $K$ must be maximal in $G$ (if $K < L$ then $N_K(C) < N_L(C)$), contradicting the fact that each $m_i > 2$. Certainly each such $K$ is transitive, as $Y$ is the only intransitive maximal subgroup in $O_C(H)$.

Assume for the moment that $H_C$ is transitive on $B$. Suppose $\pi$ is a nontrivial equipartition of $\Omega$ stabilized by $K$. Then $C \in \pi = \rho \cup \{C\}$, where $\rho = \pi \setminus \{C\}$ is an $H$-invariant equipartition of $B$. Thus by an earlier argument, $K = N_G(\pi)$ is in the same connected component of $O_C(H)'$ as $Y$, contrary to our choice of $K$. Thus $K$ is primitive. Now by [As6, (15.17)], the transitivity of $H_C$ on $B$ shows that $K$ is 2-transitive. Finally, assume that $H_C$ is intransitive on $B$. Since $[H : H_C] = 2$, we see that $H_C$ has two
orbits $B_1, B_2$ of size $\frac{|B|}{2}$ on $B$. Now $H$ preserves the partition of $B$ into $B_1 \cup B_2$ and it follows from our argument above that $|B| = 4$ and $n = 6$. Now $H_C$ acts on $B$ as an intransitive subgroup of $A_C$ that has index two in a transitive subgroup of $S_C$ that is not contained in $A_C$. It follows that $H$ is generated by an element of cycle shape $(4, 2)$. However, this gives $N_C(H) \neq H$, contradicting Lemma 6.2. \[\Box\]

We have now that $H$ is transitive but not primitive. Thus $H$ stabilizes at least one equipartition $\pi = \{P_1, \ldots, P_m\}$ of $\Omega$.

Let us begin with the case $G = S$. By Theorem 5.2, the maximal subgroups in the connected component of $O_C(H)^C$ containing $N_C(\pi)$ are $M_1^1, \ldots, M^m$, where $M^n = N_C(\pi_i)$ for an equipartition $\pi_1$ and $\pi_i$ refines $\pi_{i+1}$ for $1 \leq i < m$. Now $H = \bigcap_{i=1}^m M^i$ contains the kernel of the action of $M^1$ on the elements of $\pi_1$. From this we derive the following two facts.

(a) $H$ contains a transposition. Therefore, no element of $O_C(H)$ is primitive by 3.1.
(b) Every equipartition of $\Omega$ stabilized by $H$ is refined by $\pi_1$.

Now let $L^1, \ldots, L^l$ the maximal subgroups in a connected component of $O_C(H)^C$ that does not contain $N_C(\pi)$. Using fact (a) and Theorem 5.2, we see that we can take $L^1$ to be the stabilizer of an equipartition $\mu_1$, where $\mu_i$ refines $\mu_{i+1}$ for $1 \leq i < l$. Reasoning with the $L^1$ as we did with the $M^i$, we see that every equipartition of $\Omega$ stabilized by $H$ is refined by $\mu_1$. It follows that $\mu_1 = \pi_1$ and $L^1 = M^1$, giving a contradiction that completes our proof in this case.

Now assume $G = A$. We wish to reduce to the case in which every pair $M, L$ of maximal subgroups in the connected component $C$ of $O_C(H)^C$ containing $N_C(\pi)$ satisfies conclusion (1) of Theorem 5.2. Suppose first that some pair $M, L$ satisfies conclusion (2) of the theorem. We may assume that $M$ is the stabilizer of an $(n/2, 2)$-regular partition and that $L$ is affine. Now $C$ contains a third maximal subgroup $K$. If the pair $M, K$ satisfies conclusion (1), then $K$ is the stabilizer of an $(m, n/m)$-regular partition for some $m < n/2$, so $K, L$ satisfies none of the three conclusions of the theorem, a contradiction. If $M, K$ satisfies conclusion (2) then $K$ is affine. If $M, K$ satisfies conclusion (3) then $K$ is the stabilizer of a $(2, n/2)$-regular partition, and the pair $L, K$ satisfies none of the conclusions, again a contradiction. We see now that if $M, K$ satisfies conclusion (2), then all maximal subgroups in $C$ other than $M$ are affine, so $M = N_C(\pi)$.

Suppose next that some pair $M, L$ satisfies conclusion (3) of Theorem 5.2, so there exist a $(2, n/2)$-regular partition $\pi_M$ and an $(n/2, 2)$-regular partition $\pi_L$ such that $M = N_C(\pi_M)$ and $L = N_C(\pi_L)$. Let $K$ be as in the previous paragraph. If $K$ is affine then $K, M$ satisfies none of the conclusions of the theorem, a contradiction. Thus $K = N_C(\pi_K)$ for some $(k, n/k)$-regular partition $\pi_K$. One of $M, K$ or $L, K$ does not satisfy conclusion (3) and therefore must satisfy conclusion (1). Say $M, K$ satisfies (1) so $\pi_K < \pi_M$ in $P$. We cannot have $\pi_L < \pi_K$, since $\pi_L \not\leq \pi_M$. Thus $K, L$ does not satisfy conclusion (1). On the other hand, we have $k > 2$, since $\pi_K < \pi_M$, and it follows that $K, L$ does not satisfy conclusion (3). Thus $K, L$ satisfies none of the conclusions of Theorem 5.2, a contradiction. A similar argument leads to a contradiction if $L, K$ rather than $M, K$ is assumed to satisfy (1).

We may assume now that all maximal subgroups except $N_C(\pi)$ are affine. Let $K, L$ be two such maximal subgroups. There exist regular subgroups $V, W \cong \mathbb{Z}_2^d$ of $G$ such that $K = N_C(V) \cong AGL(V)$ and $L = N_C(W) \cong AGL(W)$. Let $Y = K \cap L \not\subseteq N_C(\pi)$. Since the only imprimitive maximal element of $O_C(H)^C$ is $N_C(\pi)$, we see that $Y$ is primitive.

If $d = \dim(V) > 3$ then $H^1(GL(V), V) = 0$, and if $d = 3$ then $H^1(GL(V), V) \cong \mathbb{Z}_2$ and a transitive complement to $V$ in $AGL(V)$ acts (primitively) as $L_2(7)$ does on 1-spaces from its natural 2-space (see [J]). Since $Y$ is an imprimitive maximal subgroup of $K$, we see that either $Y \not\subseteq Y$, in which case $F^*(Y) = V = F^*(K)$, or $n = 8$ and $Y \cong L_2(7)$. In the first case, we must also have $F^*(Y) = W = F^*(L)$ (since $Y \not\subseteq L_2(7)$), which gives the absurdity $F^*(Y) \not\subseteq G$. Thus we may assume that $Y \cong L_2(7)$ is a primitive complement.

Now $X = N_C(\pi) \cap Y$ is a transitive maximal subgroup of $Y$, from which it follows that $X \cong S_4$ acts on $Y$ as $S_4$ acts on the cosets of a Sylow 3-subgroup. A Sylow 3-subgroup is contained in two maximal subgroups of $S_4$ ($A_4$ and one $S_3$). Therefore, $X$ stabilizes exactly two nontrivial partitions. Let $\rho \neq \pi$ be stabilized by $X$. Note that $\rho$ has four parts of size two and thus the stabilizer of $\rho$
in $G$ has nontrivial center. On the other hand, $X$ has trivial center and it follows that $N_G(X) \neq X$, contradicting Lemma 6.2.

We see now that we are in the same situation we faced when we took $G = S$. The maximal subgroups in $C$ are $M^1, \ldots, M^m$, where $M^i = N_G(\pi_i)$ and the equipartition $\pi_i$ refines $\pi_{i+1}$. If the elements of $\pi_1$ have size $k > 2$ then $H$ contains a 3-cycle and no element of $O_G(H)'$ is primitive. The proof given for $G = S$ is easily adjusted to apply here. Say the elements of $\pi_1$ have size two. Then $H$ contains the product of two disjoint 2-cycles. It follows from 3.1(2) that if $n > 8$ then $O_G(H)'$ contains no primitive subgroup. Once again, we may adapt the proof for $G = S$ to this situation. Finally, if $n \leq 8$ then there do not exist proper nontrivial equipartitions $\pi_1, \pi_2, \pi_3$ of $[n]$ such that each $\pi_i$ refines the next.

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References