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On Computing the Eigenvalues of a Symplectic Pencil*

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ABSTRACT

This paper presents an algorithm for computing the eigenvalues of a symplectic pencil that arises in one of the commonly used approaches for solving the discrete-time algebraic Riccati equation. The algorithm is numerically efficient and reliable in that it employs only orthogonal transformations and makes use of the structure of the symplectic pencil. It requires about one-fourth the number of floating-point operations that the QZ algorithm uses to compute the eigenvalues of the pencil directly. The proposed method can be regarded as being analogous for the case of symplectic pencils to the method developed by Van Loan for computing the eigenvalues of Hamiltonian matrices.

1. INTRODUCTION

A matrix $S \in \mathbb{R}^{2n \times 2n}$ is called a *symplectic* matrix if

$$SJS^T = J, \tag{1.1}$$

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where

$$J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}. \tag{1.2}$$

In (1.2), I_n denotes the $n \times n$ identity matrix and 0_n denotes the $n \times n$ null matrix. A pencil $K - \lambda L$, with $K \in \mathbb{R}^{2n \times 2n}$ and $L \in \mathbb{R}^{2n \times 2n}$, is called a *symplectic pencil* if

$$KJK^T = LJL^T. \tag{1.3}$$

In this paper we consider the problem of computing the eigenvalues of a symplectic pencil $K - \lambda L$ which arises in solving the well-known discrete-time algebraic Riccati equation (DARE) [7, 8]:

$$\mathbf{DR}(P) \equiv A^T PA - P - A^T PB(D + B^T PB)^{-1} B^T PA + H = 0, \tag{1.4}$$

where $H = C^T C \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{m \times m}$ ($m \leqslant n$) are positive semidefinite and positive definite matrices respectively, and $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. A positive semidefinite matrix P satisfying (1.4) is required in solving the well-known discrete-time linear quadratic optimal-control problem [3]. It can be shown that if (A, B) is a stabilizable pair and (A, C) is a detectable pair, then (1.4) has a unique positive semidefinite solution $P = P^T \in \mathbb{R}^{n \times n}$ which is a stabilizing solution, i.e., the closed-loop state matrix

$$A_c = A - B(D + B^T PB)^{-1} B^T PA \tag{1.5}$$

has all its eigenvalues inside the unit circle with center at the origin of the complex plane.

A well-known approach to solving the above problem numerically is that proposed in [7] and is based on the properties of the regular symplectic pencil $K - \lambda L$ where

$$K = \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix} \tag{1.6a}$$

and

$$L = \begin{bmatrix} I & F \\ 0 & A^T \end{bmatrix} \tag{1.6b}$$

with $F = BD^{-1}B^T$. If (A, B) is a stabilizable pair and (A, C) is a detectable pair, then the following results can be shown [7] for such a pencil:

(a) If λ is an eigenvalue of $K - \lambda L$, then so is $1/\lambda$, i.e., the pencil has n eigenvalues *inside* the unit circle and n eigenvalues *outside*, with no eigenvalues on the unit circle.

(b) The eigenvalues of $K - \lambda L$ inside the unit circle, i.e. the stable eigenvalues, are the eigenvalues of the state matrix A_c of the closed-loop discrete-time system.

(c) The unique positive semidefinite solution P of (1.4) is given by

$$P = U_{21}U_{11}^{-1} \tag{1.7}$$

where $U_{11}, U_{21} \in \mathbb{R}^{n\times n}$ and the columns of $\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$ span the n-dimensional deflating subspace of $K - \lambda L$ corresponding to its stable eigenvalues.

The approach proposed in [7] for computing P is to use the QZ algorithm [2, 6] to solve the generalized eigenvalue problem for the pencil $K - \lambda L$. The computed eigenvalues are then reordered into groups of stable and unstable eigenvalues to obtain the columns of $\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$. The QZ algorithm does not take into account the special (symplectic) structure of the pencil $K - \lambda L$, so that the transformations employed in this algorithm destroy this structure. On the other hand, as the following result shows, it is possible to carry out the required transformations on $K - \lambda L$ so that its symplectic structure is preserved.

THEOREM 1.1. *If the regular symplectic pencil $K - \lambda L$ has no eigenvalue of modulus* 1, *then there exist an orthogonal matrix $Q \in \mathbb{R}^{2n\times 2n}$ and an orthogonal symplectic matrix*

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ -Z_{12} & Z_{11} \end{bmatrix},$$

with $Z_{11}, Z_{12} \in \mathbb{R}^{n\times n}$, such that

$$QKZ = \begin{bmatrix} T_{11} & T_{12} \\ 0_n & T_{22} \end{bmatrix} \equiv T, \tag{1.8a}$$

$$QLZ = \begin{bmatrix} R_{11} & R_{12} \\ 0_n & R_{22} \end{bmatrix} \equiv R, \tag{1.8b}$$

and $T - \lambda R$ is a symplectic pencil in generalized real Schur form (GRSF)[1], *where $T_{11} - \lambda R_{11}$ has all its eigenvalues inside the unit circle and $T_{22} - \lambda R_{22}$ has all its eigenvalues (including infinite ones) outside the unit circle.*

Proof. See [5]. ∎

There are two types of orthogonal symplectic matrices that are particularly useful in performing structure-preserving transformations on symplectic pencils. The first type consists of Householder symplectic matrices defined by

$$P(k,\underline{u}) = \begin{bmatrix} \hat{P} & 0_n \\ 0_n & \hat{P} \end{bmatrix}, \qquad \hat{P} \in \mathbb{R}^{n\times n}, \tag{1.9}$$

where

$$\hat{P} = I_n - \frac{2\underline{u}\underline{u}^T}{\underline{u}^T\underline{u}} \tag{1.10a}$$

and

$$\underline{u}^T = [0,\dots,0,u_k,\dots,u_n] \neq \underline{0}^T. \tag{1.10b}$$

The second type consists of Givens (Jacobi) symplectic matrices defined by

$$G(k,c,s) = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}, \qquad C,S \in \mathbb{R}^{n\times n}, \tag{1.11}$$

where

$$C = \operatorname{diag}[I_{k-1}, c, I_{n-k}], \tag{1.12a}$$

$$S = \operatorname{diag}[0_{k-1}, s, 0_{n-k}], \tag{1.12b}$$

and $c^2 + s^2 = 1$. Algorithms H and J in [9] show how $P(k,\underline{u})$ and $G(k,c,s)$ can be determined to zero specific entries in a vector.

Theorem 1.1 and its proof in [5] show that it is possible to reduce $K - \lambda L$ using structure-preserving *orthogonal* transformations to the GRSF (1.8), but

[1]A pencil in *generalized real Schur form* is a real upper quasitriangular pencil with 1×1 and (2×2) diagonal blocks.

it provides an *efficient* QZ-type algorithm for doing so only for a special case, namely when rank$(F) = 1$ or rank$(H) = 1$. In order to obtain a stabilizing solution of the DARE, we need to compute the eigenvalues of the pencil $K - \lambda L$ as well as its deflating subspace corresponding to the set of stable eigenvalues. The algorithm in [5] enables us to do this for the special case considered there. However, to the best of our knowledge, a numerically stable algorithm for the general case which preserves or uses the symplectic structure of the pencil $K - \lambda L$ has yet to be developed. In this context, it is worth noting that a similar situation exists for Hamiltonian matrices, i.e., for the continuous-time algebraic Riccati equation.

In this paper, we consider the problem of computing only the eigenvalues of the symplectic pencil $K - \lambda L$. To do this we employ orthogonal transformations only and make use of the special structure of the pencil. The resulting algorithm is therefore numerically backward stable and requires significantly less computation than a direct application of the QZ algorithm. The method can be regarded as the analog for the case of symplectic pencils of the method proposed by Van Loan [9] for computing the eigenvalues of Hamiltonian matrices. An efficient algorithm that uses the symplectic structure of the pencil $K - \lambda L$ to compute its eigenvalues has been proposed by Lin [4]. It requires about one-fourth the floating-point operations of the QZ algorithm. However, as pointed out in [4], the method has the disadvantage that it uses the "less favorable" $(\Gamma, \tilde{\Gamma})$ transformations, which in certain cases can cause numerical instability. Also, in order to use the $(\Gamma, \tilde{\Gamma})$ transformations, Lin's algorithm requires the use of nonorthogonal (elementary) transformations to reduce the matrix A to a diagonal matrix. The algorithm that we describe in the next section is almost as efficient as the one in [4] but without the disadvantages mentioned above.

2. COMPUTING THE EIGENVALUES OF $K - \lambda L$

As in [4], instead of computing the eigenvalues of $K - \lambda L$ we compute the eigenvalues of a pencil $\hat{K} - \lambda \hat{L}$, where

$$\hat{K} \equiv KJL^T + LJK^T, \tag{2.1a}$$

$$\hat{L} \equiv LJL^T. \tag{2.1b}$$

The pencil $\hat{K} - \lambda \hat{L}$ is called an $S + S^{-1}$ transformation of the symplectic pencil $K - \lambda L$. It can be shown [4] that if λ is an eigenvalue of $K - \lambda L$, then $\lambda + 1/\lambda$ is an eigenvalue of $\hat{K} - \lambda\hat{L}$. In order to compute the eigenvalues of $\hat{K} - \lambda\hat{L}$, we note that

$$\begin{aligned}\hat{K} - \lambda\hat{L} &= (KJL^T + LJL^T) - \lambda LJL^T \\ &= \begin{bmatrix} AF - FA^T & A^2 + FH + I \\ -(A^2 + FH + I)^T & A^TH - HA \end{bmatrix} - \lambda\begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} \\ &= \left\{\begin{bmatrix} A^2 + FH + I & FA^T - AF \\ A^TH - HA & (A^2 + FH + I)^T \end{bmatrix} - \lambda\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}\right\}J, \end{aligned} \tag{2.2}$$

which implies that the eigenvalues of $\hat{K} - \lambda\hat{L}$ are the same as those of the pencil $M - \lambda N$, where

$$M = \begin{bmatrix} A^2 + FH + I & FA^T - AF \\ A^TH - HA & (A^2 + FH + I)^T \end{bmatrix} \equiv \begin{bmatrix} Y & W \\ X & Y^T \end{bmatrix} \tag{2.3a}$$

and

$$N = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}. \tag{2.3b}$$

Note that X and W are $n \times n$ skew-symmetric matrices.

We now present an algorithm for reducing $M - \lambda N$ to a block-triangular structure, i.e.,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{11}^T \end{bmatrix} \tag{2.4a}$$

and

$$N = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix}, \tag{2.4b}$$

where $M_{11} \in \mathbb{R}^{n\times n}$ is an upper Hessenberg matrix, $N_{11} \in \mathbb{R}^{n\times n}$ is an upper triangular matrix, and $M_{12}, N_{12} \in \mathbb{R}^{n\times n}$ are skew-symmetric matrices. The eigenvalues of $M - \lambda N$ can then be computed by applying the QZ algorithm to find the eigenvalues of the pencil $M_{11} - \lambda N_{11}$.

Note that M and N can be considered to have similar structure if we regard the null matrices in N as null skew-symmetric matrices. The transformations that we will use preserve this structure. There are two types of transformations that will be used in the algorithm, both of which preserve the structure of the pencil $M - \lambda N$. The first type involves similarity transformations on M and N using Givens symplectic matrices $G(k, c, s)$, i.e.,

$$M := GMG^T, \qquad N := GNG^T. \tag{2.5}$$

The second type of transformations has the following form:

$$M := QMZ, \qquad N := QNZ, \tag{2.6}$$

where

$$Q \equiv \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \quad \text{and} \quad Z \equiv \begin{bmatrix} V^T & 0 \\ 0 & U^T \end{bmatrix},$$

where $U, V \in \mathbb{R}^{n \times n}$ are orthogonal matrices representing the application of Givens or modified Householder transformations [2]. In our algorithm, we use 2×2 modified Householder transformations instead of Givens transformations because the former are a little more efficient. We denote the second type of transformations by the pair (Q, Z). It can be easily verified that the above transformations applied to the pencil $M - \lambda N$ preserve the structure and the eigenvalues of $M - \lambda N$.

To illustrate the reductions performed by the algorithm, we consider the following case ($n = 5$):

$$M = \begin{bmatrix} Y & W \\ X & Y^T \end{bmatrix} = \begin{bmatrix} x & x & x & x & x & 0 & x & x & x & x \\ x & x & x & x & x & x & 0 & x & x & x \\ x & x & x & x & x & x & x & 0 & x & x \\ x & x & x & x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & x & x & x & 0 \\ & & & & & & & & & \\ 0 & x & x & x & x & x & x & x & x & x \\ x & 0 & x & x & x & x & x & x & x & x \\ x & x & 0 & x & x & x & x & x & x & x \\ x & x & x & 0 & x & x & x & x & x & x \\ x & x & x & x & 0 & x & x & x & x & x \end{bmatrix}, \tag{2.7a}$$

$$N=\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}=\begin{bmatrix} x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{bmatrix}. \quad (2.7b)$$

The aim is to reduce M and N to block upper triangular matrices by zeroing the 5×5 (2,1) block of M while preserving the null structure of the corresponding block of N. We do this using the orthogonal transformations mentioned above. First, we use Householder transformations to reduce A to upper triangular form. This corresponds to a (Q_1, Z_1) transformation on the pencil $M - \lambda N$, where

$$Q_1=\begin{bmatrix} U_1 & 0 \\ 0 & I_n \end{bmatrix}, \qquad Z_1=\begin{bmatrix} I_n & 0 \\ 0 & U_1^T \end{bmatrix}.$$

$$M=\begin{bmatrix} x & x & x & x & x & 0 & x & x & x & x \\ x & x & x & x & x & x & 0 & x & x & x \\ x & x & x & x & x & x & x & 0 & x & x \\ x & x & x & x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & x & x & x & 0 \\ 0 & x & x & 0 & x & x & x & x & x & x \\ x & 0 & x & x & x & x & x & x & x & x \\ x & x & 0 & x & x & x & x & x & x & x \\ x & x & x & 0 & x & x & x & x & x & x \\ x & x & x & x & 0 & x & x & x & x & x \end{bmatrix},$$

$$N=\begin{bmatrix} x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{bmatrix}. \quad (2.8)$$

Next, we reduce the elements x_{21}, x_{31}, and x_{41} in the first column of X to zero using 2×2 modified Householder transformations. First, we zero x_{21} using x_{31}. The (Q, Z) transformation required to do this introduces a nonzero entry in a_{32} [denoted by $\otimes$ in (2.9)] which can be zeroed using a_{22} by means of a Householder transformation on rows 2 and 3 of N. The elements x_{31} and x_{41} can be eliminated in the same way while maintaining the upper triangular structure of A. Note that zeros are also introduced at corresponding locations in the first row of X because the (Q, Z) transformation is structure-preserving. If we denote the Householder transformations applied in this step by V_2 and U_2, we have

$$M := \begin{bmatrix} U_2 & 0 \\ 0 & V_2 \end{bmatrix} M \begin{bmatrix} V_2^T & 0 \\ 0 & U_2^T \end{bmatrix}, \qquad N := \begin{bmatrix} U_2 & 0 \\ 0 & V_2 \end{bmatrix} N \begin{bmatrix} V^T & 0 \\ 0 & U_2^T \end{bmatrix},$$

where

$$M = \begin{bmatrix} x & x & x & x & x & 0 & x & x & x & x \\ x & x & x & x & x & x & 0 & x & x & x \\ x & x & x & x & x & x & x & 0 & x & x \\ x & x & x & x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & x & x & x & x & x & x \\ 0 & 0 & x & x & x & x & x & x & x & x \\ 0 & x & 0 & x & x & x & x & x & x & x \\ 0 & x & x & 0 & x & x & x & x & x & x \\ x & x & x & x & 0 & x & x & x & x & x \end{bmatrix},$$

$$N = \begin{bmatrix} x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & \otimes & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \otimes & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & \otimes & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & \otimes & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{bmatrix}. \qquad (2.9)$$

We now apply a Givens symplectic similarity transformation $G_1 = G(5, c, s)$ to zero x_{51} using y_{51}, i.e.,

$$M := G_1 M G_1^T, \qquad N := G_1 N G_1^T.$$

This results in

$$M = \left[\begin{array}{ccccc|ccccc} x & x & x & x & x & 0 & x & x & x & x \\ x & x & x & x & x & x & 0 & x & x & x \\ x & x & x & x & x & x & x & 0 & x & x \\ x & x & x & x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & x & x & x & 0 \\ \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & x & x & x & x & x & x & x & x \\ 0 & x & 0 & x & x & x & x & x & x & x \\ 0 & x & x & 0 & x & x & x & x & x & x \\ 0 & x & x & x & 0 & x & x & x & x & x \end{array}\right],$$

$$N = \left[\begin{array}{ccccc|ccccc} x & x & x & x & x & 0 & 0 & 0 & 0 & x \\ 0 & x & x & x & x & 0 & 0 & 0 & 0 & x \\ 0 & 0 & x & x & x & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & x & x & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & x & x & x & x & x & 0 \\ \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{array}\right]. \tag{2.10}$$

It is easy to verify that the (2,1) block of N preserves its null structure because of the Givens similarity transformation on N and the upper triangular structure of the (1,1)-block. Also, it should be noted that a row and a column of the (1,2) block of N become nonzero as a result of this transformation, although the skew-symmetric structure of the block is preserved. We have now reduced the first row and column of X to zero. In order to complete this first step, we need to zero y_{31}, y_{41}, and y_{51}. This will allow us to perform the next step without destroying the structure we have already obtained. Also, at the termination of the algorithm, it will give us Y in upper Hessenberg form and A in upper triangular form.

Elimination of y_{31}, y_{41}, and y_{51} can be carried out using 2×2 modified Householder transformations. First, the element y_{51} is zeroed using y_{41}. This results in a nonzero entry in a_{54} [denoted by $\otimes$ in (2.11)], which can be zeroed using a_{55} with a Householder transformation on columns 4 and 5 of N. The elements y_{41} and y_{31} can be zeroed in the same way while maintaining the upper triangular form of A. The procedure is similar to that used in the Hessenberg-triangular reduction, the first step in the QZ algorithm for solving the generalized eigenvalue problem; e.g., see [2]. Denoting the Householder transformations by U_3 and V_3^T, we have

$$M := \begin{bmatrix} U_3 & 0 \\ 0 & V_3 \end{bmatrix} M \begin{bmatrix} V_3^T & 0 \\ 0 & U_3^T \end{bmatrix}, \qquad N := \begin{bmatrix} U_3 & 0 \\ 0 & V_3 \end{bmatrix} N \begin{bmatrix} V_3^T & 0 \\ 0 & U_3^T \end{bmatrix},$$

where

$$M = \begin{bmatrix} x & x & x & x & x & 0 & x & x & x & x \\ x & x & x & x & x & x & 0 & x & x & x \\ 0 & x & x & x & x & x & x & 0 & x & x \\ 0 & x & x & x & x & x & x & x & 0 & x \\ 0 & x & x & x & x & x & x & x & x & 0 \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & x & x & x & x & x & x & x & x \\ 0 & x & 0 & x & x & x & x & x & x & x \\ 0 & x & x & 0 & x & x & x & x & x & x \\ 0 & x & x & x & 0 & x & x & x & x & x \end{bmatrix},$$

$$N = \begin{bmatrix} x & x & x & x & x & 0 & x & x & x & x \\ 0 & x & x & x & x & x & 0 & x & x & x \\ 0 & \otimes & x & x & x & x & x & 0 & x & x \\ 0 & 0 & \otimes & x & x & x & x & x & 0 & x \\ 0 & 0 & 0 & \otimes & x & x & x & x & x & 0 \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & \otimes & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & \otimes & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & \otimes \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{bmatrix} \qquad (2.11)$$

The problem of reducing X to a null matrix has now been essentially deflated to that of order $n = 4$. We can proceed to the next step in the reduction—to zero the second column and row of X. We use modified Householder transformations to zero x_{32} and x_{42}. This introduces nonzero entries for a_{43} and a_{54}, which can be eliminated by Householder transformations on the appropriate rows of A. The (Q, Z) transformation in this step yields the following structure:

$$M := Q_4 M Z_4 = \left[\begin{array}{ccccc c ccccc}
x & x & x & x & x & & 0 & x & x & x & x \\
x & x & x & x & x & & x & 0 & x & x & x \\
0 & x & x & x & x & & x & x & 0 & x & x \\
0 & x & x & x & x & & x & x & x & 0 & x \\
0 & x & x & x & x & & x & x & x & x & 0 \\
 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & x & x & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x & & x & x & x & x & x \\
0 & 0 & 0 & x & x & & x & x & x & x & x \\
0 & 0 & x & 0 & x & & x & x & x & x & x \\
0 & x & x & x & 0 & & x & x & x & x & x
\end{array}\right],$$

$$N := Q_4 N Z_4 = \left[\begin{array}{ccccc c ccccc}
x & x & x & x & x & & 0 & x & x & x & x \\
0 & x & x & x & x & & x & 0 & x & x & x \\
0 & 0 & x & x & x & & x & x & 0 & x & x \\
0 & 0 & \otimes & x & x & & x & x & x & 0 & x \\
0 & 0 & 0 & \otimes & x & & x & x & x & x & 0 \\
 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & x & x & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & x & x & x & \otimes & 0 \\
0 & 0 & 0 & 0 & 0 & & x & x & x & x & \otimes \\
0 & 0 & 0 & 0 & 0 & & x & x & x & x & x
\end{array}\right],$$

where

$$Q_4 = \begin{bmatrix} U_4 & 0 \\ 0 & V_4 \end{bmatrix}, \qquad Z_4 = \begin{bmatrix} V_4^T & 0 \\ 0 & U_4^T \end{bmatrix}.$$

Next, we eliminate x_{52} using y_{52} by applying a symplectic Givens transformation $G_2 = G(5, c, s)$, i.e.,

$$M = G_2 M G_2^T = \left[\begin{array}{ccccc c ccccc}
x & x & x & x & x & & 0 & x & x & x & x \\
x & x & x & x & x & & x & 0 & x & x & x \\
0 & x & x & x & x & & x & x & 0 & x & x \\
0 & x & x & x & x & & x & x & x & 0 & x \\
0 & x & x & x & x & & x & x & x & x & 0 \\
 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & x & x & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & x & x & x & x & x \\
0 & 0 & 0 & x & x & & x & x & x & x & x \\
0 & 0 & x & 0 & x & & x & x & x & x & x \\
0 & 0 & x & x & 0 & & x & x & x & x & x
\end{array}\right],$$

$$N = G_2 N G_2^T = \begin{bmatrix} x & x & x & x & x & 0 & x & x & x & x \\ 0 & x & x & x & x & x & 0 & x & x & x \\ 0 & 0 & x & x & x & x & x & 0 & x & x \\ 0 & 0 & 0 & x & x & x & x & x & 0 & x \\ 0 & 0 & 0 & 0 & x & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{bmatrix}.$$

As before, the null structure of the (2,1) block of N is not affected by this transformation. We can now zero y_{42} and y_{52} while preserving the upper triangular structure of A:

$$M = Q_5 M Z_5 = \begin{bmatrix} x & x & x & x & x & 0 & x & x & x & x \\ x & x & x & x & x & x & 0 & x & x & x \\ 0 & x & x & x & x & x & x & 0 & x & x \\ 0 & 0 & x & x & x & x & x & x & 0 & x \\ 0 & 0 & x & x & x & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & x & x & x & x & x & x & x \\ 0 & 0 & x & 0 & x & x & x & x & x & x \\ 0 & 0 & x & x & 0 & x & x & x & x & x \end{bmatrix},$$

$$N = Q_5 N Z_5 = \begin{bmatrix} x & x & x & x & x & 0 & x & x & x & x \\ 0 & x & x & x & x & x & 0 & x & x & x \\ 0 & 0 & x & x & x & x & x & 0 & x & x \\ 0 & 0 & \otimes & x & x & x & x & x & 0 & x \\ 0 & 0 & 0 & \otimes & x & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & \otimes & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & \otimes \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{bmatrix},$$

where

$$Q_5 = \begin{bmatrix} U_5 & 0 \\ 0 & V_5 \end{bmatrix}, \qquad Z_4 = \begin{bmatrix} V_5^T & 0 \\ 0 & U_5^T \end{bmatrix}.$$

The result is a further deflation of the problem. Repeating the procedure to reduce the third and fourth columns and rows of X to zero, we finally get the following structure:

$$M = \left[\begin{array}{ccccc|ccccc} x & x & x & x & x & 0 & x & x & x & x \\ x & x & x & x & x & x & 0 & x & x & x \\ 0 & x & x & x & x & x & x & 0 & x & x \\ 0 & 0 & x & x & x & x & x & x & 0 & x \\ 0 & 0 & 0 & x & x & x & x & x & x & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{array}\right],$$

$$N = \left[\begin{array}{ccccc|ccccc} x & x & x & x & x & 0 & x & x & x & x \\ 0 & x & x & x & x & x & 0 & x & x & x \\ 0 & 0 & x & x & x & x & x & 0 & x & x \\ 0 & 0 & 0 & x & x & x & x & x & 0 & x \\ 0 & 0 & 0 & 0 & x & x & x & x & x & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{array}\right].$$

It should be noted that both M and N are in block upper triangular form with a 5×5 null matrix in the (2,1) block of each. Also, Y is an upper Hessenberg matrix and A is an upper triangular matrix. Clearly, the eigenvalues of the pencil $M - \lambda N$ can be obtained from the eigenvalues of the pencil $Y - \lambda A$. The QZ algorithm can be applied to compute the eigenvalues of the $n \times n$ pencil $Y - \lambda A$. Furthermore, it should be noted that the preliminary step in the QZ algorithm involving the Hessenberg-triangular reduction is not needed, since Y and A are already in the required form. Lastly, if μ_i, $i = 1, \ldots, n$, are the eigenvalues of the pencil $Y - \lambda A$, then the eigenvalues of the pencil $K - \lambda L$ are the roots α_i, β_i ($|\alpha_i| < 1$, $|\beta_i| > 1$) of the equation $z^2 - \mu_i z + 1 = 0$, $i = 1, \ldots, n$, for $\mu_i \neq \infty$. If $\mu_i = \infty$, then $\alpha_i = 0$, $\beta_i = \infty$. To separate the stable eigenvalues of $K - \lambda L$ from the unstable ones, we let $\lambda_i = \alpha_i$ and $\lambda_{n+i} = \beta_i$, $i = 1, \ldots, n$.

ALGORITHM 2.1 (Block-triangular reduction). Given a $2n \times 2n$ symplectic pencil $M - \lambda N$ where M and N are in the form (2.3), this algorithm overwrites M and N by block upper triangular matrices QMZ and QNZ respectively with the structure shown in (2.4). The matrices Q and Z are orthogonal.

1. Compute Householder matrices $P_1, P_2, \dots, P_{n-1}$ such that if $U = P_{n-1}, \dots, P_1$, then UA is upper triangular.
2. For $i = 1, \dots, n-1$
   $M := Q_i M Z_i$,
   $N := Q_i N Z_i$,
   where $Q_i := \operatorname{diag}[P_i, I_n]$ and $Z_i := \operatorname{diag}[I_n, P_i]$.
   End
3. For $j = 1, \dots, n-2$
   For $i = j+1, \dots, n-1$
   Determine a Householder matrix $\bar{V} \in \mathbb{R}^{2\times 2}$ such that

$$\bar{V}\begin{bmatrix} x_{ij} \\ x_{i+1,j} \end{bmatrix} = \begin{bmatrix} 0 \\ * \end{bmatrix}.$$

   $M := QMZ$, $N := QNZ$,
   where

$$Q = \operatorname{diag}\left[I_{n+i-1}, \bar{V}, I_{n-i-1}\right] \quad \text{and}$$

$$Z = \operatorname{diag}\left[I_{i-1}, \bar{V}, I_{2n-i-1}\right].$$

   Determine a Householder matrix $\bar{U} \in \mathbb{R}^{2\times 2}$ such that

$$\bar{U}\begin{bmatrix} a_{ii} \\ a_{i+1,i} \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}.$$

   $M := QMZ$, $N := QNZ$,
   where

$$Q = \operatorname{diag}\left[I_{i-1}, \bar{U}, I_{2n-i-1}\right] \quad \text{and}$$

$$Z = \operatorname{diag}\left[I_{n+i-1}, \bar{U}, I_{n-i-1}\right].$$

   End

Determine constants c and s with $c^2 + s^2 = 1$ such that

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} y_{nj} \\ x_{nj} \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}.$$

$M := GMG^T$, $N := GNG^T$,
where $G = G(n, c, s)$ is a symplectic Givens transformation matrix.
For $i = n, n-1, \dots, j+2$
Determine a Householder matrix $\overline{U} \in \mathbb{R}^{2\times 2}$ such that

$$\overline{U} \begin{bmatrix} y_{i-1,j} \\ y_{ij} \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}.$$

$M := QMZ$, $N := QNZ$,
where $Q = \mathrm{diag}[I_{i-2}, \overline{U}, I_{2n-i}]$ and $Z = \mathrm{diag}[I_{n+i-2}, \overline{U}, I_{n-i}]$.
Determine a Householder matrix $\overline{V} \in \mathbb{R}^{2\times 2}$ such that

$$[a_{i,i-1} \quad a_{ii}]\overline{V} = [0 \quad *].$$

$M := QMZ$, $N := QNZ$,
where $Q = \mathrm{diag}[I_{n+i-2}, \overline{V}, I_{n-i}]$ and $Z = \mathrm{diag}[I_{i-2}, \overline{V}, I_{2n-i}]$.
End
End

This algorithm requires about $\frac{47}{3}n^3$ flops. The transformations described in the algorithm can be carried out directly on the appropriate rows and columns of M and N, so that the matrices Q and Z do not need to be constructed explicitly. Also, since only the eigenvalues are required, the transformations do not need to be accumulated. Using the roundoff properties of Householder and Givens transformations, it can be shown that Algorithm 2.1 is numerically backward stable, i.e., if we denote the output of Algorithm 2.1 by $\hat{M} - \lambda\hat{N}$, then the matrices $\hat{M}$ and $\hat{N}$ can be shown to satisfy

$$Q_0^T(M + E_M)Z_0 = \hat{M},$$

$$Q_0^T(N + E_N)Z_0 = \hat{N},$$

where Q_0 and Z_0 are exactly orthogonal, $\|E_M\|_F \approx \phi(n)\mathbf{u}\|M\|_F$, and $\|E_N\|_F \approx \psi(n)\mathbf{u}\|N\|_F$. In the above expressions, $\|(\cdot)\|_F$ denotes the Frobenius norm of the matrix $(\cdot)$, ϕ and ψ are low-degree polynomials in n (with small coefficients), and $\mathbf{u}$ is the unit roundoff [2].

We can now present the complete algorithm for computing the eigenvalues of the symplectic pencil $K - \lambda L$.

ALGORITHM 2.2 (Eigenvalues of $K - \lambda L$). Given a $2n \times 2n$ symplectic pencil $K - \lambda L$ in the form (1.6), this algorithm determines its eigenvalues λ_i, $i = 1, \dots, 2n$. The eigenvalues are separated into sets of stable and unstable eigenvalues, i.e., λ_i, $i = 1, \dots, n$, are stable, while λ_i, $1 = n + 1, \dots, 2n$, are unstable.

1. Compute the pencil $M - \lambda N$ from the submatrices of $K - \lambda L$ as in (2.3).
2. Apply Algorithm 2.1 to reduce $M - \lambda N$ to upper block triangular form and obtain the pencil $Y - \lambda A$ in Hessenberg-triangular form.
3. Using the QZ algorithm, compute the eigenvalues μ_i, $i = 1, \dots, n$, of $Y - \lambda A$.
   Comment: Since $Y - \lambda A$ is already in Hessenberg-triangular form, the first step in the QZ algorithm (which is the reduction to such a form) is not required. So, if EISPACK subroutines [1] are used for the QZ algorithm, the subroutine QZHES is not needed; only the subroutines QZIT and QZVAL are required.
4. Compute the eigenvalues of $K - \lambda L$ as follows:

```
For i = 1, ..., n
  If |μ_i| = ∞
    λ_i := 0
    λ_{n+i} := ∞
  Else
    α := ½[μ_i + (μ_i² − 4)^{1/2}]
    If |α| ≤ 1
      λ_i := α
      λ_{n+i} := 1/α
    Else
      λ_i := 1/α
      λ_{n+i} := α
    End
  End
End
```

   Comment: In the above step, the "value" of ∞ can be taken as the reciprocal of the "threshold" value of zero used in the subroutine QZIT in EISPACK.

Operation count: We now give an estimate of the amount of computation performed by Algorithm 2.2 in terms of the number of flops required.

We use the usual definition of a *flop*, i.e., the amount of work associated with computing $s := s + a_{ik}b_{kj}$:

Computing $M - \lambda N$	$4n^3$
Algorithm 2.1	$\frac{47}{3}n^3$
Computing[2] the eigenvalues of $Y - \lambda A$	$10n^3$
Total	$\frac{89}{3}n^3$

If the eigenvalues of $K - \lambda L$ are computed by a direct application of the QZ algorithm, then approximately $120n^3$ flops are needed, which implies that the work required by the proposed algorithm is about one-fourth that of the QZ algorithm. The algorithm proposed by Lin [4] involves approximately $\frac{173}{6}n^3$ flops, which is comparable to the proposed algorithm. However, as mentioned earlier, some of the transformations used in [4] for the reduction of $M - \lambda N$ to a block upper triangular form can lead to numerical instability. Such a situation will not arise in Algorithm 2, because the corresponding reduction (Algorithm 2.1) is numerically backward stable.

3. NUMERICAL EXAMPLES

In this section, we illustrate the numerical performance and properties of the algorithms proposed in the preceding section. All computations were performed on a Sun 4/370 computer using the f77 compiler.

EXAMPLE 1. This example is the same as Example 1 in [4] and Example 2 in [7]. It corresponds to a symplectic pencil $K - \lambda L$ of order 8 whose exact eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = (21 - 5\sqrt{17})/4$, $\lambda_5 = (21 + 5\sqrt{17})/4$, $\lambda_6 = \lambda_7 = \lambda_8 = \infty$. The finite eigenvalues obtained using Algorithm 2.2 and the QZ algorithm are shown in Table 1. The eigenvalues λ_4 and λ_5 evaluated to 15 significant figures are 9.61179679779241E − 02 and 1.04038820320221E + 01 respectively.

EXAMPLE 2. In this example, we used the 12×12 symplectic pencil considered in [4]. This pencil has one infinite eigenvalue and 11 finite ones.

[2] For an $n \times n$ pencil $A - \lambda B$, the QZ algorithm requires about $15n^3$ flops, of which $5n^3$ flops are needed for the Hessenberg-triangular reduction [2].

TABLE 1

	Algorithm 2.2	QZ algorithm
1	0.00000000000000E + 00	0.00000000000000E + 00
2	0.00000000000000E + 00	0.00000000000000E + 00
3	0.00000000000000E + 00	0.00000000000000E + 00
4	9.61179679779243E − 02	9.61179679779243E − 02
5	1.04038820320221E + 01	1.04038820320221E + 01

The finite eigenvalues obtained using Algorithm 2.2 and the QZ algorithm are shown in Table 2.

EXAMPLE 3. Algorithm 2.2 has been tested with several examples of 50×50 random symplectic pencils $K - \lambda L$ with reasonably well-conditioned eigenvalues. The eigenvalues were also computed by directly applying the QZ algorithm to the pencil $K - \lambda L$. In each case, the eigenvalues were arranged in order of increasing magnitude, and the largest absolute error was computed, i.e., $\max_i |a_i - b_i|$, where a_i denotes the ordered set of eigenvalues computed using Algorithm 2.2 and b_i denotes those computed using the QZ algorithm. This value was found to be of the order of 10^{-10}.

4. CONCLUSIONS

In this paper, we have presented an algorithm for computing the eigenvalues of a symplectic pencil that arises in solving the discrete-time algebraic Riccati equation. The approach is analogous to that developed by Van Loan [9] for Hamiltonian matrices. Our algorithm uses the so called $S + S^{-1}$ transformation [4] of the symplectic pencil to obtain a pencil in which the two submatrices on the diagonal are transposes of each other, and the two off-diagonal submatrices are skew-symmetric. The eigenvalues of this pencil are related in a simple way to those of the symplectic pencil and are computed using transformations that preserve its structure. However, unlike the transformations in [4], the structure-preserving transformations used in the proposed algorithm are all orthogonal (Givens symplectic and Householder). An operation count of the algorithm indicates that it is almost as efficient as that in [4], requiring about one-fourth the number of flops that the QZ algorithm would use if applied directly to the given symplectic pencil. In the numerical experiments performed on a number of random symplectic pencils of order 50 there was no noticeable difference in accuracy in the

TABLE 2

	Algorithm 2.2		QZ algorithm	
	Real	Imaginary	Real	Imaginary
1	4.39448818329700E − 17	0.0	−1.00714376330141E − 15	0.0
2	1.29516106930204E − 01	0.0	1.29516106930205E − 01	0.0
3	2.38038678693309E − 01	−1.60374636226678E − 01	2.38038678693309E − 01	1.60374636226678E − 01
4	2.38038678693309E − 01	1.60374636226678E − 01	2.38038678693310E − 01	−1.60374636226678E − 01
5	3.23100409725916E − 01	0.0	3.23100409725917E − 01	0.0
6	−4.44639422393446E − 01	0.0	−4.44639422393447E − 01	0.0
7	−2.24901335697385E + 00	0.0	−2.24901335697385E + 00	0.0
8	3.09501309778064E + 00	0.0	3.09501309778063E + 00	0.0
9	2.88943479714271E + 00	−1.94670906861109E + 00	2.88943479714272E + 00	−1.94670906861110E + 00
10	2.88943479714271E + 00	1.94670906861109E + 00	2.88943479714272E + 00	1.94670906861110E + 00
11	7.72104739481472E + 00	0.0	7.72104739481469E + 00	0.0

eigenvalues computed using our algorithm and those computed using the QZ algorithm.

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