On S-asymptotically ω-periodic functions on Banach spaces and applications

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Abstract

This paper is devoted to the study of the class of continuous and bounded functions $f : [0, \infty) \rightarrow X$ for which exists $\omega > 0$ such that $\lim_{t \rightarrow \infty} (f(t + \omega) - f(t)) = 0$ (in the sequel called $S$-asymptotically $\omega$-periodic functions). We discuss qualitative properties and establish some relationships between this type of functions and the class of asymptotically $\omega$-periodic functions. We also study the existence of $S$-asymptotically $\omega$-periodic mild solutions of the first-order abstract Cauchy problem in Banach spaces.

Keywords: $S$-asymptotically periodic functions; Asymptotically periodic functions; Asymptotically almost periodic functions; Abstract Cauchy problem; Semigroups of bounded linear operators

1. Introduction

This paper is devoted to the study of the class of continuous functions $f : [0, \infty) \rightarrow X$ for which exists $\omega > 0$ such that $\lim_{t \rightarrow \infty} (f(t + \omega) - f(t)) = 0$. These functions will be called $S$-asymptotically $\omega$-periodic.

The literature concerning $S$-asymptotically $\omega$-periodic functions is very restricted and limited essentially to the study of the existence of $S$-asymptotically $\omega$-periodic solutions of ordinary differential equations described on finite dimensional spaces. We refer to [2,4,6,15,16]. In this paper, we make an initial contribution in order to develop a theory of $S$-asymptotically $\omega$-periodic functions with values in Banach spaces. We have particular interest in the relationship between this type of functions with the classes of asymptotically periodic functions and asymptotically almost periodic functions, as well as to establish existence of $S$-asymptotically $\omega$-periodic mild solutions of first-order abstract differential equations.

This work has four sections. In the next section, we introduce some basic concepts, definitions and notations needed in the sequel. In Section 3, we study some qualitative properties of $S$-asymptotically $\omega$-periodic functions. In
particular, we establish some relationship between this type of functions and the class of asymptotically $\omega$-periodic functions. Finally, in Section 4, we study the existence of $S$-asymptotically $\omega$-periodic mild solutions for a class of first-order abstract Cauchy problem. This section is completed with an application to partial differential equations.

2. Preliminaries

In this paper $(X, \| \cdot \|)$ denotes a real or complex Banach space and $C_b([0, \infty), X)$ is the space consisting of the continuous and bounded functions from $[0, \infty)$ into $X$, endowed with the norm of the uniform convergence which is denoted by $\| \cdot \|_{\infty}$. Additionally, $C_0([0, \infty), X)$ and $C_\omega([0, \infty), X)$, for $\omega > 0$, are the subspaces of $C_b([0, \infty), X)$ defined by

$$C_0([0, \infty), X) = \{ x \in C_b([0, \infty), X) : \lim_{t \to \infty} \| x(t) \| = 0 \},$$

$$C_\omega([0, \infty), X) = \{ x \in C_b([0, \infty), X) : x \text{ is } \omega\text{-periodic} \}.$$

We abbreviate by $\mathbb{K}$ the real or complex numbers. Moreover, we denote by $B(X)$ the Banach algebra of bounded linear operators from $X$ into $X$. Next recall some concepts concerning almost periodic functions.

**Definition 2.1.** A function $f \in C_b(\mathbb{R}, X)$ is called almost periodic if for every $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, f)$, such that $\| f(t + \xi) - f(t) \| < \varepsilon$, for every $t \in \mathbb{R}$ and all $\xi \in \mathcal{H}(\varepsilon, f)$.

**Definition 2.2.** A function $f \in C_b([0, \infty), X)$ is called asymptotically almost periodic if there exists an almost periodic function $g$ and $\varphi \in C_0([0, \infty), X)$ such that $f = g + \varphi$. If $g$ is periodic (respectively $\omega$-periodic) $f$ is said asymptotically periodic (respectively asymptotically $\omega$-periodic).

For additional details on almost periodic, asymptotically almost periodic and weakly asymptotically almost periodic functions, we refer the reader to [3,11,17] and references therein.

3. On $S$-asymptotically periodic functions

In this section, we study qualitative properties of $S$-asymptotically $\omega$-periodic functions. We begin by establishing the terminology.

**Definition 3.1.** A function $f \in C_b([0, \infty), X)$ is called $S$-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t \to \infty} (f(t + \omega) - f(t)) = 0$. In this case, we say that $\omega$ is an asymptotic period of $f$ and that $f$ is $S$-asymptotically $\omega$-periodic.

In this work the notation $SAP_\omega(X)$ stands for the subspace of $C_b([0, \infty), X)$ consisting of the $S$-asymptotically $\omega$-periodic functions.

To abbreviate our developments, in the rest of this paper, for a fixed positive number $\omega$ and for every $t \geq 0$, we consider the decomposition $t = \xi(t) + \tau(t)\omega$ where $\xi(t) \in [0, \omega)$ and $\tau(t) \in \mathbb{N} \cup 0$. Additionally, for $h \geq 0$ and $f \in C_b([0, \infty), X)$, we denote by $f_h$ the function $f_h : [0, \infty) \to X$ defined by $f_h(t) = f(t + h)$.

**Definition 3.2.** A function $f \in C_b([0, \infty), X)$ is said $\omega$-normal on compact sets if for every sequence of natural numbers $(m_n)_{n \in \mathbb{N}}$ with $m_n \to \infty$ as $n \to \infty$, there exists a subsequence $(k_n)_{n \in \mathbb{N}}$ and $F \in C_b([0, \infty), X)$ such that $f_{k_n\omega} \to F$ as $n \to \infty$ uniformly on compact subsets of $[0, \infty)$.

**Remark 3.1.** We note that if $f \in C_b([0, \infty), X)$ is a uniformly continuous with relatively compact range function, then $f$ is $\omega$-normal on compact sets.

The next result is an immediate consequence of the previous definitions.

**Lemma 3.1.** Let $f : [0, \infty) \to X$ be an $S$-asymptotically $\omega$-periodic function, let $(t_n)_{n \in \mathbb{N}}$ be a sequence with $t_n \to \infty$ as $n \to \infty$ and, assume that $f_{t_n} \to F$ uniformly on compact subsets of $[0, \infty)$. Then $F \in C_\omega([0, \infty), X)$. 

Proof. It is clear that $F$ is continuous. For $t \geq 0$ and $\varepsilon > 0$ given, we select $n_0 \in \mathbb{N}$ such that
\[
\|F(s) - f(s + t_n)\| \leq \varepsilon, \quad s \in [t, t + \omega],
\]
\[
\|f(\mu + t_n + \omega) - f(\mu + t_n)\| \leq \varepsilon, \quad \mu \geq 0,
\]
for every $n \geq n_0$. Hence, for $n \geq n_0$, we have that
\[
\|F(t + \omega) - F(t)\| \leq \|F(t + \omega) - f(t + \omega + t_n)\| + \|f(t + \omega + t_n) - f(t + t_n)\| + \|f(t + t_n) - F(t)\|
\]
\[
\leq 3\varepsilon,
\]
which implies that $F(t + \omega) = F(t)$. The proof is complete. \qed

Proposition 3.1. Let $f : [0, \infty) \to X$ be a uniformly continuous, $S$-asymptotically $\omega$-periodic and $\omega$-normal on compact sets function. If $(t_n)_{n \in \mathbb{N}}$ is a sequence with $t_n \to \infty$ as $n \to \infty$, then there exists a subsequence $(s_j)_{j \in \mathbb{N}}$ of $(t_n)_{n \in \mathbb{N}}$ and a function $F \in C_\omega([0, \infty), X)$ such that $f_{s_j} \to F$ as $j \to \infty$ uniformly on compact sets.

Proof. We consider the decomposition $t_n = \xi(t_n) + \tau(t_n)\omega$ for $n \in \mathbb{N}$. It follows from the assumptions and Lemma 3.1 that there exists a function $G \in C_\omega([0, \infty); X)$ and a subsequence $(s_j)_{j \in \mathbb{N}}$ of $(t_n)_{n \in \mathbb{N}}$ such that $f_{\tau(s_j)\omega} \to G$ as $j \to \infty$ uniformly on compact sets. Moreover, we can assume that there exists $\lambda \in [0, \omega)$ such that $\xi(s_j) \to \lambda$ as $j \to \infty$.

In order to prove that $f_{s_j} \to G_\lambda$ uniformly on compacts when $j$ goes to infinity, we take a compact set $K \subseteq \mathbb{R}$. For each $\varepsilon > 0$, we can choose $j_0 \in \mathbb{N}$ such that
\[
\|G(\lambda + s) - f(\lambda + s + \tau(s_j)\omega)\| \leq \varepsilon, \quad s \in K,
\]
\[
\|f(\lambda + \mu) - f(\xi(s_j) + \mu)\| \leq \varepsilon, \quad \mu \geq 0,
\]
for every $j \geq j_0$. Therefore, for $t \in K$ and $j \geq j_0$, we can write
\[
\|G(\lambda + t) - f(s_j + t)\| = \|G(\lambda + t) - f(\xi(s_j) + \tau(s_j)\omega + t)\|
\leq \|G(\lambda + t) - f(\lambda + \tau(s_j)\omega + t)\| + \|f(\lambda + \tau(s_j)\omega + t) - f(\xi(s_j) + \tau(s_j)\omega + t)\|
\leq 2\varepsilon,
\]
which implies that $f_{s_j} \to F = G_\lambda$ as $j \to \infty$ uniformly on $K$. This completes the proof since $G_\lambda$ is $\omega$-periodic. \qed

Proposition 3.2. Let $f \in C_\omega([0, \infty), X)$ be a uniformly continuous function. Assume that for every sequence of natural numbers $(m_n)_{n \in \mathbb{N}}$ with $m_n \to \infty$ as $n \to \infty$, there exists a subsequence $(k_j)_{j \in \mathbb{N}}$ of $(m_n)_{n \in \mathbb{N}}$ and a function $F \in C_\omega([0, \infty), X)$ such that $f_{k_j\omega} \to F$ as $j \to \infty$ uniformly on compact sets. Then $f$ is $S$-asymptotically $\omega$-periodic.

Proof. Assume that the assertion is false. Then there exist $\varepsilon > 0$ and a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$ such that $\|f(t_n + \omega) - f(t_n)\| \geq \varepsilon$ for every $n \in \mathbb{N}$. Defining $m_n = \tau(t_n)$, we deduce the existence of a subsequence $(s_j)_{j \in \mathbb{N}}$ of $(t_n)_{n \in \mathbb{N}}$, of a number $\lambda \in [0, \omega)$ and, of a function $F \in C_\omega([0, \infty), X)$ such that $\xi(s_j) \to \lambda$ and $f_{k_j\omega} \to F$ as $j \to \infty$ uniformly on compact sets, where we have denoted $k_j = \tau(s_j)$. Collecting these properties with the uniform continuity of $f$, we can select $j_0 \in \mathbb{N}$ such that
\[
\|F(s) - f(s + k_j\omega)\| \leq \frac{\varepsilon}{8}, \quad s \in [\lambda + \omega, \lambda + \omega],
\]
\[
\|f(\lambda + k_j\omega + \mu) - f(\xi(s_j) + k_j\omega + \mu)\| \leq \frac{\varepsilon}{8}, \quad \mu \geq 0,
\]
for every $j \geq j_0$. Hence, employing $j > j_0$ we obtain that
\[
\|F(\lambda + \omega) - F(\lambda)\| \geq -\|F(\lambda + \omega) - f(\lambda + k_j\omega + \omega)\| - \|f(\lambda + k_j\omega + \omega) - f(\xi(s_j) + k_j\omega + \omega)\|
\]
\[
+ \|f(\xi(s_j) + k_j\omega + \omega) - f(\xi(s_j) + k_j\omega)\| - \|f(\xi(s_j) + k_j\omega) - f(\lambda + k_j\omega)\|
\]
\[
\geq \varepsilon - \frac{\varepsilon}{2},
\]
which is a contradiction since $F$ is $\omega$-periodic. This completes the proof. \qed
We next discuss the relationship between the class of $S$-asymptotically $\omega$-periodic functions and the class of asymptotically periodic functions. We initially exhibit a function that is $S$-asymptotically $\mu$-periodic for every $\mu > 0$ and is not asymptotically $\mu$-periodic.

**Example 3.1.** Let $X$ be the space $c_0 = \{(x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{R}$ and $\lim_{n \to \infty} x_n = 0\}$ endowed with the norm $\|(x_n)_{n \in \mathbb{N}}\| = \sup_{n \in \mathbb{N}}|x_n|$ and, let $f : [0, \infty) \to X$ be the function defined by $f(t) = (\frac{2nt}{t^2+n^2})_{n \in \mathbb{N}}$.

The function $f$ is bounded, uniformly continuous and $S$-asymptotically $\mu$-periodic for every $\mu > 0$. In fact, it is immediate that $\|f(t)\| \leq 1$ for every $t \geq 0$. Moreover, for $t, s \in [0, \infty)$, we have that 

$$
\|f(t+s) - f(t)\| \leq \sup_{n \geq 1} \frac{2ns|n^2 - t^2 - st|}{(t+s)^2 + n^2}(n^2 + t^2) \leq 4s,
$$

which shows that $f$ is uniformly continuous. On the other hand, for $\mu > 0$ and $t \geq 1$, we get

$$
\|f(t + \mu) - f(t)\| \leq \sup_{n \geq 1} \frac{2n|n^2 t^2 - \mu^2 - t \mu^2|}{(t + \mu)^2 + n^2}(n^2 + t^2) \leq \frac{2n^3 \mu + 2nt \mu^2 + 2n \mu^2}{n^4 + t^4} \leq \frac{3\mu}{t} + \frac{3\mu^2}{t^2},
$$

which implies that $f$ is $S$-asymptotically $\mu$-periodic for every $\mu > 0$.

However, $f$ is not asymptotically $\mu$-periodic. To show this assertion, assume that there exist $g \in C_{\mu}(\mathbb{R}, X)$ and a function $\varphi \in C_0([0, \infty), X)$ such that $f = g + \varphi$. If $f = (f_n)_{n \in \mathbb{N}}$, then each coordinate $f_n$ is asymptotically $\mu$-periodic and $f_n(t + k\mu) = g_n(t) + \varphi_n(t + k\mu)$, for every $k, n \in \mathbb{N}$ and $t > 0$. In view of that condition $\lim_{n \to \infty} f_n(t + k\mu) = 0$, it follows that $g_n(t) = 0$ for every $n \in \mathbb{N}$ and every $t \geq 0$. Consequently, the function $g = 0$ and $f = \varphi$, which is absurd since $\|f(n)\| = 1$ for every $n \in \mathbb{N}$. This proves that $f$ is not asymptotically $\mu$-periodic.

In [6, Lemma 2.1] it is stated that every $S$-asymptotically $\omega$-periodic scalar function is asymptotically $\omega$-periodic. This assertion is not true, as the following example shows.

**Example 3.2.** Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that $b_n \neq 0$ for every $n \in \mathbb{N}$, $b_n \to 0$ as $n \to \infty$, and the sequence $(a_n)_{n \in \mathbb{N}} = (\sum_{i=0}^{n} b_i)_{n \in \mathbb{N}}$ is bounded and non-convergent. Let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(n) = a_n$ for $n \in \mathbb{N}_{\geq 0}$ and

$$
\begin{align*}
  f(t) &= a_{n+1} + (a_{n+1} - a_n)(t - n - 1), \\
  \text{for } n \leq t \leq n + 1. 
\end{align*}
$$

(3.1)

Consequently, the graph of $f$ consists of the segments of line with corners the points $(n, a_n)$. It is clear from this geometrical description that $f$ is bounded and continuous. Furthermore, $f$ is uniformly continuous. In fact, we set $c = \max_{n \geq 1}|a_n - a_{n-1}|$. Employing (3.1) for $s \in [n, n+1]$ and $t \in [n, n+2]$, we obtain that

$$
|f(t) - f(s)| \leq c|t - s|.
$$

On the other hand, turning to apply (3.1), we see that

$$
|f(t+1) - f(t)| \leq |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n|,
$$

for $t \in [n, n+1]$. Therefore, $\lim_{t \to \infty}(f(t+1) - f(t)) = 0$ and $f$ is an $S$-asymptotically $1$-periodic function.

However, $f$ is not asymptotically $1$-periodic function. To establish this assertion, assume that $g = \varphi + \alpha$, where $g$ is a function $1$-periodic and $\alpha$ is a function that vanishes at infinite. In such case, $f(n) = a_n = g(n) + \alpha(n) = g(0) + \alpha(n) \to g(0)$, as $n \to \infty$, which contradicts our selection of the sequence $(a_n)_{n}$.

In the sequel, we establish conditions under which an $S$-asymptotically $\omega$-periodic function is asymptotically $\omega$-periodic.
Proposition 3.3. Let \( f \in C_b([0, \infty), X) \) be an \( \omega \)-normal on compact sets and \( S \)-asymptotically \( \omega \)-periodic function. Assume that there exist a strictly increasing sequence of natural numbers \((\tau_n)_{n \in \mathbb{N}}\) and a sequence of positive numbers \((\gamma_n)_{n \in \mathbb{N}}\) such that \( \sum_{j \geq 0} (\tau_{j+1} - \tau_j)\gamma_j < \infty. \) If \( \| f(t+\omega) - f(t) \| \leq \gamma_n \) for every \( t \in [\tau_n, \tau_{n+1}] \), then \( f \) is asymptotically \( \omega \)-periodic.

Proof. Applying Definition 3.2 and Lemma 3.1, we assert that there exist a subsequence \((m_j)_{j \in \mathbb{N}}\), with \( m_j = \tau_{n_j} \), of \((\tau_n)_{n \in \mathbb{N}}\) and a function \( F \in C_b([0, \infty), X) \) such that \( f_{m_j, \omega} \to F \) as \( j \to \infty \) uniformly on compact sets. We affirm that \( F(t) - f(t) \to 0 \) as \( t \to \infty \). To establish our claim, for each \( \epsilon > 0 \), we choose \( n_j \in \mathbb{N} \) such that \( \sum_{j \geq n_j} (\tau_{j+1} - \tau_j)\gamma_j < \epsilon \) and \( \| F(s) - f(s + m_j \omega) \| \leq \epsilon \) for every \( s \in [0, \omega) \) and \( j \geq j_0 \).

Let \( t \geq m_j \omega \). Then there exist an index \( p \in \mathbb{N} \) with \( p \geq j_0 \) such that \( t \in [m_p \omega, m_{p+1} \omega) \). The interval \([m_p \omega, m_{p+1} \omega)\) can contain another points of the original sequence \((\tau_n)_{n \in \mathbb{N}}\), which we describe in the form \( \tau_n < \tau_{n+1} < \cdots < \tau_{n+p+q} = \tau_{n+p+1} \). Similarly, each interval \([\tau_{n+i}, \tau_{n+i+1})\), with \( i = 0, \ldots, q - 1 \), can contain natural numbers \( \tau_{n+p+i} + h \), with \( h = 0, \ldots, H(i) \), so that \( \tau_{n+p+i} + H(i) = \tau_{n+p+i+1} \). We abbreviate the notation by writing \( k(i) = \tau_{n+p+i} \). Moreover, we select \( 0 \leq s < q \) such that \( t \in [\tau_{n+p+s}, \tau_{n+p+s+1}) \) and, we decompose \( t = \xi(t) + \eta(t) \omega \) with \( \xi(t) \in [0, \omega) \) and \( \eta(t) \) is a natural number such that \( \eta(t) = \tau_{n+p+s} + h(t) \) where \( 0 \leq h(t) \leq H(s) \). With these notations, we get

\[
\| F(t) - f(t) \| = \| F(\xi(t)) + \eta(t) \omega - f(\xi(t)) + \eta(t) \omega \| \\
\leq \| F(\xi(t)) - f(\xi(t)) + m_p \omega \| + \| f(\xi(t)) + m_p \omega - f(\xi(t)) + \eta(t) \omega \| \\
\leq \epsilon + \sum_{i=0}^{s-1} \sum_{j=k(i)}^{k(i)+H(i)-1} \| f(\xi(t)) + (j+1) \omega - f(\xi(t)) + j \omega \| \\
+ \sum_{j=k(s)}^{s-1} \| f(\xi(t)) + (j+1) \omega - f(\xi(t)) + j \omega \| \\
\leq \epsilon + \sum_{i=0}^{s} \sum_{j=0}^{k(i)} \gamma_{n+i} + \sum_{j=k(s)}^{s} \gamma_{n+p+s} \\
\leq \epsilon + \sum_{i=0}^{s} \gamma_{n+p+i} H(i) \\
= \epsilon + \sum_{i=0}^{s} \gamma_{n+p+i} (\tau_{n+p+i+1} - \tau_{n+p+i}) \\
\leq \epsilon + \sum_{i \geq n_p} \gamma_i (\tau_{i+1} - \tau_i) \\
\leq 2\epsilon,
\]

which shows that \( \| F(t) - f(t) \| \leq 2\epsilon \) for every \( t \geq m_j \omega \). This completes the proof. \( \square \)

Proposition 3.4. Let \( f : [0, \infty) \to X \) be an \( S \)-asymptotically \( \omega \)-periodic and asymptotically almost periodic function. Then \( f \) is asymptotically \( \omega \)-periodic.

Proof. We can decompose \( f \) as \( f = g + \varphi \) where \( g \) is an almost periodic function and \( \varphi \in C_0([0, \infty), X) \). It follows from the theory of almost periodic functions that there exists a sequence of real numbers \((t_n)_{n \in \mathbb{N}}\) such that \( t_n \to \infty \) and \( g_{t_n} \to g \) as \( n \to \infty \) uniformly on \([0, \infty)\). Therefore, \( f_{t_n} \to g \) as \( n \to \infty \) uniformly on \([0, \infty)\) and, applying Lemma 3.1, it follows that \( g \in C_0([0, \infty); X) \) which, in turn implies that the function \( f \) is asymptotically \( \omega \)-periodic. The proof is complete. \( \square \)

Corollary 3.1. Let \( f \in C_b([0, \infty), X) \). Assume that there exists a sequence of natural numbers \((n_j)_{j \in \mathbb{N}}\) with \( n_1 = 1 \) and \( n_j \to \infty \) as \( j \to \infty \) such that \( \alpha = \sup_{j \in \mathbb{N}} (n_{j+1} - n_j) < \infty \) and
\[
\lim_{t \to \infty} \left( f(t + n_j \omega) - f(t) \right) = 0,
\]
uniformly for \( j \in \mathbb{N} \). Then \( f \) is asymptotically \( \omega \)-periodic.

**Proof.** The assertion is an immediate consequence of Proposition 3.4, since the condition (3.2) implies that \( f \) is \( S \)-asymptotically \( \omega \)-periodic and asymptotically almost periodic. \( \square \)

**Theorem 3.1.** Assume that \( X \) is a reflexive space. Let \( f \in C_b([0, \infty), X) \) be a uniformly continuous function such that for each \( x' \in X' \), \( \lim_{t \to \infty} \langle x', f(t + n \omega) - f(t) \rangle = 0 \) uniformly on \( n \in \mathbb{N} \). Then there exists \( g \in C_\omega([0, \infty), X) \) and \( \varphi \in C_b([0, \infty), X) \) such that \( f = g + \varphi \) and \( \lim_{t \to \infty} \langle x', \varphi(t) \rangle = 0 \) for every \( x' \in X' \).

**Proof.** For each \( x' \in X' \), the function \( \langle x', f \rangle \) verifies the conditions in Corollary 3.1. Therefore, there exists an \( \omega \)-periodic function \( g_{x'} \in C([0, \infty), X) \) and a function \( \varphi_{x'} \in C_0([0, \infty), X) \) such that \( \langle x', f \rangle = g_{x'} + \varphi_{x'} \).

For each \( t \geq 0 \), we define the function \( A_t : X' \to X \) by \( A_t(x') = g_{x'}(t) \). It follows from the uniqueness of the decomposition of \( \langle x', f \rangle \) as the sum of a periodic function and a function that vanishes at infinity that \( A_t \) is a linear functional. Moreover, from the estimate

\[
\left| A_t(x') \right| = \left| g_{x'}(t) \right| = \left| g_{x'}(t + k \omega) \right| \\
\leq \left| \langle x', f(t + k \omega) \rangle \right| + \left| \varphi_{x'}(t + k \omega) \right| \\
\leq \| f \| \| x' \| + \| \varphi_{x'}(t + k \omega) \|,
\]

and taking the limit as \( k \to \infty \), it follows that \( \| A_t \| \leq \| f \|_\infty \) for every \( t \geq 0 \). Consequently, \( A_t \in X'' \) and, there exists a function \( g : [0, \infty) \to X \) such that \( A_t(x') = g_{x'}(t) = \langle x', f(t) \rangle \) for every \( t \geq 0 \) and \( x' \in X' \). We define \( \varphi(t) = f(t) - g(t) \). It is immediate from this construction that \( g \) is \( \omega \)-periodic and that \( w - \lim_{t \to \infty} \varphi(t) = 0 \).

To complete the proof, we prove that \( g \) is continuous. For \( \varepsilon > 0 \) given, there exists \( \delta > 0 \) such that \( \| f(t) - f(s) \| \leq \varepsilon \) for every \( t, s \in [0, \infty) \) with \( |t - s| \leq \delta \). Hence, for \( t \geq 0 \), \( 0 < |h| < \delta \) with \( t + h \geq 0 \), \( x' \in X' \) and \( k \in \mathbb{N} \), we have that

\[
\left| \langle x', g(t + h) - g(t) \rangle \right| = \left| g_{x'}(t + h) - g_{x'}(t) \right| \\
= \left| g_{x'}(t + h + k \omega) - g_{x'}(t + k \omega) \right| \\
= \left| \langle x', f(t + h + k \omega) \rangle - \varphi_{x'}(t + h + k \omega) \right| - \left| \langle x', f(t + k \omega) \rangle \right| - \left| \varphi_{x'}(t + k \omega) \right| \\
\leq \| f \| \| x' \| + \| \varphi_{x'}(t + h + k \omega) \|.
\]

Taking limit when \( k \to \infty \), we infer that \( \| \langle x', g(t + h) - g(t) \rangle \| \leq \varepsilon \| x' \| \), which implies that \( \| g(t + h) - g(t) \| \leq \varepsilon \) since \( x' \) is arbitrary. This completes the proof. \( \square \)

The following result is an immediate consequence of Theorem 3.1 and Corollary 3.1.

**Example 3.3.** Assume that \( X \) is a Hilbert space and that \( \{e_k : k \in \mathbb{N}\} \) is an orthonormal basis of \( X \). If \( f \in C_b([0, \infty), X) \) is a uniformly continuous function such that for each \( k \in \mathbb{N} \), \( \lim_{t \to \infty} \langle e_k, f(t + n \omega) - f(t) \rangle = 0 \) uniformly on \( n \in \mathbb{N} \), then there exist \( g \in C_\omega([0, \infty), X) \) and \( \varphi \in C_b([0, \infty), X) \) such that \( f = g + \varphi \) and \( \lim_{t \to \infty} \langle e_k, \varphi(t) \rangle = 0 \) for every \( k \in \mathbb{N} \).

We can also avoid the condition about the reflexivity of \( X \).

**Theorem 3.2.** Let \( f \in C_b([0, \infty), X) \) be a function \( \omega \)-normal on compact sets such that for each \( x' \in X' \), \( \lim_{t \to \infty} \langle x', f(t + n \omega) - f(t) \rangle = 0 \) uniformly on \( n \in \mathbb{N} \), then \( f \) is asymptotically \( \omega \)-periodic.

**Proof.** There exist a sequence \( (n_j)_{j \in \mathbb{N}} \in \mathbb{N} \) and \( F \in C_b([0, \infty), X) \) such that \( f_{n_j \omega} \to F \) as \( j \to \infty \) uniformly on compact sets. For each \( x' \in X' \), the function \( \langle x', f \rangle \) is asymptotically \( \omega \)-periodic and \( \langle x', f_{n_j \omega} \rangle \to \langle x', F \rangle \) as \( j \to \infty \) uniformly on compact sets. It follows from Lemma 3.1 that \( F \in C_\omega([0, \infty), X) \). Moreover, proceeding as in the
proof of Theorem 3.1, we can assert that there are functions \( g_{\chi'} \in C_\omega([0, \infty), \mathbb{K}) \) and \( \varphi_{\chi'} \in C_0([0, \infty), \mathbb{K}) \) such that 
\[
(x', f(t)) = g_{\chi'}(t) + \varphi_{\chi'}(t) \text{ for all } t \geq 0.
\]
Therefore, 
\[
(x', f(t + n_j \omega)) = g_{\chi'}(t + n_j \omega) + \varphi_{\chi'}(t + n_j \omega)
\]
which implies that 
\[
(x', F(t)) = g_{\chi'}(t) \text{ for every } t \geq 0.
\]
This shows that \( F \) is independent of the sequence \((n_j)_{j \in \mathbb{N}}\) and, as consequence, that \( f_{n_0 \omega} \to F \) uniformly on compact sets.

Let \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that \( \|f(s + j \omega) - F(s)\| \leq \varepsilon \), for every \( s \in [0, \omega] \) and all \( j \geq n_0 \). If \( t \geq n_0 \omega \), then \( \tau(t) \geq n_0 \) and
\[
\left\| F(t) - f(t) \right\| \leq \left\| F(\xi(t) + \tau(t) \omega) - f(\xi(t) + \tau(t) \omega) \right\|
\]
\[
\leq \left\| F(\xi(t)) - f(\xi(t) + \tau(t) \omega) \right\| \leq \varepsilon,
\]
which establishes that \( F(s) - f(s) \to 0 \) as \( s \to \infty \) and that \( f \) is asymptotically \( \omega \)-periodic. The proof is complete. \( \square \)

We conclude this section with the following properties of SAP\(_\omega(X)\).

**Proposition 3.5.** SAP\(_\omega(X)\) is a Banach space.

**Proof.** Let \((f_n)_n\) be a sequence in SAP\(_\omega(X)\) that converges to \( f \) when \( n \to \infty \). The decomposition
\[
F(t + \omega) - f(t) = f(t + \omega) - f_0(t + \omega) + f_0(t + \omega) - f_n(t) + f_n(t) - f(t)
\]
shows that \( \lim_{t \to \infty} (f(t + \omega) - f(t)) = 0 \). \( \square \)

**Corollary 3.2.** Let \( f : [0, \infty) \to X \) be an S-asymptotically \( \omega \)-periodic function and assume that \( f' \) is bounded and uniformly continuous. Then \( f' \) is S-asymptotically \( \omega \)-periodic.

**Proof.** It is well known that our hypotheses imply that the functions \( h^{-1}(f_h - f) \) converge to \( f' \) as \( h \to 0 \) uniformly on \([0, \infty)\). The assertion is now consequence of Proposition 3.5. \( \square \)

4. Existence of S-asymptotically \( \omega \)-periodic solutions of a first-order abstract Cauchy problem

In this section, we study the existence of S-asymptotically \( \omega \)-periodic mild solutions for the first-order abstract Cauchy problem
\[
\begin{align*}
u'(t) &= Au(t) + G(t, u(t)), \quad t \geq 0, \quad (4.1) \\
u(0) &= x_0 \in X, \quad (4.2)
\end{align*}
\]
where \( A : D(A) \subseteq X \to X \) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \((T(t))_{t \geq 0}\) on \( X \) and \( G : [0, \infty) \times X \to X \) is a continuous function.

The study of the existence of almost periodic, asymptotically almost periodic, pseudo almost periodic, almost automorphic and asymptotically almost automorphic solutions is one of the most attracting topics in the qualitative theory of differential equations due both to its mathematical interest as to their applications in physics, mathematical biology, control theory, among others.

Some recent contributions on the existence of these type of solutions for abstract differential equations and abstract functional differential equations have been made. Related with this subject, we refer the reader to the extensive bibliography in [5,8–10]. Existence results concerning S-asymptotically \( \omega \)-periodic solutions for ordinary differential equations described on finite dimensional spaces are established in [2,4,15,16]. To the best of our knowledge, the study of the existence of S-asymptotically \( \omega \)-periodic solutions of first-order abstract differential equations is an untreated topic in the literature and this fact is the main motivation of this section.

We begin this section by studying the homogeneous abstract Cauchy problem, i.e. we initially consider \( G \equiv 0 \). In this case, the question about the existence of S-asymptotically \( \omega \)-periodic mild solutions of problem (4.1)–(4.2) is reduced to study the S-asymptotically \( \omega \)-periodic semigroups.

**Definition 4.1.** A strongly continuous function \( F : [0, \infty) \to B(X) \) is said to be strongly S-asymptotically periodic if for each \( x \in X \), there is \( \omega_x > 0 \) such that \( F(\cdot)x \) is S-asymptotically \( \omega_x \)-periodic. The function \( F \) is said strongly S-asymptotically \( \omega \)-periodic if there is \( \omega > 0 \) such that \( F(\cdot)x \) is S-asymptotically \( \omega \)-periodic for all \( x \in X \).
We recall that a strongly continuous semigroup \( (T(t))_{t \geq 0} \) is called strongly stable if \( T(t)x \to 0 \) as \( t \to \infty \) for every \( x \in X \) and, uniformly stable if \( \|T(t)\| \to 0 \) as \( t \to \infty \). For additional literature on semigroups of linear operators, we refer the reader to [12].

**Theorem 4.1.** Assume that \( (T(t))_{t \geq 0} \) is a strongly \( S \)-asymptotically periodic semigroup such that \( \{T(t)x: 0 \leq t < \infty\} \) is relatively compact, for all \( x \in X \). Then there exist \( \omega > 0 \) and a decomposition of \( X = X_0 \oplus X_1 \), where \( X_i, i = 0, 1 \), is a closed subspace of \( X \) invariant under \( T(t) \), the semigroup \( T_0(t) = T(t)|_{X_0} \) is \( \omega \)-periodic and the semigroup \( T_1(t) = T(t)|_{X_1} \) is strongly stable.

**Proof.** Since \( (T(t))_{t \geq 0} \) is uniformly bounded and the sets \( \{T(t)x: 0 \leq t < \infty\} \), \( x \in X \), are relatively compact, it follows from [14, Theorem 3.1.2] that \( (T(t))_{t \geq 0} \) is asymptotically almost periodic semigroup. In addition, from [13,14] there exists a decomposition of \( X = X_0 \oplus X_1 \), where \( X_i, i = 0, 1 \), is a closed subspace of \( X \) invariant under \( T(t) \), the semigroup \( T_0(t) = T(t)|_{X_0} \) is almost periodic and the semigroup \( T_1(t) = T(t)|_{X_1} \) is strongly stable. Moreover, the function \( T(\cdot)x \) is \( S \)-asymptotically \( \omega_x \)-periodic for some \( \omega_x > 0 \) and every \( x \in X \). We deduce from Proposition 3.4 that there exist an \( \omega_x \)-periodic function \( f_x \) and a function \( q_x \) that vanishes at infinite such that \( T(t)x = f_x(t) + q_x(t) \), for all \( t \geq 0 \). In particular, for \( x \in X_0 \) we have that the function \( T_0(\cdot)x = f_x \) is almost periodic and vanishes at infinite. Consequently, \( T_0(\cdot)x = f_x \) is \( \omega \)-periodic. Thus, \( T_0 \) is a strong periodic semigroup. Applying now [1, Theorem 2.1], we can affirm that there is \( \omega > 0 \) such that \( T_0(t)x \) is \( \omega \)-periodic, for all \( x \in X_0 \).

We establish now our first result on existence of \( S \)-asymptotically \( \omega \)-periodic mild solutions for the non-homogeneous problem (4.1)–(4.2). As it is usual in the frame of the abstract Cauchy problem, we consider the following concept of mild solution.

**Definition 4.2.** A function \( u \in C_b([0, \infty), X) \) is said an \( S \)-asymptotically \( \omega \)-periodic mild solution of problem (4.1)–(4.2) if \( u(\cdot) \) is \( S \)-asymptotically \( \omega \)-periodic and

\[
    u(t) = T(t)x_0 + \int_0^t T(t-s)G(s, u(s)) \, ds, \quad t \geq 0.
\]

**Theorem 4.2.** Assume that \( (T(t))_{t \geq 0} \) is a strongly \( S \)-asymptotically \( \omega \)-periodic semigroup. Let \( G : [0, \infty) \times X \to X \) be a continuous function such that \( G(\cdot, 0) \) is integrable on \([0, \infty)\) and there exists a continuous integrable function \( L : [0, \infty) \to \mathbb{R} \) such that

\[
    \left\| G(t, x) - G(t, y) \right\| \leq L(t) \| x - y \|,
\]

for every \( t \geq 0 \) and every \( x, y \in X \). Then there exists a unique \( S \)-asymptotically \( \omega \)-periodic mild solution of the problem (4.1)–(4.2).

**Proof.** We define the operator \( \Gamma \) on the space \( \text{SAP}_{\omega}(X) \) by

\[
    \Gamma u(t) = T(t)x_0 + \int_0^t T(t-s)G(s, u(s)) \, ds = T(t)x_0 + v(t). \quad (4.3)
\]

We will show initially that \( \Gamma u \in \text{SAP}_{\omega}(X) \). In fact, since the function \( T(t)x_0 \in \text{SAP}_{\omega}(X) \), it remains only to prove that the function \( v(\cdot) \in \text{SAP}_{\omega}(X) \). In view of that the semigroup \( (T(t))_{t \geq 0} \) is uniformly bounded on \([0, \infty)\), there exists a constant \( M \geq 1 \) such that \( \| T(t) \| \leq M, \) for all \( t \geq 0 \). Moreover, it follows from the inequality \( \| G(s, u(s)) \| \leq L(s) \| u(s) \| + \| G(s, 0) \| \), that the function \( s \mapsto G(s, u(s)) \) is integrable on \([0, \infty)\). Hence, we obtain that

\[
    \int_0^t T(t-s)G(s, u(s)) \, ds \to 0, \quad a \to \infty,
\]

uniformly for \( t \geq a \). In addition, for fixed \( a \), the set \( \{ G(s, u(s)) : 0 \leq s \leq a \} \) is compact, which implies that

\[
    T(t+\omega)G(s, u(s)) - T(t)G(s, u(s)) \to 0, \quad t \to \infty,
\]
uniformly on \( s \in [0, a] \). Combining these properties with the decomposition

\[
v(t + \omega) - v(t) = \int_0^a \left[ T(t + \omega - s) - T(t - s) \right] G(s, u(s)) \, ds
\]

\[
+ \int_a^{t+\omega} T(t + \omega - s)G(s, u(s)) \, ds - \int_a^t T(t - s)G(s, u(s)) \, ds,
\]

it follows that \( v(t + \omega) - v(t) \to 0 \) as \( t \to \infty \).

Furthermore, for \( u_1, u_2 \in SAP_\omega(X) \) the inequality

\[
\| \Gamma u_1(t) - \Gamma u_2(t) \| \leq M \int_0^t L(s) \| u_1(s) - u_2(s) \| \, ds
\]

shows that \( \Gamma : SAP_\omega(X) \to SAP_\omega(X) \) is a continuous map.

On the other hand, we define the linear map \( B : C_b([0, \infty)) \to C_b([0, \infty)) \) by

\[
(B\alpha)(t) = M \int_0^t L(s)\alpha(s) \, ds
\]

(4.4)

for \( t \geq 0 \). It is clear that \( B \) is continuous. Moreover, \( B \) is completely continuous. To establish this assertion, for each \( \varepsilon > 0 \), we take \( a \geq 0 \) such that \( M \int_a^{\infty} L(s) \, ds \leq \varepsilon \) and, for each \( \alpha \in C_b([0, \infty), \mathbb{R}) \) with \( \| \alpha \|_\infty \leq 1 \), we define the functions

\[
w_1(\alpha)(t) = \begin{cases} 
M \int_0^t L(s)\alpha(s) \, ds, & 0 \leq t \leq a, \\
M \int_a^t L(s)\alpha(s) \, ds, & t \geq a,
\end{cases}
\]

and

\[
w_2(\alpha)(t) = \begin{cases} 
0, & 0 \leq t \leq a, \\
M \int_a^t L(s)\alpha(s) \, ds, & t \geq a.
\end{cases}
\]

It follows from the Ascoli–Arzela Theorem that the set \( K_\varepsilon = \{ w_1(\alpha) : \| \alpha \|_\infty \leq 1 \} \) is relatively compact. Since \( B\alpha(t) = w_1(\alpha)(t) + w_2(\alpha)(t) \) for all \( t \geq 0 \), we can affirm that

\[
\{ B(\alpha) : \| \alpha \|_\infty \leq 1 \} \subseteq K_\varepsilon + \{ \beta : \beta \in C_b([0, \infty), \mathbb{R}), \| \beta \|_\infty \leq \varepsilon \},
\]

which shows that the set \( \{ B(\alpha) : \| \alpha \|_\infty \leq 1 \} \) is relatively compact and, in turn, that \( B \) is completely continuous. Moreover, since the point spectrum \( \sigma_p(B) = \{ 0 \} \), the spectral radius of \( B \) is equal to zero.

Let \( m : C_b([0, \infty), X) \to C_b([0, \infty), \mathbb{R}) \) be the map defined by \( m(u)(t) = \sup_{0 \leq s \leq t} \| u(s) \| \). It is not difficult to verify that the maps \( \Gamma', B \) and \( m \) satisfy all the conditions of [7, Theorem 1] which implies that \( \Gamma' \) has a unique fixed point \( u \). This completes the proof. \( \Box \)

To establish our next result, we begin by introducing some definitions.

**Definition 4.3.** A continuous function \( G : [0, \infty) \times X \to X \) is said uniformly \( S \)-asymptotically \( \omega \)-periodic on bounded sets if for every bounded subset \( K \) of \( X \), the set \( \{ G(t, x) : t \geq 0, x \in K \} \) is bounded and \( \lim_{t \to \infty} (G(t, x) - G(t + \omega, x)) = 0 \) uniformly on \( x \in K \).
**Theorem 4.3.** A continuous function $G : [0, \infty) \times X \to X$ is said asymptotically uniformly continuous on bounded sets if for every $\varepsilon > 0$ and every bounded set $K \subseteq X$, there exist $L_{\varepsilon,K} > 0$ and $\delta_{\varepsilon,K} > 0$ such that $\|G(t,x) - G(t,y)\| \leq \varepsilon$, for all $t \geq L_{\varepsilon,K}$ and all $x, y \in K$ with $\|x - y\| \leq \delta_{\varepsilon,K}$.

**Lemma 4.1.** Let $G : [0, \infty) \times X \to X$ be a uniformly $S$-asymptotically $\omega$-periodic on bounded sets and asymptotically uniformly continuous on bounded sets function and, let $u : [0, \infty) \to X$ be an $S$-asymptotically $\omega$-periodic function. Then the function $v(t) = G(t, u(t))$ is $S$-asymptotically $\omega$-periodic.

**Proof.** Since the range $R(u)$ of $u(\cdot)$ is a bounded set, it follows that $v$ is a bounded function. Moreover, for $\varepsilon > 0$ there exist $\delta > 0$ and $L^1_\varepsilon > 0$ such that

$$\max\{\|G(t + \omega, z) - G(t, z)\|, \|G(t, x) - G(t, y)\|\} \leq \varepsilon,$$

for every $t \geq L^1_\varepsilon$, $z \in R(u)$ and every $x, y \in R(u)$ with $\|x - y\| < \delta$. In addition, since $u(\cdot)$ is $S$-asymptotically $\omega$-periodic, there exists $L^2_\varepsilon > 0$ such that $\|u(t + \omega) - u(t)\| \leq \delta$ for every $t \geq L^2_\varepsilon$. Thus, for $t \geq \max(L^1_\varepsilon, L^2_\varepsilon)$, we have that

$$\|v(t + \omega) - v(t)\| \leq \|G(t + \omega, u(t + \omega)) - G(t, u(t + \omega))\| + \|G(t, u(t + \omega)) - G(t, u(t))\| \leq 2\varepsilon,$$

which completes the proof. \(\Box\)

In the rest of this section, $M \geq 1$ and $\gamma > 0$ are constants such that $\|T(t)\| \leq Me^{-\gamma t}$ for every $t \geq 0$. To establish our next results on the existence of $S$-asymptotically $\omega$-periodic mild solutions of problem (4.1)–(4.2), we introduce the following assumption.

(HG) The function $G : [0, \infty) \times X \to X$ is continuous and there exists $L > 0$ such that

$$\|G(t,x) - G(t,y)\| \leq L\|x - y\|, \quad x, y \in X, \quad t \geq 0.$$

**Theorem 4.3.** Let condition (HG) be satisfied and assume that $G$ is uniformly $S$-asymptotically $\omega$-periodic on bounded sets. If $ML < \gamma$, then there exists a unique $S$-asymptotically $\omega$-periodic mild solution of problem (4.1)–(4.2).

**Proof.** Proceeding as in the proof of Theorem 4.2, we define the map $\Gamma$ on the space $SAP_\omega(X)$ by the expression (4.3). We next prove that $\Gamma$ is a contraction from $SAP_\omega(X)$ into $SAP_\omega(X)$.

We will show initially that $\Gamma$ is $SAP_\omega(X)$-valued. Let $u \in SAP_\omega(X)$. Since $T(t)x_0 \to 0$ as $t \to \infty$, then the function $T(\cdot)x_0 \in SAP_\omega(X)$ and the problem is reduced to show that the function $v$ given by (4.3) belongs to $SAP_\omega(X)$. Using the fact that $G(\cdot, u(\cdot))$ is a bounded function, it follows that $v \in C([0, \infty), X)$. On the other hand, in view of that $G$ is asymptotically uniformly continuous on bounded sets and applying Lemma 4.1, for each $\varepsilon > 0$, there is a constant $L_\varepsilon$ such that $\|G(t + \omega, u(t + \omega)) - G(t, u(t))\| \leq \varepsilon$, for every $t \geq L_\varepsilon$. Under these conditions, for $t \geq L_\varepsilon$, we can estimate

$$\|v(t + \omega) - v(t)\| \leq \int_0^\omega \|T(t + \omega - s)G(s, u(s))\| ds + \int_0^{L_\varepsilon} \|T(t - s)[G(s + \omega, u(s + \omega)) - G(s, u(s))]\| ds
\leq \frac{M\|G(\cdot, u(\cdot))\|_\infty}{\gamma} e^{-\gamma t} + 2\frac{M\|G(\cdot, u(\cdot))\|_\infty}{\gamma} e^{-\gamma(L_\varepsilon)} + \frac{\varepsilon M}{\gamma},$$

which permit to infer that $v(t + \omega) - v(t) \to 0$ as $t \to \infty$. This completes the proof that $\Gamma u \in SAP_\omega(X)$. On the other hand, for $u_1, u_2 \in SAP_\omega(X)$ we have that

$$\|\Gamma u_2(t) - \Gamma u_1(t)\| \leq \frac{ML}{\gamma} \sup_{0 \leq s \leq t} \|u_2(s) - u_1(s)\|.$$
which proves that $I^\gamma$ is a contraction. Now, the assertion is consequence of the contraction mapping principle. The proof is complete. □

To finish this section, we establish a result on existence of asymptotically $\omega$-periodic solutions for the abstract system (4.1)–(4.2).

**Proposition 4.1.** Let condition (H_3) be holds. Assume that $G(\cdot, 0)$ is a bounded function and $\lim_{t \to \infty} (G(t, x) - G(t + n\omega, x)) = 0$ uniformly for $x \in K$ and $n \in \mathbb{N}$, for every bounded subset $K$ of $X$. If $ML < \gamma$, then there exists a unique asymptotically $\omega$-periodic mild solution of problem (4.1)–(4.2).

**Proof.** We modify slightly the argument in the proof of Theorem 4.3. Let $S(X)$ be the space consisting of functions $u \in C_b([0, \infty), X)$ such that $\lim_{t \to \infty} (u(t + n\omega) - u(t)) = 0$ uniformly for $n \in \mathbb{N}$. It is easy to see that $S(X)$ is a closed subspace of $C_b([0, \infty), X)$.

Let $u \in S(X)$. Proceeding as in the proof of Lemma 4.1, it follows from our assumptions that

$$
\lim_{t \to \infty} (G(t + n\omega, u(t + n\omega)) - G(t, u(t))) = 0
$$

uniformly for $n \in \mathbb{N}$.

We keep the notations introduced in the proof of Theorem 4.3. We consider the map $I^\gamma$ defined on $S(X)$. Using the preceding property, we obtain that the estimate (4.5) holds with $n\omega$ instead of $\omega$. This implies that $v \in S(X)$ and $I^\gamma$ is $S(X)$-valued. Therefore, the fixed point of $I^\gamma$ belongs to $S(X)$ and the assertion is consequence of Corollary 3.1. The proof is complete. □

4.1. An application to partial differential equations

To complete this work, we discuss briefly the existence of $S$-asymptotically $\omega$-periodic mild solutions for the system

$$
\frac{\partial}{\partial t}u(t, \xi) = \frac{\partial^2}{\partial \xi^2}u(t, \xi) + a(t)f(u(t, \xi)), \quad t \geq 0, \quad \xi \in [0, \pi],
$$

$$
u(t, 0) = u(t, \pi) = 0, \quad t \geq 0,
$$

$$u(0, \xi) = z(\xi), \quad \xi \in [0, \pi],
$$

where $z : [0, \pi] \to \mathbb{R}, a : [0, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are appropriate functions.

To study this system in the abstract form (4.1)–(4.2), we choose the space $X = L^2([0, \pi])$ and the operator $A : D(A) \subseteq X \to X$ given by $Ax = x''$ with domain

$$D(A) = \{ x \in X : x'' \in X, \ x(0) = x(\pi) = 0 \}. $$

It is well known that $A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on $X$. Furthermore, $A$ has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) = (\frac{\pi}{n})^{1/2} \sin(n\xi)$. In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of $X$ and, for $x \in X$, $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, z_n \rangle z_n$.

It follows from this representation that $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$.

**Proposition 4.2.** Assume that $a(\cdot)$ is an $S$-asymptotically $\omega$-periodic function and that there exists $L_f > 0$ such that

$$ |f(x) - f(y)| \leq L_f |x - y|, \quad x, y \in \mathbb{R}. $$

If $\|a\|_{L^\infty} L_f < 1$, then there exists a unique $S$-asymptotically $\omega$-periodic mild solution $u(\cdot)$ of problem (4.6)–(4.8). If, in addition, $\lim_{n \to \infty} (a(t + n\omega) - a(t)) = 0$ uniformly for $n \in \mathbb{N}$, then $u(\cdot)$ is asymptotically $\omega$-periodic.

**Proof.** The existence of $u(\cdot)$ follows from Theorem 4.3. The another assertion is consequence of Proposition 4.1. □
References

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