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On the $C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ regularity for a class of electro-rheological fluids

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ABSTRACT

We are concerned with a system of nonlinear partial differential equations with $p(x)$-structure, $1 < p_\infty \leqslant p(x) \leqslant p_0 < +\infty$, and no-slip boundary conditions. We prove the existence and uniqueness of a $C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ solution corresponding to small data, without further restrictions on the bounds p_∞, p_0. In particular this result is applicable to the steady motion of shear-dependent electro-rheological fluids.

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1. Introduction

We are concerned with the following boundary value problem

$$\begin{cases} -\nabla \cdot S(\mathcal{D}u) + (u \cdot \nabla)u + \nabla\pi = f, \quad \text{in } \Omega, \\ \nabla \cdot u = 0, \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

in a bounded and suitably smooth domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. Here u denotes the velocity field, π the pressure field, f the external force, $(u \cdot \nabla)u = \sum_{j=1}^n u_j \partial_j u$ the convective term and $S(\mathcal{D}u)$ the extra-stress tensor. We assume that $S(\mathcal{D}u)$ is given by

$$S(\mathcal{D}u) = \left(1 + |\mathcal{D}u|\right)^{p(x)-2} \mathcal{D}u, \tag{1.2}$$

where p is a prescribed function, $1 < p_\infty \leqslant p(x) \leqslant p_0 < +\infty$, and $\mathcal{D}u = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the velocity gradient. We prefer to avoid the full generality in order to highlight the main ideas. Thus the system (1.1) with the choice (1.2) is just the canonical representative of a wider class of models to which our proof applies, included in the class of systems of partial differential equations with non-standard growth conditions. This kind of systems models incompressible electro-rheological fluids with shear-dependent viscosities, which are viscous fluids characterized by their ability to highly change in their mechanical properties when an electromagnetic field is applied. In the last twenty years the interest in the study of electro-rheological fluids has increased and the related literature is very wide. The basic mathematical analysis can be found in [38,39] (see also [22] for an overview of recent results). Fluids with non-constant viscosity have been treated in various settings. Without any aim of completeness, we refer for instance to [35], where Herschel–Bulkley fluids are considered, to [2,12] for different non-standard growth conditions, to [13–15,25,28,32,34] for models with a non-constant viscosity depending either on the pressure or on the temperature and, finally, to the paper [16], where different boundary conditions, related to the convex analysis, are taken into account.

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As far as the regularity problem is concerned, great part of investigations deals with the regularity of minimizers of variational integrals, in the case where the elliptic operator S depends on $|\nabla u|$ and not on $|\mathcal{D}u|$. The dependence on $|\mathcal{D}u|$ is an additional difficulty, even for p constant. Here we focus on those papers mostly connected with our interest, which is mainly the regularity up to the boundary of the solutions of (1.1) with $S = S(\mathcal{D}u)$. Recently, interior regularity in two space dimension have been obtained [24]. These are $C^{1,\alpha}_{loc}$-regularity results, in the hypothesis p of class C^1 and $\inf p > \frac{6}{5}$. For any space dimension, if $p \in C^{0,\alpha}$ and $\inf p > \frac{3n}{n+2}$, results of partial $C^{0,\alpha}$-regularity of the gradient ∇u, together with an estimate for the Hausdorff dimension of the singular set (the closed subset outside which ∇u is Hölder continuous) can be found in [1]. As far as we know, regularity results up to the boundary have been obtained only for p constant. For problems similar to (1.1), this kind of regularity is obtained in the two-dimensional framework in [29] for anisotropic fluids and in [30] for $\frac{3}{2} < p < 4$, without restrictions on the data. However in these papers the minimal regularity assumption on the extra-stress tensor $S(\mathcal{D}u)$ is the C^1-regularity, which we replace here by a Lipschitz continuity assumption. In [21], in a smallness assumption on the size of f, we prove the existence, uniqueness and $C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ regularity for the velocity field, for any $p \in (1,2)$ and $n \geqslant 2$. In the three-dimensional case, the most significant regularity result for shear-thinning fluids (i.e. $p < 2$ constant) without any restriction on the size of the data has been obtained in [9], for a smooth arbitrary domain. The author proves global regularity results for the second derivatives of the velocity and for the first derivatives of the pressure. Actually, for the problem (1.1) without the convective term, the author proves that if $f \in L^{p'}(\Omega)$, with p' conjugate exponent of $p \in (\frac{3}{2}, 2)$, and u is a weak solution, then $u \in W^{1,4p-2}(\Omega) \cap W^{2,\frac{4p-2}{p+1}}(\Omega)$. The same kind of regularity is obtained introducing the convective term, provided that one increases the lower bound of the admissible values of p. For results with a flat boundary see the papers [4,8] and [11]. Further, cylindrical domains were considered in [17,18]. In all the cited papers, the integrability exponent of the second derivatives of the solution remains strictly less than 2. Actually, if one considers a flat boundary and the simpler case where the extra-stress tensor has the form $\mu_0 \mathcal{D}u + (\mu + |\mathcal{D}u|)^{p-2}\mathcal{D}u$, with a strictly positive constant μ_0, then in [19] it is proved that the solution (u,π) belongs to $W^{2,2}(\Omega) \times W^{1,2}(\Omega)$. Moreover, in the case $p > 2$ and $n = 3$, the first regularity results go back to the pioneering paper [33] and to [3]. We also recall the $W^{2,l}$-regularity results in papers [5,7,20], for flat boundaries, and in [6] for non-flat boundaries. Here $l = l(p) < 2$. Further, in the recent paper [10], the authors improve the results obtained in [6] and extend them to any space dimensions. They show that $u \in W^{1,q} \cap W^{2,\frac{2q}{p+q-2}}$ for any $q < +\infty$ if $n = 2$, and for $q = \frac{np+2-p}{n-2}$ if $n \geqslant 3$. The previous regularity holds with $p \geqslant \max\{2, \frac{3n}{n+2}\}$ and, if the convective term is present, $p \in [3, +\infty) \cup (\frac{n}{2}, +\infty)$. Here $f \in L^2(\Omega)$. Up to now, these are the stronger results known in literature for shear-thickening fluids without a smallness assumption on f.

In the present paper we extend to fluids with non-standard growth conditions the regularity results obtained in [21] for shear-thinning fluids. Indeed our treatment is based on the same technique developed in the previous investigation [21] for generalized Newtonian fluids. More precisely we prove the existence and uniqueness of a solution u, for any $n \geqslant 2$, such that the velocity field u belongs to $C^{1,\gamma}(\overline{\Omega})$ (the pressure field π is in $C^{0,\gamma}(\overline{\Omega})$). The result is achieved if f is in $L^q(\Omega)$, $q > n$, and its L^q-norm is suitably small. If $q > 2n$ we also obtain $W^{2,2}$-regularity. Further, the technique used in the proof enable us to avoid any lower (and upper) bound for $p(x)$. As far as we know, this is the first existence result without lower bounds for p.

The smallness request on the force term arises from the technique used in the proof. Indeed we linearize the problem and we construct a sequence of approximating solutions. For the convergence of such process is mandatory a smallness assumption on the data. The main tool we use is the Hölder regularity result for solutions of elliptic systems due to Giaquinta and Modica (see [27]). As in our previous work, concerning shear-thinning fluids, a special attention is needed managing the constants in the estimates contained in [27]. The control in the growth of such constants is obtained adapting the computations made in the Appendix of [21] to the case where p is not a constant. We want to observe that, in this case too, no further pointwise regularity is expected, due to the non-differentiability of the modulus of the symmetric gradient in the equations. Nevertheless, considering a slightly different problem with an extra-stress tensor of the kind $(1 + |\mathcal{D}u(x)|^2)^{\frac{p(x)-2}{2}}\mathcal{D}u(x)$ our method would provide $C^{k+2,\gamma}$-regularity in the case of data $C^{k,\gamma}$. However we do not give any result in this sense because it needs a further evaluation of higher derivatives for the linearized problem which seems not straightforward from our previous estimate.

The plan of the paper is the following. In Section 2, we introduce the notations used throughout the paper and state our main results. In Section 3, we give all the tools needed for the regularity of the linearized problems. In Section 4 we prove the existence, uniqueness and $C^{1,\gamma}$-regularity, up to the boundary, of the solution. Finally, Section 5 is devoted to the regularity of the second derivatives of the solution.

2. Notations and statement of the main results

Throughout the paper Ω will denote a *suitably smooth* bounded domain of $\mathbb{R}^n$, $n \geqslant 2$. We shall adopt the usual notations for Lebesgue spaces $L^q(\Omega)$ and Sobolev spaces $W^{m,q}(\Omega)$ and their norms. We denote by $W^{1,q}_0(\Omega)$ the closure in $W^{1,q}(\Omega)$ of $C^\infty_0(\Omega)$ and by $W^{-1,q'}(\Omega)$, $q' = q/(q-1)$, the strong dual of $W^{1,q}_0(\Omega)$ with norm $\|\cdot\|_{-1,q'}$. The symbol $\langle\cdot,\cdot\rangle_{1,q}$ denotes the duality pairing between the spaces $W^{1,q}_0(\Omega)$ and $W^{-1,q'}(\Omega)$. Denote by $W^{\frac{1}{2},2}(\partial\Omega)$ the subspace of $L^2(\partial\Omega)$ with norm

$$\|u\|_{\frac{1}{2},2,\partial\Omega} \equiv \|u\|_{2,\partial\Omega} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^n}\, d\sigma_x\, d\sigma_y\right)^{\frac{1}{2}},$$

and by $W^{-\frac{1}{2},2}(\partial\Omega)$ its dual, where $L^2(\partial\Omega)$ is the usual Lebesgue space when we consider the $(n-1)$-dimensional measure on $\partial\Omega$. Let

$$V(\Omega)=\{v\in W_0^{1,2}(\Omega),\ \nabla\cdot v=0\}.$$

We denote by $C^{m,\gamma}(\overline{\Omega})$, m nonnegative integer and $\gamma\in(0,1)$, the subspace of $C^m(\overline{\Omega})$ consisting of functions with all derivatives of order m satisfying the following Hölder condition

$$[u]_{C^{m,\gamma}(\overline{\Omega})}\equiv\sum_{|h|=m}\sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}}\frac{|D^hu(x)-D^hu(y)|}{|x-y|^\gamma}<+\infty.$$

We recall that $C^{m,\gamma}(\overline{\Omega})$ is a Banach space with norm

$$\|u\|_{C^{m,\gamma}(\overline{\Omega})}\equiv\sum_{|h|=0}^m\|D^hu\|_\infty+[u]_{C^{m,\gamma}(\overline{\Omega})}.$$

In notation concerning duality pairings, norms and functional spaces, we shall not distinguish between scalar, vector and tensor fields.

For any given couple of vectors, a and b, and tensors, A and B, we write $a\cdot b\equiv a_ib_i$, $A\cdot B\equiv A_{ij}B_{ij}$, where we adopt the convention of summation on repeated indexes.

Finally, the letter c denotes a positive constant whose value may change even in the same equation. Sometimes the relevant dependences will be highlighted. We denote in a different way, as C, C_0, C_K or the like any occurrence of some particular constant that we shall later recall.

We make the following basic assumptions on the function $p(x)$:

$$p:\overline{\Omega}\longrightarrow(1,+\infty),\qquad p\in C(\overline{\Omega}),\qquad \min p(x)=p_\infty,\qquad \max p(x)\leqslant p_0,\quad \text{for some } p_0>2.$$

Definition 2.1. Assume that $f\in L^2(\Omega)$. We say that u is a $C^{1,\gamma}$-*solution* of problem (1.1) if $u\in C^{1,\gamma}(\overline{\Omega})$, for some $\gamma\in(0,1)$, $\nabla\cdot u=0$, $u|_{\partial\Omega}=0$ and it satisfies the integral identity

$$\int_\Omega S(\mathcal{D}u)\cdot\mathcal{D}\varphi\,dx+\int_\Omega(u\cdot\nabla)u\cdot\varphi\,dx=\int_\Omega f\cdot\varphi\,dx,\quad \forall\varphi\in V(\Omega).\tag{2.1}$$

Remark 2.1. We observe that if u is a $C^{1,\gamma}$-*solution* then we can find a pressure π at least in $L^2(\Omega)$ such that the pair (u,π) satisfies the following integral identity

$$\int_\Omega S(\mathcal{D}u)\cdot\mathcal{D}\varphi\,dx+\int_\Omega(u\cdot\nabla)u\cdot\varphi\,dx-\int_\Omega\pi\nabla\cdot\varphi\,dx=\int_\Omega f\cdot\varphi\,dx,\quad\forall\varphi\in C_0^\infty(\Omega).\tag{2.2}$$

The validity of the reverse implication is obvious. In the sequel we shall refer to u or (u,π) as solution of system (1.1) without distinction.

Our main results are the following theorems.

Theorem 2.1. *Let be $f\in L^q(\Omega)$, for some $q>n$. Let Ω be a domain of class C^{1,γ_0} and $p\in C^{0,\gamma_0}(\overline{\Omega})$, with $\gamma_0=1-\frac{n}{q}$. Then there exist two positive constants Λ and C_1, depending on $\|p\|_{C^{0,\gamma}(\overline{\Omega})}$, n, q, Ω, such that, if $\|f\|_q<\Lambda$, for any $\gamma<\gamma_0$ there exists a $C^{1,\gamma}$-solution (u,π) of problem* (1.1) *with*

$$\|u\|_{C^{1,\gamma}(\overline{\Omega})}+\|\pi\|_{C^{0,\gamma}(\overline{\Omega})}\leqslant C_1\|f\|_q.\tag{2.3}$$

Further, there exists a constant $\delta=\delta(p_\infty,\|p\|_{C^{0,\gamma}(\overline{\Omega})},n,q,\Omega)$ such that if $\|f\|_q<\delta$ the solution is also unique.

Theorem 2.2. *Let $q>2n$ and $\gamma_0=1-\frac{n}{q}$. Let Ω be a domain of class C^2, let be $p\in C^1(\overline{\Omega})$ and $f\in L^q(\Omega)$. There exists a positive constant Λ_1 such that, if $\|f\|_q<\Lambda_1$, then there exists a $C^{1,\gamma}$-solution (u,π) of problem* (1.1) *such that*

$$u\in W^{2,2}(\Omega)\cap C^{1,\gamma}(\overline{\Omega}),\qquad \pi\in W^{1,2}(\Omega)\cap C^{0,\gamma}(\overline{\Omega}),\quad\forall\gamma<\gamma_0.$$

In order to prove the above existence and regularity theorems, the idea is to approximate problem (1.1) by suitable linearized problems. The following section is concerned with the introduction and the analysis of the linearized problems.

3. Auxiliary results

Let us recall some properties of the tensor

$$S(A) \equiv \big(1+|A|\big)^{p(x)-2} A,$$

for an arbitrary second-order tensor A, with $p(x)$ as in the previous section. It is easily seen that $S(A)$ satisfies the following estimate

$$\frac{\partial S_{ij}(A)}{\partial A_{kl}} B_{ij} B_{kl} \geqslant C_S \big(1+|A|\big)^{p(x)-2} |B|^2, \tag{3.1}$$

for all tensors A and B, where

$$C_S = \begin{cases} p_\infty - 1 & \text{if } 1 < p_\infty < 2; \\ 1 & \text{if } p_\infty \geqslant 2. \end{cases}$$

Further we recall the following estimate, for which we refer to [23] (Appendix, Lemma 6.2).

Lemma 3.1. *Let $A, B \in \mathbb{R}^{n\times n}$. Then*

$$\int_0^1 \big(1+\big|\theta A + (1-\theta)B\big|\big)^{p(x)-2} d\theta \geqslant 4^{-p(x)} \big(1+|A|+|B|\big)^{p(x)-2}.$$

The previous lemma is proved for p constant, however it is easy to check that it still holds in our assumptions on p. Hence, using (3.1), the previous lemma and the bounds for $p(x)$ we get

$$\begin{aligned} \big(S(A)-S(B)\big)\cdot(A-B) &= \int_0^1 \frac{\partial S_{ij}(\theta A+(1-\theta)B)}{\partial D_{kl}} (A-B)_{ij}(A-B)_{kl}\, d\theta \\ &\geqslant C_S |A-B|^2 \int_0^1 \big(1+\big|\theta A+(1-\theta)B\big|\big)^{p(x)-2} d\theta. \end{aligned} \tag{3.2}$$

The last term can be bounded from below by

$$\begin{cases} (p_\infty - 1)4^{-p_0} \dfrac{|A-B|^2}{(1+|A|+|B|)^{2-p_\infty}} & \text{if } 1 < p_\infty < 2; \\ |A-B|^2 & \text{if } p_\infty \geqslant 2. \end{cases} \tag{3.3}$$

The remaining part of this section is concerned with the introduction of some regularity results related to the following kind of elliptic problem

$$\begin{cases} -\nabla\cdot \overline{A}(x,\nabla U) + \nabla \Pi = F, \quad \text{in } \Omega, \\ \nabla\cdot U = 0, \quad \text{in } \Omega, \\ U = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where the second-order tensor $\overline{A} = \overline{A}_{i\alpha}$ has the expression

$$\overline{A}_{i\alpha}(x,\nabla U) = A_{i\alpha j\beta}(x)\partial_\beta U_j,$$

with $A(x) = A_{i\alpha j\beta}(x)$ satisfying the conditions

$$A \text{ is continuous}, \qquad \|A\|_\infty < +\infty, \tag{3.5}$$

where $\|A\|_\infty \equiv \max_{i,\alpha,j,\beta} \|A_{i\alpha j\beta}\|_\infty$, and the coercivity condition in $W_0^{1,2}(\Omega)$

$$\int_\Omega A_{i\alpha j\beta}(x)\partial_\beta U_j \partial_\alpha U_i\, dx \geqslant \sigma \|\nabla U\|_2^2, \quad \text{for some } \sigma > 0. \tag{3.6}$$

Theorem 3.2. *Let $q > n$, $\gamma_0 = 1 - \frac{n}{q}$. Assume that Ω is a domain of class C^{1,γ_0}, $A \in C^{0,\gamma_0}(\overline{\Omega})$ satisfies* (3.5)–(3.6), *$F \in L^q(\Omega)$. Then, there exists a weak solution (U,Π) of problem* (3.4). *Moreover, $U \in C^{1,\gamma_0}(\overline{\Omega})$, $\Pi \in C^{0,\gamma_0}(\overline{\Omega})$ and*

$$\|U\|_{C^{1,\gamma_0}(\overline{\Omega})} + \|\Pi\|_{C^{0,\gamma_0}(\overline{\Omega})} \leqslant \widetilde{c}\big(\|U\|_{1,2} + \|F\|_q\big), \tag{3.7}$$

where $\widetilde{c} = \widetilde{c}(\|A\|_{C^{0,\gamma_0}}, n, q, \sigma, \Omega)$.

Theorem 3.3. *Assume that Ω is a domain of class C^2, $A \in C^1(\overline{\Omega})$ satisfies (3.5)–(3.6), $F \in L^2(\Omega)$. Then, if (U, Π) is a weak solution of problem (3.4), $U \in W^{2,2}(\Omega)$, $\Pi \in W^{1,2}(\Omega)$ and*

$$\|U\|_{2,2} + \|\Pi\|_{1,2} \leqslant \widetilde{c}_1 \big(\|U\|_{1,2} + \|F\|_2\big), \tag{3.8}$$

where $\widetilde{c}_1 = \widetilde{c}_1(\|A\|_{C^1}, n, \sigma, \Omega)$.

Remark 3.1. Theorems 3.2 and 3.3 are proved in [27] for a Neumann problem. However, as explicitly stated by the authors, the results still hold with suitable changes in the proof if one replaces the Neumann condition with a Dirichlet boundary condition.

Let us introduce the following linear problem

$$\begin{cases} -\nabla \cdot \big[\big(1 + |\mathcal{D}v|\big)^{p(x)-2} \mathcal{D}U\big] + \nabla \Pi = F, & \text{in } \Omega, \\ \nabla \cdot U = 0, & \text{in } \Omega, \\ U = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where v is a vector-valued function in $C^{1,\gamma_0}(\overline{\Omega})$.

Definition 3.1. Assume that $F \in L^2(\Omega)$. We say that U is a weak solution of problem (3.9) if $U \in V(\Omega)$ satisfies

$$\int_\Omega \big(1 + |\mathcal{D}v|\big)^{p(x)-2} \mathcal{D}U \cdot \mathcal{D}\varphi \, dx = \int_\Omega F \cdot \varphi \, dx, \quad \forall \varphi \in V(\Omega). \tag{3.10}$$

As observed in Remark 2.1, we can associate to U a pressure $\Pi \in L^2(\Omega)$, using test functions in $C_0^\infty(\Omega)$ instead of $V(\Omega)$. By setting

$$A_{i\alpha j\beta}(x) = \frac{1}{2}(\delta_{ij}\delta_{\alpha\beta} + \delta_{i\beta}\delta_{j\alpha})\big(1 + |\mathcal{D}v|\big)^{p(x)-2}, \tag{3.11}$$

we have that:

$$\begin{aligned}
&A \text{ is continuous}, \qquad \|A\|_\infty \leqslant \big(1 + \|\mathcal{D}v\|_\infty\big)^{p_0-2} < +\infty, \\
&A_{i\alpha j\beta}(x)\partial_\beta U_j = \big(1 + |\mathcal{D}v|\big)^{p(x)-2} (\mathcal{D}U)_{i\alpha}, \\
&\int_\Omega A_{i\alpha j\beta}(x)\partial_\beta U_j \partial_\alpha U_i \, dx \geqslant \int_{p(x)\geqslant 2} |\mathcal{D}U|^2 \, dx + \int_{p(x)<2} \frac{|\mathcal{D}U|^2}{(1 + \|\mathcal{D}v\|_\infty)^{2-p}} \, dx \\
&\qquad \geqslant \frac{\|\mathcal{D}U\|_2^2}{(1 + \|\mathcal{D}v\|_\infty)^{2-p_\infty}} \geqslant \sigma \|\nabla U\|_2^2, \quad \forall U \in W_0^{1,2}(\Omega),
\end{aligned} \tag{3.12}$$

where, thanks to the Korn type inequality

$$\|\nabla U\|_2^2 \leqslant 2\|\mathcal{D}U\|_2^2,$$

we have chosen

$$\sigma = \frac{1}{2(1 + \|\mathcal{D}v\|_\infty)^{2-p_\infty}}.$$

Therefore the assumptions of Theorem 3.2 are satisfied. Moreover for the particular choice (3.11) of A we are able to show explicitly the dependence of the constant $\widetilde{c}$ which appears in estimate (3.7) by $\|A\|_{C^{0,\gamma_0}(\overline{\Omega})}$. In this regard, in the Appendix of [21] we proved that, when the fourth-order tensor A is in the form (3.11) with p positive constant less than 2, Theorem 3.2 holds and the constant $\widetilde{c}$ has the expression

$$\widetilde{c} = \widehat{C}\big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{\widehat{r}}, \tag{3.13}$$

where $\widehat{C} \equiv \widehat{C}(n, q, p, \Omega)$ and the exponent $\widehat{r} \equiv \widehat{r}(n, q, p)$ is a real number greater or equal than 2 (actually, in [21] we gave a worse estimate than (3.13), in terms of the whole C^{1,γ_0}-norm of v). The proof of estimate (3.13) is quite long since it requires to follow step by step the proofs in [27] in order to give an explicit expression of $\widetilde{c}$. In particular some cautions is needed due to the differences in the boundary value problem and in the coercivity condition between our paper [21] and [27]. In the case we are now dealing with, where p is variable, following the calculations made in the Appendix of [21], it can be seen that the constant $\widetilde{c}$ has an expression similar to (3.13) that is

$$\widetilde{c} = C\big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{r}, \tag{3.14}$$

where $C \equiv C(\|p\|_{C^{0,\gamma_0}(\overline{\Omega})}, n, q, \Omega)$ is greater than p_0 and the exponent $r \equiv r(n, q, p_0)$ is a real number greater or equal than $2p_0 - 1$. This choice, which could merely be a technical assumption, has a concrete motivation that arises naturally from the computations in the Appendix of [21]. We omit the details and refer the interested reader to the last part of the Appendix of [21]. However, to give a hint on this fact, we prove the following lemma.

Lemma 3.4.

$$\|A\|_{C^{0,\gamma_0}(\overline{\Omega})} \leqslant \|p\|_{C^{0,\gamma_0}(\overline{\Omega})}\big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{p_0-1}.$$

Proof. At first observe that for any $1 < p(x) \leqslant p_0 < +\infty$, for any fixed $s, t \in [0, +\infty)$, there holds

$$\begin{aligned}\big|(1+t)^{p(x)-2} - (1+s)^{p(x)-2}\big| &\leqslant \big|p(x) - 2\big| \max\big\{(1+t)^{p(x)-3}, (1+s)^{p(x)-3}\big\}|t-s| \\ &\leqslant \big[1 + (p_0 - 2)\big(1 + \max\{t, s\}\big)^{p_0-3}\big]|t-s|. \end{aligned} \tag{3.15}$$

Moreover, if $K \geqslant 0$ and $s, t \in (-1, +\infty)$

$$\big|(1+K)^s - (1+K)^t\big| \leqslant \log(1+K)(1+K)^{\max\{t,s\}}|t-s|.$$

By $(3.12)_1$ we have

$$\|A\|_{C^{0,\gamma_0}(\overline{\Omega})} \leqslant \big(1 + \|\mathcal{D}v\|_\infty\big)^{p_0-2} + [A]_{C^{0,\gamma_0}(\overline{\Omega})}. \tag{3.16}$$

In order to evaluate the Hölder semi-norm of A we use the above estimates, obtaining

$$\begin{aligned}\frac{|A(x) - A(y)|}{|x-y|^{\gamma_0}} &\leqslant \Big[\Big|\big(1 + \big|\mathcal{D}v(x)\big|\big)^{p(x)-2} - \big(1 + \big|\mathcal{D}v(x)\big|\big)^{p(y)-2}\Big| \\ &\quad + \Big|\big(1 + \big|\mathcal{D}v(x)\big|\big)^{p(y)-2} - \big(1 + \big|\mathcal{D}v(y)\big|\big)^{p(y)-2}\Big|\Big]|x-y|^{-\gamma_0} \\ &\leqslant \log\big(1 + \|\mathcal{D}v\|_\infty\big)\big(1 + \|\mathcal{D}v\|_\infty\big)^{p_0-2}[p]_{C^{0,\gamma_0}(\overline{\Omega})} \\ &\quad + \big[1 + (p_0-2)\big(1 + \|\mathcal{D}v\|_\infty\big)^{p_0-3}\big][\mathcal{D}v]_{C^{0,\gamma_0}(\overline{\Omega})} \quad \forall x, y \in \overline{\Omega},\ x \neq y. \end{aligned} \tag{3.17}$$

Therefore, by (3.16) and (3.17)

$$\begin{aligned}\|A\|_{C^{0,\gamma_0}(\overline{\Omega})} &\leqslant \big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{p_0-2} + \big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{p_0-1}[p]_{C^{0,\gamma_0}(\overline{\Omega})} \\ &\quad + \big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{p_0-2}(p_0-2) + \big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big) \\ &\leqslant \big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{p_0-1}\big([p]_{C^{0,\gamma_0}(\overline{\Omega})} + p_0\big) \\ &\leqslant \|p\|_{C^{0,\gamma_0}(\overline{\Omega})}\big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{p_0-1}. \qquad \square \end{aligned}$$

Further, again following the computations in the Appendix of [21], the constant $\widetilde{c}$ can be decomposed in the sum of positive terms among which at least one is greater than

$$\left[\frac{1}{\sigma}\left(1 + \frac{1}{\sigma}\right)\right]^{1/2}\|A\|^2_{C^{0,\gamma_0}(\overline{\Omega})} \leqslant \sqrt{6}\big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)\|A\|^2_{C^{0,\gamma_0}(\overline{\Omega})} \leqslant \sqrt{6}\|p\|^2_{C^{0,\gamma_0}(\overline{\Omega})}\big(1 + \|\mathcal{D}v\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{2p_0-1}.$$

This motivates the assumptions $r \geqslant 2p_0 - 1$ and $C > p_0$.

Further, observe that the tensor A defined by (3.11) does not satisfy the hypothesis of Theorem 3.3, since $|\mathcal{D}v| \notin C^1(\overline{\Omega})$. However, if we replace the fourth-order tensor A by the fourth-order tensor $A^\varepsilon = A^\varepsilon_{i\alpha j\beta} = \frac{1}{2}(\delta_{ij}\delta_{\alpha\beta} + \delta_{i\beta}\delta_{j\alpha})(1 + J_\varepsilon(|\mathcal{D}v|))^{p(x)-2}$, where J_ε denotes the Friedrichs mollifier, it is clear that A^ε satisfies the hypothesis of Theorem 3.3. Obviously estimate (3.8) cannot be uniform in ε. By using an ε-approximating problem we succeed in proving Theorem 2.2 via Theorem 3.3.

Finally, we recall the following well-known results.

Lemma 3.5. *There exists a constant C_K such that*

$$\|v\|_2 + \|\nabla v\|_2 \leqslant C_K\|\mathcal{D}v\|_2, \quad \textit{for each } v \in V(\Omega).$$

Hence the two quantities above are equivalent norms in $V(\Omega)$.

For the proof we refer to [37, Proposition 1.1].

Lemma 3.6. *If a distribution g is such that* $\nabla g \in W^{-1,q}(\Omega)$*, then* $g \in L^q(\Omega)$ *and*

$$\|g\|_{L^q_\#} \leqslant c\|\nabla g\|_{-1,q},$$

where $L^q_\# = L^q/\mathbb{R}$.

For the proof we refer, for instance, to [36].

4. Proof of Theorem 2.1

Proof of the existence

We consider the following sequence of elliptic boundary value problems

$$P_m: \quad \begin{cases} -\nabla\cdot\big[\big(1+|\mathcal{D}U^m|\big)^{p-2}\mathcal{D}U\big]+\nabla\Pi = f-\big(U^m\cdot\nabla\big)U^m, & \text{in } \Omega,\\ \nabla\cdot U=0, & \text{in } \Omega,\\ U=0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where U^m is a given function belonging to $C^{1,\gamma_0}(\overline{\Omega})$. Observing that $f-(U^m\cdot\nabla)U^m \in L^q(\Omega)$ we can apply Theorem 3.2 to find a solution of problem P_m in $C^{1,\gamma_0}(\overline{\Omega})\times C^{0,\gamma_0}(\overline{\Omega})$ which we denote by (U^{m+1},Π^{m+1}). In such a way, setting $U^0\equiv 0$, $\Pi^0\equiv 0$ we build a sequence $\{(U^m,\Pi^m)\}\subset C^{1,\gamma_0}(\overline{\Omega})\times C^{0,\gamma_0}(\overline{\Omega})$.

First of all we want to prove the boundedness of the sequence $\{U^m\}$ in $C^{1,\gamma_0}(\overline{\Omega})$, provided that $\|f\|_q$ is small enough. We observe that since $U^{m+1}\in V(\Omega)$ is a weak solution of problem P_m it verifies the following integral identity for any $\varphi\in V(\Omega)$

$$\int_\Omega \big(1+|\mathcal{D}U^m|\big)^{p-2}\mathcal{D}U^{m+1}\cdot\mathcal{D}\varphi\,dx = \int_\Omega f\cdot\varphi\,dx - \int_\Omega \big(U^m\cdot\nabla\big)U^m\cdot\varphi\,dx. \tag{4.2}$$

Let us evaluate the $W^{1,2}$-norm of U^{m+1} testing Eq. (4.2) with U^{m+1} itself. By using the Hölder inequality and Lemma 3.5 we get

$$\begin{aligned}\|U^{m+1}\|_{1,2}^2 &\leqslant C_K^2\|\mathcal{D}U^{m+1}\|_2^2 \leqslant C_K^2\big(1+\|\mathcal{D}U^m\|_\infty\big)\int_\Omega\big(1+|\mathcal{D}U^m|\big)^{p-2}|\mathcal{D}U^{m+1}|^2\,dx\\ &\leqslant C_K^2\big(1+\|\mathcal{D}U^m\|_\infty\big)\big(\|f\|_2\|U^{m+1}\|_2+\|U^m\|_\infty\|\nabla U^m\|_2\|U^{m+1}\|_2\big)\\ &\leqslant C_K^2\big(1+\|\mathcal{D}U^m\|_\infty\big)\|U^{m+1}\|_{1,2}\big(\|f\|_2+|\Omega|^{1/2}\|U^m\|^2_{C^{1,\gamma_0}(\overline{\Omega})}\big),\end{aligned}$$

where by $|\Omega|$ we denote the Lebesgue measure of Ω. Recalling that $q>n\geqslant 2$ we get

$$\|U^{m+1}\|_{1,2}\leqslant C_K^2\big(1+|\Omega|^{1/2}\big)\big(1+\|\mathcal{D}U^m\|_\infty\big)\big(\|f\|_q+\|U^m\|^2_{C^{1,\gamma_0}(\overline{\Omega})}\big). \tag{4.3}$$

Now we can use Theorem 3.2 to get an estimate of the $C^{1,\gamma_0}(\overline{\Omega})$-norm of the solution U^{m+1}

$$\|U^{m+1}\|_{C^{1,\gamma_0}(\overline{\Omega})}+\|\Pi^{m+1}\|_{C^{0,\gamma_0}(\overline{\Omega})}\leqslant K_m\big(\|U^{m+1}\|_{1,2}+\|f-\big(U^m\cdot\nabla\big)U^m\|_q\big)$$

where $K_m\equiv C(1+\|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})})^r$ according to (3.14). Hence, by setting

$$I_m=\|U^m\|_{C^{1,\gamma_0}(\overline{\Omega})}+\|\Pi^m\|_{C^{0,\gamma_0}(\overline{\Omega})}\quad \forall m\in\mathbb{N}$$

we get

$$I_{m+1}\leqslant C\big(1+\|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^r\big(\|U^{m+1}\|_{1,2}+\|f\|_q+|\Omega|^{1/q}\|U^m\|^2_{C^{1,\gamma_0}(\overline{\Omega})}\big).$$

By using estimate (4.3) it is easy to obtain the following recursive inequality

$$I_{m+1}\leqslant C\big(1+C_K^2\big)\big(1+|\Omega|^{1/2}\big)(1+I_m)^{r+1}\big(\|f\|_q+I_m^2\big). \tag{4.4}$$

We shall prove the boundedness of the sequence $\{I_m\}$ by a fixed point argument. Setting, for any $t\geqslant 0$,

$$\psi(t)=C_0(1+t)^{r+1}\big(\|f\|_q+t^2\big)-t,$$

where

$$C_0\equiv C\big(1+C_K^2\big)\big(1+|\Omega|^{1/2}\big), \tag{4.5}$$

we look for a root of ψ. Let us observe that if $0\leqslant t\leqslant 1$ then

$$\psi(t)\leqslant C_0 2^{r+1}\big(\|f\|_q+t^2\big)-t \tag{4.6}$$

and the function on the right-hand side of the previous inequality has two positive roots $s_1 < s_2$ if and only if $1 - C_0^2 2^{2r+4}\|f\|_q > 0$. Moreover if

$$\|f\|_q < \frac{1}{C_0^2 2^{2r+4}} \equiv \Lambda \tag{4.7}$$

we have that

$$0 < s_1 = \frac{1 - \sqrt{1 - C_0^2\|f\|_q 2^{2r+4}}}{C_0 2^{r+2}} < 1 \tag{4.8}$$

since $C > p_0 > 1$ and consequently $C_0 2^{r+2} > 1$. We observe that $\psi(0) > 0$, ψ is convex and $\lim_{t\to+\infty}\psi(t) = +\infty$. Then estimate (4.6) shows that ψ has exactly two roots $t_1 < t_2$ such that $0 < t_1 \leqslant s_1$. Let us prove by induction that $I_m \leqslant t_1$ for any $m \in \mathbb{N}$ if condition (4.7) is satisfied. The first step of the inductive process is trivially true since

$$I_0 = \|U^0\|_{C^{1,\gamma_0}(\overline{\Omega})} + \|\Pi^0\|_{C^{0,\gamma_0}(\overline{\Omega})} = 0 < t_1.$$

If we suppose that $I_m \leqslant t_1$, by inequality (4.4) and the fact that $\psi(t_1) = 0$ we obtain

$$I_{m+1} \leqslant C_0(1+I_m)^{r+1}\left(\|f\|_q + I_m^2\right) \leqslant C_0(1+t_1)^{r+1}\left(\|f\|_q + t_1^2\right) = \psi(t_1) + t_1 = t_1$$

which proves our claim. Therefore

$$I_m \leqslant t_1 \leqslant s_1 < C_0 2^{r+2}\|f\|_q < 1 \quad \forall m \in \mathbb{N}. \tag{4.9}$$

As a consequence of the boundedness of the sequence $\{(U^m, \Pi^m)\}$ in $C^{1,\gamma_0}(\overline{\Omega}) \times C^{0,\gamma_0}(\overline{\Omega})$, by appealing to the Ascoli–Arzelà theorem we get, for any $\gamma \in (0, \gamma_0)$, the convergence of a subsequence in the norm $C^{1,\gamma}(\overline{\Omega}) \times C^{0,\gamma}(\overline{\Omega})$ to a couple (u, π). In order to prove that (u, π) is a $C^{1,\gamma}$-solution of problem (1.1) it is sufficient to show the convergence of the whole sequence $\{(U^m, \Pi^m)\}$ in $W^{1,2}(\Omega)$.

We set

$$W^{m+1}(x) = U^{m+1}(x) - U^m(x), \qquad P^{m+1} = \Pi^{m+1}(x) - \Pi^m(x) \quad \forall m \geqslant 0.$$

It is easy to check that for any $m \geqslant 1$ the pair (W^{m+1}, P^{m+1}) verifies

$$\begin{aligned}\int_\Omega \left(1+|\mathcal{D}U^m|\right)^{p-2}\mathcal{D}W^{m+1}\cdot\mathcal{D}\varphi\,dx = {}& -\int_\Omega (W^m\cdot\nabla)U^m\cdot\varphi\,dx - \int_\Omega (U^{m-1}\cdot\nabla)W^m\cdot\varphi\,dx \\ &+ \int_\Omega \left[\left(1+|\mathcal{D}U^m|\right)^{p-2} - \left(1+|\mathcal{D}U^{m-1}|\right)^{p-2}\right]\mathcal{D}U^m\cdot\mathcal{D}\varphi\,dx \\ &+ \int_\Omega P^{m+1}\nabla\cdot\varphi\,dx, \quad \forall\varphi \in C_0^\infty(\Omega).\end{aligned} \tag{4.10}$$

Recalling that $\{(U^m, P^m)\} \subset C^{1,\gamma_0}(\overline{\Omega}) \times C^{0,\gamma_0}(\overline{\Omega})$ and that $C_0^\infty(\Omega)$ is dense in $V(\Omega)$, the above equation is still valid for any $\varphi \in V(\Omega)$, in which case the last term vanishes. Hence we can choose $\varphi = W^{m+1}$ in Eq. (4.10) thus obtaining

$$\begin{aligned}\left(1+\|\mathcal{D}U^m\|_\infty\right)^{-1}\|W^{m+1}\|_{1,2}^2 &\leqslant C_K^2\left(1+\|\mathcal{D}U^m\|_\infty\right)^{-1}\|\mathcal{D}W^{m+1}\|_2^2 \\ &\leqslant C_K^2\int_\Omega \left(1+|\mathcal{D}U^m|\right)^{p-2}|\mathcal{D}W^{m+1}|^2\,dx \\ &\leqslant C_K^2\left(\|W^m\|_2\|U^m\|_{C^{1,\gamma_0}(\overline{\Omega})} + \|U^{m-1}\|_{C^{1,\gamma_0}(\overline{\Omega})}\|\nabla W^m\|_2\right)\|W^{m+1}\|_2 \\ &\quad + C_K^2\left|\int_\Omega \left[\left(1+|\mathcal{D}U^m|\right)^{p-2} - \left(1+|\mathcal{D}U^{m-1}|\right)^{p-2}\right]\mathcal{D}U^m\cdot\mathcal{D}W^{m+1}\,dx\right|.\end{aligned} \tag{4.11}$$

Hence, recalling (4.9) and (3.15) we get that the integral on the right-hand side of (4.11) can be estimated as

$$\begin{aligned}C_K^2\left(1+(p_0-2)2^{p_0-3}\right)\int_\Omega |\mathcal{D}W^m||\mathcal{D}U^m||\mathcal{D}W^{m+1}|\,dx &\leqslant C_K^2\left(1+(p_0-2)2^{p_0-3}\right)\|\mathcal{D}W^m\|_2\|U^m\|_{C^{1,\gamma_0}(\overline{\Omega})}\|\mathcal{D}W^{m+1}\|_2 \\ &\leqslant C_K^2 C_0\left(1+(p_0-2)2^{p_0-3}\right)2^{r+2}\|f\|_q\|\mathcal{D}W^m\|_2\|\mathcal{D}W^{m+1}\|_2.\end{aligned}$$

Thus from (4.11) we get

$$\begin{aligned}\|W^{m+1}\|_{1,2} &\leqslant \big[C_K^2 C_0 2^{r+3} + C_K^2 C_0\big(1+(p_0-2)2^{p_0-3}\big)2^{r+2}\big]\big(1+\|\mathcal{D}U^m\|_\infty\big)\|f\|_q\|W^m\|_{1,2}\\ &\leqslant C_K^2 C_0 p_0 2^{r+p_0+2}\|f\|_q\|W^m\|_{1,2}, \quad \forall m \geqslant 1.\end{aligned}$$

By the assumption (4.7), using (4.5) and that $C \geqslant p_0$, it follows that

$$C_K^2 C_0 p_0 2^{r+p_0+2}\|f\|_q \leqslant \frac{C_K^2 C_0 p_0 2^{r+p_0+2}}{C_0^2 2^{2r+4}} \leqslant 2^{p_0-r-2} < 1, \tag{4.12}$$

since $r \geqslant 2p_0 - 1 > p_0 - 2$. An easy induction argument ensures that

$$\|W^m\|_{1,2} \leqslant \big[C_K^2 C_0 (p_0-1) 2^{r+p_0+2}\|f\|_q\big]^{m-1}\|W^1\|_{1,2} \quad \forall m \geqslant 1. \tag{4.13}$$

Hence $\{U^m\}$ is a Cauchy sequence in $W^{1,2}(\Omega)$ and by completeness it converges to a function $\widetilde{u} \in W^{1,2}(\Omega)$. Since there exists a subsequence of $\{U^m\}$ converging to u in $C^{1,\gamma}(\overline{\Omega})$, and consequently also in $W^{1,2}(\Omega)$, by uniqueness of the limit we have that $u = \widetilde{u} \in C^{1,\gamma}(\overline{\Omega})$.

Further, let us estimate the L^2-norm of the pressure. By Eq. (4.10) and Lemma 3.6 we get

$$\begin{aligned}\|P^{m+1}\|_2 \leqslant c\|\nabla P^{m+1}\|_{-1,2} \leqslant c\big\{&\|\nabla\cdot\big[\big(1+|\mathcal{D}U^m|\big)^{p-2}\mathcal{D}W^{m+1}\big]\|_{-1,2} + \|(W^m\cdot\nabla)U^m + (U^{m-1}\cdot\nabla)W^m\|_2\\ &+ \|\nabla\cdot\big\{\big[\big(1+|\mathcal{D}U^m|\big)^{p-2} - \big(1+|\mathcal{D}U^{m-1}|\big)^{p-2}\big]\mathcal{D}U^m\big\}\|_{-1,2}\big\}.\end{aligned} \tag{4.14}$$

The first term on the right-hand side can be estimated by

$$\|\big(1+|\mathcal{D}U^m|\big)^{p-2}\mathcal{D}W^{m+1}\|_2 \leqslant (1+I_m)^{p_0-2}\|\mathcal{D}W^{m+1}\|_2 \leqslant 2^{p_0-2}\|W^{m+1}\|_{1,2}.$$

By using inequality (3.15) and the upper bound for I_m, the last term in (4.14) can be increased to

$$\|\big[\big(1+|\mathcal{D}U^m|\big)^{p-2} - \big(1+|\mathcal{D}U^{m-1}|\big)^{p-2}\big]\mathcal{D}U^m\|_2 \leqslant \big(1+(p_0-2)2^{p_0-3}\big)\|\mathcal{D}W^m\|_2 I_m \leqslant c\|W^m\|_{1,2}.$$

Hence, from (4.13) and (4.14),

$$\|P^{m+1}\|_2 \leqslant c\big(\|W^{m+1}\|_{1,2} + \|W^m\|_{1,2}\big) \leqslant c\big[C_K^2 C_0(p_0-1)2^{r+p_0+2}\|f\|_q\big]^{m-1}\|W^1\|_{1,2}.$$

Thanks to estimate (4.12), the sequence $\{\Pi^m\}$ verifies the Cauchy condition in $L^2(\Omega)$ and then converges to a function $\widetilde{\pi}$ in the L^2-norm. As before, for uniqueness, we have that $\pi = \widetilde{\pi} \in C^{0,\gamma}(\overline{\Omega})$.

As a last step we prove that the pair (u,π) solves problem (1.1). For any fixed $\varphi \in C_0^\infty(\Omega)$ we have that

$$\begin{aligned}&\left|\int_\Omega \big(1+|\mathcal{D}U^m|\big)^{p-2}\mathcal{D}U^{m+1}\cdot\mathcal{D}\varphi\,dx - \int_\Omega S(\mathcal{D}u)\cdot\mathcal{D}\varphi\,dx\right|\\ &\quad\leqslant \int_\Omega \big(1+|\mathcal{D}U^m|\big)^{p-2}\big|\mathcal{D}\big(U^{m+1}-u\big)\big||\mathcal{D}\varphi|\,dx + \int_\Omega \big|\big(1+|\mathcal{D}U^m|\big)^{p-2} - \big(1+|\mathcal{D}u|\big)^{p-2}\big||\mathcal{D}u||\mathcal{D}\varphi|\,dx\\ &\quad\leqslant (1+I_m)^{p_0-2}\|U^{m+1}-u\|_{1,2}\|\mathcal{D}\varphi\|_2 + \big(1+(p_0-2)2^{p_0-3}\big)\|\mathcal{D}\big(U^m-u\big)\|_2\|\mathcal{D}u\|_2\|\mathcal{D}\varphi\|_\infty.\end{aligned}$$

Moreover

$$\begin{aligned}\left|\int_\Omega (U^m\cdot\nabla)U^m\cdot\varphi\,dx - \int_\Omega (u\cdot\nabla)u\cdot\varphi\,dx\right| &\leqslant \int_\Omega \big|\big((U^m-u)\cdot\nabla\big)U^m\cdot\varphi\big|\,dx + \int_\Omega \big|(u\cdot\nabla)\big(U^m-u\big)\cdot\varphi\big|\,dx\\ &\leqslant \|U^m-u\|_2\|\nabla U^m\|_\infty\|\varphi\|_2 + \|u\|_2\|\nabla\big(U^m-u\big)\|_2\|\varphi\|_\infty\\ &\leqslant |U^m-u\|_{1,2}\big(I_m\|\varphi\|_2 + \|u\|_2\|\varphi\|_\infty\big) \leqslant c\|U^m-u\|_{1,2}.\end{aligned}$$

Finally

$$\left|\int_\Omega \nabla\Pi^{m+1}\cdot\varphi\,dx - \int_\Omega \nabla\pi\cdot\varphi\,dx\right| \leqslant \left|\int_\Omega \big(\Pi^{m+1}-\pi\big)\nabla\cdot\varphi\,dx\right| \leqslant \|\Pi^{m+1}-\pi\|_2\|\nabla\cdot\varphi\|_2.$$

By using the convergence of $\{U^m\}$ to u in $W^{1,2}(\Omega)$ and of $\{\Pi^m\}$ to π in $L^2(\Omega)$ we get, for any $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned}\int_\Omega f\cdot\varphi\,dx &= \lim_{m\to\infty}\left\{\int_\Omega \big(1+|\mathcal{D}U^m|\big)^{p-2}\mathcal{D}U^{m+1}\cdot\mathcal{D}\varphi\,dx + \int_\Omega (U^m\cdot\nabla)U^m\cdot\varphi\,dx + \int_\Omega \nabla\Pi^{m+1}\cdot\varphi\,dx\right\}\\ &= \int_\Omega \big(1+|\mathcal{D}u|\big)^{p-2}\mathcal{D}u\cdot\mathcal{D}\varphi\,dx + \int_\Omega (u\cdot\nabla)u\cdot\varphi\,dx + \int_\Omega \nabla\pi\cdot\varphi\,dx.\end{aligned}$$

As a final step we observe that since $U^m \in V(\Omega)$ we can argue that $u|_{\partial\Omega} = 0$ and $\nabla \cdot u = 0$, hence (u, π) solves problem (1.1). The estimate (2.3) follows passing to the limit in inequality (4.9) and setting $C_1 \equiv C_0 2^{r+2}$.

Remark 4.1. In the above proof we have carried on the expressions of the various constants, instead of denoting them by a generic constant c. This is just to use a unique smallness assumption on the L^q-norm of f, i.e. condition (4.7), both for the uniform bound (4.9) on the sequence I_m and for the $W^{1,2}$-convergence of the sequence U_m.

Proof of the uniqueness

Let us prove the uniqueness of $C^{1,\gamma}$-solutions, in the sense of Definition 2.1. Let u_1 be another $C^{1,\gamma}$-solution. Setting $w = u - u_1$, we can test with $\varphi = w$ in (2.1), written both for u and u_1. We then subtract the two equations and we find

$$\int_\Omega \big(S(\mathcal{D}u) - S(\mathcal{D}u_1)\big) \cdot (\mathcal{D}u - \mathcal{D}u_1)\, dx = \int_\Omega (w \cdot \nabla) w \cdot u \, dx. \tag{4.15}$$

If $p_\infty \geqslant 2$, recalling (3.2) and $(3.3)_2$, and then using identity (4.15), we obtain

$$\|\mathcal{D}w\|_2^2 \leqslant \int_\Omega \big(S(\mathcal{D}u) - S(\mathcal{D}u_1)\big) \cdot (\mathcal{D}u - \mathcal{D}u_1)\, dx \leqslant \int_\Omega (w \cdot \nabla) w \cdot u \, dx. \tag{4.16}$$

Increasing the last term as follows

$$\left| \int_\Omega (w \cdot \nabla) w \cdot u \, dx \right| = \left| \int_\Omega (w \cdot \nabla) u \cdot w \, dx \right| \leqslant \|\nabla u\|_\infty \|w\|_2^2 \leqslant C_K^2 \|\nabla u\|_\infty \|\mathcal{D}w\|_2^2,$$

estimate (4.16) gives

$$\big(1 - C_K^2 \|\nabla u\|_\infty\big) \|\mathcal{D}w\|_2^2 \leqslant 0.$$

Therefore, if $\|\nabla u\|_\infty < 1/C_K^2$, for $p_\infty \geqslant 2$ the uniqueness follows. Recalling that from Theorem 2.1

$$\|\nabla u\|_\infty \leqslant \|u\|_{C^{1,\gamma_0}(\overline{\Omega})} \leqslant C_1 \|f\|_q,$$

and since the last term is less than $1/C_K^2$, as it is easy to see from (4.7), condition $\|\nabla u\|_\infty < 1/C_K^2$ is always satisfied.

Let us consider the case $1 < p_\infty < 2$. By replacing φ by u_1 in Eq. (2.1) written for the solution u_1, we get

$$\int_\Omega S(\mathcal{D}u_1) \cdot \mathcal{D}u_1 \, dx = \int_\Omega f \cdot u_1 \, dx.$$

Since

$$2^{p_\infty - 2} \int_{|\mathcal{D}u_1| \geqslant 1} |\mathcal{D}u_1|^{p_\infty} \, dx \leqslant \int_{|\mathcal{D}u_1| \geqslant 1} \big(1 + |\mathcal{D}u_1|\big)^{p_\infty - 2} |\mathcal{D}u_1|^2 \, dx \leqslant \int_{|\mathcal{D}u_1| \geqslant 1} S(\mathcal{D}u_1) \cdot \mathcal{D}u_1 \, dx$$

there holds

$$\int_\Omega |\mathcal{D}u_1|^{p_\infty} \, dx = \int_{|\mathcal{D}u_1| \geqslant 1} |\mathcal{D}u_1|^{p_\infty} \, dx + \int_{|\mathcal{D}u_1| \leqslant 1} |\mathcal{D}u_1|^{p_\infty} \, dx \leqslant 2^{2 - p_\infty} \int_{|\mathcal{D}u_1| \geqslant 1} S(\mathcal{D}u_1) \cdot \mathcal{D}u_1 \, dx + \int_{|\mathcal{D}u_1| \leqslant 1} dx.$$

By Hölder's and Sobolev's inequalities and recalling Lemma 3.5, it readily follows that

$$\|\mathcal{D}u_1\|_{p_\infty}^{p_\infty} \leqslant 2^{2 - p_\infty} \|u_1\|_{\frac{np_\infty}{n - p_\infty}} \|f\|_{\frac{np_\infty}{np_\infty - n + p_\infty}} + |\Omega| \leqslant c\big(\|\mathcal{D}u_1\|_{p_\infty} \|f\|_q + 1\big),$$

where we have used the validity of the inequality $\frac{np_\infty}{np_\infty - n + p_\infty} < n < q$. Therefore, applying the Young inequality, we easily obtain the following estimate for the L^{p_∞}-norm of $\mathcal{D}u_1$

$$\|\mathcal{D}u_1\|_{p_\infty} \leqslant C_2\big(1 + \|f\|_q^{\frac{1}{p_\infty - 1}}\big), \tag{4.17}$$

where $C_2 = C_2(p_\infty, q, n, \Omega)$. By using the Hölder inequality, we can write

$$\begin{aligned} \|\mathcal{D}w\|_{p_\infty}^{p_\infty} &= \int_\Omega \left(\frac{|\mathcal{D}w|^2}{(1 + |\mathcal{D}u| + |\mathcal{D}u_1|)^{2 - p_\infty}} \right)^{\frac{p_\infty}{2}} \big(1 + |\mathcal{D}u| + |\mathcal{D}u_1|\big)^{\frac{p_\infty(2 - p_\infty)}{2}} dx \\ &\leqslant \left(\int_\Omega \frac{|\mathcal{D}w|^2}{(1 + |\mathcal{D}u| + |\mathcal{D}u_1|)^{2 - p_\infty}} \, dx \right)^{\frac{p_\infty}{2}} \left(\int_\Omega \big(1 + |\mathcal{D}u| + |\mathcal{D}u_1|\big)^{p_\infty} dx \right)^{\frac{2 - p_\infty}{2}}. \end{aligned}$$

Hence, recalling (3.2) and $(3.3)_1$, and then using identity (4.15), we obtain

$$\|\mathcal{D}w\|^2_{p_\infty} \leqslant \frac{c4^{p_0}}{p_\infty - 1}\left(\int\limits_\Omega \big(S(\mathcal{D}u) - S(\mathcal{D}u_1)\big)\cdot(\mathcal{D}u - \mathcal{D}u_1)\,dx\right)\big(1 + \|\mathcal{D}u\|^{2-p_\infty}_{p_\infty} + \|\mathcal{D}u_1\|^{2-p_\infty}_{p_\infty}\big)$$
$$= C_3(p_\infty, p_0)\left(\int\limits_\Omega (w\cdot\nabla)w\cdot u\,dx\right)\big(1 + \|\mathcal{D}u\|^{2-p_\infty}_{p_\infty} + \|\mathcal{D}u_1\|^{2-p_\infty}_{p_\infty}\big). \tag{4.18}$$

Using Hölder's and Sobolev's inequalities and then the hypothesis made on u, we have

$$\left|\int\limits_\Omega (w\cdot\nabla)w\cdot u\,dx\right| = \left|\int\limits_\Omega (w\cdot\nabla)u\cdot w\,dx\right| \leqslant \|\nabla u\|_{\frac{np_\infty}{np_\infty - 2n + 2p_\infty}}\|w\|^2_{\frac{np_\infty}{n-p_\infty}} \leqslant c\|u\|_{C^{1,\gamma}(\overline{\Omega})}\|\nabla w\|^2_{p_\infty} \leqslant cC_1\|f\|_q\|\mathcal{D}w\|^2_{p_\infty},$$

where in the last step we have also used a Korn type inequality. Taking into account estimate (4.17) and the above estimate, recalling that $\|f\|_q < 1$, (4.18) gives

$$\|\mathcal{D}w\|^2_{p_\infty} \leqslant C_4\|f\|_q\big(1 + \|f\|^{2-p_\infty}_q\big)\|\mathcal{D}w\|^2_{p_\infty}$$

where $C_4 = C_4(p_\infty, p_0, q, n, \Omega)$. The previous estimate provides the uniqueness if $\|f\|_q < \delta$, for a sufficiently small δ.

5. Proof of Theorem 2.2

The proof can be obtained, with minor changes, by the proof of the same regularity theorem in paper [21] (see Theorem 2.2, pp. 470–473), concerning shear-thinning flows. However, for completeness, we perform it here. We assume that $\|f\|_q < \Lambda < 1$, with Λ given by (4.7). In this way all the hypotheses of Theorem 2.1 are satisfied and we can find the sequences (U^m), (Π^m) converging to the solution (u, π), as in the proof of the previous theorem.

In order to get $D^2u \in L^2(\Omega)$ we proceed by induction on m. The first step is trivial, as $U_0 \equiv 0$. Assume that $D^2U^m \in L^2(\Omega)$ and consider the following boundary value problem

$$\begin{cases} -\nabla\cdot\big[\big(1 + J_\varepsilon\big(|\mathcal{D}U^m|\big)\big)^{p-2}\mathcal{D}U^{m+1}_\varepsilon\big] + \nabla\Pi^{m+1}_\varepsilon = f - \big(U^m\cdot\nabla\big)U^m, & \text{in } \Omega,\\ \nabla\cdot U^{m+1}_\varepsilon = 0, & \text{in } \Omega,\\ U^{m+1}_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where J_ε denotes the Friedrichs mollifier. Since $f - (U^m\cdot\nabla)U^m$ belongs to $L^q(\Omega)$, the hypotheses of Theorem 3.2 are verified and there exists a solution $(U^{m+1}_\varepsilon, \Pi^{m+1}_\varepsilon) \in C^{1,\gamma_0}(\overline{\Omega}) \times C^{0,\gamma_0}(\overline{\Omega})$ satisfying the following estimate

$$\big\|U^{m+1}_\varepsilon\big\|_{C^{1,\gamma_0}(\overline{\Omega})} + \big\|\Pi^{m+1}_\varepsilon\big\|_{C^{0,\gamma_0}(\overline{\Omega})} \leqslant \widetilde{c}\big(\big\|U^{m+1}_\varepsilon\big\|_{1,2} + \big\|f - \big(U^m\cdot\nabla\big)U^m\big\|_q\big),$$

where $\widetilde{c}$, in particular, depends on the $C^{0,\gamma_0}(\overline{\Omega})$-norms of p and $J_\varepsilon(|\mathcal{D}U^m|)$. We observe that the constant $\widetilde{c}$ cannot be expressed in the form (3.14), since the tensor of problem (5.1) is not in the form (3.11). However, with minor changes to the proof given in the Appendix of [21] we can achieve that

$$\widetilde{c} = C\big(1 + \big\|J_\varepsilon\big(|\mathcal{D}U^m|\big)\big\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^r,$$

where we recall that C depends on the $C^{0,\gamma_0}(\overline{\Omega})$-norm of p but not on the $C^{0,\gamma_0}(\overline{\Omega})$-norm of $J_\varepsilon(|\mathcal{D}U^m|)$. Proceeding as in Theorem 2.1 (recall (4.3)), observing that $\|J_\varepsilon(|\mathcal{D}U^m|)\|_{C^{0,\gamma_0}(\overline{\Omega})} \leqslant \|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})}$ and taking into account (4.9), we get, uniformly in ε and m,

$$\begin{aligned} &\big\|U^{m+1}_\varepsilon\big\|_{C^{1,\gamma_0}(\overline{\Omega})} + \big\|\Pi^{m+1}_\varepsilon\big\|_{C^{0,\gamma_0}(\overline{\Omega})}\\ &\quad\leqslant C\big(1 + \big\|\mathcal{D}U^m\big\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^r\big[c\big(1 + \big\|\mathcal{D}U^m\big\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)\big(\|f\|_q + \big\|U^m\big\|^2_{C^{1,\gamma_0}(\overline{\Omega})}\big) + \|f\|_q + \big\|\big(U^m\cdot\nabla\big)U^m\big\|_q\big]\\ &\quad\leqslant cC(1 + I_m)^{r+1}\big(\|f\|_q + I^2_m\big) \leqslant cC_1\|f\|_q. \end{aligned} \tag{5.2}$$

Further, as the tensor $A^\varepsilon_{i\alpha j\beta} = \frac{1}{2}(\delta_{ij}\delta_{\alpha\beta} + \delta_{i\beta}\delta_{j\alpha})(1 + J_\varepsilon(|\mathcal{D}U^m|))^{p-2}$ satisfies the hypotheses of Theorem 3.3, then $U^{m+1}_\varepsilon \in W^{2,2}(\Omega)$ and $\Pi^{m+1}_\varepsilon \in W^{1,2}(\Omega)$ (and verifies an estimate of the kind (3.8) not uniformly in ε). Let us multiply (5.1) by $\Delta U^{m+1}_\varepsilon$ and integrate on $\Omega_\eta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \eta\}$, for some $\varepsilon < \eta$. We get

$$\begin{aligned} \int\limits_{\Omega_\eta}\big(1 + J_\varepsilon\big(|\mathcal{D}U^m|\big)\big)^{p-2}\big|\Delta U^{m+1}_\varepsilon\big|^2\,dx &= \int\limits_{\Omega_\eta}\big(2 - p(x)\big)\big(1 + J_\varepsilon\big(|\mathcal{D}U^m|\big)\big)^{p-3}\mathcal{D}U^{m+1}_\varepsilon\cdot\big(\Delta U^{m+1}_\varepsilon \otimes \nabla J_\varepsilon\big(|\mathcal{D}U^m|\big)\big)\,dx\\ &\quad - \int\limits_{\Omega_\eta}\big(1 + J_\varepsilon\big(|\mathcal{D}U^m|\big)\big)^{p-2}\log\big(1 + J_\varepsilon\big(|\mathcal{D}U^m|\big)\big)\mathcal{D}U^{m+1}_\varepsilon\cdot\big(\Delta U^{m+1}_\varepsilon \otimes \nabla p\big)\,dx\\ &\quad + \int\limits_{\Omega_\eta}\nabla\Pi^{m+1}_\varepsilon\cdot\Delta U^{m+1}_\varepsilon\,dx - \int\limits_{\Omega_\eta}\big(f - \big(U^m\cdot\nabla\big)U^m\big)\cdot\Delta U^{m+1}_\varepsilon\,dx = \sum_{i=1}^4 K_i. \end{aligned} \tag{5.3}$$

Let us estimate each term on the right-hand side. Using the induction hypothesis on D^2U^m, recalling that

$$\|\nabla J_\varepsilon(|\mathcal{D}U^m|)\|_{2,\Omega_\eta} = \|J_\varepsilon(\nabla|\mathcal{D}U^m|)\|_{2,\Omega_\eta} \leqslant \|\nabla|\mathcal{D}U^m|\|_{2,\Omega_\eta},$$

we get

$$\begin{aligned}|K_1| &\leqslant \bigl(1+(p_0-2)\bigl(1+\|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})}\bigr)^{p_0-3}\bigr)\|\mathcal{D}U_\varepsilon^{m+1}\|_{\infty,\Omega_\eta}\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta}\|\nabla J_\varepsilon(|\mathcal{D}U^m|)\|_{2,\Omega_\eta}\\ &\leqslant cC_1\|f\|_q\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta}\|\nabla|\mathcal{D}U^m|\|_{2,\Omega}.\end{aligned}$$

Further

$$\begin{aligned}|K_2| &\leqslant \bigl(1+\|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})}\bigr)^{p_0-1}\|\mathcal{D}U_\varepsilon^{m+1}\|_{\infty,\Omega_\eta}\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta}\|\nabla p\|_{2,\Omega_\eta}\\ &\leqslant cC_1\|f\|_q\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta}\|\nabla p\|_{2,\Omega}.\end{aligned}$$

By using the divergence theorem,

$$|K_3| = \left|\int_{\partial\Omega_\eta} \Pi_\varepsilon^{m+1}\Delta U_\varepsilon^{m+1}\cdot n\,d\sigma\right| \leqslant \|\Pi_\varepsilon^{m+1}\|_{W^{\frac12,2}(\partial\Omega_\eta)}\|\Delta U_\varepsilon^{m+1}\cdot n\|_{W^{-\frac12,2}(\partial\Omega_\eta)},$$

where n denotes the unit outer normal to $\partial\Omega_\eta$ and $d\sigma$ the $(n-1)$-dimensional measure. It is known that (see [31]) if $\gamma_0 > \frac12$ then

$$\|\Pi_\varepsilon^{m+1}\|_{W^{\frac12,2}(\partial\Omega_\eta)} \leqslant c\|\Pi_\varepsilon^{m+1}\|_{C^{0,\gamma_0}(\overline{\Omega})}.$$

In our hypotheses, since $q > 2n$ and $\gamma_0 = 1-\frac{n}{q}$, the condition $\gamma_0 > \frac12$ is satisfied. Moreover (see [26, Chapter III])

$$\|\Delta U_\varepsilon^{m+1}\cdot n\|_{W^{-\frac12,2}(\partial\Omega_\eta)} \leqslant \|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta} + \|\nabla\cdot\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta} = \|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta}.$$

Hence, recalling (5.2), for the term K_3 we obtain

$$|K_3| \leqslant cC_1\|f\|_q\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta}.$$

Finally, using the Cauchy–Schwartz inequality and the result of Theorem 2.1,

$$|K_4| \leqslant \|f\|_2\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta} + \|(U^m\cdot\nabla)U^m\|_2\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta} \leqslant c\bigl(\|f\|_q + C_1^2\|f\|_q^2\bigr)\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta}.$$

Bounding the left-hand side of (5.3) from below as

$$\int_{\Omega_\eta}\bigl(1+J_\varepsilon(|\mathcal{D}U^m|)\bigr)^{p-2}|\Delta U_\varepsilon^{m+1}|^2\,dx \geqslant \frac{\|\Delta U_\varepsilon^{m+1}\|_2^2}{1+\|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})}},$$

the above estimates imply that $\Delta U_\varepsilon^{m+1} \in L^2(\Omega_\eta)$ and

$$\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta} \leqslant cC_1\|f\|_q\bigl(1+C_1\|f\|_q\bigr)\bigl(1+\|\nabla p\|_2+\|\nabla|\mathcal{D}U^m|\|_{2,\Omega}\bigr). \tag{5.4}$$

Since the previous estimate holds for any $\eta > 0$, we can replace $\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega_\eta}$ with $\|\Delta U_\varepsilon^{m+1}\|_{2,\Omega}$. The above boundedness of $\Delta U_\varepsilon^{m+1}$ in $L^2(\Omega)$, uniformly in ε, ensures the existence of a subsequence weakly converging in $L^2(\Omega)$.

On the other hand, we prove that, for any fixed $m \in \mathbb{N}$, U_ε^{m+1} tends to U^{m+1} in $W^{1,2}(\Omega)$ as ε goes to zero. Indeed, using (4.2) and the following definition of weak solution for U_ε^{m+1}

$$\int_\Omega\bigl(1+J_\varepsilon(|\mathcal{D}U^m|)\bigr)^{p-2}\mathcal{D}U_\varepsilon^{m+1}\cdot\mathcal{D}\varphi\,dx = \int_\Omega\bigl(f-(U^m\cdot\nabla)U^m\bigr)\cdot\varphi\,dx,\quad \forall\varphi\in V(\Omega),$$

then subtracting the second identity by the first and finally choosing $\varphi = U^{m+1}-U_\varepsilon^{m+1}$, we get

$$\begin{aligned}&\int_\Omega\bigl(1+|\mathcal{D}U^m|\bigr)^{p-2}|\mathcal{D}U^{m+1}-\mathcal{D}U_\varepsilon^{m+1}|^2\,dx\\ &\quad= -\int_\Omega\bigl[\bigl(1+|\mathcal{D}U^m|\bigr)^{p-2}-\bigl(1+J_\varepsilon(|\mathcal{D}U^m|)\bigr)^{p-2}\bigr]\mathcal{D}U_\varepsilon^{m+1}\cdot\bigl(\mathcal{D}U^{m+1}-\mathcal{D}U_\varepsilon^{m+1}\bigr)\,dx.\end{aligned}$$

Hence, estimating the left-hand side as usual and employing (3.15), we have

$$\begin{aligned}\|\mathcal{D}U^{m+1}-\mathcal{D}U_\varepsilon^{m+1}\|_2^2 &\leqslant \big(1+\|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)\big[1+(p_0-2)\big(1+\|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{p_0-3}\big]\\ &\quad\times\int_\Omega \big|J_\varepsilon\big(|\mathcal{D}U^m|\big)-|\mathcal{D}U^m|\big|\big|\mathcal{D}U_\varepsilon^{m+1}\big|\big|\mathcal{D}U^{m+1}-\mathcal{D}U_\varepsilon^{m+1}\big|\,dx\\ &\leqslant c(p_0-1)\big(1+\|\mathcal{D}U^m\|_{C^{0,\gamma_0}(\overline{\Omega})}\big)^{p_0-1}\|\mathcal{D}U_\varepsilon^{m+1}\|_{C^{0,\gamma_0}(\overline{\Omega})}\\ &\quad\times\big\|J_\varepsilon\big(|\mathcal{D}U^m|\big)-|\mathcal{D}U^m|\big\|_2\|\mathcal{D}U^{m+1}-\mathcal{D}U_\varepsilon^{m+1}\|_2,\end{aligned}$$

which, dividing by $\|\mathcal{D}U^{m+1}-\mathcal{D}U_\varepsilon^{m+1}\|_2$ and letting ε tend to zero, ensures that $\mathcal{D}U_\varepsilon^{m+1}$ converges to $\mathcal{D}U^{m+1}$ in $L^2(\Omega)$.

By using the strong convergence of U_ε^{m+1} to U^{m+1} in $W^{1,2}(\Omega)$, we also deduce that the limit point of the subsequence of $\Delta U_\varepsilon^{m+1}$ in $L^2(\Omega)$ is ΔU^{m+1}. Since $\|D^2U^{m+1}\|_2\leqslant c\|\Delta U^{m+1}\|_2$, by using (5.4) we get

$$\|D^2U^{m+1}\|_2\leqslant C_5\|f\|_q\big(1+\|D^2U^m\|_2\big), \tag{5.5}$$

where C_5 depends on $\|p\|_{C^{0,\gamma_0}(\overline{\Omega})}$, n, q, Ω. Set

$$\Psi(z)=C_5\|f\|_q(1+z),\quad z\geqslant 0.$$

If

$$C_5\|f\|_q<1, \tag{5.6}$$

then there exists $z_0>0$ such that $\Psi(z_0)=z_0$. By induction one has $\|D^2U^m\|_2\leqslant z_0$ for any $m\in\mathbb{N}$. Indeed $\|D^2U^0\|_2=0\leqslant z_0$ and, since Ψ is increasing, for any $m\in\mathbb{N}$, using (5.5), we have

$$\|D^2U^{m+1}\|_2\leqslant\Psi\big(\|D^2U^m\|_2\big)\leqslant\Psi(z_0)=z_0$$

for any $m\in\mathbb{N}$. By the uniform boundedness of the L^2-norm of D^2U^m, using the strong convergence in $W^{1,2}(\Omega)$ of U^m to the solution u of problem (1.1), we deduce that if the force term satisfies (5.6) and $\|f\|_q\leqslant\Lambda$, then $u\in W^{2,2}(\Omega)$. By (1.1), we have that

$$\nabla\pi=\nabla\cdot\big[\big(1+|\mathcal{D}u|\big)^{p-2}\mathcal{D}u\big]+f-(u\cdot\nabla)u$$

in the distribution sense. Observing that the right-hand side of the previous identity belongs to $L^2(\Omega)$, we also obtain $\nabla\pi\in L^2(\Omega)$.

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