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JOURNAL OF COMBINATORIAL THEORY (A) 17, 252–253 (1974)

Note

A Simple Proof of the Kruskal–Katona Theorem

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Communicated by the Managing Editors

Received September 26, 1972

This note is a continuation of the preceeding paper [1] in this journal.

Let n, ℓ be positive integers, and let $\mathscr{S}$ be the collection of the first n sets of cardinality ℓ. Also let $\mathcal{A}$ be any other family of n sets of cardinality ℓ. Then $\mathscr{S}$ is Kruskal's cascade and the theorem which we are about to prove ([1, Theorem 3]) says that $|\Delta\mathscr{S}| \leqslant |\Delta\mathcal{A}|$.

Let m be the minimum value of $|W\backslash X|$ over all sets W of cardinality ℓ not in $\mathcal{A}$ and sets X in $\mathcal{A}$ with $W < X$. Since $\mathcal{A} \neq \mathscr{S}$ we have $m \geqslant 1$ and a particular choice W_0, X_0 for W, X with $m = |W_0\backslash X_0|$. We put

$$M = W_0\backslash X_0 \quad \text{and} \quad N = X_0\backslash W_0$$

so $M \cap N = \phi$ and $m = |M| = |N|$ and $M < N$. Then we put

$$\begin{aligned} \mathscr{B} &= \{X : X \in \mathcal{A}, M \cap X = \phi, N \subset X, (X\backslash N) \cup M \notin \mathcal{A}\}, \\ \mathscr{C} &= \{(X\backslash N) \cup M : X \in \mathscr{B}\}, \\ \mathscr{D} &= \mathscr{C} \cup (\mathcal{A}\backslash\mathscr{B}), \end{aligned}$$

so $|\mathcal{A}| = |\mathscr{D}|$. Roughly speaking, we form $\mathscr{D}$ from $\mathcal{A}$ by replacing N as a subset of a set X of $\mathcal{A}$ whenever possible by M. If this replacement is made in a set X, the resulting set $(X\backslash N) \cup M$ is $< X$. In particular, X_0 is replaced by W_0 which is $< X_0$. Thus $\mathscr{D}$ is nearer to $\mathscr{S}$ than is $\mathcal{A}$, and repetition of the process would gradually transform $\mathcal{A}$ into $\mathscr{S}$. Hence the theorem will follow if we can show that $|\Delta\mathscr{D}| \leqslant |\Delta\mathcal{A}|$.

Let $Z \in (\Delta\mathscr{D})\backslash(\Delta\mathcal{A})$. Then there is a set $Y \in \mathscr{D}\backslash\mathcal{A} = \mathscr{C}$ with $Z \subset Y$. By definition of $\mathscr{C}$, there is a set $X \in \mathscr{B}$ with $Y = (X\backslash N) \cup M$ and so $N \cap Z = \phi$. Suppose now that $M \not\subset Z$. Because we get Z by deleting one element from Y and $M \subset Y$, if $P = M \cap Z$ then $|P| = m - 1$. Moreover, if Q is the set obtained by deleting the smallest integer β in N from N, then $|Q| = m - 1$ and $P \cap Q = \phi$ and either $P = Q = \phi$ or $P < Q$. Since $X \in \mathscr{B}$ we have $M \cap X = \phi$ and $N \subset X$, so $P \cap X = \phi$ and $Q \subset X$. If

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$W_1 = (X \backslash Q) \cup P$, then $|W_1| = \ell$ and $|W_1 \backslash X| = |P| = m - 1$ and $W_1 \leqslant X \in \mathcal{O}\!\mathcal{l}$. Using the definition of m, we see that $W_1 \in \mathcal{O}\!\mathcal{l}$. But $Q \cap Z = N \cap Z = \phi$ and $M \cap Z = P$ so $Z \subset W_1$, giving the contradiction $Z \in \Delta\mathcal{O}\!\mathcal{l}$. Thus we have proved that $N \cap Z = \phi$ and $M \subset Z$.

We now write

$$\psi Z = (Z \backslash M) \cup N.$$

Then $\psi Z \subset X$ and so ψ defines an injection of $(\Delta\mathcal{D}) \backslash (\Delta\mathcal{O}\!\mathcal{l})$ into $\Delta\mathcal{O}\!\mathcal{l}$. Next we claim that $\psi Z \notin \Delta\mathcal{D}$. For if $\psi Z \subset V \in \mathcal{D}$, then $N \subset V$ so $V \notin \mathcal{C}$ but $V \in \mathcal{O}\!\mathcal{l} \backslash \mathcal{B}$. Since $N \subset V \notin B$, either (i) $M \cap V \neq \phi$ or (ii) $M \cap V = \phi$ and $(V \backslash N) \cup M \in \mathcal{O}\!\mathcal{l}$. In case (ii) we have $Z \subset (V \backslash N) \cup M$ so $Z \in \Delta\mathcal{O}\!\mathcal{l}$, a contradiction. Thus we must have case (i) $M \cap V \neq \phi$. Now there is an integer α such that $V = \alpha \cup \psi Z$ and $M \cap \psi Z = \phi$ so in fact $M \cap V = \alpha$. Let $P = M \backslash \alpha$ and $Q = N \backslash \beta$ as before. Then $Q \subset N \subset V$ and $P \cap V = \phi$ so we let $W_2 = (V \backslash Q) \cup P$. Now $|W_2| = \ell$ and $|W_2 \backslash V| = |P| = m - 1$ and $W_2 \leqslant V \in \mathcal{O}\!\mathcal{l}$ so using the definition of m again $W_2 \in \mathcal{O}\!\mathcal{l}$. Moreover, because $\alpha \in V$, we have $M \subset W_2$ and hence $Z \subset W_2$. However, this implies the contradiction $Z \in \Delta\mathcal{O}\!\mathcal{l}$. Thus we have proved that ψ injects $(\Delta\mathcal{D}) \backslash (\Delta\mathcal{O}\!\mathcal{l})$ into $(\Delta\mathcal{O}\!\mathcal{l}) \backslash (\Delta\mathcal{D})$ and hence $|\Delta\mathcal{D}| \leqslant |\Delta\mathcal{O}\!\mathcal{l}|$ as required.

Acknowledgment

The author thanks Jean Godfrey for kindly pointing out an error in the first version of this note.

Reference

1. D. E. Daykin, J. Godfrey, and A. J. W. Hilton, Existence theorems for Sperner families, *J. Combinatorial Theory* (*A*) **16** (1974), 245–251.