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Strongly Regular Designs and Coherent Configurations of Type $\left[\begin{smallmatrix} 3 & 2 \\ & 3 \end{smallmatrix}\right]$

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1. Introduction

The *strongly regular designs* (srd's) considered in this paper are a class of 1-designs which arise in the investigation of coherent configurations (cc's) of 'small type'. We refer to [7] for basics about cc's, where it is seen that the nontrivial types of cc's with two fibers each of rank at most three are $\left[\begin{smallmatrix} 2 & 2 \\ & 2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 2 & 2 \\ & 3 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 3 & 2 \\ & 3 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 3 & 3 \\ & 3 \end{smallmatrix}\right]$. These can be interpreted as classes of designs, the first type corresponding to symmetric designs, the second to quasi-symmetric designs introduced by Goethals and Seidel [6], and the third to srd's. (The last type will be considered elsewhere.) Srd's are involved, e.g., in connection with derived configurations of certain quasisymmetric designs [8], and in the *triality configurations* introduced in [9]. Here we consider them in some detail in their own right. We are interested in the connection with cc's and use the method of [7] to obtain parameter conditions. The discussion of srd's in this paper can be made independent of cc's by the reader willing to provide counting and matrix theoretic proofs of the parameter conditions of section 3.

Srd's, defined in Section 2, are $1\frac{1}{2}$-designs in the sense of Neumaier [11], and form a self-dual class. They include those partial geometries which are neither 2-designs nor dual 2-designs. We refer to Brouwer and van Lint [4] for recent work on partial geometries and strongly regular designs. An srd has a *point graph* and a *block graph* both of which are strongly regular. In section 3 we apply the same procedure to srd's as was applied to quasisymmetric designs in [7] to obtain parameter conditions and the connection with cc's. Our procedure gives a comprehensive list of parameter conditions and has the advantage of being a routine one (to the extent that most details can be safely omitted here) applicable to a variety of situations. The conditions amount to the existence of a feasible intersection algebra in the sense of [7] for the associated c.c. Moreover, we want the precise equivalence with srd's, e.g., in [9]. The Krein conditions and Calderbank's inequality [5; Theorem 1] are effective in eliminating candidates for parameters for srd's. In Section 4 the parameters for an srd are determined in terms of the number of points, the number of blocks, the block size, the block intersections sizes, and one of the point join sizes, and the case of an equal number of points and blocks is examined. There is a natural definition of *symmetric* srd, and because the srd's form a self-dual class, the questions considered in Section 5 of existence of polarities, dualities and absolute points arise. Some generalities about groups associated with cc's in Section 6 place the considerations of Section 5 in a more general setting (this will be useful, e.g. in [9]). The analogue for srd's of the Geothal–Seidel problem (Goethals and Seidel [6], Neumaier [12]) for quasi-symmetric designs, namely the determination of the srd's with given point graph, is considered in Section 7. In the final Section 8 we describe some examples, including the srd's which we know on at most 50 points and families of nonsymmetric srd's and self-polar srd's, which are not partial geometries, but we do not attempt to give a complete list of known examples.

The referee, to whom we are indebted for a number of remarks, points out that there is earlier related work in [1–3, 10, 11, 13–15]; more explicit references will be made at appropriate points in the text.

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2. Strongly Regular Designs

The incidence structures $(\mathscr{P}, \mathscr{B}, F)$ considered here will consist of a set $\mathscr{P}$ of *points*, a set $\mathscr{B}$ of *blocks*, and a set F of *flags*, such that $\mathscr{P} \cap \mathscr{B} = \varnothing$ and $F \subseteq \mathscr{P} \times \mathscr{B}$. A point x and a block y are *incident* if $(x, y) \in F$. The *dual* incidence structure is $(\mathscr{B}, \mathscr{P}, F^{\mathrm{T}})$, $F^{\mathrm{T}} = \{(y, x) | (x, y) \in F\}$. A *morphism* $\sigma:(\mathscr{P}, \mathscr{B}, F) \geqslant (\mathscr{P}', \mathscr{B}', F')$ of incidence structures is a map $\sigma: \mathscr{P} \cup \mathscr{B} \rightarrow \mathscr{P}' \cup \mathscr{B}'$ such that $\sigma(\mathscr{P}) \subseteq \mathscr{P}'$, $\sigma(\mathscr{B}) \subseteq \mathscr{B}'$ and $\sigma(F) \subseteq F'$. A *duality* is an isomorphism of $(\mathscr{P}, \mathscr{B}, F)$ onto its dual, and a *polarity* is a duality of period 2. A class of incidence structures is *self-dual* if it contains the dual of every one of its members. An incidence structure is *self-dual* or *self-polar* if it admits a duality or polarity, respectively.

We define a *strongly regular design* to be a finite incidence structure with n_1 points and n_2 blocks which satisfies the following conditions (1), (2) and (3) together with their duals (1′), (2′) and (3′).

(1) Each block is incident with S_1 points.

(2) Two distinct blocks are incident with either a_1 or b_1 points, $a_1 > b_1$, and both cases occur.

Given (2) we define the *block graph* Γ_2 to be the graph whose vertices are the blocks, two distinct blocks being adjacent if they are incident with a_1 common points.

(3) The number of blocks incident with a point x and adjacent to a block y is N_2 or P_2 according as x and y are incident or not.

Our convention is to use a subscript 1 in connection with a number of points and a subscript 2 in connection with a number of blocks. The duals (1′), (2′) and (3′) give rise to parameters $S_2, a_2 > b_2$, N_1 and P_1. Condition (3′) refers to the *point graph* Γ_1 defined by declaring two distinct points to be adjacent if they are incident with a_2 common blocks. We call an srd *symmetric* if the subscripts can be dropped from the parameters n_i, S_i, a_i, b_i, N_i, P_i (see sections 4 and 5 below).

The class of srd's is included in the class of $1\frac{1}{2}$-*designs* as defined by Neumaier [11], and the srd's with $a_1 = 1$, or equivalently with $a_2 = 1$, are precisely those partial geometries which are neither 2-designs nor dual 2-designs. The srd's form a self-dual class of designs, so results about srd's come in dual pairs of which we frequently only mention one member.

2.1. *Let* $\mathbf{C}$ *be the incidence matrix of an incidence structure (rows indexed by points, columns by blocks). Then* (1) *is equivalent to*
(4) $\mathbf{JC} = S_1\mathbf{J}$.
Assuming (1), (2) is equivalent to
(5) $\mathbf{C}^{\mathrm{T}}\mathbf{C} = (S_1 - b_1)\mathbf{I} + (a_1 - b_1)\mathbf{A}_2 + b_1\mathbf{J}$,
where $\mathbf{A}_2$ *is a* $(0, 1)$*-matrix with* 0 *diagonal, and then* $\mathbf{A}_2$ *is the adjacency matrix of* Γ_2.
Assuming (1) and (2), (3) is equivalent to
(6) $\mathbf{CA}_2 = (N_2 - P_2)\,\mathbf{C} + P_2\mathbf{J}$.
(Here as always in what follows, $\mathbf{J}$ *denotes the 'all 1' matrix of appropriate size).*

Now we can see that:

2.2. *Assuming (1), (1′), (2) and (3), the block graph* Γ_2 *is strongly regular.*

Namely, by (4) and (5), $\mathbf{A}_2$ has constant column sum and $\mathbf{A}_2^2 \in \langle(\mathbf{C}^{\mathrm{T}}\mathbf{C})^2, \mathbf{I}, \mathbf{A}_2, \mathbf{J}\rangle$. But by (4), (5) and (6), $(\mathbf{C}^{\mathrm{T}}\mathbf{C})^2 = \mathbf{C}^{\mathrm{T}}((S_1 - b_1)\mathbf{C} + (a_1 - b_1)\mathbf{CA}_2 + b_1\mathbf{CJ}) \in \mathbf{C}^{\mathrm{T}}\langle\mathbf{C}, \mathbf{J}\rangle \subseteq \langle\mathbf{I}, \mathbf{A}_2, \mathbf{J}\rangle$.

These facts show that an srd is precisely the same as a $1\frac{1}{2}$-design satisfying (2) and (2′). Let us now observe that

2.3. *If (1) and (2) and their duals (1′) and (2′) hold, then (3) and (3′) are equivalent.*

For we then have $(a_1 - b_1)\mathbf{CA}_2 = \mathbf{CC}^T\mathbf{C} - (S_1 - b_1)\mathbf{C} - b_1 S_2 \mathbf{J}$ and dually $(a_2 - b_2)\mathbf{A}_1\mathbf{C} = \mathbf{CC}^T\mathbf{C} - (S_2 - b_2)\mathbf{C} - b_2 S_1 \mathbf{J}$ whence $(a_1 - b_1)\mathbf{CA}_2 - (a_2 - b_2)\mathbf{A}_1\mathbf{C} \in \langle \mathbf{C}, \mathbf{J} \rangle$. Hence $\mathbf{CA}_2 \in \langle \mathbf{C}, \mathbf{J} \rangle$ if and only if $\mathbf{A}_1\mathbf{C} \in \langle \mathbf{C}, \mathbf{J} \rangle$.

Note that (3) is not a consequence of (1), (2), (1′) and (2′) as is seen by the example of lines and colines in $PG_d(q)$, $d \geq 4$, with inclusion as incidence. Designs satisfying (1), (2), (3), (1′) and (3′) are in fact just the *special balanced partial designs* (Bridges and Shrikhande [3]; see also Bose and Shrikhande [2]).

The parameter conditions implicit in the proofs of 2.2 and 2.3 are not recorded here because 3.2 in section 3 includes conditions equivalent to these.

From our discussion so far we know in particular that

2.4. *If* $\mathbf{C}$ *is the adjacency matrix of an* srd, *then (4), (5) and (6) hold together with their duals*
(4′) $\mathbf{CJ} = S_2\mathbf{J}$,
(5′) $\mathbf{CC}^T = (S_2 - b_2)\mathbf{I} + (a_2 - b_2)\mathbf{A}_2 + b_2\mathbf{J}$,
and
(6′) $\mathbf{A}_1\mathbf{C} = (N_1 - P_1)\mathbf{C} + P_1\mathbf{J}$,
where $\mathbf{A}_1$ *is the adjacency matrix of the point graph and* $\mathbf{A}_2$ *is the adjacency matrix of the block graph.*

For an srd, the usual parameters of Γ_i will be denoted by k_i, l_i, λ_i, μ_i, r_i, s_i, f_i and g_i ($i = 1, 2$); we refer to Brauer and van Lint [4] for the meaning of these parameters and for recent work on strongly regular graphs and partial geometries. As we will see in Section 4, the subscripts can be dropped for symmetric srd's.

It will be convenient to call an srd *primitive* if its point and block graphs are both primitive, i.e., connected with connected complements. In Section 9 the primitive srd's that we know with at most 50 points are described in 'Examples (A)' of Section 8, and infinite families of primitive srd's which are not partial geometries are described in 'Examples (B) and (C)', the srd's of 'Examples (C)' being self-dual. The sporadic partial geometry pg(6, 6, 2) (which is primitive in the present sense) is self-dual; see Smits and van Vroonhoven [15].

Concerning imprimitivity we remark first that an srd has repeated points, i.e., $S_2 = a_2$, if and only if its point graph Γ_1 is not connected, i.e., $\mu_1 = \emptyset$. In that case $(\mathscr{B}, B)$, $B = \{F(x) \mid x \in \mathscr{P}\}$, is a dual quasisymmetric design, so these srd's are obtained by repeating the points of dual quasi-symmetric designs in the obvious way. Second, an extensive class of symmetric srd's for which Γ_1 is not connected is provided by the *symmetric nets* for which we refer to Beth, Jungnickle and Lenz [1; 7.18/19 in ch. I, with further theory and constructions in ch. II, sect. 8, ch. VII, sect. 3 and ch. XII, sect. 6]. Also there is a family of partial geometries of this kind consisting of the points off and the lines not meeting a given coline in $PG_d(q)$, $d \geq 3$.

The complement of an srd with adjacency matrix $\mathbf{C}$ is an srd with adjacency matric $\mathbf{J} - \mathbf{C}$ and the same adjacency matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ for the point and block graphs respectively as the original srd. The parameters are $n_i' = n_i$, $S_i' = n_i - S_i$, $a_i' = n_i - 2S_i + a_i$, $b_i' = n_i - 2S_i + b_i$, $N_i' = k_i - P_i$ and $P_i' = k_i - N_i$ ($i = 1, 2$).

3. Coherent Configurations of Type $\left[\begin{smallmatrix} 3 & 2 \\ & 3 \end{smallmatrix}\right]$

The essential equivalence of srd's with cc's is given in 3.1 and 3.3. The terminology and notation for cc's is that if [7]. For srd's the main results of this section are the parameter conditions in 3.2 and 3.4.

We begin with the observation that given an srd $(\mathscr{P}, \mathscr{B}, F)$, the obvious relations on $X = \mathscr{P} \cup \mathscr{B}$ constitute a configuration which is coherent. Sepcifically we have the configuration $C = (X, (f_i)_{i \in \mathscr{I}})$, $\mathscr{I} = \{1, 2, \ldots, 10\}$, with

$f_1 = \operatorname{diag} \mathscr{P}^2$, f_2 and f_3 the subsets of $\mathscr{P}^2$ consisting of the (x, y) such that x and y are incident with a_2 and b_2 common blocks respectively,

f_4, f_5 and f_6 the dually defined relations on $\mathscr{B}$, and

$f_7 = F, f_8 = \mathscr{P} \times \mathscr{B} - F, f_9 = f_7^{\mathrm{T}}$ and $f_{10} = f_8^{\mathrm{T}}$. Then by 2.4 we have

3.1. *C is a cc of type* $[^3\ ^2_3]$.

We remark in passing that C is the coherent closure of $(X, \{F\})$ in the sense of [7]. It is interesting to think of distance-biregular graphs of diameter 4, which are equivalent to srd's with $b_1 = b_2 = 0$, from this point of view; for distance-biregular graphs we refer to Mohar and Shawe-Taylor [10].

In the notation of [7], $\Omega = \{1, 4\}$, $X_1 = \mathscr{P}$, $X_4 = \mathscr{B}$, $\mathscr{I}^{11} = \{1, 2, 3\}$, $\mathscr{I}^{44} = \{4, 5, 6\}$, $\mathscr{I}^{14} = \{7, 8\}$, $\mathscr{I}^{41} = \{9, 10\}$, and $7^* = 9$, $8^* = 10$. The point and block diagrams are $\Gamma_1 = (X_1, f_2)$ and $\Gamma_2 = (X_4, f_5)$. The intersection numbers of C are determined by the parameters of the srd. In particular $v_1 = 1$, $v_2 = k_1$, $v_3 = l_1$, $v_4 = 1$, $v_5 = k_2$, $v_6 = l_2$, $v_7 = S_2$, $v_8 = n_2 - S_2$, $v_9 = S_1$, $v_{10} = n_1 - S_1$, $p_{92}^{9} = N_1$, $p_{92}^{10} = P_1$, $p_{97}^{5} = a_1$, $p_{97}^{6} = b_1$, $p_{75}^{5} = N_2$, $p_{75}^{8} = P_2$, $p_{79}^{2} = a_2$ and $p_{79}^{3} = b_2$.

Now consider a cc of type $[^3\ ^2_3]$ with notation consistent with that of the preceding paragraph. Using the formulas following 3.1 to define k_i, l_i, $S_i, \ldots$, etc., apply the procedure of section 9B of [7] (applied there to cc's of type $[^2\ ^2_3]$). This involves working out the intersection matrices and the irreducible representations (of which there are two of degree 2 and two of degree 1, with multiplicities $1, f_1 = f_2$, g_1 and g_2, respectively. The details, which are routine, will be omitted here. The result is the list of parameter conditions 3.2 below (so these hold for srd's by 3.1). It is an easy consequence that $a_i > b_i$, $i = 1, 2$, and therefore (X_1, X_4, f_j), $j \in \mathscr{I}^{14}$, is an srd, giving 3.3 below. The point here is that we must have two distinct block intersection sizes and two distinct point join sizes in the design (X_1, X_4, f_j). Our convention that Γ_i for an srd is defined by a_i, $a_i > b_i$, $i = 1, 2$, corresponds to the condition $f_1 = f_2$.

3.2. *The parameters of an srd satisfy the following equations and their duals*:

(1) $f_1 = f_2$,

(2) $n_1 S_2 = n_2 S_1$,

(3) $P_1(n_1 - S_1) = (k_1 - N_1)S_1$,

(4) $a_2 k_1 = N_1 S_2$,

(5) $b_2 l_1 = (S_1 - N_1 - 1)S_2$,

(6) $N_1^2 + P_1(k_1 - N_1) = k_1 + \lambda_1 N_1 + \mu_1(S_1 - N_1 - 1)$,

(7) $N_1 P_1 + P_1(k_1 - P_1) = \lambda_1 P_1 + \mu_1(S_1 - P_1)$,

(8) $N_1 a_2 + P_1(S_2 - a_2) = S_2 + a_2\lambda_1 + b_2(k_1 - \lambda_1 - 1)$,

(9) $N_1 b_2 + P_1(S_2 - b_2) = a_2\mu_1 + b_2(k_1 - \mu_1)$,

(10) $S_1 + a_1 N_2 + b_1(S_2 - N_2 - 1) = S_2 + a_2 N_1 + b_2(S_1 - N_1 - 1)$,

(11) $a_1 P_2 + b_1(S_2 - P_2) = a_2 P_1 + b_2(S_1 - P_1)$,

(12) $P_1(k_2 - N_2) = P_2(k_1 - N_1)$,

(13) $S_1 + a_1 k_2 + b_1 l_2 = S_2 + a_2 k_1 + b_2 l_1 = S_1 S_2$,

(14) $S_1 + a_1 r_2 - b_1(r_2 + 1) = S_2 + a_2 r_1 - b_2(r_1 + 1)$,

(15) $S_1 + a_1 S_2 - b_1(s_2 + 1) = S_2 + a_2 S_1 - b_2(s_1 + 1) = 0$.

3.3. *Every cc of type* $[^3\ ^2_3]$ *arises from an srd by the above construction.*

An srd and its complement give rise to the same cc. We usually identify a complementary pair of srd's with the corresponding cc.

An action of type $[^3\ ^2_3]$ of a group G on a finite set X affords a cc of type $[^3\ ^2_3]$ and hence gives rise to an srd. This is the *group case.*

To the list of conditions in 3.2 we add the following.

3.4. *For an srd*:
(*16*) $r_1 = N_1 - P_1$
(*17*) $b_2(n_1 - 1) < S_2(S_1 - 1)$
(*18*) $a_2 n_1 \geqslant S_1 S_2$ *with equality if and only if* $\mu_1 = k_1$, *i.e., the srd is group divisible,*
(*19*) $n_2 L \leqslant n_1(n_1 - 1)(S_1 - a_1)(S_1 - b_1)$, *where* $L = a_1 b_1 n_1^2 - ((a_1 + b_1 - 1)S_1^2 + a_1 b_1)n_1 + S_1^2(S_1^2 - 2S_1 + a_1 + b_1)$.

PROOF. (16), (17) and (18) can be derived from 3.2.
To prove (16) put $\alpha = N_1 - P_1$, then by (6)

$$(P_1 + \alpha)^2 + P_1(k_1 - P_1 - \alpha) = (\lambda_1 - \mu_1)P_1 + (\lambda_1 - \mu_1)\alpha + \mu_1 S_1 + k_1 - \mu_1$$

and by (7) $(P_1 + \alpha)P_1 + P_1(k_1 - P_1) = (\lambda_1 - \mu_1)P_1 + \mu_1 S_1$ from which follows $\alpha^2 - (\lambda_1 - \mu_1)\alpha - (k_1 - \mu_1) = 0$ and hence $\alpha = r_1$ or s_1. If $\alpha = s_1$, then by (15), $(a_2 - b_2)(N_1 - P_1) = -(S_2 - b_2)$. But by (8), $(N_1 - P_1)a_2 + P_1 S_2 = S_2 - b_2 + (a_2 - b_2)\mu_1 + b_2 k_1$, and by (9), $(N_1 - P_1)b_2 + P_1 S_2 = (a_2 - b_2)\mu_1 + b_2 k_1$, so $(a_2 - b_2)(N_1 - P_1) = S_2 - b_2 + (a_2 - b_2)(\lambda_1 - \mu_1)$, and hence $s_1 = (a_2 - b_2)r_1$, which is impossible. This proves (16).

By (13), $(a_2 - b_2)k_1 = S_2(S_1 - 1) - b_2(n - 1)$ from which the inequality (17) follows.

By (16), (4) and (3), $r_1 = N_1 - P_1 = [(a_2 n_1 - S_1 S_2)k_1 / S_1(S_2 - a_2)]$, which implies (18).

(19) is equivalent to Calderbank's inequality [5; Theorem 1] in the context of srd's.

4. Analysis of Parameters

By (4), (10) we have

$$S_1 + (a_1 - b_1)\frac{a_1 k_2}{S_1} + b_1(S_2 - 1) = S_2 + (a_2 - b_2)\frac{a_2 k_1}{S_2} + b_2(S_1 - 1)$$

which by (13) becomes

(20) $a_2[S_2(S_1 - 1) - b_2(n_1 - 1)]S_1$
$= a_1[(S_1(S_2 - 1) - b_1(n_2 - 1)]S_2 + [S_1 - S_2 + b_1(S_2 - 1) - b_2(S_1 - 1)]S_1 S_2.$

Now we see that

4.1. *The parameters of an srd are determined by* n_1, n_2, S_1, a_1, b_1 *and one of* a_2, b_2.

More precisely, the feasible parameter sets for primitive srd's (up to duality and complements) can be generated as follows:

Start with integers $n_1, n_2, S_1 a_1, b_1$ and a_2 such that $n_1 \leqslant n_2$, $S_1 \leqslant n_1/2$, $0 \leqslant b_1 < a_1 < S_1$ and $0 \leqslant b_2$. Put $S_2 = n_2 S_1 / n_2$, and determine a_2 from (20). Determine the rest of the parameters from the following formulas and their duals.

$$k_1 = \frac{1}{a_2 - b_2}[S_2(S_1 - 1) - b_2(n_1 - 1)], \qquad N_1 = \frac{a_2 k_1}{S_2}, \qquad P_1 = \frac{(k_1 - N_1)S_1}{n_1 - S_1},$$

$$r_1 = N_1 - P_1, \qquad s_1 = -\frac{S_2 - b_2}{a_2 - b_2}, \qquad \mu_1 = k_1 + r_1 s_1, \qquad \lambda_1 = \mu_1 + r_1 + s_1,$$

$$f_1 = \frac{(n_1 - 1)(-s_1) - k_1}{r_1 - s_1}.$$

The feasibility conditions are (a) integrality of b_2, S_2, k_i, N_i, P_i and f_1; (b) the inequalities $a_2 > b_2$, $a_1 n_1 > S_1^2$, $b_1(n_2 - 1) < S_1(S_2 - 1)$, $\lambda_1 \geqslant 0$, $\tilde{\lambda}_1 = n - 2k + \mu_1 - 2 \geqslant 0$, $\mu_1 > 0$, $\mu_1 = n - 2k + \lambda_1 > 0$ and their duals, (c) Calderbank's inequality (19) and its dual, and (d) the Krein conditions applied to the strongly regular parameters.

Calderbank's inequality (19) gives a bound on n_2 in terms of n_1, S_1, a_1 and b_1 in case $L > 0$. We know no such practical bound in case $L \leqslant 0$.

Now consider the case on which $n_1 = n_2$, so that $S_1 = S_2$, and denote these numbers by n and S, respectively. Then (20) becomes

(21) $(a_2 - a_1 + b_2 - b_1)S(S - 1) = (a_2 b_2 - a_1 b_1)(n - 1)$.

Assuming n, S, a_1, b_1 and b_2 are given, we can determine a_2 from (A) and the rest of the parameters as before. We remark without proof that when $n_1 = n_2$, the intersection numbers are interlaced, namely, if $a_2 \geqslant a_1$, then $a_2 \geqslant a_1 \geqslant b_2 \geqslant b_1$.

It is an easy consequence of 3.2 that an srd is symmetric as defined in Section 2 if and only if the subscripts can be dropped from all the parameters. An initial examination of feasible parameter sets suggested that an srd with an equal number of points and blocks must be symmetric. But it turns out that this cannot be a consequence of our parameter conditions. There are exactly the three sets of feasible srd parameters given in Table 1 with $n_1 = n_2 \leqslant 500$ which are not symmetric.

TABLE 1.
The three sets of feasible srd parameters with $n = n_2 < 500$ which are not symmetric.

n	S	a_i	b_i	N_i	P_i	k_i	l_i	μ_i	λ_i	r_i	s_i	f	g
351	126	63	42	85	70	210	140	129	120	15	−6	90	260
		51	36	25	14	50	300	13	6	11	−4		
352	144	64	56	130	126	312	39	276	280	4	−8	208	143
		60	48	52	45	117	234	36	40	7	−11		
496	216	104	90	196	189	441	54	392	392	7	−7	216	279
		96	76	65	54	135	360	38	36	11	−9		

We have no information about the existence of strongly regular graphs for any of the parameter sets in Table 1 nor do we know any srd with $m_1 = m_2$ which is not symmetric.

5. Symmetric Strongly Regular Designs

We observe that

5.1. *If* $n_1 = n_2$, *then any one of*
(i) $a_1 = a_2$,
(ii) $b_1 = b_2$, *or*
(iii) $k_1 = k_2$ *and* $r_1 = r_2$
implies that the srd is symmetric

Proof. Drop subscripts as appropriate in the respective cases. Assume (i), then by (21)

$$(b_2 - b_1)[S(S - 1) - a(b_2 - b_1)](n - 1) = 0,$$

so $b_1 = b_2$ by (18). The rest of the required equalities follow easily. The sufficiency of (ii) is seen similarly. Assume (iii), then by (4), (3) and (16), $N_1 = a_2 k/S$, $P_1 = k(S - a_2)/(n - S)$, and dually, and $r = N_1 - P_1 = N_2 - P_2$. It follows that $a_2 = a_1$.

The feasible parameters of symmetric srd's can be generated by the formulas of Section 4 without the subscripts (of course (20) is trivial here). For symmetric srd's (19) becomes $abn^2 - [(a + b)S^2 - (a + b)S + 2ab]n + (S^2 - S + a)(S^2 - S + b) \leqslant 0$, which is a consequence of the other feasibility conditions.

We will have occasion, e.g., in [9], to consider symmetric srd's in which the blocks are cliques in Γ_1, i.e., $b = 0$. These are the *thin near octagons*; see Shad [14]. In this case we start with integers n, m and a such that $2 \leqslant m \leqslant \sqrt{n}$, $1 \leqslant a \leqslant n/m^2$, and put:

$$S = am, \quad K = m(am - 1), \quad N = S - 1,$$

$$P = \frac{am(am - 1)(m - 1)}{n - am}, \quad r = N - P = \frac{am(am - 1)(n - am^2)}{n - am},$$

$$s = -m, \quad \mu = mP, \quad f = \frac{(n + am - 2)(n - am)}{a(am - 1)(n - am^2) + (n - am)}.$$

Finally in this section we remark on the absolute points of a polarity, and more generally of an isomorphism of the point graph onto the block graph.

A polarity amounts to an arrangement of the points and blocks so that $\mathbf{A}_1 = \mathbf{A}_2$ and $\mathbf{C}$ is symmetric. Then the eigenvalues of $\mathbf{C}$ are S, $\theta^{1/2}$, $-\theta^{1/2}$ and 0, with the respective multiplicities 1, $\alpha, f - \alpha$ and g, where $\theta = (a - b)(r - s)$ and $0 \leqslant \alpha \leqslant f$. The *absolute points* of the polarity δ are the points x such that x and $\delta(x)$ are incident. With $\mathbf{C}$ as above, their number is equal to the trace of $\mathbf{C}$, and we have

5.2. *The number of absolute points of a polarity is* $S + (f - 2\alpha)\theta^{1/2}$, *where* $\theta = (a - b)(r + m)$ *and* α *is the multiplicity of* $\theta^{1/2}$ *as an eigenvalue of* $\mathbf{C}$. *In particular, therefore, a self-polar srd with odd* f *must have* θ *a square.*

More generally we can define the *absolute* (or *auto-conjugate*) *points* of an isomorphism of the point graph onto the block graph as the points which are incident with their images, so that in case σ is included by a polarity or duality these are the usual absolute points. An isomorphism σ of Γ_1 onto Γ_2 amounts to an arrangement of the points and the blocks so that $\mathbf{A}_1 = \mathbf{A}_2$, and then since the srd is symmetric by 5.1, the incidence matrix $\mathbf{C}$ must be normal. This situation will arise in [9].

6. The Chromatic Group

This section is not essential for the rest of the present paper, but it places Section 5 above in a more general setting and will be useful in [9]. By an *isochromism* of one coherent algebra onto another we mean an algebra isomorphism which preserves hadamard multiplication and maps the all 1 matrix onto the all 1 matrix. By an *isochromism* of a cc C onto a cc D we mean an isochromism of the adjacency algebra A of C onto the adjacency algebra B of D. Thus the isochromisms σ of C onto D are the isomorphisms of A onto B which map the standard basis $(A_i)_{i \in \mathscr{I}}$ of A onto the standard basis $(B_j)_{j \in \mathscr{J}}$ of B, i.e., with the property that
(i) $\sigma(A_i) = B_{\pi(i)}, \quad i \in \mathscr{I}$,
where $\pi : \mathscr{I} \to \mathscr{J}$ is a bijection such that
(ii) $p_{ij}^k = q_{\pi(i)\pi(j)}^{\pi(k)}, \quad i, j, k \leqslant \mathscr{I}$,
(p_{ij}^k) and (q_{ab}^c) being the intersection numbers of C and D. Conversely, given a bijection $\pi : \mathscr{I} \to \mathscr{J}$ satisfying (ii), (i) defines an isochromism of C onto D; in this sense we can speak of isochromic coherent algebras or configurations as those having the same intersection numbers. Our terminology seems reasonably natural in terms of the interpretation of $\mathscr{I}$ and $\mathscr{J}$ as sets of colors (see e.g., [7]).

We call the group of autochromisms of C the *chormatic group* of C, and denote it by $\operatorname{Chr} C$. Observe that $\operatorname{Chr} C \approx \{\pi \in \Sigma_{\mathscr{I}} \mid p_{ij}^k = p_{\pi(i)\pi(j)}^{\pi(k)}\}$ and that an autochromism induces a permutation of the indexing set $\Omega \subseteq \mathscr{I}$ of the standard partition. We have the group $\operatorname{Aut}_1 C$ of automorphisms of C which fix the fibres, the group $\operatorname{Chr}_1 C$ of autochromisms of C

which induce the trivial permutation of Ω, and the following commutative diagram of group homorphisms with exact rows and columns

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{Aut}_0 C & = & \mathrm{Aut}_0 C & & \\ & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Aut}_1 C & \to & \mathrm{Aut} C & \to & \Sigma_\Omega \\ & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Chr}_1 C & \to & \mathrm{Chr} C & \to & \Sigma_\Omega \end{array}$$

Certainly an element of Aut$C$ which is not in $\mathrm{Aut}_1 C$ induces an automorphism of $\mathrm{Aut}_0 C$ whichis not inner.

An srd $\mathcal{D}$ can be identified with the corresponding cc C, and then dualities of $\mathcal{D}$ are the automorphisms of C which interchange the fibers, and polarities are the automorphisms of period 2 with this property. Clearly $\mathrm{Aut}_1 C = \mathrm{Aut}_0 C$, so $\mathcal{D}$ is self dual if and only if $\mathrm{Aut}C/\mathrm{Aut}_0 C \approx \mathbb{Z}_2$. For $\mathcal{D}$ to be symmetric means the existence of an autochromism of C which interchanges the elements of Ω. Since $\mathrm{Chr}_1 C$ is the trivial group, this means $\mathrm{Chr}C \approx \mathbb{Z}_2$. An isomorphism of Γ_1 onto Γ_2 is an isomorphism between the corresponding fibers of C. For Γ_1 and Γ_2 to have the same parameters means the existence of an isochromism between the fibers of C.

7. Strongly Regular Designs with Given Point Graph

Analogous to the problem initiated by Goethals and Seidel [6] for quasisymmetric designs (see also Neumaier [12]) we have the problem of determining the srd's with a given strongly regular graph Γ_1 as point graph. Examples are given in 'Examples (A)' of Section 8 of nonisomorphic rds's with the same point graph. Furthermore the parameter conditions show that it is relatively rare for a strongly regular graph to be the point graph of an srd (see also 7.2 below).

We remark that if the parameters of Γ_1 together with the block size S_1 and the number n_2 of blocks are specified, then the remaining srd parameters are determined. Namely by Section 3 we have $S_2 = n_2 S_1/n_1$, $P_1 = (k_1 - r_1)S_1/n_1$, $N_1 = P_1 + r_1$, $a_2 = N_1 S_2/k_1$, and $b_2 = (S_1 - N_1 - 1)S_2/l_1$. According to the dual of 5.2, the rest of the parameters are determined. Unfortunately we do not have a practical bound for n_2 in terms of the parameters of Γ_1 and the block size. Observe that the parameters of an symmetric srd are determined by the parameters of the point graph and the block size; for instance, the strongly regular parameters can be given in terms of k, r, s, then specifying S determines the rest starting with

$$P = \frac{(k - r)}{n} S = \frac{(k + rs)(k - r)}{(k + r)(k + s)} S.$$

Let us now determine the srd's with the $d \times d$ lattice graph $L_2(d)$ point graph. A block of such an srd is a regular subgraph of $L_2(d)$ with valency N_1 such that each point outside it is adjacent to P_1 of its points. We see that a connected component of a block must be either a clique of size $N_1 + 1$ or an $L_2(e)$, $e = N_1/2 + 1$. If a clique of size $< d$ is a component, then $P_1 \geqslant N_1 + 1$, which is impossible, hence there are no such components. If a clique of size d is a component of a block, then the blocks are the cliques of size d and the srd is the grid. The remaining possibility is that all the components are $L_2(e)$'s, and clearly $e < d$ in this case. But there at least two components and $P_1 \geqslant 2[(N_1/2) + 1]$, which is impossible. Thus:

7.1. *The only srd with point graph* $L_2(d)$, $d \geqslant 2$, *is the* $d \times d$ *grid.*

For the same question for the complement of $L_2(d)$, the components of a block viewed as subgraphs of $L_2(d)$ are again either cliques or $L_2(e)$s. A somewhat tedious but straight forward analysis of the case gives

7.2. *If an srd has the complement of* $L_2(d)$ *as point graph, then either* $d = 2$ *or* 4. *If* $d = 4$, *then there are* 18 *blocks of size* 8 *and these are the subgraphs of* $L_2(4)$ *having two components isomorphic with* $L_2(2)$.

8. Examples

We describe here some of the examples referred to at various points in the text.

Examples (A). Up to complements and duals we know just 13 examples of primitive srd's with $n_1 \leqslant n_2$ and $n_1 \leqslant 50$. Of these, all but the two partial geometries pg(5, 7, 3) belong to the group case, coming from the group actions given in Table 2. Concerning the pg(5, 7, 3)'s see Brouwer and van Lint [4].

Table 2.
Group case examples $n_1 \leqslant n_2$, $n_1 \leqslant 50$.

Group	Orbit lengths and type
$Sp_4(2) \approx \Sigma_6$	$\begin{matrix} 15 & 15 \end{matrix}$ $\begin{bmatrix} 3 & 3 \\ & 2 \end{bmatrix}$
$O_6^-(2) \approx U_4(2) \approx Sp_4(3) \approx O_5(3)$	$\begin{matrix} 27 & 36 & 45 & 40 & 40 \end{matrix}$ $\begin{bmatrix} 3 & 2 & 2 & 1 & 1 \\ & 3 & 2 & 2 & 2 \\ & & 3 & 2 & 2 \\ & & & 3 & 2 \\ & & & & 3 \end{bmatrix}$
$L_4(2) \approx A_8$	$\begin{matrix} 28 & 35 \end{matrix}$ $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$
$U_3(5)$	$\begin{matrix} 50 & 50 & 50 \end{matrix}$ $\begin{bmatrix} 3 & 2 & 2 \\ & 3 & 2 \\ & & 3 \end{bmatrix}$

Concerning the examples of Table 2 see Neumaier [13]. The action of $U_3(5)$ is on two of the three classes of A_7's. Since these are automorphic, there is just one srd up to isomorphism. The configuration afforded by the action on all three classes is a *triality configuration* as defined in [9].

The parameters are given in Table 3.

TABLE 3.
Known primitive srd's, $n_1 \leqslant n_2$, $n_1 \leqslant 50$

n	S	a	b	N	P	k	l	λ	μ	r	s	f	g	
15	3	1	0	2	1	6	8	1	3	1	−3	9	5	GQ(2, 2)
27	12	6	4	5	4	10	16	1	5	1	−5	20	6	
36	16	8	6	10	8	20	15	10	12	2	−4	20	15	
27	3	1	0	2	1	10	16	1	5	1	−5	20	6	GQ(2, 4)
45	5	1	0	4	1	12	32	3	3	3	−3	20	24	
28	12	6	4	7	6	15	12	6	10	1	−5	20	7	
35	15	7	5	8	6	16	18	6	8	2	−4	20	14	
36	9	3	0	6	3	15	20	6	6	3	−3	15	20	TWO
40	10	4	1	9	6	27	12	18	18	3	−3	25	24	
36	12	6	3	8	6	20	15	10	12	2	−4	20	15	
45	15	6	3	6	3	12	32	3	3	3	−3	20	24	
40	4	1	0	3	1	12	27	2	4	2	−4	24	15	GQ(3, 3)
40	16	7	4	4	4	12	27	2	4	2	−4	24	15	TWO
45	18	9	6	12	12	32	12	22	24	2	−4	24	20	
45	5	1	0	4	3	28	16	15	21	1	−7	35	9	pg(5, 7, 3)
63	7	1	0	6	3	30	32	13	15	3	−5	35	27	TWO
50	15	5	0	14	12	42	7	35	36	2	−3	21	28	

EXAMPLES (B). One of the two examples with $n_1 = 40$ and $n_2 = 45$ can be viewed as the dual of the srd afforded by the action of $U_4(2)$ on the 45 absolute points and the 40 ordinary points. More generally, the action of $U_m(2)$, $m \geqslant 4$, on the absolute and ordinary points of unitary $PG_{m-1}(4)$ provides a generic family of primitive srd's which are not partial geometries, and which can realized by taking the absolute points as the points, the ordinary points as the blocks, and orthogonality as incidence. Equivalently we can take the non-degenerate hyperplanes as the blocks, with incidence as in $PG_{m-1}(2)$. We give the parameters.

The numbers of absolute points and ordinary points are respectively $u_m = (2^m - (-1)^m)(2^{m-2} - (-1)^{m-1})/3$ and $v_m = 2^{m-1}(2^m - (-1)^m)/3$. The parameters for the orthogonality graphs on the absolute the ordinary points are as given in Table 4. These

TABLE 4.
Parameters of the orthogonality graphs.

		Absolute points	Ordinary points
n		u_m	v_m
k		$4u_{m-2}$	v_{m-1}
l		2^{2m-3}	$3v_{m-1} - 1$
λ		$4u_{m-2} - (2^{2m-5} + 1)$	v_{m-2}
μ		u_{m-2}	$4v_{m-3}$
r	m even	$2^{m-2} - 1$	2^{m-3}
	m odd	$2^{m-3} - 1$	2^{m-2}
s	m even	$-(2^{m-3} + 1)$	-2^{m-2}
	m odd	$-(2^{m-2} + 1)$	-2^{m-3}
f	m even	$\frac{4}{9}(2^{m-1} - 1)(2^{m-3} + 1)$	$\frac{8}{9}(2^{m-1} + 1)(2^{m-2} - 1)$
	m odd	$\frac{8}{9}(2^{m-1} - 1)(2^{m-2} + 1)$	$\frac{1}{9}(2^m + 1)(2^{m-1} - 1)$
g	m even	$\frac{8}{9}(2^{m-1} + 1)(2^{m-2} - 1)$	$\frac{1}{9}(2^m - 1)(2^{m-1} + 1)$
	m odd	$\frac{4}{9}(2^m + 1)(2^{m-3} - 1)$	$\frac{8}{9}(2^{m-1} - 1)(2^{m-2} + 1)$

TABLE 5.
Remaining parameters of srd.

		$i = 1$	$i = 2$
S_i		u_{m-1}	$4v_{m-2}$
a_i		u_{m-2}	v_{m-2}
b_i		$(2^{2m-5} - (-1)^m 2^{m-2} - 1)/3$	$16v_{m-4}$
N_i	m even	2^{m-5}	$2^{m-2}(2^{m-3} + 1)$
	m odd	$4u_{m-3}$	$4v_{m-3} - 1$
P_i	m even	$2^{m-3}(2^{m-2} - 1)$	$2^{m-3}(2^{m-2} - 1)$
	m odd	u_{m-2}	v_{m-2}

graphs are respectively the complement of the point graph and the block graph if m is even and the point graph and complement of the block graph if m is odd. The remaining parameters of the srd are given in Table 5.

We can easily reconstruct unitary $PG_{m-1}(4)$ from the above srd. This suggests consideration of srd's for which the analogous construction produces a self-polar design. This is done in [8].

EXAMPLES (C). Two families of self-polar srd's are obtained by taking as points and blocks the points in $PG_{2d-1}(3)$, $d \geqslant 3$, represented by vectors x with $Q_\varepsilon(x) = 1$ and -1, respectively, where $Q_\varepsilon(x)$ is a nondegerate quadratic form of maximal or nonmaximal index according as $\varepsilon = 1$ or -1. Incidence is taken to be orthogonality. The case $d = 2, \varepsilon = -1$ is isomorphic with the generalized quadrangle G(2, 2) corresponding to the isomorphism $O_4^-(3) \approx Sp(2)'$.

The orthogonality graphs on the points $\langle x \rangle$, with $Q_\varepsilon(x) = 1$ and -1, respectively, are isomorphic strongly regular graphs with the following parameters:

$$n_\varepsilon = \tfrac{1}{2}3^{d-1}(3^d - \varepsilon), \quad k_\varepsilon = \tfrac{1}{2}3^{d-1}(3^{d-1} - \varepsilon), \quad l_\varepsilon = 3^{2d-2} - 1,$$

$$\lambda_\varepsilon = \tfrac{1}{2}3^{d-2}(3^{d-1} + \varepsilon), \quad \mu_\varepsilon = \tfrac{1}{2}3^{d-1}(3^{d-2} - \varepsilon),$$

$$r_+ = -s_- = 3^{d-1}, \quad r_- = -s_+ = 3^{d-2},$$

$$f_+ = \tfrac{1}{8}(3^{d-1} - 1)(3^d - 1), \quad f_- = \tfrac{9}{8}(3^{2d-2} - 1),$$

$$g_+ = \tfrac{9}{8}(3^{2d-2} - 1), \quad g_- = \tfrac{1}{8}(3^{d-1} + 1)(3^d + 1).$$

In the case $\varepsilon = +1$, the point and block graphs are the complements of the orthogonality graphs, and the remaining parameters of the srd are

$$S_+ = \tfrac{1}{2}3^{d-1}(3^{d-1} + 1),$$

$$a_+ = \tfrac{1}{2}3^{d-1}(3^{d-2} + 1), \quad b_+ = \tfrac{1}{2}3^{d-2}(3^{d-1} + 1),$$

$$N_+ = (3^{d-1} - 1)(3^{d-2} + 1), \quad P_+ = 3^{d-2}(3^{d-1} + 1).$$

In the case $\varepsilon = -1$, the point graphs are the orthogonality graphs, and

$$S_- = \tfrac{1}{2}3^{d-1}(3^{d-1} - 1),$$

$$a_- = \tfrac{1}{2}3^{d-2}(3^{d-1} - 1), \quad b_- = \tfrac{1}{2}3^{d-1}(3^{d-2} - 1),$$

$$N_- = \tfrac{1}{2}3^{d-2}(3^{d-1} + 1), \quad P_- = \tfrac{1}{2}3^{d-2}(3^{d-1} - 1).$$

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