On Structural Instability of Normal Forms of Affine Control Systems Subject to Static State Feedback

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ABSTRACT

Smooth affine control systems acted on by the feedback group are dealt with, from the viewpoint of the existence of structurally stable normal forms. Detailed analysis of the action of the approximate feedback group of order 1 has revealed that in most cases an obstacle to structural stability appears just in the action of this group.

1. INTRODUCTION

The problem of classifying nonlinear control systems by certain nice normal forms is of considerable importance, both from theoretical and from the applied point of view. For linear systems a celebrated classification by orbits of the state feedback group was done by Brunovsky [1, 2]. The orbits have been described by simple normal forms called the Brunovsky canonical forms. Among the forms there exists one, attributed to the open dense orbit, which is clearly structurally stable [3].

As might be expected, the classification of nonlinear systems by feedback is very hard, and the existing results refer to a few particular cases [4–6]. In this paper we study a particular aspect of the problem, namely the existence of structurally stable local normal forms of nonlinear systems subject to feedback. A similar approach has already been adopted in [6]; however, here we are able to improve and revise some results stated in that reference.

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To be more specific, we shall deal with *smooth affine control systems* of the form

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \qquad m \leqslant n, \quad f(0) = 0, \quad (1.1)$$

defined on $\mathbb{R}^n$ or an open neighborhood of $0 \in \mathbb{R}^n$. Vector fields $f, g_1, \ldots, g_m$ are assumed to be smooth, i.e. of class C^∞. Smooth affine systems will be identified with tuples $\sigma = (f, g) \in \Sigma$, where $\Sigma \cong C^\infty(\mathbb{R}^n, \mathbb{R}^{(m+1)n})$, along with the C^∞ Whitney topology [7].

Two affine systems $\sigma = (f, g)$, $\sigma' = (f', g') \in \Sigma$ are said to be *feedback equivalent* (locally, around $0 \in \mathbb{R}^n$), if there exists an open neighborhood U of $0 \in \mathbb{R}^n$, and a triple (φ, η, ψ) of smooth maps defined on U, where $\varphi \in \mathrm{Diff}(U)$, $\varphi(0) = 0$, represents a local change of coordinates in the state space, $\eta \in C^\infty(U, \mathbb{R}^n)$, $\eta(0) = 0$, denotes the proper feedback, and $\psi \in C^\infty(U, \mathrm{GL}_m(\mathbb{R}))$ is a state-dependent linear change of input coordinates, such that

$$f' = \left(\frac{\partial \varphi}{\partial x}\right)^{-1} (f \circ \varphi + g \circ \varphi \cdot \eta), \qquad g' = \left(\frac{\partial \varphi}{\partial x}\right)^{-1} g \circ \varphi \cdot \psi.$$

Within singularity theory the above equivalence can be formalized using the concept of germs of maps [7, 8]. Thus, the feedback group G for affine control systems consists of triples of germs (φ, η, ψ) at $0 \in \mathbb{R}^n$ such that φ is the germ of local diffeomorphisms preserving $0 \in \mathbb{R}^n$, $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $\eta : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, and $\psi : (\mathbb{R}^n, 0) \to \mathrm{GL}_m(\mathbb{R})$. The group multiplication is given by the formula

$$(\varphi', \eta', \psi')(\varphi, \eta, \psi) = (\varphi \circ \varphi', \eta \circ \varphi' + \psi \circ \varphi' \cdot \eta, \psi \circ \varphi' \cdot \psi'). \qquad (1.2)$$

G acts on germs of affine systems according to the following expression:

$$\gamma : (\varphi, \eta, \psi)(f, g) \mapsto \left(\left(\frac{\partial \varphi}{\partial x}\right)^{-1} (f \circ \varphi + g \circ \varphi \cdot \eta), \left(\frac{\partial \varphi}{\partial x}\right)^{-1} g \circ \varphi \cdot \psi \right), \quad (1.3)$$

which should be read in terms of germs at $0 \in \mathbb{R}^n$.

Having introduced the action (1.3), we call a "nice" affine system $\sigma \in \Sigma$ a *structurally stable local normal form* of affine systems if there exists an open neighborhood V of σ (w.r.t. the Whitney topology) such that, given any $\sigma' \in V$, the germs of σ and σ' lie on the same orbit of G.

This paper is composed as follows. In Section 2 we introduce a basic algebraic tool which simplifies considerably the analysis of the equivalence

(1.3), viz. a collection of approximate feedback groups. Then, in Section 3, we analyze in detail the action of the approximate group of order 1 and establish a range of dimensions (m, n) within which an obstacle to the existence of structurally stable normal forms comes from the action of this group. Section 4 contains conclusions.

2. APPROXIMATE FEEDBACK GROUPS

Given a smooth map $f: \mathbb{R}^n \to \mathbb{R}^n$, by the *k-jet* ($k \geqslant 0$) of f at $0 \in \mathbb{R}^n$ we mean the collection of Taylor coefficients of f at 0 up to order k, i.e.

$$j^k f(0) = \left(f(0), \frac{\partial f}{\partial x}(0), \dots, \frac{\partial^k f}{\partial x^k}(0) \right).$$

From now on all the jets of maps or germs will be considered at $0 \in \mathbb{R}^n$, so "0" will be omitted everywhere.

With the feedback group G defined in the previous section we associate a family of *approximate feedback groups* G_k, $k \geqslant 0$ [6, 9]. The group G_k consists of triples $(j^{k+1}\varphi, j^k\eta, j^k\psi)$ of jets for $(\varphi, \eta, \psi) \in G$. The group multiplication in G_k is inherited naturally from (1.2), i.e.

$$\begin{aligned} &(j^{k+1}\varphi', j^k\eta', j^k\psi')(j^{k+1}\varphi, j^k\eta, j^k\psi) \\ &\quad = (j^{k+1}\varphi \circ \varphi', j^k(\eta \circ \varphi' + \psi \circ \varphi' \cdot \eta), j^k\psi \circ \varphi' \cdot \psi'). \end{aligned} \tag{2.1}$$

Let $\Sigma_k = \{(j^k f, j^k g) | (f, g) \in \Sigma\}$ denote the set of k-jets of affine systems. Then G_k acts on Σ_k in accordance with the following formula:

$$\begin{aligned} &\gamma_k: (j^{k+1}\varphi, j^k\eta, j^k\psi)(j^k f, j^k g) \\ &\quad \mapsto \left(j^k \left(\frac{\partial \varphi}{\partial x} \right)^{-1} (f \circ \varphi + g \circ \varphi \cdot \eta), j^k \left(\frac{\partial \varphi}{\partial x} \right)^{-1} g \circ \varphi \cdot \psi \right). \end{aligned} \tag{2.2}$$

It can be proved that G_k is a Lie group acting on the analytic manifold Σ_k; hence (2.2) is indeed much easier to handle than (1.3). On the other hand, the following lemma from [6] establishes an important connection between openness of orbits of γ_k and the existence of structurally stable normal forms.

LEMMA 2.1. *Let $\sigma \in \Sigma$ be a structurally stable local normal form of affine systems. Then the orbit $G_k(j^k\sigma)$ of G_k through $j^k\sigma$ has nonempty interior in Σ_k for any $k \geqslant 0$.*

Now let us look at actions γ_0 and γ_1 of approximate feedback groups G_0, G_1. The case of γ_0 is easy. Due to the assumptions $f(0)=0$, $\varphi(0)=0$, $\eta(0)=0$, we have $\Sigma_0 \cong \mathrm{Mat}(n,m)=$ the set of all real $n\times m$ matrices, and $G_0 \cong \mathrm{GL}_n(\mathbb{R})\times \mathrm{GL}_m(\mathbb{R})$. Hence $\gamma_0:(\partial\varphi/\partial x,\psi)g \mapsto (\partial\varphi/\partial x)^{-1}\cdot g\cdot\psi$ leads to the classification of rectangular matrices by their rank. Clearly, the number of orbits is finite and there exists an open dense orbit described by a normal form like

$$\begin{bmatrix} I_m \\ 0 \end{bmatrix} \qquad (m \leqslant n).$$

The case of γ_1 is much more involved. First observe that $\Sigma_1 \cong \mathrm{Mat}(n,m) \times \mathrm{Mat}(n)\times \mathrm{Mat}(n,m)^n$, while $G_1 \cong \mathrm{GL}_n(\mathbb{R})\times \mathrm{Sym}(n)^n \times \mathrm{Mat}(m,n)\times \mathrm{GL}_m(\mathbb{R})\times \mathrm{Mat}(m)^n$, where $\mathrm{Mat}(n)$ is the set of $n\times n$ square matrices, $\mathrm{Sym}(n)$ is the set of $n\times n$ symmetric matrices, and superscripts denote Cartesian powers. Then after some calculations, the action γ_1 can be given the following form:

$$\gamma_1:\left(\frac{\partial\varphi}{\partial x},\frac{\partial^2\varphi_1}{\partial x^2},\dots,\frac{\partial^2\varphi_n}{\partial x^2},\frac{\partial\eta}{\partial x},\psi,\frac{\partial\psi}{\partial x_1},\dots,\frac{\partial\psi}{\partial x_n}\right)\left(g,\frac{\partial f}{\partial x},\frac{\partial g^1}{\partial x},\dots,\frac{\partial g^n}{\partial x}\right)$$

$$\mapsto\left(\left(\frac{\partial\varphi}{\partial x}\right)^{-1} g\psi,\left(\frac{\partial\varphi}{\partial x}\right)^{-1}\frac{\partial f}{\partial x}\frac{\partial\varphi}{\partial x}+\left(\frac{\partial\varphi}{\partial x}\right)^{-1} g\frac{\partial\eta}{\partial x},\right.$$

$$\left(\frac{\partial\varphi}{\partial x}\right)^{T}\frac{\partial^2\varphi_i^{-1}}{\partial x^2}g\psi+\sum_{s=1}^{n}\frac{\partial\varphi_i^{-1}}{\partial x_s}\left(\frac{\partial\varphi}{\partial x}\right)^{T}\frac{\partial g^s}{\partial x}\psi$$

$$\left.+\sum_{s=1}^{n}\frac{\partial\varphi_i^{-1}}{\partial x_s}\left(g\frac{\partial\psi}{\partial x}\right)^{s},\ i=1,2,\dots,n\right). \tag{2.3}$$

Hereabove $(\partial\psi/\partial x_s)_{ij}=\partial\psi_{ij}/\partial x_s$, $(\partial g^s/\partial x)_{ij}=\partial g_{sj}/\partial x_i$, $(g\,\partial\psi/\partial x)^s_{ij}=(g\,\partial\psi/\partial x_i)_{sj}$, $i,s=1,2,\dots,n$, $j=1,2,\dots,m$, $g=[g_1,\dots,g_m]$, all the terms being calculated at $0\in\mathbb{R}^n$.

The expression (2.3) clearly contains the action γ_0; moreover, one immediately recognizes the first two terms on the r.h.s. of (2.3) as resulting from the action of the linear-feedback group. Therefore, due to Lemma 2.1, if there exists a structurally stable normal form of affine systems, it must be manifested in γ_1 by an open orbit passing through $(\partial f/\partial x, g)$ in the so-called generic Brunovsky form. Because of that, we can assume in (2.3) that

$\partial f/\partial x = \mathrm{diag}\{J_1,\dots,J_m\}$, $g = \mathrm{diag}\{e_1,\dots,e_m\}$, with

$$J_i = \begin{bmatrix} 0 & I_{\varkappa_i - 1} \\ 0 & 0 \end{bmatrix}_{\varkappa_i \times \varkappa_i}, \qquad e_i = (0,\dots,0,1)^T_{\varkappa_i \times 1},$$

where, if $n = km + r$, $0 \leqslant r < m$, then $\varkappa_1 = \cdots = \varkappa_r = k+1$, $\varkappa_{r+1} = \cdots = \varkappa_m = k$. This being so, the third group of terms on the r.h.s. of (2.3) is acted on by the stabilizer of the generic form and by new agents like $\partial^2\varphi_i/\partial x^2$, $\partial\psi/\partial x_i$.

We choose $\sigma_1 = (g, \partial f/\partial x, \partial g^1/\partial x,\dots, \partial g^n/\partial x) \in \Sigma_1$ with $(\partial f/\partial x, g)$ as above and $\partial g^i/\partial x$ arbitrary. It is a standard fact that $\dim G_1\sigma_1 = \operatorname{codim} \operatorname{Stab} \sigma_1$ [10], so the orbit $G_1\sigma_1$ cannot be open whenever $\operatorname{codim}\operatorname{Stab}\sigma_1 < \dim \Sigma_1$. Thus all we need is to calculate $\operatorname{codim}\operatorname{Stab}\sigma_1$, i.e. to count independent equations defining the stabilizer. This will be done in the next section.

3. MAIN RESULT

For σ_1 described in Section 2, $\operatorname{Stab}\sigma_1$ is determined by the following equations:

$$\left(\frac{\partial\varphi}{\partial x}\right)^{-1} g\psi = g, \qquad \left(\frac{\partial\varphi}{\partial x}\right)^{-1} \frac{\partial f}{\partial x}\frac{\partial\varphi}{\partial x} + \left(\frac{\partial\varphi}{\partial x}\right)^{-1} g\frac{\partial\eta}{\partial x} = \frac{\partial f}{\partial x}, \quad (3.1a)$$

$$\left(\frac{\partial\varphi}{\partial x}\right)^{T} \frac{\partial^2\varphi_i^{-1}}{\partial x^2} g\psi + \sum_{s=1}^{n} \frac{\partial\varphi_i^{-1}}{\partial x_s}\left(\frac{\partial\varphi}{\partial x}\right)^{T}\frac{\partial g^s}{\partial x}\psi + \sum_{s=1}^{n}\frac{\partial\varphi_i^{-1}}{\partial x_s}\left(g\frac{\partial\psi}{\partial x}\right)^{s} = \frac{\partial g^i}{\partial x},$$

$$i = 1,2,\dots,n. \quad (3.1b)$$

The solution of (3.1a) for $(\partial f/\partial x, g)$ in the generic form is well known [11, 12]. With regard to the form of (3.1b) it is convenient to write down the solution (stabilizer of the linear feedback group) as follows:

$$\left(\frac{\partial\varphi}{\partial x}\right)^{-1} = \left(\begin{array}{c|c} \Phi_{11} & 0 \\ \hline \Phi_{21} & \Phi_{22} \end{array}\right)$$

with

$$\Phi_{11}=\begin{bmatrix} a_{11} & 0 & \cdots & 0 & \cdots & a_{1r} & 0 & \cdots & 0 \\ 0 & a_{11} & & \vdots & \cdots & 0 & a_{1r} & & \vdots \\ \vdots & & \ddots & 0 & \cdots & \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{11} & \cdots & 0 & \cdots & 0 & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & 0 & \cdots & 0 & \cdots & a_{rr} & 0 & \cdots & 0 \\ 0 & a_{r1} & & \vdots & \cdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \cdots & \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{r1} & \cdots & 0 & \cdots & 0 & a_{rr} \end{bmatrix},$$

$$\Phi_{21}=\begin{bmatrix} c_{11} & d_{11} & 0 & \cdots & 0 & \cdots & c_{1r} & d_{1r} & 0 & \cdots & 0 \\ 0 & c_{11} & d_{11} & \cdots & 0 & \cdots & 0 & c_{1r} & d_{1r} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{11} & d_{11} & \cdots & 0 & \cdots & 0 & c_{1r} & d_{1r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m-r1} & d_{m-r1} & 0 & \cdots & 0 & \cdots & c_{m-rr} & d_{m-rr} & 0 & \cdots & 0 \\ 0 & c_{m-r1} & d_{m-r1} & \cdots & 0 & \cdots & 0 & c_{m-rr} & d_{m-rr} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \cdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{m-r1} & d_{m-r1} & \cdots & 0 & \cdots & 0 & c_{m-rr} & d_{m-rr} \end{bmatrix},$$

$$\Phi_{22}=\begin{bmatrix} b_{11} & 0 & \cdots & 0 & \cdots & b_{1m-r} & 0 & \cdots & 0 \\ 0 & b_{11} & \cdots & 0 & \cdots & 0 & b_{1m-r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{11} & \cdots & 0 & 0 & \cdots & b_{1m-r} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ b_{m-r1} & 0 & \cdots & 0 & \cdots & b_{m-rm-r} & 0 & \cdots & 0 \\ 0 & b_{m-r1} & \cdots & 0 & \cdots & 0 & b_{m-rm-r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{m-r1} & \cdots & 0 & 0 & \cdots & b_{m-rm-r} \end{bmatrix},$$

and

$$\psi^{-1}=\left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1r} & & & \\ \vdots & & \vdots & & \mathbf{0} & \\ a_{r1} & \cdots & a_{rr} & & & \\ \hline d_{11} & \cdots & d_{1r} & b_{11} & \cdots & b_{1m-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{m-r1} & \cdots & d_{m-rr} & b_{m-r1} & \cdots & b_{m-rm-r} \end{array}\right], \quad (3.2)$$

where the entries a, b, c, d are arbitrary, and where $\partial\varphi/\partial x, \psi$ are invertible. We have disregarded the term $\partial\eta/\partial x$, as it does not appear in (3.1b).

Now, thanks to (3.2), the equations (3.1b) can be transformed further. Recall that $n = km + r$, $0 \leqslant r < m$. Then (3.1b) is equivalent to the following:

(a) $i = l(k+1)$, $l = 1, 2, \ldots, r$:

$$\left(\frac{\partial\varphi}{\partial x}\right)^T \frac{\partial^2\varphi_i^{-1}}{\partial x^2} g\psi + \sum_{t=1}^{r} a_{lt}\left(\frac{\partial\varphi}{\partial x}\right)^T \frac{\partial g^{t(k+1)}}{\partial x}\psi$$

$$+ \sum_{t=1}^{r} a_{lt} \begin{bmatrix} \dfrac{\partial\psi_{t1}}{\partial x_1} & \cdots & \dfrac{\partial\psi_{tm}}{\partial x_1} \\ \vdots & & \vdots \\ \dfrac{\partial\psi_{t1}}{\partial x_n} & \cdots & \dfrac{\partial\psi_{tm}}{\partial x_n} \end{bmatrix} = \frac{\partial g^i}{\partial x};$$

(b) $i = r(k+1) + pk$, $p = 1, 2, \ldots, m - r$:

$$\left(\frac{\partial\varphi}{\partial x}\right)^T \frac{\partial^2\varphi_i^{-1}}{\partial x^2} g\psi + \sum_{t=1}^{r} c_{pt}\left(\frac{\partial\varphi}{\partial x}\right)^T \frac{\partial g^{t(k+1)-1}}{\partial x}\psi$$

$$+ \sum_{t=1}^{r} d_{pt}\left(\frac{\partial\varphi}{\partial x}\right)^T \frac{\partial g^{t(k+1)}}{\partial x}\psi + \sum_{s=1}^{m-r} b_{ps}\left(\frac{\partial\varphi}{\partial x}\right)^T \frac{\partial g^{r(k+1)+sk}}{\partial x}\psi$$

$$+ \sum_{t=1}^{r} d_{pt} \begin{bmatrix} \dfrac{\partial\psi_{t1}}{\partial x_1} & \cdots & \dfrac{\partial\psi_{tm}}{\partial x_1} \\ \vdots & & \vdots \\ \dfrac{\partial\psi_{t1}}{\partial x_n} & \cdots & \dfrac{\partial\psi_{tm}}{\partial x_n} \end{bmatrix}$$

$$+ \sum_{s=1}^{m-r} b_{ps} \begin{bmatrix} \dfrac{\partial\psi_{r+s1}}{\partial x_1} & \cdots & \dfrac{\partial\psi_{r+sm}}{\partial x_1} \\ \dfrac{\partial\psi_{r+s1}}{\partial x_n} & \cdots & \dfrac{\partial\psi_{r+sm}}{\partial x_n} \end{bmatrix}$$

$$= \frac{\partial g^i}{\partial x};$$

(c) $i=(l-1)(k+1)+t,\ l=1,2,\dots,r,\ t=1,2,\dots,k$:

$$\left(\frac{\partial\varphi}{\partial x}\right)^T\frac{\partial^2\varphi_i^{-1}}{\partial x^2}g\psi+\sum_{j=1}^{r}a_{lj}\left(\frac{\partial\varphi}{\partial x}\right)^T\frac{\partial g^{(j-1)(k+1)+t}}{\partial x}\psi=\frac{\partial g^i}{\partial x};$$

(d) $i=r(k+1)+(s-1)k+w,\ w=1,2,\dots,k-1,\ s=1,2,\dots,m-r$:

$$\left(\frac{\partial\varphi}{\partial x}\right)^T\frac{\partial^2\varphi_i^{-1}}{\partial x^2}g\psi+\sum_{j=1}^{r}c_{sj}\left(\frac{\partial\varphi}{\partial x}\right)^T\frac{\partial g^{(j-1)(k+1)+w}}{\partial x}\psi$$

$$+\sum_{j=1}^{r}d_{sj}\left(\frac{\partial\varphi}{\partial x}\right)^T\frac{\partial g^{(j-1)(k+1)+w+1}}{\partial x}\psi$$

$$+\sum_{t=1}^{m-r}b_{st}\left(\frac{\partial\varphi}{\partial x}\right)^T\frac{\partial g^{r(k+1)+(t-1)k+w}}{\partial x}\psi$$

$$=\frac{\partial g^i}{\partial x}.$$

A careful analysis of (a)–(d) and (3.1a) yields the main result of this paper.

THEOREM 3.1. *Let*

$$\sigma_1=\left(g,\frac{\partial f}{\partial x},\frac{\partial g^1}{\partial x},\dots,\frac{\partial g^n}{\partial x}\right)$$

with $(\partial f/\partial x, g)$ *in the generic Brunovsky form,* $\partial g^i/\partial x$ *arbitrary. Then* $\operatorname{codim} G_1\sigma_1 \geqslant (m/2)(m-1)(n-m)-m^2$.

By combining the theorem with Lemma 2.1 one easily derives the next.

COROLLARY 3.2. *Let $n > m(m+1)/(m-1)$, $m \geqslant 2$. Then the orbits of G_1 have positive codimensions in Σ_1; hence there are no structurally stable local normal forms of affine systems under feedback.*

4. CONCLUSION

We have found a condition for the nonexistence of structurally stable normal forms of affine systems (Corollary 3.2) which is much stronger than that stated in [6, Proposition 4.3]. This can be seen from the following example. Let $n = 10$, $m = 5$. Then according to [6, Proposition 4.3] an obstacle to the structural stability appears in the action of G_5, while Corollary 3.2 implies that an obstacle exists already in the action of G_1. Furthermore, we believe that Theorem 3.1 could be of some independent interest for a local classification of bilinear systems.

Being stronger, Corollary 3.2 does not apply to all the cases covered by Proposition 4.3 in [6]. In particular, to use the corollary we must assume $m \geqslant 2$ and $n \geqslant m+3$, so it does not help us in the investigation of structurally stable normal forms for $n = m+1$ (the case left undecided by Proposition 4.3 in [6] and settled erroneously by Proposition 4.5 therein).

If $n \leqslant m(m+1)/(m-1)$, $m \geqslant 2$ or $m = 1$, n arbitrary, we conjecture that G_1 acts on Σ_1 with open orbits. Having found normal forms for the orbits, we might be able to extend the region of dimensions (m, n) without structurally stable normal forms by analyzing the action of G_2, etc. We conjecture that even the case $n = m+1$ may be decided using approximate feedback groups of relatively low orders.

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