Electroencephalography in ellipsoidal geometry

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Received 27 May 2003
Submitted by T. Fokas

Abstract

The human brain is shaped in the form of an ellipsoid with average semiaxes equal to 6, 6.5 and 9 cm. This is a genuine 3-D shape that reflects the anisotropic characteristics of the brain as a conductive body. The direct electroencephalography problem in such anisotropic geometry is studied in the present work. The results, which are obtained through successively solving an interior and an exterior boundary value problem, are expressed in terms of elliptic integrals and ellipsoidal harmonics, both in Jacobian as well as in Cartesian form. Reduction of our results to spheroidal as well as to spherical geometry is included. In contrast to the spherical case where the boundary does not appear in the solution, the boundary of the realistic conductive brain enters explicitly in the relative expressions for the electric field. Moreover, the results in all three geometrical models reveal that to some extend the strength of the electric source is more important than its location.

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1. Introduction

The study of electrical phenomena that occur within the human organs concerns the interdisciplinary field of bioelectricity, which extends from mathematics to electrical engineering, computer sciences, physics and medicine. Among the most widely used noninvasive methods, for studying electrical human brain activity, electroencephalography has the greatest acceptance.

With this method the electrochemical activity of the brain is mapped on an electroencephalogram (EEG). The data of an EEG are electric potentials registered on the head surface by properly placed electrodes [8]. The interpretation of the EEG data for the purpose of locating the electrochemical source, inside the brain, which generates the externally
measured electric potential, identifies the inverse EEG problem. But before the inverse EEG problem can be solved, one has to deal with the electric potential field generated by a given source, inside the brain. Such a problem is known as the forward EEG problem.

In order to work with both, the inverse and the forward, EEG problems, one has to make certain assumptions, concerning the electrochemical source and the conductor that models the human brain. In bioelectricity the most popular source model that has been used since 1967 is a current dipole [7,8,12] with fixed moment and location inside the brain. As far as the brain itself is concerned, in most of the work that has been published [6–9,11], it is considered to be a homogeneous or a partially homogeneous conductor. The most popular geometrical model is the spherical one [11,15,17], although the spheroidal model [6,15], and a model that allows for small perturbations of the sphere [16], have also been studied.

In the work at hand, the forward EEG problem is solved for the case of an electric current dipole source with a given moment which is located inside a homogeneous ellipsoidal conductor. The ellipsoidal geometry has been chosen because it incorporates the complete anisotropy of the 3-D space, and it best fits the anatomical model of the human brain [18].

Section 2 states the boundary value problems that the electric potential has to solve in the interior and in the exterior of a given homogeneous ellipsoid. A brief introduction to the ellipsoidal geometry, as well as the basic notation for the spectral decomposition of the Laplace operator in ellipsoidal coordinates is included here in order to help the reader with this rather unfamiliar material. The solution of these boundary value problems in ellipsoidal geometry forms the content of Section 3, where relative expressions in ellipsoidal, in tensorial and in Cartesian form are provided as multipole electric potential fields. Sections 4 and 5 include the laborious and by no means easy task of reducing the ellipsoidal results to the spheroidal and to the spherical environment. Their Cartesian counterparts are also included.

2. Statement of the problem

Let \( S \) denotes the triaxial ellipsoidal surface, which in rectangular coordinates is specified by

\[
\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1, \tag{1}
\]

where \( 0 < \alpha_3 < \alpha_2 < \alpha_1 < +\infty \), are its semiaxes. The ellipsoid (1) introduces an ellipsoidal system \([10]\) with coordinates \( \rho, \mu, \nu \) and semifocal distances \( h_1, h_2, h_3 \), where

\[
h_1^2 = \alpha_2^2 - \alpha_3^2, \tag{2}
\]
\[
h_2^2 = \alpha_1^2 - \alpha_3^2, \tag{3}
\]
\[
h_3^2 = \alpha_1^2 - \alpha_2^2. \tag{4}
\]

The ellipsoidal coordinates \( \rho, \mu, \nu \) are connected to the Cartesian ones via the following formulae \([10]\)

\[
x_1 = \frac{\rho \mu \nu}{h_2 h_3}, \tag{5}
\]
and vary in the intervals $[h_2, +\infty)$, $[h_3, h_2)$ and $[-h_3, h_3]$, respectively.

In ellipsoidal coordinates, the surface $S$, given in (1), corresponds to $\rho = \alpha_1$ and it represents the boundary of the brain. The interior to $S$ space $V^-$, corresponds to the interval $\rho \in [h_2, \alpha_1)$ and it is characterized by the conductivity $\sigma$ and the magnetic permeability $\mu_0$.

The exterior nonconductive space $V^+$ is characterized by the same magnetic permeability $\mu_0$ and corresponds to the interval $\rho \in (\alpha_1, +\infty)$.

At the point $r_0 \in V^-$ there exists a primary current dipole source with moment $Q$ and density function defined by

$$J^p(r) = Q\delta(r - r_0),$$

where $\delta$ stands for the Dirac measure.

The primary current $J^p$ induces an electric field $E$ in the interior conductive space $V^-$ which in turn causes the induction of a volume current with density $J^V$,

$$J^V(r) = \sigma E(r),$$

resulting the total current density

$$J(r) = J^p(r) + \sigma E(r).$$

The current $J$ generates an electromagnetic wave which propagates in the interior as well as in the exterior to the conductor space.

It can be shown that the values of the characteristic parameters of the human brain, i.e., the values of the electric conductivity, the dielectric constant and the magnetic permeability, which specify the wave length of the electromagnetic wave, allow for both the electric and the magnetic field to be considered as quasistatic [8,12,17,19].

Therefore the electric field $E$ and the magnetic field $B$ satisfy the following quasistatic approximation of Maxwell equations [19]:

$$\nabla \times E = 0,$$

$$\nabla \times B = \mu_0 J,$$

$$\nabla \cdot E = 0,$$

$$\nabla \cdot B = 0.$$

Since $E$ is irrotational, it is represented by an electric potential $u$, such that

$$E(r) = -\nabla u(r).$$

The potential $u$ is the field recorded during the electroencephalographic process. In particular, we denote the electric potential in the interior of the ellipsoid $\rho = \alpha_1$ by $u^-$ and in
the exterior of $\rho = \alpha_1$ by $u^+$. Taking the divergence of (10) and using (12) and (15), we obtain Poisson’s equation

$$\Delta u^-(r) = \frac{1}{\sigma} \nabla \cdot J^p(r), \quad r \in V^-,$$

which the interior potential $u^-$ must satisfy in $V^-$. The exterior potential $u^+$ solves Laplace’s equation

$$\Delta u^+(r) = 0, \quad r \in V^+,$$

in the source-free space $V^+$.

On the boundary surface $S$, the continuity of the electric potential field demands that

$$u^-(r) = u^+(r), \quad r \in S,$$

and since the conductivity vanishes outside $V^-$, the continuity of the normal component of the electric field implies that

$$\frac{\partial u^-(r)}{\partial n} = 0, \quad r \in S,$$

where outward normal differentiation on $S$ is considered. In addition, the asymptotic behaviour at infinity

$$u^+(r) = O\left(\frac{1}{r}\right), \quad r \to \infty,$$

has to be assumed if the exterior problem is taken to be well-posed. The interior potential problem is given by (16) and (19), while the exterior potential problem is specified by (17), (18) and (20).

Separation of variables for Laplace’s equation in the ellipsoidal coordinate system leads to the Lamé equation [10]

$$\left( x^2 - h_2^2 \right) \left( x^2 - h_3^2 \right) E''(x) + x \left( 2x^2 - h_2^2 - h_3^2 \right) E'(x) + \left[ (h_2^2 + h_3^2)P - n(n+1)x^2 \right] E(x) = 0$$

for each one of the factors $E(\rho), E(\mu)$ and $E(\nu)$ that form the interior harmonic function

$$E_n^{m}(\rho, \mu, \nu) = E_n^{m}(\rho)E_n^{m}(\mu)E_n^{m}(\nu).$$

In Eq. (21) the parameters $P$ and $n$ are constants that define, in a complicated way, the degree $n$ and the order $m$ of the interior ellipsoidal harmonic (22).

The corresponding exterior ellipsoidal harmonic assumes the form

$$F_n^{m}(\rho, \mu, \nu) = (2n + 1)E_n^{m}(\rho, \mu, \nu)I_n^{m}(\rho)$$

$$= (2n + 1)I_n^{m}(\rho)E_n^{m}(\rho)E_n^{m}(\mu)E_n^{m}(\nu).$$

The $\rho$-dependent functions $I_n^{m}(\rho)$ are elliptic integrals of the form
\[ I_n^m (\rho) = \int_{\rho}^{+\infty} \frac{dt}{\sqrt{t^2 - h_1^2 \sqrt{t^2 - h_2^2}}}, \quad (24) \]

for each \( n = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots, 2n + 1 \).

The products \( E_n^m (\mu) E_n^m (\nu) \), defined on the surface of any specific ellipsoid, are known as surface ellipsoidal harmonics and they form a complete orthogonal set of surface eigenfunctions, with respect to the weighting function

\[ l_{\rho_0} (\mu, \nu) = \left[ (\rho_0^2 - \mu^2) (\rho_0^2 - \nu^2) \right]^{-1/2}, \quad (25) \]

corresponding to the ellipsoid \( \rho = \rho_0 \).

We define the normalization constants \( \gamma_n^m \) as

\[ \gamma_n^m = \int_{\rho=\rho_0} \left[ E_n^m (\mu) E_n^m (\nu) \right]^2 l_{\rho_0} (\mu, \nu) \, ds, \quad (26) \]

i.e., the squares of the \( L^2 \)-norms of the corresponding surface ellipsoidal harmonics.

In the present work the following normalization constants are to be used [5]:

\[ \gamma_0^1 = 4\pi, \quad (27) \]

\[ \gamma_1^m = \frac{4\pi h_1^2 h_2^2 h_3^2}{h_m^2}, \quad m = 1, 2, 3, \quad (28) \]

\[ \gamma_2^1 = -\frac{8\pi}{5} (\Lambda - \Lambda') (\Lambda - \alpha_1^2) (\Lambda - \alpha_2^2), \quad (29) \]

\[ \gamma_2^2 = \frac{8\pi}{5} (\Lambda - \Lambda') (\Lambda' - \alpha_1^2) (\Lambda' - \alpha_2^2), \quad (30) \]

\[ \gamma_2^{6-m} = \frac{4\pi}{15} h_1^2 h_2^2 h_3^2 h_m^2, \quad m = 1, 2, 3, \quad (31) \]

where

\[ \Lambda = \frac{1}{3} \sum_{i=1}^{3} \alpha_i^2 \pm \frac{1}{3} \left[ \sum_{i=1}^{3} \left( \alpha_i^4 - \frac{\alpha_1^2 \alpha_2^2 \alpha_3^2}{\alpha_0^2} \right) \right]^{1/2}, \quad (32) \]

Although the form of the ellipsoidal harmonics is known, an exact analytic expression in terms of the ellipsoid’s semiaxes \( \alpha_1, \alpha_2, \alpha_3 \) is possible only up to the third degree, as the parameters of harmonics of degree higher than three solve irreducible polynomial equations of the cubic or higher degree. This means that an analytic form will be either not practical in use, or will not exist. However, in the present work, harmonics only up to the second degree are needed.

The Cartesian form of the ellipsoidal harmonics of degree less or equal to two are

\[ E_0^1 (\rho, \mu, \nu) = 1, \quad (33) \]

\[ E_1^m (\rho, \mu, \nu) = \frac{h_1 h_2 h_3}{h_m} x_m, \quad m = 1, 2, 3, \quad (34) \]

\[ E_2^1 (\rho, \mu, \nu) = (\Lambda - \alpha_1^2) (\Lambda - \alpha_2^2) (\Lambda - \alpha_3^2) \left( \sum_{n=1}^{3} \frac{x_n^2}{\Lambda - \alpha_n^2} + 1 \right), \quad (35) \]
In the following section the interior boundary value problem (16), (19) is solved in ellipsoidal geometry and the interior electric potential \( u^- \) is then used to evaluate the exterior electric potential \( u^+ \), by solving the boundary value problem (17), (18) and (20).

### 3. The ellipsoidal model of the brain

First, we are going to solve the problem (16), (19) in the interior of the ellipsoidal conductor \( V^- \). The general solution of Poisson’s equation (16) in \( V^- \), is a superposition of an interior harmonic function \( \Phi(r) \) and the function

\[
V(r) = -\frac{1}{4\pi\sigma} \frac{Q}{|r - r_0|} \tag{38}
\]

which, obviously, is a particular solution of (16).

Using the ellipsoidal expansion of the interior harmonic function \( \Phi(r) \),

\[
\Phi(r) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} b_n^m E_n^m(\rho, \mu, \nu), \tag{39}
\]

the interior potential \( u^- \), that satisfies Eq. (16), assumes the form

\[
u^- (r) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \left[ b_n^m + \frac{1}{\sigma\gamma_n^m} \left( Q \cdot \nabla r_0 E_n^m(\rho_0) \right) I_n^m(\rho) \right] E_n^m(\rho, \mu, \nu).
\tag{40}
\]

In order to force the potential \( u^- \), given in (40), to satisfy the boundary condition (19) on \( S \), we use the proper ellipsoidal expansion of the fundamental solution of the Laplace operator [13]

\[
\frac{1}{|r - r_0|} = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{4\pi}{2n + 1} \gamma_n^m(\rho_0, \mu_0, \nu_0) \frac{1}{\sigma\gamma_n^m} E_n^m(\rho, \mu, v) \tag{41}
\]

on the ellipsoidal surface \( \rho = \alpha_1 > \rho_0 \).

Applying properly the gradient operator on (41), and using (40), (22) and (23), the interior potential \( u^- (r) \) is expressed in the following form:

\[
u^- (\rho, \mu, v) = b_0^1 + \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \left[ b_n^m + \frac{1}{\sigma\gamma_n^m} \left( Q \cdot \nabla r_0 E_n^m(\rho_0) \right) I_n^m(\rho) \right] E_n^m(\rho, \mu, v). \tag{42}
\]

The potential \( u^- (r) \) has to satisfy the boundary condition (19), which because of the orthogonality of the ellipsoidal system is expressed as

\[
\frac{\partial u^- (\rho, \mu, v)}{\partial \rho} = 0, \quad \rho = \alpha_1. \tag{43}
\]
Introducing (42) in (43) and using the orthogonality properties of the surface ellipsoidal harmonics, the constants $b^m_n$ are calculated as

$$b^m_n = \frac{1}{\sigma \gamma^m_n} \left( Q \cdot \nabla_{r_0} u^m_n(r_0) \right) \left[ \frac{1}{\alpha_2 \alpha_3 E^m_n(\alpha_1)} E^m_n(\alpha_1) - I^m_n(\alpha_1) \right]$$

(44)

for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, 2n + 1$.

In view of (44) and (23), Eq. (40) provides the electric potential $u^-(r)$ in the interior of the ellipsoid $\rho = \alpha_1$ as

$$u^-_{\alpha_1}(r) = u^-_{\alpha_1}(\rho, \mu, v) = b^1_{0} + \frac{1}{4\pi \sigma} Q \frac{r - r_0}{|r - r_0|}$$

$$- \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{(2n+1)\sigma \gamma^m_n} \left( Q \cdot \nabla_{r_0} u^m_n(r_0) \right) \frac{E^m_n(\alpha_1)}{E^m_{n-1}(\alpha_1)} E^m_n(r),$$

(45)

where the prime denotes differentiation with respect to the argument and $b^1_{0}$ is an arbitrary constant.

Next, we want to turn to the exterior space $V^+$ and evaluate the exterior potential $u^+$, by solving the corresponding boundary value problem (17), (18), (20). To this end we observe that condition (18) is given through the values of $u^-$ on the boundary and in order to apply this condition, we need to express these values in terms of surface ellipsoidal eigenfunctions.

Working with the expression (45) further and using the ellipsoidal expansion (41) for the fundamental solution of Laplace’s operator we reach at the following form of $u^-$ which holds for $\rho_0 < \rho < \alpha_1$:

$$u^-_{\alpha_1}(\rho, \mu, v) = b^1_{0} + \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\sigma \gamma^m_n} \left( Q \cdot \nabla_{r_0} u^m_n(r_0) \right) \left[ I^m_n(\rho) - I^m_n(\alpha_1) \right]$$

$$+ \frac{1}{\alpha_2 \alpha_3 E^m_n(\alpha_1)} \frac{E^m_n(\rho)}{E^m_{n-1}(\alpha_1)} E^m_n(\mu) E^m_n(v).$$

(46)

Applying the gradient operator on $E^m_n$ and differentiating the corresponding Lamé functions $E^m_{n}$, we arrive at the following ellipsoidal expression for the interior electric field:

$$u^-_{\alpha_1}(r) = b^1_{0} + \frac{3}{4\pi \sigma} \sum_{m=1}^{3} Q_m x_m \left( I^m_1(\rho) - I^m_1(\alpha_1) + \frac{1}{\alpha_1 \alpha_2 \alpha_3} \right)$$

$$- \frac{5}{4\pi \sigma (\Lambda - \Lambda')} \sum_{m=1}^{3} Q_m x_m \left( I^m_2(\rho) - I^m_2(\alpha_1) + \frac{1}{2\alpha_1 \alpha_2 \alpha_3 \Lambda} \right) \frac{E^m_1(r)}{(\Lambda - \alpha^2_m)}$$

$$+ \frac{5}{4\pi \sigma (\Lambda - \Lambda')} \sum_{m=1}^{3} Q_m x_m \left( I^m_2(\rho) - I^m_2(\alpha_1) + \frac{1}{2\alpha_1 \alpha_2 \alpha_3 \Lambda'} \right) \frac{E^m_2(r)}{(\Lambda' - \alpha^2_m)}.$$
\[
\begin{align*}
+ \frac{15}{4\pi a} & \sum_{i,j=1}^{3} Q_i x_0 j x_j \left( I_2^{i+j} (\rho) - I_2^{i+j} (\alpha_1) \right) + \frac{1}{\alpha_1 \alpha_2 \alpha_3 (\alpha_1^2 + \alpha_2^2)} \\
+ O(\varepsilon_3),
\end{align*}
\]  

where the notation \( O(\varepsilon_3) \) denotes ellipsoidal terms that are of order greater or equal to three. The factors \( Q_m, x_m, x_{0m}, m = 1, 2, 3 \), denote the Cartesian components of the vectors \( \mathbf{Q}, \mathbf{r} \) and \( \mathbf{r}_0 \), respectively.

For the purpose of easing the algebraic manipulations we introduce the following notation, where the single wiggle on the top denotes a dyadic and the double wiggle on the top denotes a tetradic [1,5]:

\[
\begin{align*}
\tilde{\mathbf{I}} &= \sum_{m=1}^{3} \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m, \\
\tilde{\mathbf{I}} &= \sum_{i,j=1}^{3} \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j, \\
\tilde{\mathbf{M}}(\alpha_1) &= \sum_{m=1}^{3} \alpha_2^2 \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m, \\
\tilde{\mathbf{A}} &= \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\lambda - \alpha_m^2}, \\
\tilde{\mathbf{A}}' &= \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\lambda' - \alpha_m^2}, \\
\tilde{\mathbf{H}}_1(\rho) &= \sum_{m=1}^{3} I_1^m (\rho) \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m, \\
\tilde{\mathbf{H}}_2(\rho) &= \sum_{i,j=1}^{3} \frac{I_2^{i+j} (\rho) \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j}{\alpha_i^2 + \alpha_j^2}, \\
\tilde{\mathbf{N}}_1 &= \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_m^2}, \\
\tilde{\mathbf{N}}_2 &= \sum_{i,j=1}^{3} \frac{\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j}{\alpha_i^2 + \alpha_j^2}.
\end{align*}
\]

It is of interest to observe in (50)–(56) how the anisotropy of the ellipsoidal geometry modifies the polyadic identities of the corresponding isotropic space. Using these polyadic identities...
modifications (50)–(56) in expression (47) as well as straightforward arguments, the interior potential in the ellipsoidal shell \( \rho_0 < \rho < \alpha_1 \) is rewritten in the following form [5]:

\[
u^- (r) = b_1^0 + Q \cdot \tilde{A} (\rho) \cdot r + Q \otimes r_0 : \tilde{B} (r) + Q \otimes r_0 : \tilde{\Gamma} (\rho) : r \otimes r + O (e_3), \tag{57}
\]

where

\[
\tilde{A} (\rho) = \frac{3}{4 \pi \sigma} (\tilde{H}_1 (\rho) - \tilde{H}_1 (\alpha_1)) + \frac{1}{\sigma V} \tilde{I}, \tag{58}
\]

\[
\tilde{B} (r) = - \frac{5}{4 \pi \sigma (A - A')} \left[ \left( I_2^1 (\rho) - I_2^1 (\alpha_1) + \frac{2 \pi}{3 V A} \right) \tilde{A} E_2^1 (r) \right.
\]

\[\left. - \left( I_2^2 (\rho) - I_2^2 (\alpha_1) + \frac{2 \pi}{3 V A'} \right) \tilde{A} E_2^2 (r) \right], \tag{59}
\]

\[
\tilde{\Gamma} (\rho) = \frac{15}{4 \pi \sigma} (\tilde{H}_2 (\rho) - \tilde{H}_2 (\alpha_1)) + \frac{5}{\sigma V} \tilde{N}_2, \tag{60}
\]

where \( V \) denotes the volume of the ellipsoid \( 3 V = 4 \pi \alpha_1 \alpha_2 \alpha_3 \) and the double contraction is defined as

\[
a \otimes b : c \otimes d = (a \cdot c) (b \cdot d). \tag{61}
\]

We are ready now to proceed to the evaluation of the exterior potential \( u^+ \) which in accordance with (17) and (20) is an exterior harmonic function that assumes the following ellipsoidal expansion:

\[
u^+ (\rho, \mu, \nu) = \sum_{n=0}^{\infty} \sum_{m=1}^{n} e_n^m \nu^m_n (\rho, \mu, \nu). \tag{62}
\]

Applying the Dirichlet condition (18) to the expressions (46) and (62) for \( u^- \) and \( u^+ \), respectively, and using properly Eq. (23) as well as orthogonality arguments, the exterior potential \( u^+ \) in \( V^+ \) is reduced to the form

\[
u^+ (\rho, \mu, \nu) = b_1^0 \frac{I_0^1 (\rho)}{I_0^1 (\alpha_1)} + \sum_{n=0}^{\infty} \sum_{m=1}^{n} \frac{(Q \cdot \nabla_n \nu^m_n (r_0))}{\sigma \gamma_n^m \alpha_1 \alpha_2 \alpha_3} \frac{I_n^m (\rho)}{I_n^m (\alpha_1)} \nu_n^m (\rho, \mu, \nu). \tag{63}
\]

Elaborating further on expression (63), using the gradient operator on \( \nu_n^m \) and also differentiating the corresponding Lamé functions \( E_n^m \), we obtain the following analytic form of \( u^+ \):

\[
u^+ (\rho, \mu, \nu) = b_1^0 \frac{I_0^1 (\rho)}{I_0^1 (\alpha_1)} + \frac{3}{4 \pi \sigma \alpha_1 \alpha_2 \alpha_3} \sum_{m=1}^{3} Q_m x_m \frac{I_n^m (\rho)}{I_n^m (\alpha_1)}
\]

\[\frac{3}{8 \pi \sigma \alpha_1 \alpha_2 \alpha_3 (A - A')} \times \sum_{m=1}^{3} Q_m x_m \left[ \frac{I_2^1 (\rho)}{I_2^1 (\alpha_1)} \frac{E_2^1 (r)}{A (A - \alpha^m_1)} - \frac{I_2^2 (\rho)}{I_2^2 (\alpha_1)} \frac{E_2^2 (r)}{A' (A' - \alpha^m_2)} \right]. \tag{64}
\]

\[ + \frac{15}{4\pi\sigma\alpha_1\alpha_2\alpha_3} \sum_{i,j=1 \atop i \neq j}^3 \frac{Q_i x_0 x_i x_j}{\alpha_i^2 + \alpha_j^2} I_2^{i+j}(\rho) + O(\epsilon) \]. \quad (64)

Furthermore, introducing in (64) the following modifications of the corresponding polyadic identities

\[ \tilde{L}_1(\rho) = \frac{3}{4\pi\sigma\alpha_1\alpha_2\alpha_3} \sum_{m=1}^3 \frac{I_m^{3}(\rho)}{I_1^{3}(\alpha_1)} \tilde{x}_m \otimes \tilde{x}_m, \]

\[ \tilde{L}_2(\rho) = \sum_{i,j=1 \atop i \neq j}^3 \frac{I_2^{i+j}(\rho)}{I_2^{i+j}(\alpha_1)} \frac{\tilde{x}_i \otimes \tilde{x}_j \otimes \tilde{x}_i \otimes \tilde{x}_j}{\alpha_i^2 + \alpha_j^2} \]

as well as the polyadic quantities

\[ \tilde{A}_1(\rho) = \frac{3}{4\pi\sigma\alpha_1\alpha_2\alpha_3} \tilde{L}_1(\rho), \]

\[ \tilde{B}_1(\mathbf{r}) = -\frac{5}{8\pi\sigma\alpha_1\alpha_2\alpha_3(\Lambda - \Lambda')} \left[ \frac{I_1^{1}(\rho)}{I_1^{1}(\alpha_1)} \tilde{A}_1^{1}(\mathbf{r}) - \frac{I_2^{2}(\rho)}{I_2^{2}(\alpha_1)} \tilde{A}_2^{2}(\mathbf{r}) \right], \]

and

\[ \tilde{\Gamma}_1(\rho) = \frac{5}{\sigma V} \tilde{L}_2(\rho), \]

we rewrite \( u^+ \) as

\[ u^+(\rho, \mu, \nu) = b_0 \frac{I_0^{1}(\rho)}{I_0^{1}(\alpha_1)} + \mathbf{Q} \cdot \tilde{A}_1(\rho) \cdot \mathbf{r} \]

\[ + \mathbf{Q} \otimes \mathbf{r}_0 : \tilde{B}_1(\mathbf{r}) + \mathbf{Q} \otimes \mathbf{r}_0 : \tilde{\Gamma}_1(\rho) : \mathbf{r} \otimes \mathbf{r} + O(\epsilon). \]

(70)

The use of the polyadic notation in expressing the interior and the exterior electric potential offers the advantage of a compact form in which the role of geometry and physics is distinctive and clear.

Comparing expressions (57) and (70) for \( u^- \) and \( u^+ \), respectively, we observe that the Physics enter to the solution through the moment \( \mathbf{Q} \) and the position vector \( \mathbf{r}_0 \) of the source. On the other hand, the geometry is encoded in the polyadics \( \tilde{I}_1, \tilde{A}_1(\rho), \tilde{B}_1(\mathbf{r}) \) and \( \tilde{\Gamma}_1(\rho) \) for \( u^- \), and in the polyadics \( (I_0^{1}(\rho)/I_0^{1}(\alpha_1)) \tilde{I}_1, \tilde{A}_1(\rho), \tilde{B}_1(\mathbf{r}) \) and \( \tilde{\Gamma}_1(\rho) \) for the exterior field \( u^+ \).

We note that the existence of \( b_0^1 \) in (57) reflects the lack of uniqueness of the Neumann problem. A remarkable observation is associated with the fact that the source enters the potentials \( u^+ \) and \( u^- \) in the monopole and the dipole terms only through its dipole moment, while its position vector appears for the first time in the quadrupole term. Hence, to some extent, its location is less important than its strength.

In the following sections we will reduce the results (57) and (70) to the special cases of spheroidal and of spherical geometries.
4. The spheroidal case

The reduction of the ellipsoidal to the spheroidal geometry, as well as to the spherical one, is by no means trivial, since as the symmetries increase many individual terms become undetermined.

The increase of the symmetries causes a simultaneous diminishing of the focal sets on which each system is based, starting from the 2-D focal ellipse for the ellipsoidal case, which is compressed to the 1-D focal distance for the prolate spheroidal system and collapses to the 0-D focal centre for the spherical system.

This reduction leads to the manifestation of certain indeterminate forms in all ellipsoidal results, which have not unique limits as the ellipsoid formally reduces to a spheroid or to a sphere.

In order to deal with these indeterminacies one has to perform complicated algebraic manipulations first working with appropriate groups of terms which eliminate the indeterminacies before the limiting process is applied. Finding the appropriate grouping is not a straightforward task and can only be faced as an art applying to each individual case.

Now, let us consider the prolate spheroid

\[ \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1 \]  

(71)

with semiaxes

\[ 0 < \alpha_2 < \alpha_1 < +\infty \]  

(72)

and semifocal distance

\[ c = \sqrt{\alpha_1^2 - \alpha_2^2}. \]  

(73)

The prolate spheroidal coordinates \((\xi, \eta, \phi)\) [4] are connected to the Cartesian and to the ellipsoidal ones via the relations

\[ x_1 = c \cosh \xi \cos \eta = \rho \frac{\mu \nu}{h_2 h_3}, \]  

(74)

\[ x_2 = c \sinh \xi \sin \eta \cos \phi = \sqrt{\rho^2 - h_2^2} \frac{\sqrt{\mu^2 - h_3^2} \sqrt{h_3^2 - \nu^2}}{h_1 h_3}, \]  

(75)

\[ x_3 = c \sinh \xi \sin \eta \sin \phi = \sqrt{\rho^2 - h_2^2} \frac{\sqrt{\mu^2} \sqrt{h_3^2} - \nu^2}{h_1 h_2}. \]  

(76)

As the ellipsoid (1) is reduced to the prolate spheroid (71) and to the sphere

\[ x_1^2 + x_2^2 + x_3^2 = \alpha^2, \]  

(77)

the following limits hold true:

\[ \rho \xrightarrow{el \to sd} c \cosh \xi \xrightarrow{sd \to sr} r, \]  

(78)

\[ \frac{\mu \nu}{h_2 h_3} \xrightarrow{el \to sd} \cos \eta \xrightarrow{sd \to sr} \cos \theta, \]  

(79)
\[ \sqrt{\mu^2 - h_3^2} / h_1 \rightarrow \sin \eta \cos \phi \rightarrow \sin \theta \cos \phi, \]  
\[ \sqrt{h_2^2 - \mu^2} / h_1 \rightarrow \sin \eta \sin \phi \rightarrow \sin \theta \sin \phi, \]  
(80)

where the notations \( \text{el} \rightarrow \text{sd} \) and \( \text{sd} \rightarrow \text{sr} \) stand for the limit when the ellipsoid is reduced to the prolate spheroid and when the spheroid approaches the sphere, respectively. The above spherical system is taken with \( x_1 \) as the polar axis and the azimuthal angle is measured from \( x_2 \) axis.

Introducing the representation [4]

\[ \tau = \cosh \xi \]  
(82)

we easily see that

\[ \tau = \frac{1}{2c} \left( \sqrt{x_1^2 + x_2^2 + x_3^2 + c^2} + \sqrt{x_1^2 + x_2^2 + x_3^2 + c^2 - 2cx_1} \right). \]  
(83)

Then, Eqs. (78) and (82) yield

\[ \lim_{\text{el} \rightarrow \text{sd}} \rho = c \tau. \]  
(84)

As far as the ellipsoidal parameters are concerned, at the spheroidal limit they are reduced to

\[ \lim_{\text{el} \rightarrow \text{sd}} \Lambda = \frac{2\alpha_1^2 + \alpha_2^2}{3}, \]  
(85)

\[ \lim_{\text{el} \rightarrow \text{sd}} \Lambda' = \alpha_2^2, \]  
(86)

while the semifocal distances provide

\[ \lim_{\text{el} \rightarrow \text{sd}} h_2 = \lim_{\text{el} \rightarrow \text{sd}} h_3 = c, \]  
(87)

\[ \lim_{\text{el} \rightarrow \text{sd}} h_1 = 0. \]  
(88)

The limiting integrals \( I_n^m \) are not elliptic anymore since

\[ \lim_{\text{el} \rightarrow \text{sd}} I_n^m (\rho) = \int_{\rho}^{+\infty} \frac{dt}{\left[ \left| \lim_{\text{el} \rightarrow \text{sd}} E_n^m (t) \right| \right]^2 |t^2 - \alpha_1^2 + \alpha_2^2|} \]  
(89)

for each \( n = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots, 2n + 1 \). Consequently, we obtain [2]

\[ \lim_{\text{el} \rightarrow \text{sd}} I_0^1 (\rho) = \frac{1}{2c} \ln \frac{\tau + 1}{\tau - 1}, \]  
(90)

\[ \lim_{\text{el} \rightarrow \text{sd}} I_1^1 (\rho) = \frac{1}{2c^3} \left( \ln \frac{\tau + 1}{\tau - 1} - \frac{2}{\tau} \right), \]  
(91)

\[ \lim_{\text{el} \rightarrow \text{sd}} I_2^1 (\rho) = \lim_{\text{el} \rightarrow \text{sd}} I_3^1 (\rho) = \frac{1}{4c^3} \left( \frac{2\tau}{(\tau^2 - 1)} - \ln \frac{\tau + 1}{\tau - 1} \right), \]  
(92)
and the following identities will be used:

Replacing (33)–(37), (85)–(88) and (91)–(95) into (47) and using (96)–(98), the interior electric potential in the spheroidal shell between the source’s location and the conductor’s location becomes

\[
\lim_\text{el-\text{sd}} I^1_\text{el}(\rho) = \frac{9}{8c^3} \left( \ln \frac{\tau + 1}{\tau - 1} - \frac{6\tau}{3(\tau^2 - 1)} \right),
\]

\[
\lim_\text{el-\text{sd}} I^2_\text{el}(\rho) = \lim_\text{el-\text{sd}} I^2_\text{el}(\rho) = \frac{3}{16c^3} \left( \ln \frac{\tau + 1}{\tau - 1} - \frac{2\tau(3\tau^2 - 5)}{3(\tau^2 - 1)^2} \right),
\]

\[
\lim_\text{el-\text{sd}} I^3_\text{el}(\rho) = \lim_\text{el-\text{sd}} I^3_\text{el}(\rho) = \frac{3}{4\pi c^3} \left( -\ln \frac{\tau + 1}{\tau - 1} + \frac{2(3\tau^2 - 2)}{3\tau(\tau^2 - 1)} \right).
\]

In order to evaluate the reduced form of the interior electric potential to the spheroidal case, the following identities will be used:

\[
\lim_\text{el-\text{sd}} \left\{ \frac{(A - \alpha_1^2)(A - \alpha_2^2)(A - \alpha_3^2)}{(A - A')} \left[ \hat{x}_1 \otimes \hat{x}_1 + \hat{x}_2 \otimes \hat{x}_2 + \hat{x}_3 \otimes \hat{x}_3 \right] \right\} = -\frac{c^2}{3} (\hat{I} - 3\hat{x}_1 \otimes \hat{x}_1),
\]

\[
\lim_\text{el-\text{sd}} \left\{ \frac{(A' - \alpha_1^2)(A' - \alpha_2^2)(A' - \alpha_3^2)}{(A' - A')} \left[ \hat{x}_1 \otimes \hat{x}_1 + \hat{x}_2 \otimes \hat{x}_2 + \hat{x}_3 \otimes \hat{x}_3 \right] \right\} = 0,
\]

and

\[
\lim_\text{el-\text{sd}} \left\{ \frac{(A' - \alpha_1^2)(A' - \alpha_2^2)(A' - \alpha_3^2)}{(A' - A')} \left[ \hat{x}_1 \otimes \hat{x}_1 + \hat{x}_2 \otimes \hat{x}_2 + \hat{x}_3 \otimes \hat{x}_3 \right] \right\} \times \left( \frac{x_1^2}{A' - \alpha_1^2} + \frac{x_2^2}{A' - \alpha_2^2} + \frac{x_3^2}{A' - \alpha_3^2} \right) = \frac{3}{2} (\hat{x}_2 \otimes \hat{x}_2 - \hat{x}_3 \otimes \hat{x}_3). \]

The use of these identities into the reduction of the form (47) annihilates all indeterminate forms. Actually, the reduction which is based on the above identities follows a continuous transformation of the whole ellipsoidal system to the corresponding prolate spheroidal one. Replacing (33)–(37), (85)–(88) and (91)–(95) into (47) and using (96)–(98), the interior electric potential in the spheroidal shell between the source’s location and the conductor’s boundary is expressed in the form

\[
u_{\text{sd}}(\tau) = \lim_\text{el-\text{sd}} u_{\text{el}}(\tau) = b_0 + \frac{3Q \otimes r}{4\pi \sigma} \left[ \hat{x}_1 \otimes \hat{x}_1 f_1(\tau) + (\hat{I} - \hat{x}_1 \otimes \hat{x}_1) f_2(\tau) \right]
\]

\[
+ \frac{15Q \otimes r_0}{4\pi \sigma} : (\hat{I} - 3\hat{x}_1 \otimes \hat{x}_1) \left( \frac{2x_1^2 - x_2^2 - x_3^2}{2} - \frac{c^2}{3} \right) f_3(\tau)
\]

\[
+ \frac{15Q \otimes r_0}{4\pi \sigma} : \left[ (\hat{x}_2 \otimes \hat{x}_2 - \hat{x}_1 \otimes \hat{x}_1) (\frac{x_2^2 - x_3^2}{2}) \right. \left. + (\hat{x}_2 \otimes \hat{x}_3 + \hat{x}_3 \otimes \hat{x}_2) x_2 x_3 \right] f_4(\tau)
\]

\[
+ \frac{15Q \otimes r_0}{4\pi \sigma} : \left[ (\hat{x}_1 \otimes \hat{x}_2 + \hat{x}_2 \otimes \hat{x}_1) x_1 x_2 + (\hat{x}_1 \otimes \hat{x}_3 + \hat{x}_3 \otimes \hat{x}_1) x_1 x_3 \right] f_5(\tau)
\]

\[
+ O \left( \frac{x^3}{\tau} \right).
\]
where the symbol $O((x/\alpha)^3)$ offers a measure of the ratio between the coordinates of the point of observation and the geometrical characteristics of the spheroidal conductor. The double contraction is defined as in (61) and the functions $f_i(\tau)$, $i = 1, 2, 3, 4, 5$, are given by

$$f_1(\tau) = \frac{1}{c^3} \left[ \ln \left( \frac{\tau + 1}{\tau - 1} \right) - \frac{1}{\tau} - \ln \left( \frac{\alpha_1 + c}{\alpha_2} \right) + \frac{\alpha_1 c^2}{\alpha_2^2} \right],$$

$$f_2(\tau) = \frac{1}{2c^3} \left[ - \frac{1}{2} \ln \left( \frac{\tau + 1}{\tau - 1} \right) - \frac{\tau}{\tau^2 - 1} + \ln \left( \frac{\alpha_1 + c}{\alpha_2} \right) + \frac{(\alpha_1^2 - 2\alpha_2^2)}{\alpha_1 \alpha_2^2} c \right],$$

$$f_3(\tau) = -\frac{3}{8c^3} \left[ \ln \left( \frac{\tau + 1}{\tau - 1} \right) - \frac{6\tau}{3\tau^2 - 1} \right] + \frac{3}{4c^3} \left[ \ln \left( \frac{\alpha_1 + c}{\alpha_2} \right) - \frac{c(2\alpha_1^4 + 2\alpha_2^4 + 5\alpha_1^2 \alpha_2^2)}{3\alpha_1 \alpha_2^2 (2\alpha_1^2 + \alpha_2^2)} \right].$$

$$f_4(\tau) = -\frac{3}{16c^3} \left[ \ln \left( \frac{\tau + 1}{\tau - 1} \right) - \frac{2\tau (3\tau^2 - 5)}{3(\tau^2 - 1)^2} \right] - \frac{3}{8c^3} \left[ \ln \left( \frac{\alpha_1 + c}{\alpha_2} \right) - \frac{c(2\alpha_1^4 + 4\alpha_2^4 - 3\alpha_1^2 \alpha_2^2)}{3\alpha_1 \alpha_2^2 (2\alpha_1^2 + \alpha_2^2)} \right].$$

$$f_5(\tau) = \frac{3}{4c^3} \left[ - \ln \left( \frac{\tau + 1}{\tau - 1} \right) + \frac{2(3\tau^2 - 2)}{3(\tau^2 - 1)^2} \right] + \frac{3}{2c^3} \left[ \ln \left( \frac{\alpha_1 + c}{\alpha_2} \right) + \frac{c\alpha_1 (\alpha_1^2 - 7\alpha_2^2)}{3\alpha_1 \alpha_2^2 (2\alpha_1^2 + \alpha_2^2)} \right].$$

On the other hand, the reduction of the exterior electric potential $u^+$ from the ellipsoidal form (64) to the corresponding prolate spheroidal one utilizes the identities (96)–(98) together with the Cartesian form (33)–(37) of ellipsoidal harmonics. Using also the limits (84)–(95) for the ellipsoidal parameters and the elliptic integrals, we conclude the following reduced form for $u^+$ in the exterior of the prolate spheroidal conductor

$$u^+_{sd}(\tau) = \lim_{\epsilon \to 0^+} u^+_{sd}(\tau) = b^o_{10} g_0(\tau) + \frac{3Q \otimes r}{4\pi \sigma} : \left( \hat{x}_1 \otimes \hat{x}_1 \right) g_1(\tau) + \left( \hat{r} - \hat{x}_1 \otimes \hat{x}_1 \right) g_2(\tau)$$

$$+ \frac{15Q \otimes r_0}{4\pi \sigma} : \left( \hat{r} - 3\hat{x}_1 \otimes \hat{x}_1 \right) \left[ \frac{2x_1^2 - x_2^2 - x_3^2}{2} - \frac{c^2}{3} \right] g_3(\tau)$$

$$+ \frac{15Q \otimes r_0}{4\pi \sigma} : \left[ \hat{x}_2 \otimes \hat{x}_2 - \hat{x}_3 \otimes \hat{x}_3 \right] \left( \frac{x_2^2 - x_3^2}{2} \right)$$

$$+ \left[ \hat{x}_2 \otimes \hat{x}_3 + \hat{x}_3 \otimes \hat{x}_2 \right] x_2 x_3 \right) g_4(\tau)$$

$$+ \frac{15Q \otimes r_0}{4\pi \sigma} : \left[ \hat{x}_1 \otimes \hat{x}_2 + \hat{x}_2 \otimes \hat{x}_1 \right] x_1 x_2 + \left[ \hat{x}_1 \otimes \hat{x}_3 + \hat{x}_3 \otimes \hat{x}_1 \right] x_1 x_3 \right) g_5(\tau)$$

$$+ O \left( \left( \frac{\alpha}{\lambda} \right)^4 \right).$$

(105)
The notation used in (105) is the same as that in (99), while the functions \( g_i(\tau), i = 0, 1, 2, 3, 4, 5 \), are listed below:

\[
g_0(\tau) = \frac{\ln \frac{\tau + 1}{\tau - 1}}{2 \ln \frac{\alpha_1 + \alpha_2}{\alpha_1}}, \tag{106}
\]

\[
g_1(\tau) = \frac{1}{2\alpha_1\alpha_2^2} \left( \ln \frac{\tau + 1}{\tau - 1} - \frac{\alpha_1}{\alpha_2} \right), \tag{107}
\]

\[
g_2(\tau) = \frac{1}{2\alpha_1\alpha_2^2} \ln \left( \frac{\alpha_1 + \alpha_2}{\alpha_2} \right), \tag{108}
\]

\[
g_3(\tau) = -\frac{1}{4\alpha_1\alpha_2^4} \left( \ln \frac{\tau + 1}{\tau - 1} - \frac{6\tau}{3\tau^2 - 1} \right), \tag{109}
\]

\[
g_4(\tau) = \frac{1}{4\alpha_1\alpha_2^4} \left( \ln \frac{\tau + 1}{\tau - 1} - \frac{2\tau(3\tau^2 - 5)}{3\tau^2 - 12} \right). \tag{110}
\]

The expressions (99) and (105) for the interior and exterior electric potential provide a clear view of the way that physics and geometry enter the scene. Actually, the projections of \( Q, r \) and \( r_0 \) on the axes are weighted by the different functions \( f_i(\tau) \) and \( g_i(\tau) \) in order to assign the results to the interior or to the exterior space, respectively. In both expressions, the effect that the axis of symmetry has on the resulting potential is distinguished from the corresponding effect that the other two axes have.

We finally discuss the case of an oblate spheroid which is defined also by (71) but with \( \alpha_1 < \alpha_2 \). The oblate spheroidal coordinates are given by (74)–(76) after an interchange between sinh \( \xi \) and cosh \( \xi \) is performed [14].

If \( \bar{c} \) denotes the semifocal distance of an oblate spheroid, then formal expressions can be obtained from the corresponding prolate ones through the substitution [3]

\[
c \rightarrow -i\bar{c}, \tag{111}
\]

\[
\tau = \cosh \xi \rightarrow i \sinh \xi. \tag{112}
\]

5. The spherical case

We turn now to the reduction of the ellipsoidal results to the corresponding spherical ones, as the ellipsoid (1) tends to the sphere (77).

In this case

\[
(\alpha_1, \alpha_2, \alpha_3) \rightarrow (\alpha, \alpha, \alpha), \tag{113}
\]

while

\[
\lim_{\text{el} \to \text{sr}} A = \lim_{\text{el} \to \text{sr}} A' = \alpha^2 \]. \tag{114}
and
\[
\lim_{\epsilon \to 0} h_i = 0, \quad i = 1, 2, 3.
\] (115)

The ellipsoidal coordinates \((\rho, \mu, \nu)\) at the spherical limit are given by (78)–(81), while
\[
\lim_{\epsilon \to 0} \mu = \lim_{\epsilon \to 0} v = 0.
\] (116)

The reduction of the elliptic integrals (24) leads to the limit
\[
\lim_{\epsilon \to 0} I''_m(\rho) = \frac{1}{(2n + 1)r^{2n+1}}
\] (117)

for each \(n = 0, 1, 2, \ldots\) and \(m = 1, 2, \ldots, 2n + 1\), and the combined identities needed to handle the indeterminacies are the following:
\[
\frac{E_1(\rho, \mu, \nu)}{\Lambda(\Lambda - \Lambda')(\Lambda - \alpha_m^2)} = \frac{E_2(\rho, \mu, \nu)}{\Lambda'(\Lambda - \Lambda')(\Lambda' - \alpha_m^2)}
\]
\[
= \frac{1}{\Lambda \Lambda'} \sum_{k=1}^{3} x_k^2 \left[ \frac{3}{2} (\Lambda + \Lambda') - \alpha_k^2 - \alpha_m^2 + 3 \delta_{km}(\alpha_k^2 - \Lambda - \Lambda') \right] + 1 - \frac{\alpha_1^2 \alpha_2^2 \alpha_3^2}{\alpha_m^2 \Lambda \Lambda'}
\] (118)

and
\[
\frac{E_1(\rho, \mu, \nu)}{\Lambda - \alpha_1^2} = (\Lambda' - \alpha_2^2)(x_3^2 - x_1^2) + (\Lambda' - \alpha_3^2)(x_2^2 - x_1^2) + (\Lambda' - \alpha_1^2)(\Lambda' - \alpha_3^2),
\] (120)

\[
\frac{E_1(\rho, \mu, \nu)}{\Lambda - \alpha_2^2} = (\Lambda' - \alpha_1^2)(x_3^2 - x_1^2) + (\Lambda' - \alpha_3^2)(x_1^2 - x_2^2) + (\Lambda' - \alpha_1^2)(\Lambda' - \alpha_3^2),
\] (121)

\[
\frac{E_1(\rho, \mu, \nu)}{\Lambda - \alpha_3^2} = (\Lambda' - \alpha_1^2)(x_2^2 - x_3^2) + (\Lambda' - \alpha_2^2)(x_1^2 - x_2^2) + (\Lambda' - \alpha_1^2)(\Lambda' - \alpha_2^2),
\] (122)

where
\[
I'_1(\rho) = \int_{t}^{+\infty} \frac{dt}{t^2(h_2^2 - \alpha_1^2)^{3/2}(t^2 - h_3^2)^{3/2}}.
\] (123)
Taking the spherical limit of (118)–(122) and using (113)–(117), the polyadics \( \hat{A}(\rho) \) and \( \hat{B}(\mathbf{r}) \) and \( \tilde{\Gamma}(\rho) \) that carry the ellipsoidal geometry in expression (57) for the interior electric potential \( u^- \) enjoy the following unique limits:

\[
\lim_{\alpha \to \rho} \hat{A}(\rho) = \frac{1}{4\pi \sigma} \left( \frac{1}{r^3} + \frac{2}{\alpha^2} \right) \mathbf{I},
\]

\[
\lim_{\alpha \to \rho} \hat{B}(\mathbf{r}) = \frac{3}{4\pi \sigma} \left[ -\left( \frac{1}{3r^3} + \frac{r^2}{2\alpha^2} \right) \mathbf{I} + \left( \frac{1}{r^3} + \frac{3}{2\alpha^2} \right) \sum_{m=1}^{3} \mathbf{x}_m \otimes \mathbf{x}_m \right],
\]

and

\[
\lim_{\alpha \to \rho} \tilde{\Gamma}(\rho) = \frac{3}{4\pi \sigma} \left( \frac{1}{r^3} + \frac{3}{2\alpha^2} \right) \sum_{i,j=1}^{3} \mathbf{x}_i \otimes \mathbf{x}_j \otimes \mathbf{x}_i \otimes \mathbf{x}_j.
\]

Then, the interior electric potential in the ellipsoidal shell is reduced to the corresponding field in the interior of the spherical shell \( r_0 < r < \alpha \),

\[
u^-_{\alpha}(\mathbf{r}) = \lim_{\alpha \to \rho} \nu^-_{\alpha}(\mathbf{r}) = b_0 + \frac{3}{4\pi \sigma} \mathbf{Q} \cdot \hat{\mathbf{r}} \left[ -\left( \frac{1}{3r^3} + \frac{r^2}{2\alpha^2} \right) \mathbf{I} + \left( \frac{1}{r^3} + \frac{3}{2\alpha^2} \right) \mathbf{r} \otimes \mathbf{r} \right]
\]

\[
+ \frac{3}{4\pi \sigma} \mathbf{Q} \otimes \mathbf{r}_0 : \left[ -\left( \frac{1}{3r^3} + \frac{r^2}{2\alpha^2} \right) \mathbf{I} + \left( \frac{1}{r^3} + \frac{3}{2\alpha^2} \right) \mathbf{r} \otimes \mathbf{r} \right]
\]

\[
+ O\left( \frac{1}{r^4} \right) + O\left( \left( \frac{r}{\alpha} \right)^3 \right).
\]

where the symbol \( r/\alpha \) provides a measure of the approximation. The reduction of the exterior electric potential, given in (70), to the spherical limit demands the use of more complicated identities combining ellipsoidal harmonics and elliptic integrals, that lead to the following key formula:

\[
\lim_{\alpha \to \rho} \left\{ \frac{1}{T_2(\rho)} \frac{\mathcal{E}_1(\rho, \mu, \nu)}{T_2(\alpha)} \left[ \lambda(A - A')(A - A') \right] - \frac{1}{T_2(\rho)} \frac{T_2(\mu)}{T_2(\alpha)} \left[ \lambda'(A - A')(A' - A) \right] \right\}
\]

\[
= \frac{\alpha^3}{r^3} - \frac{3\alpha^3}{r^3} \mathbf{r}_m^2.
\]

Then, using (113)–(117) and (128) to evaluate the limiting forms of (67), (68) and (69) in (70) we obtain

\[
u^-_{\alpha}(\mathbf{r}) = \lim_{\alpha \to \rho} \nu^-_{\alpha}(\mathbf{r}) = b_0 \frac{\alpha}{r} + \frac{3}{4\pi \sigma} \frac{\mathbf{Q} \cdot \hat{\mathbf{r}}}{r^2}
\]

\[
+ \frac{5}{8\pi \sigma} \left( \frac{\mathbf{Q} \cdot \mathbf{r}_0}{r^3} + \frac{3}{r^3} \mathbf{Q} \otimes \mathbf{r}_0 : \mathbf{r} \right) + O\left( \frac{1}{r^4} \right).
\]
At this stage, we have concluded the evaluation of the interior and the exterior electric potential produced by a current dipole source inside the human brain, when this is modeled as an ellipsoid, as a spheroid and as a sphere.

A comparison between the corresponding results for the three geometrical head models, the ellipsoidal (57) and (70), the prolate spheroidal (99) and (105), and the spherical (127) and (129), reveals the existence of a geometric operator that describes the reduction, while the dipole operates in an invariable way for all models. It is clear that through this reduction process the geometric operator, that defines the exterior electric potential, gradually looses the influence of the boundary of the conductor, as this tends to become a spherical surface. This means that as the symmetry of the geometrical model used is increased, the boundary of the conductive domain becomes more invisible in the evaluation of the exterior electric potential. It is also clear that the dipole enters into the first terms of the multipole expansion, of both the interior and the exterior electric potential, through only its moment, while in the following terms the position vector also makes its appearance.

Obviously, an analytic expression in terms of Cartesian coordinates provides a more familiar notation and therefore a more easy to understand form for both results.

To this end the interior electric potential for the spherical brain model can be expressed in Cartesian coordinates as

\[
\begin{aligned}
\tilde{u}_{\text{int}}(r) &= b_1 + \frac{1}{2\pi \sigma} \left( \frac{1}{2r^3} + \frac{1}{\alpha^3} \right) \sum_{i=1}^{3} Q_i x_i - \frac{3}{4\pi \sigma} \left( \frac{1}{3r^3} + \frac{r^2}{2\alpha^5} \right) \sum_{i=1}^{3} Q_i x_0_i \\
&+ \frac{3}{4\pi \sigma} \left( \frac{1}{3r^3} + \frac{3}{2\alpha^5} \right) \sum_{i=1}^{3} Q_i x_0_i x_i^2 + \frac{3}{4\pi \sigma} \left( \frac{1}{r^5} + \frac{3}{2\alpha^5} \right) \sum_{i,j=1, i \neq j}^{3} Q_i x_0_j x_i x_j \\
&+ O \left( \frac{1}{r^4} \right).
\end{aligned}
\]  

for \( r_0 < r < \alpha \), where the notation is defined as in (127).

The corresponding Cartesian form of the exterior electric potential for the spherical case is written as

\[
\begin{aligned}
\tilde{u}_{\text{ext}}(r) &= b_1 + \frac{3}{4\pi \sigma \alpha r^3} \sum_{i=1}^{3} Q_i x_i - \frac{5}{8\pi \sigma \alpha r^3} \sum_{i=1}^{3} Q_i x_0_i \\
&+ \frac{15}{8\pi \sigma \alpha r^3} \sum_{i=1}^{3} Q_i x_0_i x_i^2 + \frac{15}{16\pi \sigma \alpha r^5} \sum_{i,j=1, i \neq j}^{3} Q_i x_0_j x_i x_j \\
&+ O \left( \frac{1}{r^4} \right).
\end{aligned}
\]  

Obviously, the expressions (130) and (131) can also be obtained by solving independently the corresponding boundary value problems in spherical geometry.

We end this report by noting that the inverse EEG and MEG problems in ellipsoidal geometry is under current investigation.
Acknowledgments

The author thanks Professor George Dassios of the University of Patras and Professor Athanassios Fokas of Cambridge University for fruitful discussion during the preparation of this work.

References