ECT-B-splines defined by generalized divided differences

G. Mühlbach

Institut für Angewandte Mathematik, Universität Hannover, Germany

Received 4 October 2004

Abstract

ECT-spline curves are generated from different local ECT-systems via connection matrices. If they are nonsingular, lower triangular and totally positive there is a basis of the space of ECT-splines consisting of functions having minimal compact supports, normalized either to form a nonnegative partition of unity or to have integral one. In this paper such ECT-B-splines are defined by generalized divided differences. This definition reduces to the classical one in case of a Schoenberg space. Under suitable assumptions it leads to a recursive method for computing the ECT-B-splines that reduces to the de Boor–Mansion–Cox recursion in case of ordinary polynomial splines and to Lyche’s recursion in case of Tchebycheff splines [Mühlbach and Tang, Calculation of ECT-B-splines and of ECT-spline curves recursively, in preparation].

There is an ECT-spline space naturally adjoint to every ECT-spline space. We also construct B-splines via generalized divided differences for this space and study relations between the two adjoint spaces.

© 2005 Elsevier B.V. All rights reserved.

MSC: 41A15; 41A05

Keywords: ECT-systems; ECT-B-splines; Generalized divided differences

1. ECT-systems and their duals, rET- and lET-systems

Let \( J \) be a subinterval of the real line \( \mathbb{R} \) that is open to the right. For \( n \in \mathbb{N}_0 \) let

\[
C^n_r(J; \mathbb{R}) := \{ f \in C(J; \mathbb{R}) : \text{for every } x \in J \text{ and for } v = 1, \ldots, n \text{ there exists the right derivative of } f \text{ of order } v \text{ at } x \text{ and } J \ni x \mapsto D^v_f(x) \text{ is right continuous}. \]

E-mail address: mb@ifam.uni-hannover.de.
A system of functions \( U = (u_1, \ldots, u_n) \) in \( C_r^{n-1}(J; \mathbb{R}) \) is called a right-sided extended Tchebycheff system (rET-system, for short) of order \( n \) on \( J \) provided for all \( T = (t_1, \ldots, t_n), t_1 \leq \cdots \leq t_n, \ t_j \in J \),

\[
V \begin{vmatrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{vmatrix}_r := \det(D_r^{v_j} u_i(t_j))|_{i,j=1,\ldots,n} > 0,
\]

with \( v_j := \max\{l : t_j = t_{j-1} = \cdots = t_{j-l} \geq t_1\} , \ j = 1, \ldots, n \), where \( D_r f(x) := \lim_{h \to 0^+} (f(x + h) - f(x))/h \) denotes the operator of ordinary right differentiation. Then \( \text{span} \ U \) will be called an rET-space of dimension \( n \) on \( J \).

If \( q \in \text{span} \ U \), where \( U \) is an rET-system of order \( n \) on \( J \), a point \( x_0 \in J \) is called a zero of \( q \) of right multiplicity \( v_0 \) iff \( q(x_0) = 0, D_r^1q(x_0) = 0, \ldots, D_r^{v_0-1}q(x_0) = 0, D_r^{v_0}q(x_0) \neq 0 \).

If \( J \) is any subinterval of \( \mathbb{R} \) and above everywhere except at an endpoint of \( J \) the right derivative \( D_r \) can be replaced by the ordinary derivative \( D = \frac{d}{dx} \) where at a right end point \( D \) is replaced by the left derivative then \( U \) is called an extended Tchebycheff System (ET-system) of order \( n \) on \( J \).

The following characterization of rET-spaces is an immediate consequence of the Alternative Theorem of Linear Algebra, as is the corresponding well known characterization for ET-spaces (cf. [7], p. 376).

**Proposition 1.1.** Let \( U = (u_1, \ldots, u_n) \) be elements of class \( C_r^{n-1}(J; \mathbb{R}) \). Then the following assertions are equivalent:

(i) \( (u_1, \ldots, u_{n-1}, u_n) \) or \( (u_1, \ldots, u_{n-1}, -u_n) \) is an rET-system of order \( n \) on \( J \).

(ii) Every nontrivial element of \( \text{span}(u_1, \ldots, u_n) \) has at most \( n - 1 \) zeros in \( J \) counting right multiplicities.

(iii) Every problem of right sided Hermite interpolation

\[
H(U, T+, f): \begin{cases} \text{given points } t_1 \leq \cdots \leq t_n \text{ in } J, \\ \text{given } f \in C_r^{n-1}(J; \mathbb{R}), \\ \text{find } q \in \text{span} \ U \text{ such that} \\ D_r^{v_j} q(t_j) = D_r^{v_j} f(t_j) \ j = 1, \ldots, n \end{cases}
\]

has a unique solution.

Analogously, left sided ET-systems and ET-spaces and related concepts as the problem of left sided Hermite interpolation \( H(U, T-, f) \) are defined. In the analysis of dual functionals to ECT-B-splines naturally certain rET-and ET-spaces arise (see (37) and (38) below) that are no ET-spaces.

If \( U = (u_1, \ldots, u_n) \) is an rET-system on \( J \) then the leading coefficient (that before \( u_n \)) of the unique \( q \in \text{span} \ U \) that solves \( H(U, T+, f) \) is called the right sided generalized divided difference of \( f \) with respect to \( u_1, \ldots, u_n \) and with nodes \( t_1, \ldots, t_n \). By Cramer’s rule it is

\[
\begin{vmatrix} u_1, \ldots, u_n \\
\hline t_1, \ldots, t_n \end{vmatrix}_r f = \frac{\begin{vmatrix} u_1, \ldots, u_{n-1}, f \\ t_1, \ldots, t_{n-1}, t_n \end{vmatrix}_r}{\begin{vmatrix} u_1, \ldots, u_{n-1}, u_n \\ t_1, \ldots, t_{n-1}, t_n \end{vmatrix}_r}.
\]
Developing the numerator determinant along its last column one sees

\[
\begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_n
\end{bmatrix}_{r} = f(t_{n+1}) = \sum_{j=1}^{n} c_j D_{r}^{v_j} f(t_j), \quad c_n = \frac{V \begin{bmatrix}
    u_1, \ldots, u_{n-1} \\
    t_1, \ldots, t_{n-1}
\end{bmatrix}_{r}}{V \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_n
\end{bmatrix}_{r}},
\]

(1)

with coefficients \(c_j\) that do not depend on \(f\).

It is known \([10]\) that if \((u_1, \ldots, u_{n+1}), (u_1, \ldots, u_n)\) are ECT-systems, and, if \(n \geq 2\), also \((u_1, \ldots, u_{n-1})\) is an ECT-system, then if \(t_1 \neq t_{n+1}\)

\[
\begin{bmatrix}
    u_1, \ldots, u_{n+1} \\
    t_1, \ldots, t_{n+1}
\end{bmatrix}_{r} = \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_2, \ldots, t_{n+1}
\end{bmatrix}_{r} f(t_{n+1}) - \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_{n}
\end{bmatrix}_{r} u_{n+1}.
\]

(2)

This formula holds for the right sided generalized divided differences as well: if \((u_1, \ldots, u_{n+1}), (u_1, \ldots, u_n)\) are rET-systems, and, if \(n \geq 2\), also \((u_1, \ldots, u_{n-1})\) is an rECT-systems, then if \(t_1 \neq t_{n+1}\)

\[
\begin{bmatrix}
    u_1, \ldots, u_{n+1} \\
    t_1, \ldots, t_{n+1}
\end{bmatrix}_{r} = \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_2, \ldots, t_{n+1}
\end{bmatrix}_{r} f(t_{n+1}) - \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_{n}
\end{bmatrix}_{r} u_{n+1}.
\]

(3)

The condition that \((u_1, \ldots, u_j)\) for \(j = n-1, n, n+1\) are rET-systems on \(J\) is sufficient but not necessary for (2) to hold. If \(t_1 \neq t_{n+1}\) and

\[
V \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_n
\end{bmatrix}_{r} V \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_2, \ldots, t_{n+1}
\end{bmatrix}_{r} \neq 0,
\]

then

\[
V \begin{bmatrix}
    u_1, \ldots, u_n, u_{n+1} \\
    t_1, \ldots, t_{n+1}
\end{bmatrix}_{r} V \begin{bmatrix}
    u_1, \ldots, u_{n-1} \\
    t_1, \ldots, t_n
\end{bmatrix}_{r} \neq 0
\]

\[
\Leftrightarrow \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_2, \ldots, t_{n+1}
\end{bmatrix}_{r} u_{n+1} = \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_n
\end{bmatrix}_{r} u_{n+1} \neq 0
\]

(4)

and for every function \(f\) (2) holds. Indeed, by Sylvester’s identity on determinants \([11]\) with \(1 \leq i := \max \{j : t_1 = \cdots = t_j\} \leq n\) there holds

\[
\begin{align*}
V \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_n
\end{bmatrix}_{r} & V \begin{bmatrix}
    u_1, \ldots, u_{n-1}, u_{n+1} \\
    t_1, \ldots, t_n
\end{bmatrix}_{r} \\
V \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_{n+1}
\end{bmatrix}_{r} & V \begin{bmatrix}
    u_1, \ldots, u_{n-1}, u_{n+1} \\
    t_1, \ldots, t_{n+1}
\end{bmatrix}_{r} \\
V \begin{bmatrix}
    u_1, \ldots, u_n \\
    t_1, \ldots, t_{n+1}
\end{bmatrix}_{r} & V \begin{bmatrix}
    u_1, \ldots, u_{n-1} \\
    t_1, \ldots, /t_i \ldots, t_n
\end{bmatrix}_{r}
\end{align*}
\]

(5)
Here we use the notation $/t_i$ meaning that the node $t_i$ does not occur. Computing the determinant of 4 determinants and dividing by (3) yields (4) since

$$
(t_1, \ldots, t_{i-1}) = (t_1, \ldots, t_1) = (t_2, \ldots, t_i).
$$

The recurrence relation itself is obtained if in (5) $u_{n+1}$ is replaced by $f$ and this result is divided by the members of Eq. (5).

Analogously, left sided generalized divided differences

$$
\left[ u_1, \ldots, u_n \atop t_1, \ldots, t_n \right]_l
$$

are defined. Of course, for them also a recurrence relation (2) holds with $r$ replaced by $l$, provided (3) is assumed to hold with $r$ replaced by $l$.


**Proposition 1.2.** Let $u_1, \ldots, u_n$ be of class $C^{n-1}(J; \mathbb{R})$. Then the following assertions are equivalent:

(i) $(u_1, \ldots, u_n)$ is an ECT-system of order $n$ on $J$.

(ii) All Wronskian determinants

$$
W(u_1, \ldots, u_k)(x) = \det(D^{j-1}u_i(x))_{i=1, \ldots, j=1, \ldots, k} > 0, \quad k = 1, \ldots, n; \quad x \in J
$$

are positive on $J$. As before $D := d/dx$ denotes the operator of ordinary differentiation.

(iii) There exist positive weight functions $w_j \in C^{n-j}(J; \mathbb{R})$, $j = 1, \ldots, n$, and for every $c \in J$ coefficients $c_{j,i} \in \mathbb{R}$ such that

$$
u_j(x) = w_1(x) \int_c^x w_2(t_2) \int_c^{t_2} w_3(t_3) \int_c^{t_3} \cdots \int_c^{t_{j-1}} w_j(t_j) \, dt_j \cdots dt_2
+ \sum_{i=1}^{j-1} c_{j,i}u_i(x), \quad j = 1, \ldots, n, \quad x \in J.
$$

Clearly, the functions $s_j(x, c) := u_j(x) - \sum_{i=1}^{j-1} c_{j,i}u_i(x)$ $j = 1, \ldots, n$ satisfy

$$
s_j(x, c) = w_1(x)h_{j-1}(x, c; w_2, \ldots, w_j) \quad j = 1, \ldots, n,
$$

where $h_0(x, c) := 1$ and for $1 \leq m \leq n$

$$
h_m(x, c; w_1, \ldots, w_m) := \int_c^x w_1(t)h_{m-1}(t, c; w_2, \ldots, w_m) \, dt.
$$

The system (8) $(s_1, \ldots, s_n)$ forms a special basis of span$(u_1, \ldots, u_n)$ which we call an ECT-system in canonical form with respect to $c$. 

Example 1.1. If \( w_j = 1 \) for \( j = 1, \ldots, n \), where \( 1 \) denotes the constant function equal to one then

\[
h_m(x, c; 1, \ldots, 1) = \frac{(x - c)^m}{m!}, \quad m = 0, \ldots, n \quad \text{and}
\]

\[
s_j(x, c) = \frac{(x - c)^{j-1}}{(j-1)!}, \quad j = 1, \ldots, n
\]

and \( \text{span}\{s_1, \ldots, s_n\} = \mathbb{P}_{n-1} \), the space of ordinary polynomials of degree \( n - 1 \) or of order \( n \) at most.

Example 1.2 (cf. also Buchwald and Mühlbach [3], Tang and Mühlbach [14]). If \( n \geq 3 \) and \( w_j = 1 \) for \( j = 1, \ldots, n - 2 \),

\[
w_{n-1}(x) = \frac{(n - 2)!}{(x - a + \varepsilon)^{n-1}}, \quad w_n(x) = \frac{(n - 1)(b - a + 2\varepsilon)(x - a + \varepsilon)^{n-2}}{(b + \varepsilon - x)^n},
\]

with \( \varepsilon > 0 \) a parameter, then for any \( c \in [a, b] \)

\[
s_j(x, c) = \frac{(x - c)^{j-1}}{(j-1)!}, \quad j = 1, \ldots, n - 2
\]

\[
s_{n-1}(x, c) = \frac{(x - c)^{n-2}}{(x - a + \varepsilon)(c - a + \varepsilon)^{n-2}}
\]

\[
s_n(x, c) = \frac{(x - c)^{n-1}(b - a + 2\varepsilon)}{(x - a + \varepsilon)(b + \varepsilon - x)(b + \varepsilon - c)^{n-1}}
\]

is a \textit{Cauchy–Vandermonde-system} in canonical form with respect to \( c \) whose first \( n - 2 \) functions are polynomials and the last two are proper rational functions, \( s_{n-1} \) having a pole of order 1 at \( x = a - \varepsilon \) and \( s_n \) having poles of order 1 at \( x = a - \varepsilon \) and at \( x = b + \varepsilon \).

Associated with an ECT-system (7) or (8) are the linear differential operators

\[
D_0 u = u, \quad D_j u = D \left( \frac{u}{w_j} \right), \quad j = 1, \ldots, n,
\]

\[
\hat{L}_j u = D_j \cdots D_0 u, \quad j = 0, \ldots, n,
\]

\[
L_j u = \frac{1}{w_{j+1}} \hat{L}_j u, \quad j = 0, \ldots, n - 1.
\]

Obviously, \( \ker \hat{L}_j = \text{span}\{u_1, \ldots, u_j\}, \quad j = 1, \ldots, n \), and

\[
L_j s_{j+1}(x, c) = 1, \quad j = 0, \ldots, n - 1,
\]

\[
L_j s_{l+1}(c, c) = \delta_{j,l}, \quad j, l = 0, \ldots, n - 1.
\]
There is a Taylor’s Theorem with respect to ECT-systems. The initial value problem
\[
\hat{L}_n u(x) = f(x), \quad x \in J, \\
L_j u(c) = c_j, \quad j = 0, \ldots, n - 1,
\]
with \( f \in C(J; \mathbb{R}) \) and \( c_j \in \mathbb{R} \) given has the solution
\[
u(x) = \sum_{j=0}^{n-1} c_j s_{j+1}(x, c) + \int_c^x f(t) s_n(x, t) \, dt.
\]
This is easily checked by differentiating (13) according to Leibniz’ rule in view of (10). Accordingly, the Taylor interpolation problem \( H(U, T, f) \) for \( T = \{c, \ldots, c\} \) with \( c \) repeated \( n \) times and with \( f \in C^n(J; \mathbb{R}) \) has the solution
\[
p_f(x) = \sum_{j=0}^{n-1} L_j(c) s_{j+1}(x, c),
\]
with remainder term (cf. [13], p. 425)
\[
f(x) = p_f(x) + \int_c^x \hat{L}_n f(t) s_n(x, t) \, dt.
\]

For later use we note
\[
\frac{\partial}{\partial x} h_m(x, c; w_1, \ldots, w_m) = w_1(x) h_{m-1}(x, c; w_2, \ldots, w_m),
\]
\[
\frac{\partial}{\partial c} h_m(x, c; w_1, \ldots, w_m) = -w_m(c) h_{m-1}(x, c; w_1, \ldots, w_{m-1}),
\]
\[
\left( \frac{\partial}{\partial x} \right)^i \left( \frac{\partial}{\partial c} \right)^j h_m(x, c; w_1, \ldots, w_m)|_{x=c} = \begin{cases} 0, & i + j < m, \\ (-1)^i w_1(c) \cdots w_m(c), & i + j = m. \end{cases}
\]

Formulas (16), (17) and (18) follow from the definition (9) using Leibniz’ formula for differentiating an integral which for (17) and (18) must be applied repeatedly.

It is important that for \( j = 0, \ldots, n \) the \( j \)th reduced system of \( u_1, \ldots, u_n \) consisting of the functions \( u_{j,1}, \ldots, u_{j,n-j} \)
\[
u_{j,l-j} = L_j u_l, \quad l = j + 1, \ldots, n,
\]
again is an ECT-system on \( J \) with corresponding weight functions \( 1, w_{j+2}, \ldots, w_n \).

Associated with any ECT-system \( U = (s_j)_{j=1}^n \) of order \( n \) on \( J \) in canonical form with respect to \( c \in J \) with weights \( w_1, \ldots, w_n \) we define its dual canonical system \( U^* = (s_i^*)_{i=1}^n \) with respect to \( c \in J \) by
\[
s_{j,n}^*(x, c) := h_{j-1}(x, c; w_n, \ldots, w_{n+2-j}), \quad j = 1, \ldots, n.
\]
It is again an ECT-system of order $n$ on $J$ with weights $(w_1^*, \ldots, w_n^*) = (1, w_n, \ldots, w_2)$ provided
\[ w_j \in C^{\max\{n-j+2\}}(J; \mathbb{R}), \quad j = 2, \ldots, n. \] (21)

Assuming this, with the dual canonical ECT-system with respect to $c$ associated are the linear differential operators
\[ D_0 f = f, \quad D_1 f = Df, \quad D_j f = D \left( \frac{f}{w_{n+2-j}} \right), \quad j = 2, \ldots, n, \]
\[ \hat{L}_j^* f = D_j \cdots D_0 f, \quad j = 0, \ldots, n, \]
\[ L_0^* f = f, \quad L_j^* f = \frac{1}{w_{n+1-j}} \hat{L}_j^* f, \quad j = 1, \ldots, n. \]

Clearly, \( \ker \hat{L}_j^* = \text{span}\{s_1^*, \ldots, s_j^*, \ldots, s_n^*\} \), \( j = 1, \ldots, n \) is the null space of \( \hat{L}_j^* \) and
\[ L_j^* s_{j+1}^*(x, c) = 1, \quad j = 0, \ldots, n, \]
\[ L_j^* s_{j+1}^*(c, c) = \delta_{j,l}, \quad j, l = 0, \ldots, n. \]

The operator \((-1)^n L_n^*\) is the formal adjoint of the operator \( \hat{L}_n \) since
\[ \int_a^b \psi(x) \hat{L}_n \varphi(x) \, dx = \int_a^b \varphi(x)(-1)^n L_n^* \psi(x) \, dx \quad \text{for all } \varphi, \psi \in C_0^n(J; \mathbb{R}) \]
having compact supports in \( \text{int} J \).

If the initial value problem (11), (12) is considered as a boundary value problem on an interval \([c, d] \subset J\), its adjoint boundary value problem [4] is
\[ (-1)^n L_n^* v(y) = f(y), \]
\[ (-1)^j L_j^* v(d) = d_j, \quad j = 0, \ldots, n-1, \]
with the solution
\[ v(y) = \sum_{j=0}^{n-1} d_j (-1)^j s_{j+1,n}^*(y, d) + \int_d^y f(t)(-1)^n s_n^*(y, t) \, dt. \]

This follows directly from (13). Therefore, the Taylor interpolation problem \( H(U^*, T, f) \) for \( T = \{d, \ldots, d\} \) with \( d \) repeated \( n \) times and \( f \in C^n(J; \mathbb{R}) \) has the solution
\[ q f(y) = \sum_{j=0}^{n-1} L_j f(d) s_{j+1,n}^*(y, d), \] (22)
with remainder term
\[ f(y) = q f(y) + \int_y^d L_n f(t)(-1)^{n-1} s_{n,n}^*(y, t) \, dt. \] (23)
If
\[
g_j(x, y) := \begin{cases} w_1(x)h_{j-1}(x, y; w_2, \ldots, w_j), & x \geq y, \\ 0, & \text{otherwise}, \end{cases}
\]
then \( g(x, y) := g_n(x, y) \) has the characteristic behaviour of a Green’s function for the differential operator \( L_{n-1} \) acting on the variable \( x \), i.e.
\[
L_{n-1}g_n(x, y)|_{x=y-} = 0,
\]
\[
L_{n-1}g_n(x, y)|_{x=y+} = L_{n-1}s_n(x, y)|_{x=y} = 1.
\]
In particular, for \( x, y, c \in J \)
\[
h(x, y) := s_n(x, y) = w_1(x)h_{n-1}(x, y; w_2, \ldots, w_n)
\]
\[
= \sum_{k=1}^{n} (-1)^{n-k} s_k(x, c)s^*_{n+1-k, n}(y, c)
\]
\[
= (-1)^{n-1} w_1(x)h_{n-1}(y, x; w_n, \ldots, w_2)
\]
\[
= (-1)^{n-1} w_1(x)s^*_{n, n}(y, x),
\]
where the right-hand side of (24) is independent of \( c \) [8]. Consequences of (24) are
\[
s_{j+1}(x, c) = w_1(x)h_j(x, c; w_2, \ldots, w_{j+1}) = (-1)^j w_1(x)s^*_{j+1, j+1}(c, x)
\]
\[
= (-1)^j w_1(x)h_j(c, x; w_{j+1}, \ldots, w_2),
\]
\( j = 0, \ldots, n - 1 \)
and
\[
(-1)^{m-1-j}L_j s_m(x, y) = s^*_{m-j, m}(y, x), \quad 0 \leq j < m \leq n.
\]
In view of (24) if \( w_1 = 1 \) the two Taylor remainders (15) and (23) may be written in the form
\[
f(x) - p_f(x) = \int_c^d \hat{L}_n f(t)g(x, t) \, dt, \quad f(y) - q_f(y) = \int_c^d L^*_n f(t)g(t, y) \, dt.
\]

Example 1.1 (continued). If \( w_1 = \cdots = w_n = 1 \), then \( s^*_j(x, c) = (x - c)^{j-1}/(j - 1)! \), \( j = 1, \ldots, n \), and (24) reduces to the Binomial Theorem
\[
s_n(x, y) = h(x, y) = \frac{(x - y)^{n-1}}{(n-1)!} = \sum_{k=1}^{n} (-1)^{n-k} \frac{(x - c)^{k-1}}{(k-1)!} \cdot \frac{(y-c)^{n-k}}{(n-k)!}.
\]

Example 1.2 (continued, cf. Tang and Mühlbach [14]). If \( n \geq 3 \) and the weight functions are taken as in Example 1.2 then \( (w^*_1, \ldots, w^*_n) = (1, w_n, w_{n-1}, \ldots, w_2) \), and if for any \( c \in [a, b] \)
\[
\gamma(k, n, c) := \frac{(n-2)!}{(k-3)!} (c - a + e)^{k-1-n} \sum_{\kappa=0}^{k-3} \binom{k-3}{\kappa} \frac{(-1)^{k-3-\kappa}}{n-2-\kappa},
\]
where
\[ \delta(k, n) := (b - a + 2\varepsilon) \frac{(n - 1)!}{(k - 3)!}, \]

\[ \lambda(v, n) := (-1)^{n-1-v} \left( \frac{n - 1}{v} \right) (b - a + 2\varepsilon)^v, \quad 1 \leq v \leq n - 1, \]

\[ \mu(k, n, v, c) := \frac{1}{v} \left( \frac{k - 3}{n - v - 1} \right)^{v+k-n-2} \sum_{i=0}^{v+k-n-2} (-1)^{k-i} \left( \frac{v + k - n - 2}{i} \right) \]
\[ \times \left( b - a + 2\varepsilon \right)^i \left( c - a + i \right)^{v+k-n-2-i} \frac{1}{v - 1 - i}, \quad n - k - 2 \leq v \leq n - 1, \]

\[ \psi_v := \psi_v(x, b, c, \varepsilon) := \frac{1}{(b + \varepsilon - x)^v} - \frac{1}{(b + \varepsilon - c)^v}, \quad 1 \leq v \leq n - 1, \]

then

\[ s^{*}_{1,n}(x, c) = 1, \]

\[ s^{*}_{2,n}(x, c) = \sum_{v=1}^{n-1} \psi_v \lambda(2, n, v, c), \quad \lambda(2, n, v, c) = \lambda(v, n) \] (26)

and for \( 3 \leq k \leq n \)

\[ s^{*}_{k,n}(x, c) = \sum_{v=1}^{n-1} \psi_v \lambda(k, n, v, c), \] (27)

where

\[ \lambda(k, n, v, c) = \begin{cases} 
\gamma(k, n, c) \lambda(v, n), & 1 \leq v \leq n - k + 1, \\
\gamma(k, n, c) \lambda(v, n) + \delta(k, n) \mu(k, n, v, c), & n - k + 2 \leq v \leq n - 1. 
\end{cases} \]

The representations (26) and (27) are proved by calculating the integrals according to the definition of the dual system in its canonical form with respect to \( c \). In Example 1.2 according to (24)

\[ h(x, y) = (-1)^{n-1} s^{*}_{n,n}(y, x). \]

For convenience for \( \mu \in \mathbb{N} \) we will use the notations

\[ L^\mu[f](t) := \begin{pmatrix} L_0 f(t) \\
\vdots \\
L_{\mu-1} f(t) \end{pmatrix}, \quad L^\mu[f](t) := \begin{pmatrix} L_0^* f(t) \\
\vdots \\
L_{\mu-1}^* f(t) \end{pmatrix} \] (28)

for the ECT-derivative vectors of dimension \( \mu \) of a sufficiently smooth function \( f \). Also, we will use the limits

\[ L^\mu[f](t-) := \lim_{\tau \to t-0} L^\mu[f](\tau), \quad L^\mu[f](t+) := \lim_{\tau \to t+0} L^\mu[f](\tau), \]

\[ L^\mu[f](t-) := \lim_{\tau \to t-0} L^\mu[f](\tau), \quad L^\mu[f](t+) := \lim_{\tau \to t+0} L^\mu[f](\tau). \]
2. ECT-splines; the spaces $S_n(\mathcal{A}, \mathcal{A}^+, M, X)$ and $S_n(\mathcal{A}, \mathcal{A}^+, \xi_{\text{ext}})$

Assume that $x$ is a real number and that in nontrivial closed intervals $J_0 = [a, x]$ and $J_1 = [x, b]$ left and right to $x$ there are given two ECT-systems of order $n$

$$U^{[0]} := U^{[0]}_n := (u^{[0]}_1, \ldots, u^{[0]}_n), \quad U^{[1]} := U^{[1]}_n := (u^{[1]}_1, \ldots, u^{[1]}_n),$$

with weights $w_j^{[i]}$ ($i = 1, \ldots, n$; $i = 0, 1$) and associated linear differential operators $L^{[i]}_j$ ($j = 0, \ldots, n-1$; $i = 0, 1$), correspondingly. Suppose that $\mu$ is an integer, $0 \leq \mu \leq n$, and that $A$ is a square $(n-\mu)$-dimensional real matrix which is nonsingular. A function $s: [a, b] \mapsto \mathbb{R}$ such that $s|_{[a,x)} \in \text{span}\ U^{[0]}$ and $s|_{[x,b]} \in \text{span}\ U^{[1]}$ and

$$L^{[1]n-\mu}[s](x+) = A \cdot L^{[0]n-\mu}[s](x-),$$

where $L^{[i]n-\mu}[s](t)$ ($i = 0, 1$) denote the ECT-derivative vectors of $s$ at $t$ of dimension $n-\mu$ (28) is called $(U^{[0]}, U^{[1]}, A)$-smooth of order $n-\mu$ at $x$. Eq. (29) are called the connection equations of $s$ at the knot $x$ and $A$ is called a connection matrix at $x$. We allow $0 \leq \mu \leq n$, where in case $\mu = n$ there is no condition on $s$ at $x$. In case $\mu = 0$ the knot $x$ is a knot with no freedom. If $1 \leq \mu \leq n$ at $x$, given $s$ on $[a, x]$, there are $\mu$ degrees of freedom in extending $s$ to $[x, b]$ as a function belonging to span $U^{[1]}$ such that $s \in C^{n-1}_r([a, b]; \mathbb{R})$. Symmetrically, if $1 \leq \mu \leq n$ at $x$, given $s$ on $(x, b)$, there are $\mu$ degrees of freedom in extending $s$ to $[a, x]$ as a function belonging to span $U^{[0]}$ such that $s \in C^{n-1}_r([a, b]; \mathbb{R})$.

It should be observed that $(U^{[0]}, U^{[1]}, A)$-smoothness in general does not imply smoothness in the ordinary sense [9].

Let $[a, b] \subset \mathbb{R}$ be either a nontrivial compact interval or the real line. By $X$ we denote a finite or a bi-infinite partition of $[a, b]$ respectively, i.e.,

$$X = \{x_0, \ldots, x_{k+1}\} \quad \text{with} \quad a = x_0 < x_1 < \cdots < x_{k+1} = b \quad \text{or}$$

$$X = (x_i)_{i \in \mathbb{Z}} \quad \text{with} \quad \cdots < x_{-1} < x_0 < x_1 < \cdots \quad \text{and} \quad \lim_{i \to -\infty} x_i = -\infty, \quad \lim_{i \to \infty} x_i = \infty.$$

The points of $X$ which are not endpoints are called inner knots and endpoints are called auxiliary knots. The index sets for inner knots are

$$K_X := \begin{cases} \{1, \ldots, k\} & \text{if } X = \{x_0, \ldots, x_{k+1}\}, \\ \mathbb{Z} & \text{if } X = (x_i)_{i \in \mathbb{Z}}. \end{cases}$$

In any case by $A = (J_i)$, $J_i := [x_i, x_{i+1})$ for all $i$ except the last we denote the corresponding partition of $[a, b]$ into subintervals called $r$-knot intervals where in case of a finite partition of a compact interval the last $r$-knot interval is $J_k := [x_k, x_{k+1}]$.

Assume that on each closed interval $J_i = [x_i, x_{i+1}]$ the system $U^{[i]}_n = (u^{[i]}_1, \ldots, u^{[i]}_n)$ is an ECT-system of order $n$ with associated weight functions $w_j^{[i]} \in C^{n-j}(J_i; (0, \infty))$, $j = 1, \ldots, n$ and associated linear differential operators $L^{[i]}_j$ and ECT-derivative vectors of dimension $\mu$, $L^{[i]n}[f](t) = (L^{[i]}_0 f(t), \ldots, L^{[i]}_{n-1} f(t))^T$. 


By \( \mathcal{U} = \mathcal{U}_n = (U^{[i]})_i \) we denote the sequence of ECT-systems. Assume that corresponding to the inner knots we are given a sequence of integers \( M = (\mu_i), \ 0 \leq \mu_i \leq n \), and a sequence of nonsingular matrices

\[
\mathcal{A} = \mathcal{A}_n = (A^{[i]}), \quad A^{[i]} \in \mathbb{R}^{(n-\mu_i) \times (n-\mu_i)}.
\]

A function \( s : [a, b] \mapsto \mathbb{R} \) is called an rECT-spline function on \([a, b]\) with respect to the generating sequences \( \mathcal{U}, \ \mathcal{A}, M, X \) provided

\[
\begin{align*}
\text{\( s |_{J_i} \in \text{span} \ U^{[i]} \) for all \( i \), and} \\
\text{\( s \) is \((U^{[i]-1}, U^{[i]}, A^{[i]})\)-smooth at } x_i \text{ for all inner knots.} 
\end{align*}
\]

(30)

The set of all such functions will be denoted by \( \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) \).

Clearly, every rECT-spline function is right continuous everywhere and jumps may occur only at the knots. If all ECT-systems \( U^{[i]} \) have the first weight function

\[
\begin{align*}
w_1^{[i]}(x) &= 1, \quad x \in \tilde{J}_i, \quad \text{for all } i
\end{align*}
\]

and all connection matrices \( A^{[i]} \) have the form

\[
A^{[i]} = \text{diag}(1, \tilde{A}^{[i]}),
\]

(32)

where \( \mu_i \leq n - 1 \) and \( \tilde{A}^{[i]} \in \mathbb{R}^{(n-1-\mu_i) \times (n-1-\mu_i)} \) is nonsingular for all \( i \) then \( \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) \subset C([a, b]; \mathbb{R}) \). It is not hard to give conditions that are necessary and sufficient for \( \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) \subset C^m([a, b]; \mathbb{R}) \).

Symmetrically, a partition of \([a, b]\) into \( l \)-knot intervals \( \mathcal{A} = (\tilde{J}_i)_i \), \( \tilde{J}_i := (x_i, x_{i+1}] \) for all \( i \) except the first, \( \tilde{J}_0 := [x_0, x_1] \), can be used. Given sequences \( \mathcal{U}, \mathcal{A}, M, X \) as before, lECT-spline functions are defined analogously. They are left continuous everywhere and jumps may occur only at the knots. The space of all lECT-splines will be denoted by \( \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) \).

Under the assumption that \( \mathcal{A} = \mathcal{A}^+ := (A^{[i]})_i \), where for every \( i \) the connection matrix

\[
A^{[i]} \text{ is nonsingular, lower triangular, totally positive}
\]

(33)

it is possible to construct for the space \( \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \) a local support basis \( (N_j) \) that is normalized to form a nonnegative partition of unity. In order to give the definitions the following notation is useful. For any partition \( X = (x_i) \) of \([a, b]\), finite or biinfinite, with corresponding sequence of multiplicities of inner knots \( M = (\mu_i) \) such that \( 1 \leq \mu_i \leq n \) for all \( i \), we denote by \( \xi \) resp. by \( \xi_{\text{ext}} \) the weakly increasing sequence of inner resp. of all knots where auxiliary knots by definition have multiplicity \( n \), each repeated according to its multiplicity, the enumeration being fixed by the convention \( \xi_1 = \xi_2 = \cdots = \xi_{\mu_1} = x_1 \). We will also use the notation \( \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) = \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \xi_{\text{ext}}) \).

By \( \varphi : j \mapsto i_j \) we denote the mapping which assigns to each \( \xi_j \) the unique knot \( x_{ij} \) such that \( \xi_j = x_{ij} \). Then

\[
X = \varphi(\xi_{\text{ext}}), \quad M_{\text{ext}} := (\mu_i) \text{ with } \mu_i = \text{card } \varphi^{-1}([x_i]).
\]

It will be convenient to use the index set

\[
\begin{align*}
J_\varphi = J_\varphi^n := \left\{ \{ -n + 1, \ldots, \mu \} \right\}_Z \quad \text{if } [a, b] \text{ is compact,}
\end{align*}
\]

if \([a, b] = \mathbb{R} \).
Observe that the sequences $\xi$ or $\xi_{\text{ext}}$ are well defined as nonvoid sequences of $\mu := \sum \mu_i$ terms also in case $0 \leq \mu_i \leq n$ for all $i$ provided $1 \leq \mu_1 \leq n$. Only in case that all inner knots have multiplicities zero, $M = M^0 = (0)_j$, we have $\xi = ( )$, a void sequence.

For simplicity, we represent a spline function $s \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ by

$$ s = \sum_{i} \sum_{j=1}^{n} c_{j}^{[i]} u_{j}^{[i]}, \quad (34) $$

meaning that

$$ s|_{J_i} = \sum_{j=1}^{n} c_{j}^{[i]} u_{j}^{[i]}, \quad \text{all } i \quad (35) $$

with coefficients $c_{j}^{[i]} (j = 1, \ldots, n - \mu_i)$ that are related by the connection equations (30). If in $J_{i-1}$ and in $J_i$ the canonical ECT-systems with respect to $c = x_i$ in both cases are used as bases, then in the general case $0 \leq \mu_i \leq n$ the coefficients $c_{j}^{[i]} (j = 1, \ldots, n - \mu_i)$ are connected by the connection equations (30) as

$$ \begin{pmatrix} c_{1}^{[i]} \\ \vdots \\ c_{n - \mu_i}^{[i]} \end{pmatrix} = A^{[i]} \begin{pmatrix} c_{1}^{[i-1]} \\ \vdots \\ c_{n - \mu_i}^{[i-1]} \end{pmatrix}. $$

This is immediate from the biorthogonality conditions (10). Then there remain $\mu_i$ degrees of freedom for $s$ right to $x_i$. If, in particular, $A^{[i]}$ is the identity matrix of dimension $n - \mu_i$, then the connection equations at $x_i$ require that those first $n - \mu_i$ coefficients are identical, correspondingly.

**Remark 2.1.**

(i) The space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^{+}, M, X)$ was introduced by Barry [1], p. 396. Barry has constructed de Boor–Fix functionals first and used them to derive existence of a local support basis for ECT-splines.

(ii) If $U^{[i]} = U_{n}^{[i]}|_{\bar{J}_i}$, where $U_{n}$ is a fixed global ECT-system of order $n$ on $[a, b]$ and $A^{[i]}$ is the $(n - \mu_i)$-dimensional identity matrix then $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ is the space of Tchebycheff splines of order $n$ on $[a, b]$ with knots $x_1, \ldots, x_k$ of multiplicities $\mu_1, \ldots, \mu_k$, respectively.

(iii) If $U^{[i]} = (1, x, \ldots, x^{n-1})|_{\bar{J}_i}$ for $i = 0, \ldots, k$ then $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) = \mathcal{S}_n(x_1, \ldots, x_k|A^{[1]}, \ldots, A^{[k]})$ is the space of piecewise ordinary polynomials of order $n$ generated by connection matrices $A^{[i]}$ considered by Dyn and Micchelli [6], p. 321, and by Barry et al. [2]. If moreover each $A^{[i]}$ is an identity matrix then $\mathcal{S}_n$ is the well known Schoenberg space of ordinary polynomial spline functions of order $n$ with knots $x_i$ of multiplicity $\mu_i$, $i = 1, \ldots, k$.

It is easily seen that with the usual pointwise defined algebraic operations $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^{+}, \xi_{\text{ext}})$ as defined by (34), (35) and (30) is a linear space over the reals whose dimension is $d := \dim \mathcal{S}_n(\mathcal{U}, \mathcal{A}^{+}, M, X) = n + \mu$, $\mu := \sum \mu_i$.

**Proposition 2.1.** For $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^{+}, \xi_{\text{ext}})$ in any of the two considered cases under the assumptions (31)–(33) for all $i$, there exists a local support basis $(N_{j})_{j \in I^n}$ consisting of functions having the
properties

\[ N_j(x) = N_j^n(x) = N_j(x|\xi_j, \ldots, \xi_{j+n}), \]
\[ N_j(x) > 0, \quad x \in (\xi_j, \xi_{j+n}), \]
\[ N_j(x) = 0, \quad x \notin [\xi_j, \xi_{j+n}], \]
\[ N_j^{(l)}(\xi_j^+) = 0 \quad \text{for} \quad l = 0, \ldots, n - 1 - \mu_j^+, \quad D_r^{(n-\mu_j^+)} N_j(\xi_j) > 0, \]
\[ N_j^{(l)}(\xi_j^{-n}) = 0 \quad \text{for} \quad l = 0, \ldots, n - 1 - \mu_{j+n}^-, \quad D_l^{(n-\mu_{j+n}^-)} N_j(\xi_{j+n}) < 0, \]
\[ \sum_j N_j(x) = 1, \quad x \in [a, b]. \]

Here \( \mu_j^+ := \#\{l \geq 0 : \xi_{j+l} = \xi_j\} \) and \( \mu_j^- := \#\{l \geq 0 : \xi_{j-l} = \xi_j\} \) denote the \textit{right} and \textit{left multiplicities} of a knot \( \xi_j \) in the sequence \( \xi \). Symmetrically, there is a basis \( (\tilde{N}_j^n(y))_{j \in J} \) of the space \( \tilde{S}_n(\mathcal{U}, \mathcal{A}^+, \xi_{\text{ext}}) \), where each \( \tilde{N}_j^n(y) \) has similar properties.

**Proof.** Existence of such a basis has been proved by Barry [1]. On the other side it is not hard to extend the explicit construction of polynomial B-splines via connection matrices due to Dyn and Micchelli [6] to get the ECT-B-splines \( (N_j^n)_{j \in J} \) and their properties as stated in Proposition 2.1. \( \Box \)

In this paper we will give explicit representations of ECT-B-splines in terms of generalized left sided divided differences with respect to certain IET-systems. Therefore a generalization of Marsden’s identity is basic.

### 3. Pólya-polynomials and Marsden’s identity generalized to ECT-splines

In this section and in the rest of the paper we take (31), (32) and (33) as general assumptions. Let \( \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \xi_{\text{ext}}) = \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \) be an ECT-spline space as in Section 2. Assuming for the weights of every local ECT-system (21) we set

\[ \mathcal{C}_{\mathcal{A}} := (C[i])_{i \in K} \quad \text{with} \quad C[i] = \text{diag}(A[i], I_{\mu_i}), \]
\[ \mathcal{E}_{\mathcal{A}} := (E[i])_{i \in K} \quad \text{with} \quad E[i]^T := R^{-1}(C[i])^{-1} R, \]

where \( R = R_n \) is the \( n \)-dimensional orthogonal matrix defined by

\[
R^T := \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & \ldots & 0 & (-1) & 0 \\
0 & 0 & \ldots & (-1)^2 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(-1)^{n-1} & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}.
\]
We use the spaces
\[ \mathcal{P}_n := \mathcal{P}_n(\mathcal{U}, \mathcal{C}_{sf}^+, X) := \mathcal{I}_n(\mathcal{U}, \mathcal{C}_{sf}^+, M^0, X) := \{ f : f \in C^1_p([a, b]; \mathbb{R}) \}, \]
\[ \times f|_J \in \text{span } U^{[i]}, f \text{ is } (U^{[i]-1}, U^{[i]}, C^{[i]})\text{-smooth at } x_i, i \in K \} \text{ and } \]
\[ \mathcal{P}_n^* := \mathcal{P}_n(\mathcal{U}^*, \mathcal{C}_{sf}^+, X) := \mathcal{J}_n(\mathcal{U}^*, \mathcal{C}_{sf}^+, M^0, X) := \{ f : f \in C^1_p([a, b]; \mathbb{R}) \}, \]
\[ f|_J \in \text{span } U^{[i]*}, f \text{ is } (U^{[i]-1}, U^{[i]}, E^{[i]})\text{-smooth at } x_i, i \in K \}. \] (37)

Clearly, \( \mathcal{P}_n \subset \mathcal{I}_n(\mathcal{U}, \mathcal{C}_{sf}^+, M, X) \). If \( A[i] \in \mathbb{R}^{(n-\mu_i) \times (n-\mu_i)} \) may be partitioned as in (32) then it is easily seen that
\[ E^{[i]} = \text{diag}(I_{\mu_i}, E^{[i]-1}_{n-1-\mu_i}, 1), \quad E^{[i]}_{n-1-\mu_i} = R^{[-1]}_{n-1-\mu_i}(\tilde{A}^{[i]})^{-T}R^{[1]}_{n-1-\mu_i}. \]

Hence every \( E^{[i]} \) is a \( n \)-dimensional square matrix which is nonsingular, lower triangular and totally positive.

We will also use spaces
\[ \mathcal{P}_{n+1} := \mathcal{I}_{n+1}(\mathcal{U}, \mathcal{C}_{sf}^+, M^0, X) \]
having \( \mathcal{P}_n \) as a subspace and
\[ \mathcal{P}_{n+1}^* := \mathcal{J}_{n+1}(\mathcal{U}^*, \mathcal{C}_{sf}^+, M^0, X) \]
having \( \mathcal{P}_n^* \) as a subspace. They are defined by the extensions
\[ \hat{\mathcal{U}} = (\hat{U}^{[i]})_{i \in K}, \quad \hat{\mathcal{C}}_{sf}^+ = (\hat{C}^{[i]})_{i \in K}, \quad \hat{\mathcal{U}}^* = (\hat{U}^{[i]*})_{i \in K}, \quad \hat{\mathcal{C}}_{sf}^+ = (\hat{E}^{[i]})_{i \in K}, \]
where \( \hat{U}^{[i]} = (u_1^{[i]}, \ldots, u_n^{[i]}, u_{n+1}^{[i]}) \) is an ECT-system generated by the weights \( (w_1^{[i]}, \ldots, w_{n+1}^{[i]}) = (1, w_2^{[i]}, \ldots, w_n^{[i]}, \hat{w}_{n+1}^{[i]}) \) and \( \hat{U}^{[i]*} = (u_1^{[i]*}, \ldots, u_n^{[i]*}, u_{n+1}^{[i]*}) \) is an ECT-system generated by the weights \( (w_1^{[i]*}, \ldots, w_{n+1}^{[i]*}) = (1, w_2^{[i]}, \ldots, w_n^{[i]}, w_{n+1}^{[i]}) \). Here \( 0 < w_{n+1}^{[i]} \in C^0(\hat{J}_i; \mathbb{R}) \) and \( 0 < u_{n+1}^{[i]*} \in C^0(\hat{J}_i; \mathbb{R}) \) may be chosen arbitrarily, where now we assume that
\[ w_j^{[i]} \in C^{\max(n+1-j, j-1)}(J_i; \mathbb{R}), \quad j = 2, \ldots, n. \]

The connection matrices for \( \mathcal{P}_n \) resp. for \( \mathcal{P}^*_n \) for every \( i \in K \) are defined by \( \hat{C}^{[i]} := \text{diag}(C^{[i]}, 1) \) resp. \( \hat{E}^{[i]} := \text{diag}(E^{[i]}, 1) \). Here \( \hat{E}^{[i]} := \text{diag}(I_{\mu_i}, E^{[i]-1}_{n-1-\mu_i}, 1, 1) \) if \( \tilde{A}^{[i]} \) may be partitioned as in (32).

According to Corollary 4.1.1 of [9] \( \mathcal{P}_n \) resp. \( \mathcal{P}^*_n \) is an rET- resp. lET-space of order \( n \) each and \( \mathcal{P}^*_n \) is an ECT-space of order \( n + 1 \). By Corollary 4.1.4 of [9] the space \( \mathcal{P}^*_n \) has a basis
\[ q_1, \ldots, q_{n+1} \] (39)
such that for \( v = 0, 1, 2 \) the system \( q_1, \ldots, q_{n-1+v} \) is an IET-system of order \( n - 1 + v \). Such a basis is obtained by fixing in any knot interval \( J_i \) the local ECT-system (20) in canonical form with respect to any \( c \in [x_i, x_{i+1}] \) \( (s^*[i,n+1]_1(x, c), \ldots, s^*[i,n+1]_n(x, c)), \ x \in (x_i, x_{i+1}], \) and extending these functions
by the connection equations of $\mathcal{P}_{n+1}^*$ to the left and right of $J_i$. Since $s_{j,n}^{[i]} = s_{j,n+1}^{[i]}$, $j = 1, \ldots, n$, the basis $q_1, \ldots, q_{n+1}$ of $\mathcal{P}_{n+1}^*$ constructed this way under the hypothesis (32) indeed yields in the sections $q_1, \ldots, q_{n-1}$ and $q_1, \ldots, q_n$ IET-systems of orders $n - 1$ and $n$, respectively.

Generalized Pólya polynomials are defined for $j \in J$ by

$$M_j(y) = M_j^n(x) = M_j(y|\xi_{j+1}, \ldots, \xi_{j+n-1}) = (-1)^{n-1}r q_n\left[\begin{array}{c} q_1, \ldots, q_{n-1} \\ \xi_{j+1}, \ldots, \xi_{j+n-1} \end{array}\right]_l(y), \quad (40)$$

denoting by $rf^{[u_1, \ldots, u_n]}_l(y) := f(y) - pf^{[u_1, \ldots, u_n]}_l(y)$ the interpolation remainder, where $pf^{[u_1, \ldots, u_n]}_l(y)$ is the solution of the Hermite interpolation problem $H(U, T^-, f)$.

$M_j \in \mathcal{P}_n^*$ has exactly $n - 1$ zeros $\xi_{j+1}, \ldots, \xi_{j+n-1}$, counting left multiplicities, and no other zeros, and $M_j$ has leading coefficient $(-1)^{n-1}$ in every interval $\tilde{J}_i$. Therefore $M_j$ is positive for $x < \xi_{j+1}$. It is not hard to show that every $n$ consecutive generalized Pólya polynomials $(M_j)_l^{l+n-1}$, $l \in J$, form a basis of span{$q_1, \ldots, q_n$}.

Barry [1] has constructed dual functionals for the ECT-B-spline basis

$$A_j(x)[f] := \sum_{p=0}^{n-1} (-1)^{n-1-p} L_p^{[r]} f(x) L_{n-1-p}^{[r]} M_j(x), \quad x \in J_r, \, \xi_j < x < \xi_{j+n}.$$ 

Actually, it is easily derived from (24) that under our general assumptions (31)–(33) for every $j \in J$, the function

$$x \mapsto A_j(x)[f]$$

is a constant function of $x \in (\xi_j, \xi_{j+n})$. As a consequence,

$$\mathcal{E}_n(U, \mathcal{G}_n^+, M, X) \ni f \mapsto A_j(f) := A_j(x)[f], \quad x \in (\xi_j, \xi_{j+n})$$

is a well defined linear functional for every $j \in J$.

Barry [1] has proved.

**Theorem 3.1.** Under the assumptions (31)–(33)

$$A_j(N_i) = \delta_{i,j}, \quad i, j \in J.$$ 

The following theorem will be basic for the definition of ECT-B-splines via generalized divided differences.

**Theorem 3.2.** Under the assumptions (31)–(33) there exists a unique function $[a, b] \times [a, b] \ni (x, y) \mapsto h(x, y)$ such that

(i) for each $y \in [a, b]$ $h(\cdot, y) \in \mathcal{P}_n,$

(ii) for each $x \in [a, b]$ $h(x, \cdot) \in \mathcal{P}_n^*,$
(iii) whenever for some \( l \in K_X \) \( x \in J_l \) and \( y \in \tilde{J}_l \) then
\[
h(x, y) = w^{[l]}_1(x) h_{n-1}(x, y; w^{[l]}_2, \ldots, w^{[l]}_n) = \sum_{k=1}^{n} (-1)^{n-k} s^{[l]}_k(x, c) \cdot s_{n+1-k,n}^{[l]}(y, c)
\]
\[
= w^{[l]}_1(x) \sum_{k=1}^{n} h_{k-1}(x, c; w^{[l]}_2, \ldots, w^{[l]}_k) h_{n-k}(y, c; w^{[l]}_{k+1}, \ldots, w^{[l]}_n)
\]
with \( c \in \tilde{J}_l \) arbitrary.

(iv) For \( i \in K_X \) fixed, \( c \in \tilde{J}_i \) and \( j = 1, \ldots, n \) let \( p_j(\cdot, c) \in P_n \) be defined by
\[
p_j(x, c) = w^{[i]}_1(x) h_{j-1}(x, c; w^{[i]}_2, \ldots, w^{[i]}_j) = s^{[i]}_j(x, c), \ x \in J_i
\]
and for \( i \in K_X \) fixed, \( c \in \tilde{J}_i \) and \( j = 1, \ldots, n \) let \( q_j(\cdot, c) \in P_n^* \) be defined by
\[
q_j(y, c) = h_{j-1}(y, c; w^{[i]}_n, \ldots, w^{[i]}_{n+2-j}) = s^{[i]}_j(y, c), \ y \in \tilde{J}_i.
\]

Then the function \( h \) has the representation
\[
h(x, y) = \sum_{k=1}^{n} (-1)^{n-k} p_k(x, c) q_{n+1-k}(y, c), \quad (x, y) \in [a, b] \times [a, b], \ c \in \tilde{J}_i, \quad (41)
\]
where the right-hand side is independent of \( i \) and of \( c \in \tilde{J}_i \).

Remark 3.2. If \( U^{[i]} = (1, x, \ldots, x^{n-1})|_{J_i} \) for all \( i \) then \( h(x, y) = (x - y)^{n-1}/(n-1)! \) whenever \( x \in J_l \) and \( y \in \tilde{J}_l \) for some \( l \) and (41) reduces to formula (3.67) of [6]. If moreover \( A^{[i]} = I_{n-\mu_i} \) for all \( i \) then (41) reduces to the binomial theorem
\[
\frac{(x - y)^{n-1}}{(n-1)!} = \sum_{k=1}^{n} (-1)^{n-k} \frac{(x - c)^{k-1}}{(k-1)!} \frac{(y - c)^{n-k}}{(n-k)!}.
\]
When \( U^{[i]} = U|_{J_i} \), where \( U = (u_1, \ldots, u_n) \) is an ECT-system on \([a, b]\) and for all \( i \in \mathbb{Z} \) \( A^{[i]} = I_{n-\mu_i} \) is an identity matrix then (41) reduces to Marsden’s identity for Tchebycheff splines [13] p. 382.

Proof. The proof of Theorem 3.2 is a somewhat tedious but straightforward extension of the proof due to Dyn and Micchelli [6] of the similar result for polynomial splines via connection matrices. Essential for this extension are the Taylor formulas (14) and (22). \( \Box \)

The next theorem is a generalization of Marsden’s identity to ECT-B-splines.

Theorem 3.3. Under the assumptions (31)–(33) for the function \( h \) of Theorem 3.2 there holds
\[
h(x, y) = \sum_{i \in I_o} N_i(x) M_i(y) \quad \text{for all } (x, y) \in [a, b] \times [a, b], \quad (42)
\]
Proof. Since $\mathcal{P}_n(\mathcal{U}, \mathcal{A}^+, X) \subset \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ and since $(N_j)$ is a basis of $\mathcal{S}_n$, for each $y \in (a, b]$
\[ h(x, y) = \sum_{i \in \mathbb{Z}} c_i(y)N_i(x) \quad \text{for all } x \in [a, b], \]
with certain functions $c_i(y)$ that must belong to $\mathcal{P}_n(\mathcal{U}_n^*, \mathcal{A}^+, X)$ since for fixed $x$, as function of $y$, by Theorem 3.2 $h(x, \cdot) \in \mathcal{P}_n(\mathcal{U}_n^*, \mathcal{A}^+, X)$. Hence, for $x \in J_l$, $y \in \tilde{J}_l$ 
\[ h(x, y) = s_{n}[l](x, y) = \sum_{i=l-n+1}^{l} c_i(y)N_i(x). \]
By applying the linear functional $A_l(y)[\cdot]$ acting on functions of $x$ we get 
\[ A_l(y)[h(\cdot, y)] = \sum_{p=0}^{n-1} (-1)^{n-1-p} L_p^{[l]}(y)L_{n-1-p}^{[l]} M_l(y) = M_l(y) \]
\[ = \sum_{i=l-n+1}^{l} c_i(y)A_l(y)[N_i] = c_l(y), \quad y \in \tilde{J}_l. \]
Hence, for all $l$, $c_l = M_l$ agree on $\tilde{J}_l$ and therefore on $[a, b]$ since both are elements of $\mathcal{P}_n(\mathcal{U}_n^*, \mathcal{A}^+, X)$. □

Remark 3.3. Theorems 3.1 and 3.3 are equivalent in the sense that each is a consequence of the other. Indeed, Theorem 3.3 was derived from Theorem 3.1 via 3.2. Conversely, by applying $A_j(x)[\cdot]$ to both sides of the identity (42), considered as functions of $x$, for $x \in J_l$, $y \in \tilde{J}_l$ by Theorem 3.2 on the left-hand side we get 
\[ A_j(x)[h(\cdot, y)] = \sum_{p=0}^{n-1} (-1)^{n-1-p} L_p^{[l]}(x)h(\cdot,y)L_{n-1-p}^{[l]} M_j(y) = M_j(y) \]
Since by (25) applied with $m = n$
\[ (-1)^{n-1-p} L_p^{[l]}(y)L_{n-1-p}^{[l]} M_j(x) = s_{n-p,n}^{[l]}(y,x), \quad p = 0, \ldots, n - 1, \]
by Taylor’s expansion (23) this equals $M_i(y)$. The right-hand side equals 
\[ A_j(x) \left[ \sum_{i=l-n+1}^{l} N_i(x)M_i(y) \right] = \sum_{i=l-n+1}^{l} M_i(y)A_j(x)[N_i]. \]
Since the functions $M_i$ ($i = l-n+1, \ldots, l$) are linearly independent on $\tilde{J}_l$ it follows 
\[ A_j(x)[N_i] = \delta_{i,j}, \quad i, j \in J_0. \]

4. Explicit representations of the generalized B-splines $N^n_j$ and $Q^n_j$

Let
\[ g(x, y) := \begin{cases} h(x, y), & x \geq y, \\ 0, & \text{otherwise}, \end{cases} \quad (43) \]
where $h$ is the function of Theorem 3.2. By the properties of $h$, with $y$ fixed, as a function of $x$, $g(x, y)$ belongs piecewise, for $x \geq y$ and for $x < y$, to $\mathcal{P}_n \subset C^1_n(U, A^+, M, X)$. If $x$ is fixed, as a function of $y$, $g(x, y)$ belongs piecewise, for $x \geq y$ and for $x < y$, to $\mathcal{P}_n^*$. The function $g$ is separately, with respect to $x$, in $C^{n-1}_n(J; \mathbb{R})$, and with respect to $y$, in $C^n(J; \mathbb{R})$, since $h$ has this property, and the $(n-1)$st ECT-derivative of $g$ with respect to $x$ at $x = y$ has the characteristic jump discontinuity of a Green’s function: for every $i$

$$
\lim_{x \to y^-} L^{(v)} g(\cdot, y)|_{x=y} = 0, \quad v = 0, \ldots, n - 1, \quad \text{if} \ x_i < y \leq x_{i+1},
$$

$$
\lim_{x \to y^+} L^{(v)} g(\cdot, y)|_{x=y} = \left\{
\begin{array}{ll}
0, & v = 0, \ldots, n - 2, \\
1, & v = n - 1 \quad \text{if} \ x_i \leq y < x_{i+1}.
\end{array}
\right.
$$

We will call (43) the Green’s function with respect to the spaces $\mathcal{P}_n(U, C^1_+, X)$ and $\mathcal{P}_n^*(U, C^1_+, X)$.

**Definition 4.1.** For $j \in J_{\varphi}$ and $x \in [a, b]$

$$
\tilde{N}_j^n(x) := \tilde{N}(x|\tilde{\xi}_j, \ldots, \tilde{\xi}_{j+n})
\quad := (-1)^n \left[ \begin{array}{c}
q_1, \ldots, q_n \\
\tilde{\xi}_{j+1}, \ldots, \tilde{\xi}_{j+n}
\end{array} \right]_l g(x, \cdot) - \left[ \begin{array}{c}
q_1, \ldots, q_n \\
\tilde{\xi}_j, \ldots, \tilde{\xi}_{j+n-1}
\end{array} \right]_l g(x, \cdot).
$$

(44)

Here we have made use of the notation (6) for the left sided generalized divided differences of a function, and the functions $q_1, \ldots, q_n$ are those defined in Theorem 3.2.

**Proposition 4.1.** For $j \in J_{\varphi}$ and $x \in [a, b]$

$$
\tilde{N}_j^n(x) = (-1)^n f_{n, j} \left[ \begin{array}{c}
q_1, \ldots, q_{n+1} \\
\tilde{\xi}_{j+1}, \ldots, \tilde{\xi}_{j+n}
\end{array} \right]_l g(x, \cdot),
$$

$$
f_{n, j} = \left[ \begin{array}{c}
q_1, \ldots, q_n \\
\tilde{\xi}_{j+1}, \ldots, \tilde{\xi}_{j+n}
\end{array} \right]_l q_{n+1} - \left[ \begin{array}{c}
q_1, \ldots, q_n \\
\tilde{\xi}_j, \ldots, \tilde{\xi}_{j+n-1}
\end{array} \right]_l q_{n+1}.
$$

(45)

Here $q_1, \ldots, q_{n+1} \in \mathcal{P}_{n+1}$ are the functions defined by (39) and $q_1, \ldots, q_n$ are those of Theorem 3.2.

**Proof.** The equivalence of (45) with (44) is an immediate consequence of the recurrence relation (2) for left sided divided differences. □

**Remark 4.1.** In case of ordinary polynomial splines of order $n$ where all connection matrices are identity matrices (45) simplifies to

$$
\tilde{N}_j^n(x) = (-1)^n (\tilde{\xi}_j - \tilde{\xi}_{j+n}) [\tilde{\xi}_j, \ldots, \tilde{\xi}_{j+n}]_l (x - \cdot)^{n-1}_+, 
$$

(46)

where $[\tilde{\xi}_j, \ldots, \tilde{\xi}_{j+n}]_l f$ denotes the ordinary left sided divided difference of the function $f \in C^{n-1}_n(J; \mathbb{R})$ with respect to the polynomials of degree $n$ at most. In case of Tchebycheffian splines of order $n$ where all connection matrices are identity matrices (45) extends Lyche’s definition (44) of Tchebycheffian B-splines [8].
Proposition 4.2. For \( j \in J_0 \)

(i) \( \tilde{N}_j^n \) is right continuous,
(ii) \( \text{supp} \tilde{N}_j^n = [\xi_j, \xi_j+n] \),
(iii) \( \tilde{N}_j^n(x) > 0 \) if \( x \in (\xi_j, \xi_j+n) \),
(iv) \( \tilde{N}_j^n \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \),
(v) \( \sum_{j \in J_0} \tilde{N}_j^n(x) = 1, \quad x \in [a, b] \),
(vi) \( \tilde{N}_j^n \) has a zero at \( x = \xi_j \) of order \( n - \mu_j^+ \) with \( D_r^{n-\mu_j^+} \tilde{N}_j^n(\xi_j) > 0 \),
(vii) \( \tilde{N}_j^n \) has a zero at \( x = \xi_j+n \) of order \( n - \mu_j^{-} \) with \( D_r^{n-\mu_j^{-}+n} \tilde{N}_j^n(\xi_j+n) < 0 \),
(viii) \( \tilde{N}_j^n = N_j^n \).

Proof. (i) We are going to show that the function

\[
x \mapsto \begin{bmatrix} q_1, \ldots, q_n \\ \xi_j, \ldots, \xi_j+n-1 \end{bmatrix} g(x, \cdot)
\]

is right continuous. For \( x \) fixed, the function \( y \mapsto g(x, y) \) is in \( C^{n-1}_I(J; \mathbb{R}) \). The left sided generalized divided difference is applied to \( g \) as a function of \( y \), with \( x \) fixed. From (1) and from (41) we see that it is a linear combination of derivatives

\[
\left( \frac{\partial}{\partial y} \right)^{v_p} h(x, \xi_p-) = \sum_{k=1}^{n} (-1)^{n-k} p_k(x, x_i) \left( \frac{\partial}{\partial y} \right)^{v_p} q_{n+1-k}(y, x_i) |_{y=\xi_p-}
\]

if \( \xi_p < x \) and \( v_p := \text{multiplicity of} \ \xi_p \) in \((\xi_j, \ldots, \xi_{p-1})\). They all are right continuous as functions of \( x \).

(ii) If \( x < \xi_j \) then by Definition 4.1 \( g(x, \cdot) \) vanishes at all knots \( \xi_j, \ldots, \xi_{j+n} \) (counting multiplicities) and \( \tilde{N}_j^n(x) = 0 \). If \( x \geq y \) then

\[
g(x, y) = \sum_{k=1}^{n} (-1)^{n-k} p_k(x, \xi_r) q_{n+1-k}(y, \xi_r), \tag{47}
\]

where according to Theorem 3.2 we start the construction of \( h \) in an interval \((\xi_r, \xi_{r+1})\) with \( \xi_r \leq x < \xi_{r+1} \). In case \( \xi_j+n < x \) the two divided differences involved in the Definition 4.1 of \( \tilde{N}_j \) are well defined as the leading coefficients of the interpolants of \( g(x, \cdot) \) at the nodes \( \xi_{j+1}, \ldots, \xi_{j+n} \) or at \( \xi_j, \ldots, \xi_{j+n-1} \), respectively. But the two interpolants, as functions of \( y \), restricted to \( y < x \), must coincide with \( g(x, y) \), since \( g(x, y) \) belongs to span \((q_1(y, \xi_r), \ldots, q_n(y, \xi_r))\) in the interval \( a < y < x \). Hence the difference of their leading coefficients is 0. Now (ii) becomes a consequence of (iii) to be proved below.

(iv) From (1) with \( r \) replaced by \( l \) and from (47) it is clear that, as a function of \( x \), \( \tilde{N}_j^n \) is a linear combination of \( p_1(x, \xi_r), \ldots, p_n(x, \xi_r) \). Since \( \mathcal{P}_n \subset \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \) we have \( \tilde{N}_j^n \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \).
(v) Assume first $\xi_r < x < \xi_{r+1}$. By Definition 4.1 then only for $j = r - n + 1, \ldots, r \tilde{N}_j^n(x) \neq 0$. Hence by (44)

$$
\sum_{j \in J_0} \tilde{N}_j^n(x) = (-1)^n \sum_{j = r-n+1}^r \left( \begin{bmatrix} q_1, \ldots, q_n \\ \xi_{j+1}, \ldots, \xi_{j+n} \end{bmatrix} g(x, \cdot) - \begin{bmatrix} q_1, \ldots, q_n \\ \xi_j, \ldots, \xi_{j+n-1} \end{bmatrix} g(x, \cdot) \right)
$$

$$= (-1)^n \left( \begin{bmatrix} q_1, \ldots, q_n \\ \xi_{r+1}, \ldots, \xi_{r+n} \end{bmatrix} g(x, \cdot) - \begin{bmatrix} q_1, \ldots, q_n \\ \xi_{r-n+1}, \ldots, \xi_r \end{bmatrix} g(x, \cdot) \right)
$$

since the sum is telescoping. According to (ii) the first divided difference is zero and according to (iv) of Theorem 3.2 the second one is $-(-1)^{n-1} p_1(x, \xi_r) = (-1)^n w_1^{[r]}(x) = (-1)^n$. Consequently, the sum is equal to one. Since all $\tilde{N}_j^n$ are right continuous this holds for $\xi_r \leq x < \xi_{r+1}$.

(vi) Since by (18) and by (43)

$$
\left( \frac{\partial}{\partial x} \right)^{\lambda} \left( \frac{\partial}{\partial y} \right)^{\lambda} g(x, y)|_{x = \xi_j} = \begin{cases} 
0 & \text{if } \xi_j < y, \\
0 & \text{if } \xi_j = x_i = y \text{ and } v + \lambda < n, \\
(-1)^\lambda \frac{1}{w_1^{[r]}(x_i) \cdots w_n^{[r]}(x_i)} & \text{if } \xi_j = x_i = y \text{ and } v + \lambda = n,
\end{cases}
$$

from (44) we infer with $D = d/dx$

$$
D^\lambda \tilde{N}_j(\xi_j)
$$

$$= (-1)^n \left( \begin{bmatrix} q_1, \ldots, q_n \\ \xi_{j+1}, \ldots, \xi_{j+n} \end{bmatrix} \left( \frac{\partial}{\partial x} \right)^v g(x, \cdot) - \begin{bmatrix} q_1, \ldots, q_n \\ \xi_j, \ldots, \xi_{j+n-1} \end{bmatrix} \left( \frac{\partial}{\partial x} \right)^v g(x, \cdot) \right)|_{x = \xi_j}
$$

$$= \begin{cases} 
0 & v = 0, \ldots, n - 1 - \mu^+_j \\
V \left| \begin{bmatrix} q_1, \ldots, q_n \\ \xi_j, \ldots, \xi_{j+n} \end{bmatrix} \right| w_1^{[r]}(\xi_j) \cdots w_n^{[r]}(\xi_j) > 0 & v = n - \mu^+_j.
\end{cases}
$$

(vii) Is proved similarly.

(iii) From (vi) we infer $\tilde{N}_j^n(x) > 0$ if $x > \xi_j$ is sufficiently close to $\xi_j$. On the other hand from (i),(ii),(iv),(vi) we see that $\tilde{N}_j^n$ belongs to

$$
\mathcal{S}^0_n(\xi_j, \ldots, \xi_{j+n}) := \{ s \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \xi_{\text{ext}}) : s(x) \equiv 0 \text{ for } x \notin [\xi_j, \xi_{j+n}] \\
\quad s^{(v)}(\xi_{j+n}) = 0 \text{ for } v = 0, \ldots, n - 1 - \mu^-_j, \\
\quad s^{(v)}(\xi_{j+n}) = 0 \text{ for } v = 0, \ldots, n - 1 - \mu^-_{j+n} \},
$$

which is a one-dimensional subspace of $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$. From the zero bound $Z(\tilde{N}_j^n(\xi_j, \xi_{j+n})) \leq 0$ (see corollary 3.1.1 of [9]), it follows $\tilde{N}_j^n(x) > 0$ for $\xi_j < x < \xi_{j+n}$.

(viii) For every $j \in J_\varphi$ we must have $\tilde{N}_j = c_j N_j$ for some $c_j > 0$ since $\mathcal{S}^0_n(\xi_j, \ldots, \xi_{j+n})$ has dimension one. Now $c_j = 1$ for all $j$ follows from the linear independency of $(N_j)$ since both systems form a partition of unity. □
We add a second proof of Proposition 4.2 reducing it to Proposition 2.1. Actually, we are going to show that \( \tilde{N}_j(x) = N_j(x) \) for all \( j \in J_\Theta \) and \( x \in [a, b] \) by applying Definition 4.1 to the Green’s function (43) where for \( h \) the representation (42) is used. Let \( y \in [a, b] \) and \( x \in [a, b] \) be arbitrary such that \( \xi_r \leq x < \xi_{r+1} \) with \( r \in \{0, \ldots, \mu\} \) resp. \( r \in \mathbb{Z} \). According to (42) and to Proposition 2.1 then
\[
h(x, y) = \sum_{i=-n+r+1}^{r} N_i(x) M_i(y)
\]
and for all \( l \)
\[
g(x, \xi_l) = \begin{cases} 0, & r < l, \\ \sum_{\max[l, r-n+1]}^{r} N_i(x) M_i(\xi_l), & l \leq r. \end{cases}
\]
(48)

By Definition 4.1 \( \tilde{N}_j(x) \) for \( j \in \{-n + r + 1, \ldots, r\} \) is the product of \((-1)^n\) and the coefficient of \( q_n \) in the difference
\[
\delta_j := pg(x, \cdot) \left[ \begin{array}{c} q_1, \ldots, q_n \\ \xi_{j+1}, \ldots, \xi_{j+n} \end{array} \right]_l - pg(x, \cdot) \left[ \begin{array}{c} q_1, \ldots, q_n \\ \xi_{j}, \ldots, \xi_{j+n-1} \end{array} \right]_l,
\]
where as before by \( pf[\begin{array}{c} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{array} \] we denote the solution of the Hermite interpolation problem \( H(U, T \cdot, f) \). Since \( \delta_j \in \text{span} \{q_1, \ldots, q_n\} \) has the zeros \( \xi_{j+1}, \ldots, \xi_{j+n-1} \) counting left multiplicities it must be a constant multiple of \( M_j \): \( \delta_j = a_j \cdot M_j \). Then, by Definition 4.1 and (40) \( \tilde{N}_j^n(x) = -a_j \). It remains to show that \( a_j = -\tilde{N}_j(x) \). This will be done by computing \( \delta_j(\xi_j) = -\tilde{N}_j(\xi_j)M_j(\xi_j) \). In fact,
\[
pg(x, \cdot) \left[ \begin{array}{c} q_1, \ldots, q_n \\ \xi_{j}, \ldots, \xi_{j+n-1} \end{array} \right]_l(\xi_j) = \sum_{i=j+1}^{r} N_i(x) M_i(\xi_j),
\]
by (48) and the interpolation conditions. On the other side it is easily seen that
\[
pg(x, \cdot) \left[ \begin{array}{c} q_1, \ldots, q_n \\ \xi_{j}, \ldots, \xi_{j+n-1} \end{array} \right]_l(y) = \sum_{i=j+1}^{r} c_{i,j+1} M_i(y).
\]
From the interpolation properties of this interpolant one infers \( c_{i,j+1} = N_i(x), \ i = j + 1, \ldots, r \), and a subtraction yields the result claimed.

We will have a closer look to the terms involved in the definition (44) of \( \tilde{N}_j^n \). They have similar properties as in the case of Tchebycheff splines (see [8], Lemma 4.1).

Lemma 4.3. For \( n \in \mathbb{N}, \ j \in J_\Theta \) and \( x \in [a, b] \)

(i)
\[
d_j^n(x) := (-1)^n \left[ \begin{array}{c} q_1, \ldots, q_n \\ \xi_{j}, \ldots, \xi_{j+n-1} \end{array} \right]_l g(x, \cdot) = \begin{cases} 0, & x < \xi_j, \\ -p_1(x) = -1, & x \geq \xi_{j+n-1}. \end{cases}
\]
(49)

(ii)
\[
d_j^n(\xi_j) = 0 \quad \text{if} \quad \xi_j < \xi_{j+n-1}.
\]
Proof. (i) \( d^n_j(x) = 0 \) for \( x < \xi_j \) follows directly from the definition of \( g \). To prove the second claim observe that \( g(x, y) = h(x, y) \) for \( x \geq y \). If \( x \geq \xi_{j+n-1} \) from (41) we get

\[
d^n_j(x) = \sum_{k=1}^{n} (-1)^k p_k(x) \left[ q_1, \ldots, q_n \atop \xi_j, \ldots, \xi_{j+n-1} \right] q_{n+1-k} = - p_1(x) = -1.
\]

(ii) Suppose \( x = \xi_j \). Since \( \xi_j < \xi_{j+n-1} \) is assumed, \( \mu_j^+ \leq n - 1 \). Therefore the numerator determinant of the divided difference defining \( d^n_j(\xi_j) \) has a last column with only zero entries. □

Lemma 4.4. Let \( n \geq 2 \). For every \( j \in J_\varphi \) if \( \xi_j < \xi_{j+n-1} \) the function

\[
\mathbb{R} \ni x \mapsto Q^{n-1}_j(x) = -L_1^{[i]} d^n_j(x), \quad x \in J_i, \ i \in J_\varphi,
\]

where \( d^n_j(x) \) is defined by (49) is the basic function of \( S^0_{n-1}(\xi_j, \ldots, \xi_{j+n-1}) \subseteq S_{n-1}(\mathcal{U}, \mathcal{A}^+, M, X) \) normalized by

\[
\int_a^b Q^{n-1}_j(t) \, dt = 1,
\]

where \( \mathcal{U} = (U_i[i])_{i \in \mathbb{Z}}, \ U_i[i] = (u_1[i], \ldots, u_{n-1}[i]) : = (L_1[i]u_2[i], \ldots, L_1[i]u_n[i]) \) is the first reduced system of \( U[i] \) (cf. (19)) and where \( \mathcal{A}^+ = (\bar{A}_{i}[i])_{i \in \mathbb{Z}} \) and \( \bar{A}[i] \) is defined by (32).

Proof. In view of the general assumptions (31) and (32) it follows from Lemma 4.3 that \( \text{supp} \ Q^{n-1}_j = [\xi_j, \xi_{j+n-1}] \). It remains to show

(i) \( Q^{n-1}_j \in S_{n-1}(\mathcal{U}, \mathcal{A}^+, M, X) \) and

(ii)

\[
\int_a^b Q^{n-1}_j(t) \, dt = 1.
\]

To prove (i) observe that

\[
Q^{n-1}_j(x) = (-1)^{n-1} \left[ q_1, \ldots, q_n \atop \xi_j, \ldots, \xi_{j+n-1} \right] L_1^{[i]} g(x, \cdot), \quad x \in J_i, \ i \in J_\varphi,
\]

where

\[
L_1^{[i]} g(x, y) = \begin{cases} w_2^{[i]} h_{n-2}(x, y; w_3^{[i]}, \ldots, w_n^{[i]}), & x \geq y, \ x \in J_i, \ y \in \bar{J}_i, \\ 0, & x < y, \end{cases}
\]

\[= g(x, y) \]
is the Green’s function corresponding to the spaces \( \mathcal{P}_{n-1}(\mathcal{U}, \mathcal{A}_x^+, X) \) and \( \mathcal{P}_{n-1}^*(\mathcal{U}^*, \mathcal{A}_x^+, X) \) where \( \mathcal{U} \) has been defined above and

\[
\mathcal{U}^* = (U_{j}^{[i]*})_{i \in \mathbb{Z}}, \quad U_{j}^{[i]*} = (s_{1,n}^{[i]*}(x, c), \ldots, s_{n-1,n}^{[i]*}(x, c)), \quad x, c \in \bar{J}_i,
\]

generated by \( 1, w_1^{[i]}, \ldots, w_3^{[i]} \),

\[
\mathcal{C}_x^+ = (\tilde{C}^{[i]})_{i \in \mathbb{Z}}, \quad \tilde{C}^{[i]} = \text{diag}(\tilde{A}^{[i]}, I_{\mu_i}),
\]

\[
\mathcal{C}_{x}^* = (E^{[i]})_{i \in \mathbb{Z}}, \quad E^{[i]} = R_{n-1}^{-1} \tilde{C}^{[i]} R_{n-1} = \text{diag} (I_{\mu_i}, \tilde{E}_{n-1-\mu_i}^{[i]})
\]

and \( R_{n-1} \) is defined by (36) with \( n \) replaced by \( n - 1 \). This proves (i).

(ii) Follows easily from

\[
\int_a^x Q_{j}^{n-1}(t) \, dt = - \int_a^x \frac{d}{dt} d_j^n(t) \, dt = -d_j^n(x), \quad x \in [a, b],
\]

which according to Lemma 4.3 equals 1 if \( x \geq \bar{x}_{j+n-1} \). \( \square \)

In Lemma 4.4 we have found the ECT-B-splines \( Q_{j}^{n-1} \) of the reduced space \( \mathcal{S}_{n-1}(\mathcal{U}, \mathcal{A}_x^+, M, X) \) of order \( n - 1 \), normalized to have integral one over its support. In order to get the corresponding basis \( Q_{j}^{n} \) of the space \( \mathcal{S}_n(\mathcal{U}, \mathcal{A}_x^+, \bar{\xi}_n) \) we only have to find a spline space \( \mathcal{S}_{n+1}(\hat{\mathcal{U}}, \hat{\mathcal{A}}_x^+, \bar{\xi}_n) \) of order \( n + 1 \) such that its reduced spline space is \( \mathcal{S}_{n}(\mathcal{U}, \mathcal{A}_x^+, \bar{\xi}_n) \). Of course, we also have to consider the adjoint of \( \mathcal{S}_{n+1}(\hat{\mathcal{U}}, \hat{\mathcal{A}}_x^+, M, X) \) and the associated spaces \( \mathcal{P}_{n+1}, \mathcal{P}^*_{n+1} \). This construction is as follows:

\[
\hat{\mathcal{U}} = (\hat{U}^{[i]}) = (1, x - c, s_3^{[i]}(x, c), \ldots, s_{n+1}^{[i]}(x, c))
\]

generated by the weights \( (\hat{w}_1^{[i]*}, \hat{w}_2^{[i]*}, \ldots, \hat{w}_{n+1}^{[i]*}) = (1, 1, w_2^{[i]}, \ldots, w_n^{[i]}) \).

Since its dual ECT-system is

\[
\hat{U}^{[i]*} = (1, s_1^{[i]*}(y, c), \ldots, s_{n,n}^{[i]*}(y, c), s_{n+1,n}^{[i]*}(y, c))
\]

generated by \( (\hat{w}_1^{[i]*}, \ldots, \hat{w}_{n+1}^{[i]*}) = (1, w_1^{[i]}, \ldots, w_2^{[i]}, 1) \),

we can put \( \hat{\mathcal{U}}^* = (\hat{U}^{[i]*})^* \). As connection matrices for the integrated systems we take \( \hat{\mathcal{A}}_x^+ = (\hat{A}^{[i]*}) \), \( \hat{\mathcal{A}}_x^+ := \text{diag}(1, 1, \tilde{A}^{[i]}) \). Then \( \hat{\mathcal{C}} = (\hat{C}^{[i]}) \) with \( \hat{C}^{[i]} = \text{diag}(1, 1, \tilde{A}^{[i]}, I_{\mu_i}) \) and \( \hat{\mathcal{C}}^* = (\hat{E}^{[i]}) \) with \( \hat{E}^{[i]} = R_{n+1}^{-1}(\tilde{C}^{[i]})^{-T} \) \( R_{n+1} = \text{diag}(I_{\mu_i}, \tilde{E}_{n-\mu_i-1}^{[i]}, 1, 1) \). According to Theorem 3.1 the corresponding function \( \hat{h} \) reads

\[
\hat{h}(x, y) = \sum_{k=1}^{n+1} (-1)^{n+1-k} \hat{p}_k(x, c) \hat{q}_{n+2-k}(y, c), \quad (x, y) \in [a, b] \times [a, b], c \in \bar{J}_i.
\]

It is easily seen that

\[
D \hat{p}_1(x, c) = 0, \quad D \hat{p}_2(x, c) = 1, \quad D \hat{p}_k(x, c) = p_{k-1}(x, c), \quad k = 3, \ldots, n + 1
\]
and that $\hat{q}_{n+1-l}(y, c) = q_{n+1-l}(y, c), \ l = 1, \ldots, n$. As a consequence, $D\hat{g}(x, \cdot) = g(x, \cdot)$. Hence, from Lemmas 4.3 and 4.4 we infer

$$Q^n_j(x) = -D\tilde{\hat{q}}^{n+1}_j(x) \ \ x \in J_i, \ i \in J^n_\varphi,$$

$$= (-1)^n \left[q_1, \ldots, q_{n+1}\right]_{i} g(x, \cdot)$$

(50)
is the ECT-B-spline with support $[\xi_j, \xi_{j+n}]$ normalized to have integral one over its support. As a consequence of (50) and of (45)

$$\int_a^b N^n_j(x) \, dx = 1/f_n.$$

**Remark 4.2.** In case of ordinary polynomial splines where all connection matrices are identity matrices (50) for $n \geq 1$ reduces to

$$Q^n_j(x) = (-1)^n \left[\xi_j, \ldots, \xi_{j+n}\right]_r (x - \cdot)^{n-1}_+$$

the well known right continuous polynomial B-spline of order $n$ normalized to have integral 1 over the real line. For $n \geq 1$ (44) reduces to

$$\tilde{N}^n_j(x) = (-1)^n (\xi_{j+n} - \xi_j) [\xi_j, \ldots, \xi_{j+n}]_r (x - \cdot)^{n-1}_+$$

the classical right continuous polynomial B-spline of order $n$ with knots $\xi_j, \ldots, \xi_{j+n}$. Polynomial B-splines often [13], p. 129, [5], p. 108 are defined by

$$\tilde{N}^n_j(x) := (\xi_{n+j} - \xi_j) [\xi_j, \ldots, \xi_{j+n}]_r (\cdot - x)^{n-1}_+, \ x \in \mathbb{R}. \ (51)$$

By making use of the identity

$$(x - y)^{n-1}_+ + (-1)^{n-1} (y - x)^{n-1}_+ = (x - y)^{n-1}_+, \ x, y \in \mathbb{R}, \ (52)$$

which is valid for $n > 1$, the equality $\tilde{N}^n_j(x) = \tilde{N}^n_j(x)$ results for $n > 1$ and for all $x$ except at knots of multiplicity $n$. A certain symmetry of this kind with regard to applying the generalized divided difference to the first or to the second variable of the Green’s function holds in general.

**Remark 4.3.** It is Definition 4.1 which, under suitable assumptions, leads to a recursive method for computing ECT-B-splines and ECT-spline curves developed in [12]. This method reduces to the de Boor–Mansion–Cox recursion and to the de Boor algorithm in case of ordinary polynomial splines, and it reduces to Lyche’s recursion [8] in case of Tchebycheff-splines.

In [14] cardinal ECT-B-splines defined by connection matrices are computed directly according to Proposition 4.1. Actually, there the generalized divided differences are computed directly via a certain characteristic polynomial wherein also the Taylor’s expansions (14) and (22) are involved.

5. The adjoint spline space and its ECT-B-splines

Observe that the identity (52) fails to hold in case $n = 1$ if its right-hand side for $n = 1$ is interpreted to be the constant function equal to 1, whether $0^0$ is defined either to be 1 or to be 0. An identity which
allows application of a right sided generalized divided difference with respect to an rECT-systems with
respect to the other variable in Green’s function is obtained by using the cuts of a continuous real function
\( f(x, y) \).

**Definition 5.1.** If \([a, b] \times [a, b] \ni (x, y) \mapsto f(x, y) \in \mathbb{R} \) is a continuous real function of two real
variables, its first resp. second cut are the functions

\[
[a, b] \times [a, b] \ni (x, y) \mapsto f(1)(x, y) := \begin{cases} f(x, y), & x \geq y, \\ 0, & x < y, \end{cases}
\]

\[
[a, b] \times [a, b] \ni (x, y) \mapsto f(2)(x, y) := \begin{cases} 0, & x \geq y, \\ f(x, y), & x < y. \end{cases}
\]

Clearly, \( f(1) \) and \( f(2) \) are separately piecewise continuous and add up to \( f \):

\[
f(1)(x, y) + f(2)(x, y) = f(x, y) \quad \text{for all } x, y.
\]

More precisely, as functions of \( x \), with \( y \) fixed, both cuts are right continuous, and as functions of \( y \), with
\( x \) fixed, both are left continuous. If a new function \( \tilde{f} \) is defined by \( f \) by interchanging its arguments,

\[
\tilde{f}(x, y) := f(y, x) \quad \text{for all } x, y,
\]

then its cuts

\[
\tilde{f}(1)(x, y) = \begin{cases} f(y, x), & x \geq y, \\ 0, & x < y, \end{cases}
\]

\[
\tilde{f}(2)(x, y) = \begin{cases} 0, & x \geq y, \\ f(y, x), & x < y, \end{cases}
\]

do not coincide with the interchanged cuts of \( f \) with interchanged arguments

\[
f(1)(y, x) = \begin{cases} f(y, x), & y \geq x, \\ 0, & y < x, \end{cases}
\]

\[
f(2)(y, x) = \begin{cases} 0, & y \geq x, \\ f(y, x), & y < x, \end{cases}
\]

actually they do, except on the diagonal \( y = x \).

In Definition 4.1 the ECT-B-splines \( N_n^b(x) \) of order \( n \) (now we drop the tilde) for the rECT-spline
space \( \mathcal{S}_n = \mathcal{S}_n(\mathbb{U}, \Sigma, M, X) \) are defined by applying generalized left sided divided differences to
the Green’s function (43), i.e. to the first cut of \( h, h(1)(x, y) \), with respect to its second variable \( y \), with \( x \) fixed, and with respect to the space \( \mathcal{P}_n = \mathcal{S}_n(\mathbb{U}, \Sigma, M, X) \) whose elements are left continuous. Also involved in the definition of Green’s function (43) is the space of generalized polynomials \( \mathcal{P}_n = \mathcal{S}_n(\mathbb{U}, \Sigma, M, X) \). We recall that \( \Sigma = (C[i], E[i]) = \text{diag}(A[i], I_{\mu_i}) = \text{diag}(1, \bar{A}[i], I_{\mu_i}) \), \( \Sigma = (E[i], [E[i]]^{-1} - C[i]) R_n = \text{diag}(I_{\mu_i}, \bar{E}[i], \bar{E}[i], 1) =: \text{diag}(\tilde{E}[i], \tilde{E}[i]) \) with \( 1 \leq \bar{\mu}_i \leq \mu_i \) maximal. Consider now the lECT-spline space \( \mathcal{S}_n := \mathcal{S}_n(\mathbb{U}, \tilde{\Sigma}, \tilde{M}, X) = \mathcal{S}_n(\mathbb{U}, \tilde{\Sigma}, \tilde{\xi}) \) with \( \tilde{M} := (\tilde{\mu}_i) \). It is referred to as the adjoint of \( \mathcal{S}_n \). We want to define its B-splines \( \tilde{N}_n^b(y) \). Associated with \( \mathcal{S}_n \) are the spaces of generalized polynomials \( \tilde{\mathcal{P}}_n = \mathcal{S}_n(\mathbb{U}, \tilde{\Sigma}, \tilde{M}, X) \). We recall that \( \tilde{\Sigma} = (\tilde{C}[i], \tilde{E}[i]) = \text{diag}(\bar{C}[i], \bar{E}[i], \tilde{C}[i], \tilde{E}[i]) = \text{diag}(1, \tilde{A}[i], \bar{E}[i], \bar{E}[i]) \), \( \tilde{\mathcal{P}}_n = \mathcal{S}_n(\mathbb{U}, \tilde{\Sigma}, \tilde{M}, X) \), \( \tilde{\mathcal{P}}_n = \mathcal{S}_n(\mathbb{U}, \tilde{\Sigma}, \tilde{\xi}) \).

The function \( (x, y) \mapsto h(x, y) \) defined in Theorem 3.2 was constructed with respect to the spaces \( \mathcal{S}_n, \mathcal{P}_n, \mathcal{P}_n^* \). When applied with respect to the spaces \( \mathcal{S}_n, \tilde{\mathcal{P}}_n, \tilde{\mathcal{P}}_n^* \) the pendant of Theorem 3.2 for lECT-spline spaces yields a function
(y, x) \mapsto \tilde{h}(y, x) = \sum_{k=1}^{n} (-1)^{n-k} q_k(y, c) p_{n+1-k}(x, c) = (-1)^{n-1} h(x, y),

which, by the preceding considerations, is identified to be $(-1)^{n-1} h(x, y)$. According to the analogues of Definition 4.1 and of Proposition 4.2 for IECT-splines then for $j \in J_\phi$ and $y \in [a, b]$

$$\tilde{N}_j^n(y) = (-1)^n \left( \begin{bmatrix} p_1, \ldots, p_n \\ \tilde{\zeta}_{j+1}, \ldots, \tilde{\zeta}_{j+n} \end{bmatrix} \right)_r \tilde{g}(y, \cdot) - \left( \begin{bmatrix} p_1, \ldots, p_n \\ \tilde{\zeta}_j, \ldots, \tilde{\zeta}_{j+n-1} \end{bmatrix} \right)_r \tilde{g}(y, \cdot),$$

(54)

where $p_1, \ldots, p_n$ is the basis of $P_n$ constructed in Theorem 3.2 and

$$\tilde{g}(y, x) := \begin{cases} \tilde{h}(y, x), & y > x \\ 0, & y \leq x \end{cases}$$

is the Green’s function with respect to the spaces $P_n^*$ and $P_n$. From (53)

$$(-1)^{n-1} h_2(y, x) = (-1)^{n-1} h(x, y) + (-1)^n h_1(x, y)$$

and therefore, for all $n \geq 1$, $j \in J_\phi$ and all $y$

$$\tilde{N}_j^n(y) = (-1)^n \tilde{f}_{n,j} \cdot \left[ \begin{bmatrix} p_1, \ldots, p_{n+1} \\ \tilde{\zeta}_{j+1}, \ldots, \tilde{\zeta}_{j+n} \end{bmatrix} \right] \tilde{g}(y, \cdot),$$

Similarly to Proposition 4.1 we also have

$$\tilde{f}_{n,j} = \left[ \begin{bmatrix} p_1, \ldots, p_n \\ \tilde{\zeta}_{j+1}, \ldots, \tilde{\zeta}_{j+n} \end{bmatrix} \right] p_{n+1} - \left[ \begin{bmatrix} p_1, \ldots, p_n \\ \tilde{\zeta}_j, \ldots, \tilde{\zeta}_{j+n-1} \end{bmatrix} \right] p_{n+1},$$

where $p_1, \ldots, p_n, p_{n+1} \in P_{n+1}$ are functions defined to be a basis of the space $P_{n+1} = V_{n+1}(\hat{\mathcal{H}}, \hat{\mathcal{C}}_+, M^0, X)$ as defined above. According to Corollary 4.1.1 of [9] $P_n^*$ resp. $P_n$ is an IET- resp. an rET-space of order $n$ each, and $P_{n+1}$ is an rET-space of order $n + 1$. By Corollary 4.1.4 of [9] the space $P_{n+1}$ has a basis $p_1, \ldots, p_n, p_{n+1}$ such that for $v = 0$, $1$ the system $p^1, \ldots, p_{n+v}$ is an rET-system of order $n + v$.

Remark 5.1. There is an interesting case of selfadjointness. We call an ECT-spline space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}_+, M, X)$ selfadjoint provided $M = \hat{\mathcal{M}}$, and for all $i$ span $U^{[i]} = \text{span} U^{[i], \mathcal{C}} = C^{[i]}$. In general this requires $\mu_i = 1$ and $(\hat{A}^{[i]})^{-1} = R_{n-2} \hat{A}^{[i]} R_{n-2}^{-1}$ for all $i$.

Remark 5.2. What does this tell us in the particular case of ordinary polynomial splines with simple knots? Obviously, this is a spline space which is selfadjoint. Therefore besides (46) according to (54) we have for all $n \geq 1$ the representation

$$\tilde{N}_j^n(y) = (-1)^n (\xi_{j+n} - \xi_j) [\xi_j, \ldots, \xi_{j+n}]_r \tilde{g}(y, \cdot) = (\xi_{j+n} - \xi_j) [\xi_j, \ldots, \xi_{j+n}]_r (\cdot - y)^{n-1},$$

which for $n > 1$ is (51) if $y$ is replaced by $x$. 

References