Sigma-fragmentability and the property SLD in $C(K)$ spaces

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We characterize two topological properties in Banach spaces of type $C(K)$, namely, being $\sigma$-fragmented by the norm metric and having a countable cover by sets of small local norm-diameter (briefly, the property norm-SLD). We apply our results to deduce that $C_p(K)$ is $\sigma$-fragmented by the norm metric when $K$ belongs to a certain class of Rosenthal compacta as well as to characterize the property norm-SLD in $C_p(K)$ in case $K$ is scattered.

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1. Introduction

In the present paper we are concerned with two topological notions introduced by J.E. Jayne, I. Namioka and C.A. Rogers in [5,6] to study the topological structure of Banach spaces. A topological space $(X,T)$ is said to be $\sigma$-fragmented by a metric $\rho$ defined on $X$ if for every $\epsilon > 0$ we can write

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

in such a way that for every $n \in \mathbb{N}$ and every nonempty subset $A$ of $X_n$ there exists a $T$-open set $U$ such that $U \cap A \neq \emptyset$ and $\rho$-diam$(U \cap A) < \epsilon$. On the other hand, $X$ is said to have a countable cover by sets of small local $\rho$-diameter ($X$ has $\rho$-SLD for short) if for every $\epsilon > 0$ we can write

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

in such a way that for every $n \in \mathbb{N}$ and every $x \in X_n$ there exists a $T$-open set $U$ such that $x \in U$ and $\rho$-diam$(U \cap X_n) < \epsilon$. Clearly, having $\rho$-SLD implies being $\sigma$-fragmented by $\rho$. The converse remains open in general although it holds, for instance, in case where $X$ is a metric space, $T$ the metric topology and $\rho$ any metric on $X$ [15].

Banach spaces taken with their weak topologies are a natural and interesting context to study the notions above, see e.g. [5–8]. Specially relevant for the present paper are the connections between these concepts and some renormings in Banach spaces, for instance, in [5] it was showed that a Banach space $X$ endowed with its weak topology has a countable

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cover by sets of small local norm-diameter (X has norm-SLD for short) if the norm of X is Kadec, i.e., the norm and the weak topologies coincide on the unit sphere of X. For the converse the reader is referred to [16]. On the other hand, a Banach space X is said to be σ-fragmentable if X endowed with its weak topology is σ-fragmented by the norm metric. It is known [3] that a σ-fragmentable $C_0(Y)$ space, where $Y$ is a Hausdorff tree, admits an equivalent Kadec norm but, in general, it is not clear whether a σ-fragmentable Banach space has such sort of renorming [3, Problem 11.4].

Of particular concern in the present paper is the Banach space $C(K)$ of all real-valued continuous functions on K with the supremum norm $\|x\|_\infty = \sup\{|x(t)|: t \in K\}, x \in C(K)$. The topology of pointwise convergence in $C(K)$ will be denoted by $T_p$ and the topological space $(C(K), T_p)$ will be written as $C_p(K)$ for short. Let us point out that all compacta K for which $C_p(K)$ is σ-fragmented by the norm metric share a property involving separate and joint continuity of maps of the form $f: B \times K \to \mathbb{R}$, where B is a Baire space. A compact K is said to have the Namioka property if for every Baire space B and every separately continuous map $f: B \times K \to \mathbb{R}$, there exists a dense $G_\delta$ subset D of B such that f is jointly continuous at every point of $D \times K$. In [5] it was proved that K has the Namioka property if $C_p(K)$ is σ-fragmented by the norm metric. The converse is false in some set-theoretic models [14]. Let us recall that $C_p(K)$ is σ-fragmented by the norm metric if $C(K)$ admits an equivalent $T_p$-lower semicontinuous norm $|\cdot|$ with the property that, the norm topology and $T_p$ coincide on the $|\cdot|$-unit sphere of $C(K)$ [5].

A norm $|\cdot|$ on a Banach space X is said to be locally uniformly rotund (LUR for short) if $\lim_{n \to \infty} |x - x_n| = 0$ whenever $x, x_n \in X$ and $\lim_{n \to \infty} (2|x|^2 + 2|x_n|^2 - |x + x_n|^2) = 0$. The connection between LUR norms and a particular case of the property norm-SLD was exposed in [12,17] where it was proved that a Banach space X has a LUR renorming if, and only if, for every $\varepsilon > 0$ we can write $X = \bigcup_{n \in \mathbb{N}} X_n$ in such a way that for every $n \in \mathbb{N}$ and every $x \in X_n$ there exists an open half space H such that $x \in H$ and $\text{diam}(H \cap X_n) < \varepsilon$. Our goal in the present paper is to characterize the σ-fragmentability and the property norm-SLD in $C(K)$ spaces (see Theorems 1 and 4 below) in the spirit of [11], where it has been recently showed that the existence of a LUR renorming in such sort of spaces is equivalent to, roughly speaking, describing regularly the members of a covering of $K$ on which every $x \in C(K)$ has arbitrarily small oscillation. An alternative version of this result for pointwise compact subsets K of $\mathbb{R}^I$ is also presented in [11] in terms of regular descriptions of the coordinates of $\Gamma_\varepsilon$ that $\varepsilon$-control every $x \in C(K)$, see (1) below. Our results about σ-fragmentability and the property norm-SLD in $C(K)$ are also enounced in terms of coverings and control coordinates as above since our approach is similar to that given in [11]. As an application we characterize the property norm-SLD in $C_p(K)$ when K is scattered and we deduce a result in [4,9] about the σ-fragmentability of $C_p(K)$ when K is a Rosenthal compact made up by functions that have at most countably many discontinuities. The contents of this paper were announced by the author in [10].

2. $C(K)$ spaces σ-fragmented by the norm metric

The main result of this section is a characterization of $C_p(K)$ spaces that are σ-fragmented by the norm metric, see Theorem 1 below. For its proof, that is based on a result about LUR renormings in $C(K)$ appeared in [11], we introduce some terminology. It is well known that any compact space K is homeomorphic to a subspace of a cube $[0,1]^I$ for some set $I_\varepsilon$. Without loss of generality we can and do assume that $K \subseteq [0,1]^{I_\varepsilon}$. Since any continuous function on K is uniformly continuous, given $\varepsilon > 0$ and $x \in C(K)$ there is a finite subset $N$ of $I_\varepsilon$ and there is $\delta > 0$ such that

$$|x(s) - x(t)| < \varepsilon \quad \text{whenever } s, t \in K \quad \text{and} \quad \sup_{y \in N} |s(y) - t(y)| < \delta.$$  \hspace{1cm} (1)

Following [13] we say that $N \varepsilon$-controls $x$ with $\delta$ whenever (1) holds. We simply say that $N \varepsilon$-controls $x$ if there is some $\delta > 0$ such that $N \varepsilon$-controls $x$ with $\delta$. In case N is infinite we say that $N \varepsilon$-controls $x$ if some finite subset of $N \varepsilon$-controls $x$. If $L \subseteq K$ we write $\text{osc}(x, L)$ for the diameter of $x(L)$. As usual, $B(x, r)$ stands for the open ball in $C(K)$ centered at $x$ with radius $r$.

**Theorem 1.** Let $K \subseteq [0,1]^{I_\varepsilon}$ be a compact Hausdorff space. The following assertions are equivalent:

(i) $C_p(K)$ is σ-fragmented by the norm metric;
(ii) for every $\varepsilon > 0$ we can write $C(K) = \bigcup_{n \in \mathbb{N}} C_n$ in such a way that for every $n \in \mathbb{N}$ and every nonempty subset A of $C_n$ there exists a $T_p$-open set U which meets A and there is a finite subset $N$ of $I_\varepsilon$ that $\varepsilon$-controls every $x \in U \cap A$;
(iii) for every $\varepsilon > 0$ we can write $C(K) = \bigcup_{n \in \mathbb{N}} C_n$ in such a way that for every $n \in \mathbb{N}$ and every nonempty subset A of $C_n$ there exists a $T_p$-open set U which meets A and there is a countable subset $N$ of $I_\varepsilon$ that $\varepsilon$-controls every $x \in U \cap A$;
(iv) for every $\varepsilon > 0$ we can write $C(K) = \bigcup_{n \in \mathbb{N}} C_n$ in such a way that for every $n \in \mathbb{N}$ and every nonempty subset A of $C_n$ there exists a $T_p$-open set U with meets A and there is a finite covering $L$ of K such that $\text{osc}(x, L) < \varepsilon$ for every $x \in U \cap A$ and every $L \subseteq L$;
(v) for every $\varepsilon > 0$ we can write $C(K) = \bigcup_{n \in \mathbb{N}} C_n$ in such a way that for every $n \in \mathbb{N}$ and every nonempty subset A of $C_n$ there exists a $T_p$-open set U with meets A and there are countably many finite coverings $|L_\varepsilon|$ of K with the property that for every $x \in U \cap A$ there is $j \in \mathbb{N}$ such that $\text{osc}(x, L) < \varepsilon$ for every $L \subseteq L_j$.

**Proof.** (i) $\rightarrow$ (ii) Let $\varepsilon > 0$. Since $C_p(K)$ is σ-fragmented by the norm metric we can write $C(K) = \bigcup_{n \in \mathbb{N}} C_n$ in such a way that for every $n \in \mathbb{N}$ and every nonempty subset A of $C_n$ there is a $T_p$-open set U such that $U \cap A \neq \emptyset$ and
diam(U ∩ A) < ε/3. Given n, A and U as above we choose a finite subset N of Γ that ε/3-controls some element of U ∩ A. It is easily checked that N ε-controls every x ∈ U ∩ A.

(ii) → (iii) It is obvious.

(iii) → (ii) Given ε > 0 let C(K) = ∪ n∈N Cn be the decomposition of (iii) for ε/2. Then given n ∈ N and A ⊆ Cn, A ≠ ∅, there exists a T̃ρ-open set U which meets A and there is a countable subset N of Γ that ε/2-controls every x ∈ U ∩ A. For a finite set F ⊆ N and i ∈ N we let X(F, i) = {x ∈ U ∩ A: F ε/2-controls x with 1/i}. For each s ∈ K we write V_s for the open neighbourhood of s made up by all t ∈ K such that sup{|x(y) − t(y)|: γ ∈ F} < 1/2i. By compactness it is easily seen that there is a finite covering L′ of K with the property that osc(x, L′) < ε for every x ∈ X(F, i) and every L ∈ L′. Hence, (v) follows from the fact that U ∩ A can be written as a countable union of the sets X(F, i) above.

(v) → (i) We follow [4, Proposition 5]. Given ε > 0 let C(K) = ∪ n∈N Cn be the decomposition of (v) for ε/9. Then given n ∈ N and A ⊆ Cn, A ≠ ∅, there exists a T̃ρ-open set U which meets A and there are countably many finite coverings {C_l} l∈N of K with the property that for every x ∈ U ∩ A there is j ∈ N such that osc(x, L) < ε/9 for every L ∈ L_j. Given J ∈ N we are going to show that the set X_j = {x ∈ U ∩ A: osc(x, L) < ε/9 for every L ∈ L_j} can be covered by countably many sets of diameter less than ε. We first take a covering {I_l} l∈N of R made up by sets of diameter less than ε/9. Since L_j can be written as {I_k} k∈J for some m ∈ N, we choose a point s_k ∈ L_k for all k, 1 ≤ k ≤ m. Finally, for every map π : {1, ..., m} → N we fix x_π ∈ X_j satisfying x_π(s_k) = I_π(k) for all k, 1 ≤ k ≤ m. Whenever this is possible. We now claim that for these π we have X_j ⊆ B(x_π, ε/3). Indeed, if x ∈ X_j then for every k ∈ {1, ..., m} there exists I_k ∈ N such that x(s_k) ∈ I_k. Denote by π the map k ↦ I_k. Thus, if s ∈ K there is k such that s ∈ I_k so

\[|x(s) - x_π(s)| ≤ |x(s) - x(s_k)| + |x(s_k) - x_π(s_k)| + |x_π(s_k) - x_π(s)| ≤ osc(x, I_k) + diam(I_π(k)) + osc(x_π, I_k) ≤ ε/3.

Hence, x ∈ B(x_π, ε/3) and our claim is as required. Since U ∩ A = ∪ I∈N X_j, we conclude that U ∩ A can be covered by countably many sets of diameter less than ε. That C_p(K) is σ-foliated by the norm metric follows from [6, Theorem 4.1].

(i) → (iv) As in (ii) → (i), given ε > 0 we can write C(K) = ∪ n∈N C_n in such a way that for every n ∈ N and every A ⊆ C_n, A ≠ ∅, there exists a T̃ρ-open set U such that U ∩ A ≠ ∅ and diam(U ∩ A) < ε/3. Given n, A and U as above and x_0 ∈ U ∩ A, we choose a finite covering L′ of K with the property that osc(x_0, L′) < ε/3 for every L ∈ L′. It is easily checked that osc(x, L′) < ε for every x ∈ U ∩ A and every L ∈ L′.

(iv) → (v) It is obvious. □

**Remark 2.** Given a normed space X we say that a linear subspace F of X^* is norming if the map x ↦ sup{|f(x)|: f ∈ X^* ∩ F}, x ∈ X, provides an equivalent norm on X. The topology in X of convergence on the elements of F is denoted by σ(X, F). Theorem 1 also holds if C(K) is replaced by any linear subspace X of it and if the pointwise topology in X is changed by σ(X, F), F being a norming linear subspace of X^*.

Using topological games and strengthening some results of A. Bouziad in [1] concerning the Namioka property, I. Kortezov proved in [9] that if K is a pointwise compact set of functions on a Polish space with the property that each s ∈ K has at most countably many discontinuities then C_p(K) is σ-foliated by the norm metric. This result has been recently shown in [4] with an alternative approach involving control coordinates and we may now deduce it as an application of Theorem 1.

**Corollary 3. ([4,9])** Let Γ be a Polish space and K be a pointwise compact set of functions on Γ such that each s ∈ K has only countably many discontinuities. Then C_p(K) is σ-foliated by the norm metric.

**Proof.** In [4, Lemma 12] it was proved that for every ε > 0 there is a decomposition C(K) = ∪ n∈N C_n in such a way that for every n ∈ N and every x ∈ C_n there exists a T̃ρ-open set U that contains x and there exists a countable set N ⊆ Γ which ε-controls every y ∈ U ∩ C_n. The result follows from Theorem 1(iii). □

3. C(K) spaces with the property norm-SLD

In this section we present an analogous result to Theorem 1 about the property norm-SLD in C_p(K), see Theorem 4 below, whose proof is closely modeled on a characterization of existence of LUR renormings in C(K) appeared in [11]. Let us introduce some notation. Given a bounded subset A of a normed space, the Kuratowski index of non-compactness of A, α(A), is defined by α(A) = inf n∈N α(A, n) where

\[α(A, n) = \inf \{ε > 0: A can be covered by n sets of diameter less than ε \}.

In [2] it was pointed out that if X is a normed space and F is a norming subspace of X^* then (X, α(X, F)) has norm-SLD if, and only if, for every ε > 0 we can write X = ∪ n∈N X_n in such a way that for every n ∈ N and every x ∈ X_n there exists a σ(X, F)-open set U such that x ∈ U and α(U ∩ X_n) < ε. Indeed, since α(U ∩ X_n) ≤ diam(U ∩ X_n), the condition above is necessary. Conversely, given ε > 0 let X = ∪ n∈N X_n be a decomposition with the property that, for every n ∈ N and every x ∈ X_n there exists a σ(X, F)-open set U that contains x and there is a finite covering of U ∩ X_n made up by norm-closed...
balls \(|B_i|_{i \in I}; I \text{ finite, of diameter less than } \varepsilon/2\). Taking \(I_0 \subseteq I\) such that \(x \notin \bigcup_{i \in I_0} B_i\) and \(x \in \bigcap_{i \notin I_0} B_i\) and having in mind that each norm-closed ball is \(\sigma(X, F)-\text{closed}, it is easily seen that the } \sigma(X, F)-\text{open set } W = X \setminus \bigcup_{i \in I_0} B_i \text{ contains } x \text{ and satisfies diam}(W \cap U \cap X_n) < \varepsilon.}

**Theorem 4.** Let \(K \subseteq [0, 1]^\Gamma\) be a compact Hausdorff space. The following assertions are equivalent:

(i) \(C_p(K)\) has the property norm-SLD;

(ii) for every \(\varepsilon > 0\) we can write \(C(K) = \bigcup_{n \in \mathbb{N}} C_n\) in such a way that for every \(n \in \mathbb{N}\) and every \(x \in C_n\) there is a \(T_p\)-open set \(U\) that contains \(x\), a finite subset \(N\) of \(\Gamma\) and \(\delta > 0\) such that \(N\varepsilon\)-controls every \(y \in U \cap C_n\) with \(\delta\);

(iii) for every \(\varepsilon > 0\) we can write \(C(K) = \bigcup_{n \in \mathbb{N}} C_n\) in such a way that for every \(n \in \mathbb{N}\) and every \(x \in C_n\) there is a \(T_p\)-open set \(U\) that contains \(x\) and a finite covering \(L\) of \(K\) such that \(osc(y, L) < \varepsilon\) for every \(y \in U \cap C_n\) and every \(L \in L\).

**Proof.** (i) \(\rightarrow\) (ii) Since \(C_p(K)\) has norm-SLD, given \(\varepsilon > 0\) there is a decomposition \(C(K) = \bigcup_{n \in \mathbb{N}} C_n\) in such a way that for every \(n \in \mathbb{N}\) and every \(x \in C_n\) there is a \(T_p\)-open set \(U\) such that \(x \in U\) and \(diam(U \cap C_n) < \varepsilon/3\). If we choose a finite set \(N \subseteq \Gamma\) and \(\delta > 0\) with \(\varepsilon\)-controls every \(y \in U \cap C_n\) with \(\delta\),

(ii) \(\rightarrow\) (iii) Given \(\varepsilon > 0\) let \(C(K) = \bigcup_{n \in \mathbb{N}} C_n\) be the decomposition of (ii) for \(\varepsilon/2\). Then given \(n \in \mathbb{N}\) and \(x \in C_n\) there is a \(T_p\)-open set \(U\) that contains \(x\), a finite subset \(N\) of \(\Gamma\) and a positive \(\delta\) such that \(N\varepsilon\)-controls every \(y \in U \cap C_n\) with \(\delta\). For every \(s \in K\) we write \(V_s\) for the open neighbourhood of \(s\) made up by all \(t \in K\) such that \(sup\{|s(y) - t(y)|: y \in N\} < \delta/2\).

Applying compactness it is easy to check that there is a finite covering \(L\) of \(K\) with the property that \(osc(y, L) < \varepsilon\) for every \(y \in U \cap C_n\) and every \(L \in L\).

(iii) \(\rightarrow\) (i) We follow [4, Proposition 5]. Given \(\varepsilon > 0\) let \(C(K) = \bigcup_{n \in \mathbb{N}} C_n\) be the decomposition of (iii) for \(\varepsilon/9\). Then given \(n \in \mathbb{N}\) and \(x \in C_n\) there is a \(T_p\)-open set \(U\) that contains \(x\) and a finite covering \(L\) of \(K\) such that \(osc(y, L) < \varepsilon/9\) for every \(y \in U \cap C_n\) and every \(L \in L\). We are going to show that \(U \cap C_n\) can be covered by finitely many sets of diameter less than \(\varepsilon\). Without loss of generality we can and do assume that there is \(r \in \mathbb{N}\) such that \(\|y\| \leq r\) for all \(y \in C_n\). Let \(\{I_j\}_{j=1}^n\) be a finite covering of \([-r, r]\) made up by sets of diameter less than \(\varepsilon/9\). Since \(L\) can be written as \(\{I_j\}_{j=1}^m\) for some \(m \in \mathbb{N}\), we choose a point \(\tau \in I_l\) for all \(i, 1 \leq i \leq m\). Finally, for each map \(\pi : \{1, \ldots, m\} \rightarrow \{1, \ldots, \ell\}\) we fix \(x_{\pi} \in U \cap C_n\), satisfying \(x_{\pi}(s_j) \in I_{\pi(j)}\) for all \(i, 1 \leq i \leq m\), whenever this is possible. We now claim that for these \(x_{\pi}\) we have \(U \cap C_n \subseteq \bigcup B(x_{\pi}, \varepsilon/3)\).

Indeed, if \(y \in U \cap C_n\) then for every \(i \in \{1, \ldots, m\}\) there is \(k_i \in \{1, \ldots, \ell\}\) such that \(y(s_j) \in I_{k_i}\). Denote by \(\pi\) the map \(i \mapsto k_i\). Thus, if \(x \in K\) there exists \(i \in I_l\) such that \(s_j \in I_l\), so

\[
\left|y(s) - x_{\pi}(s)\right| \leq \left|y(s) - y(s_j)\right| + \left|y(s_j) - x_{\pi}(s_j)\right| + \left|x_{\pi}(s_j) - x_{\pi}(s)\right|
< osc(y, I_l) + diam(I_{\pi(j)}) + osc(x_{\pi}, L_l) < \varepsilon/3.
\]

Hence, \(y \in B(x_{\pi}, \varepsilon/3)\) and our claim is true as required. Since the Kuratowski index of non-compactness of \(U \cap C_n\) is less than \(\varepsilon\), it follows that \(C_p(K)\) has norm-SLD. □

**Remark 5.** Remark 2 is also valid for Theorem 4.

As an application of Theorem 4 we characterize the property norm-SLD in \(C_p(K)\) when \(K\) is scattered, see Theorem 6 below. Let us recall that a topological space is said to be scattered (respectively, \(\sigma\)-discrete) if each of its nonempty closed subsets has an isolated point (respectively, if it is a countable union of relatively discrete subsets).

**Theorem 6.** Let \(K\) be a scattered compact space. Then \(C_p(K)\) has the property norm-SLD if, and only if, \(C_p(K; [0, 1])\) is \(\sigma\)-discrete.

**Proof.** If we write \(A\) for the family of all compact and open subsets of \(K\) it is clear that \(C(K; [0, 1]) = \{\mathbb{0}_A: A \in A\}\) where \(\mathbb{0}_A\) denotes the indicator function.

Suppose that \(C_p(K)\) has norm-SLD. Then there is a decomposition \(A = \bigcup_{i \in I} A_i\) with the property that, given \(i \in \mathbb{N}\) and \(A \in A_i\) there exists a \(T_p\)-open set \(U\) such that \(\mathbb{0}_A \in U \text{ and } diam(U \cap \mathbb{0}_A) < 1\). Since the last inequality yields \(U \cap \mathbb{0}_A \cap \mathbb{0}_{A_i} = \{0\}\), the set \(\{\mathbb{0}_A: B \in A_i\}\) is \(T_p\)-discrete.

Conversely, suppose that there exists a decomposition \(A = \bigcup_{i \in I} A_i\) with the property that, for every \(i \in \mathbb{N}\) the set \(\{\mathbb{0}_A: A \in A_i\}\) is \(T_p\)-discrete. Then for every \(i \in \mathbb{N}\) and every \(A \in A_i\) there is a finite set \(F(i, A) \subseteq K\) such that

\[
\mathbb{B} = A \text{ whenever } B \in A_i \text{ and } 1_B(t) = \mathbb{0}_A(t) \text{ for all } t \in F(i, A).
\]

Let \(\varepsilon > 0\). Given a natural number \(n\), a map \(\pi : \{1, \ldots, n\} \rightarrow \mathbb{N}\) and a family \(\pi_n = \{I_j\}_{j=1}^n\) of open real intervals with rational end points and length less than \(\varepsilon\) such that \(I_j \cap I_k = \emptyset\) when \(j \neq k\), let \(C_n, \pi_n, \pi_n\) be the set of all \(x \in C(K)\) such that \(x(K) \subseteq \bigcup_{j=1}^n I_j\) and \(x^{-1}(I_j) \in A_{\pi(j)}\) for all \(j, 1 \leq j \leq n\). Since \(K\) is scattered \(C(K)\) can be written as a countable union of the sets \(C_n, \pi_n, \pi_n\).

Given \(n, \pi\) and \(\pi_n\) as above we take \(x \in C_n, \pi, \pi_n\). If we write \(L_j = x^{-1}(I_j)\) for all \(j, 1 \leq j \leq n\), then \(L_j \in A_{\pi(j)}\) and the
family $\mathcal{L} = \{L_i\}_{i=1}^n$ is a finite covering of $K$. Setting $d = \min|a - b|:\ a \in I_j,\ b \in I_k,\ 1 \leq j \neq k \leq n$, we define $U$ to be the $T_p$-open set

$$U = \bigcap_{j=1}^n \left\{ y \in C(K): \max_{t \in F(\pi(j), I_j)} |x(t) - y(t)| < d \right\}.$$ 

According to Theorem 4(iii), to show that $C_p(K)$ has norm-SLD it suffices to prove that $\text{osc}(y, L_j) < \varepsilon$ for every $y \in U \cap C_n(\pi, I_n)$ and every $j \in \{1, \ldots, n\}$. Indeed, let $y \in U \cap C_n(\pi, I_n)$ and $j \in \{1, \ldots, n\}$. By the definition of $d$ it follows that, if $t \in K$ and $|x(t) - y(t)| < d$ then $x(t) \in I_j$ if, and only if, $y(t) \in I_j$. Hence, for every $t \in F(\pi(j), L_j)$ we have $\mathbb{1}_{L_j}(t) = \mathbb{1}_{y^{-1}(I_j)}(t)$. Since $y^{-1}(I_j) \in A_{\pi(j)}$, we conclude from (2) that $L_j = y^{-1}(I_j)$. Thus, $\text{osc}(y, L_j) \leq \text{diam}(I_j) < \varepsilon$ and the proof is complete. \[\Box\]

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