ELSEVIER Stochastic Processes and their Applications 68 (1997) 209–228

stochastic processes and their applications

On solutions of one-dimensional stochastic differential equations driven by stable Lévy motion

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Received 25 December 1995; received in revised form 3 February 1997

Abstract

We consider the stochastic differential equation

$$dX_t = b(X_{t-})\,dZ_t, \quad t \geqslant 0,$$

where b is a Borel measurable real function and Z is a symmetric α-stable Lévy motion. In Section 1 we study the convergence of certain functionals of Z and in particular, we extend Engelbert and Schmidt 0–1 law (for functionals of the Wiener process) to functionals of a symmetric α-stable Lévy motion with $1 < \alpha \leqslant 2$. In Section 2 we study the existence of weak solutions for the above equation. When $0 < \alpha < 1$ or $1 < \alpha \leqslant 2$ we prove a sufficient existence condition. In the case $1 < \alpha \leqslant 2$, we extend Engelbert and Schmidt's necessary and sufficient existence condition (for the equation driven by a Wiener process) to the above equation: we prove that, for every x there exists a nontrivial solution starting from x, if and only if $|b|^{-\alpha}$ is locally integrable. In Section 3 we study "local" solutions. We also prove a result relating "local" and "global" solutions.

Keywords: α-Stable Lévy motions; 0–1 Law; Stochastic differential equations; Existence; "Local" existence; Stable integrals; Purely discontinuous martingales; Random measures; Time change

0. Introduction

In the first section of the present paper we study the convergence of certain functionals of a symmetric α-stable Lévy motion (with $0 < \alpha < 1$ or $1 < \alpha \leqslant 2$) which we always denote by Z and simply call an α-stable motion. In particular, when $1 < \alpha \leqslant 2$, we prove a 0–1 law (Theorem 1.4) analogous to Engelbert and Schmidt (1981) 0–1 law for functionals of the Wiener process. A 2-stable motion being a Wiener process, the present 0–1 law extends that of Engelbert and Schmidt to the class of α-stable motions with $1 < \alpha \leqslant 2$.

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PII S0304-4149(97)00030-6

As in the Wiener case, the principal tool of the proof is the use of a local time (i.e. an occupation time density with respect to Lebesgue measure) for α-stable motions with $1 < \alpha \leqslant 2$: in particular, see the proof of Lemma 1.6.

In the subsequent sections some of these results are applied to the case of certain functionals of Z which are associated with one-dimensional stochastic differential equations driven by stable motions. Such functionals arise when applying the method of random time change to solving the equations we consider.

These are of the form

$$X_t = x + \int_{]0,t]} b(X_{s^-})\, \mathrm{d}Z_s, \quad t \geqslant 0, \tag{1}$$

where x is a real starting point, b is a Borel measurable real function and Z denotes an α-stable motion.

In Section 2 for such equations we study the problem of the existence of weak solutions which we want to be nontrivial i.e. which are to move away from the starting point x with probability > 0. We prove a general sufficient condition (Theorem 2.5) that, in a sense, unifies the cases $0 < \alpha < 1$ and $1 < \alpha \leqslant 2$.

In the latter case we also obtain a necessary condition (Proposition 2.30) for the existence of a nontrivial solution.

As a corollary of these two conditions we have the following result about solutions of Eq. (1) driven by an α-stable motion Z with $1 < \alpha \leqslant 2$ (see Theorem 2.32):
For every real number x there exists a nontrivial solution starting from x if and only if the function $|b|^{-\alpha}$ is locally integrable.

This necessary and sufficient condition extends an Engelbert and Schmidt (1981) result for equation (1) driven by a Wiener process (i.e. a 2-stable motion) to the class of equations (1) driven by Z with $1 < \alpha \leqslant 2$.

Besides some results in Section 1, the principal tools we use are various results in Kallenberg (1992) such as the integrability criterion with respect to stable Lévy motion, the time change representation of stable integrals as well as the predictable reduction property of integer-valued random measures to Poisson ones (see Theorems 3.1, 4.1, 2.1 in Kallenberg).

We also use some properties of time changed processes and random measures as studied in a general setting in Jacod (1979, Section 1, Chap. X).

In Section 3 we study the existence of "local" solutions i.e. of processes that solve Eq. (1) up to the first exit time of an interval. We obtain a sufficient condition for local existence (Theorem 3.4) that completely unifies the cases $0 < \alpha < 1$ and $1 < \alpha \leqslant 2$.

In the case $1 < \alpha \leqslant 2$ we also state a necessary condition (Proposition 3.13) very natural if compared with that in Proposition 2.30: it is a consequence of that condition indeed.

Still in the case we have a result (Theorem 3.17) which relates "local" solutions with "global" solutions (i.e. solutions defined for all t). Roughly, it states the following: For Eq. (1) the existence for any real x of a "local" nontrivial solution starting from x and the existence for any x of a "global" nontrivial solution starting from x, are equivalent properties.

1. Some properties of symmetric stable Lévy motions. A 0–1 law

First of all we recall a few basic notions. For all other notions the reader is referred to well-known treatises as e.g. Dellacherie (1972), Dellacherie and Meyer (1980), Jacod (1979).

Let $(\Omega, \mathscr{F}, \boldsymbol{P})$ be a complete probability space. An increasing family $\mathbb{F} = (\mathscr{F}_t)_{t \geqslant 0}$ of sub-σ-algebras of $\mathscr{F}$ is called a filtration. We term stochastic basis a space $(\Omega, \mathscr{F}, \boldsymbol{P})$ equipped with a filtration $\mathbb{F}$: for such an object we use the notation $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$. All the stochastic bases $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$ we consider in this paper, are supposed to satisfy the usual conditions, i.e. $\mathbb{F}$ is right-continuous and $\mathscr{F}_0$ contains all $\mathscr{F}$-sets of $\boldsymbol{P}$-measure 0.

For a process $X = (X_t)_{t \geqslant 0}$ defined on $(\Omega, \mathscr{F}, \boldsymbol{P})$ we write $(X, \mathbb{F})$ to mean that X is $\mathbb{F}$-adapted.

All the processes we consider here are supposed to be real-valued and to have all the sample paths in $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R})$, the space of all càdlàg mappings from $\mathbb{R}_+ = [0, +\infty[$ into $\mathbb{R}$.

Also we shall adopt here the usual convention $0 \cdot \infty = 0/0 = 0$.

Now let $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$ be a stochastic basis: throughout the paper the notation $(Z, \mathbb{F})$ will stand for a process on this basis which is symmetric $\mathbb{F}$-α-stable Lévy motion, $0 < \alpha \leqslant 2$, i.e. an $\mathbb{F}$-adapted process $(Z_t)_{t \geqslant 0}$ with all sample paths in $\mathbb{D}$, such that $Z_0 = 0$ and for any $0 \leqslant s < t$, $\vartheta \in \mathbb{R}$

$$\boldsymbol{E}[\exp\{i\vartheta(Z_t - Z_s)\} | \mathscr{F}_s] = \exp\{-(t-s)|\vartheta|^\alpha\}.$$

This assumption implies that the increments $Z_t - Z_s$, $s \leqslant t$, are independent of $\mathscr{F}_s$. Hence a symmetric $\mathbb{F}$-α-stable Lévy motion has stationary α-stable symmetric increments that are independent of the past for the filtration $\mathbb{F}$.

In the sequel such a process $(Z, \mathbb{F})$ will be called an $\mathbb{F}$-α-stable motion; we will, however, suppress the prefix "$\mathbb{F}$" whenever its appearance is not essential.

Also denote by $v(\omega, t, \cdot)$ the occupation time measure of Z i.e. the measure defined as follows, for all Borel sets A in $\mathbb{R}$ and all ω, $t \geqslant 0$

$$v(\omega, t, A) = \int_0^t 1_A(Z_s(\omega))\,\mathrm{d}s = \lambda(\{s \leqslant t : Z_s(\omega) \in A\}), \tag{1.1}$$

λ denoting Lebesgue measure.

As it's well known (see, e.g. Kesten, 1969), Z has a local time if and only if $1 < \alpha \leqslant 2$: by local time we mean here an occupation time density i.e. a function $L(\omega, t, y)$ such that, $\boldsymbol{P}$-a.s.

$$v(t, A) = \int_{\mathbb{R}} 1_A(y) L(t, y)\,\mathrm{d}y \tag{1.2}$$

for any t and any Borel set A. And a version of $L(\omega, t, y)$ exists which is jointly measurable in (ω, t, y). Moreover, as it was proved in Boylan (1964), $(t, y) \mapsto L(\omega, t, y)$ can be chosen to be jointly continuous for all ω. In the sequel we shall always employ a version of the local time L satisfying these regularity conditions.

Proposition 1.3. *Let Z denote an α-stable motion and ν its occupation time measure defined in* (1.1).

(a) *Assume* $0 < \alpha < 1$. *Then we have*

$$P\left(\left\{\lim_{t\to\infty} \nu(t, A) < \infty\right\}\right) = 1$$

for every Borel set A such that $\lambda(A) < \infty$.

(b) *Assume* $1 < \alpha \leqslant 2$. *Then we have*

$$P\left(\left\{\lim_{t\to\infty} \nu(t, A) = \infty\right\}\right) = 1$$

for every Borel set A such that $\lambda(A) > 0$.

Proof. (a) In this case one has for every A with $\int_A |y|^{\alpha-1}\,\mathrm{d}y < \infty$

$$E\left[\int_0^\infty 1_A(Z_s)\,\mathrm{d}s\right] < \infty$$

(see Blumenthal and Getoor, 1968, Exercise 1.7, p.71). Thus the same relation holds for every A with $\lambda(A) < \infty$.

(b) Since, P-a.s., $L(t, y)$ is increasing in t and $\lim_{t\to\infty} L(t, y) = \infty$ for all y (cf. Stone, 1963, Theorem 1, p. 633) the property follows from (1.2) because of the monotone convergence theorem. □

Now for α-stable motions with $1 < \alpha \leqslant 2$ we prove a 0–1 law (Theorem 1.4) analogous to the Engelbert and Schmidt 0–1 law for functionals of the Wiener process: see Theorem 1 in Engelbert and Schmidt (1981). As in that case the idea of the proof is based on the use of local time of α-stable motions (with $1 < \alpha \leqslant 2$).

Theorem 1.4. *Fix* α, $1 < \alpha \leqslant 2$ *and let* $(Z, \mathbb{F})$ *denote an α-stable motion defined on* $(\Omega, \mathscr{F}, \mathbb{F}, P)$. *Let f be a Borel measurable function of the real line into* $[0, \infty]$.

Then the following conditions are equivalent.

(a) $P(\{\int_0^t f(Z_s)\,\mathrm{d}s < \infty$ for every $t \geqslant 0\}) > 0$.
(b) $P(\{\int_0^t f(Z_s)\,\mathrm{d}s < \infty$ for every $t \geqslant 0\}) = 1$.
(c) $\int_K f(y)\,\mathrm{d}y < \infty$ *for every compact subset K of the real line.*

Remark 1.5. Each of the following two conditions is also equivalent to the relations in the preceding theorem.

(d) For all $x \in \mathbb{R}$

$$P\left(\left\{\int_0^t f(x + Z_s)\,\mathrm{d}s < \infty \text{ for every } t \geqslant 0\right\}\right) = 1.$$

(e) For every $x \in \mathbb{R}$ there exists an α-stable motion $(\zeta, \mathbb{H})$ and a finite, strictly positive random variable τ on a suitable stochastic basis $(\Xi, \mathscr{H}, \mathbb{H}, \boldsymbol{Q})$ such that

$$\boldsymbol{Q}\left(\left\{\int_0^\tau f(x+\zeta_s)\,\mathrm{d}s < \infty\right\}\right) > 0.$$

First we prove the following:

Lemma 1.6. *Let Z, f be resp. a process, a function as in the above theorem. For a fixed $x \in \mathbb{R}$ assume that there exists a finite and strictly positive random variable τ such that*

$$\boldsymbol{P}\left(\left\{\int_0^\tau f(x+Z_s)\,\mathrm{d}s < \infty\right\}\right) > 0.$$

Then there is a real number $\delta > 0$ such that

$$\int_{|y| \leqslant \delta} f(x+y)\,\mathrm{d}y < \infty.$$

Proof. It is known that, for each $t > 0$, $L(t,0) > 0$, $\boldsymbol{P}$-a.s.: see Stone (1963, Theorem 1, p. 633). Owing to the strict positivity of τ and the fact that $L(\omega,\cdot,0)$ is increasing, it follows that $L(\omega,\tau(\omega),0) > 0$ for $\boldsymbol{P}$-almost all ω. Also, for $\boldsymbol{P}$-almost all ω

$$\int_0^{\tau(\omega)} f(x+Z_s(\omega))\,\mathrm{d}s = \int_{\mathbb{R}} f(x+y)L(\omega,\tau(\omega),y)\,\mathrm{d}y; \tag{1.7}$$

thus, by assumption, we may choose ω such that the last relation holds as well as $\int_0^{\tau(\omega)} f(x+Z_s(\omega))\,\mathrm{d}s < \infty$ and $L(\omega,\tau(\omega),0) > 0$. With this choice of ω, we may appeal to the continuity of $L(\omega,\tau(\omega),\cdot)$ to choose strictly positive numbers δ and k such that $L(\omega,\tau(\omega),y) \geqslant k$ whenever $|y| \leqslant \delta$. By (1.7)

$$k\int_{-\delta}^{\delta} f(x+y)\,\mathrm{d}y \leqslant \int_0^{\tau(\omega)} f(x+Z_s(\omega))\,\mathrm{d}s < \infty. \qquad \square$$

Proof of Theorem 1.4. The proof that condition (a) implies (c) is carried out as in the brownian case using the above Lemma and the following known fact about α-stable motions Z with $1 < \alpha \leqslant 2$ (cf. Port, 1967):

For any $x \in \mathbb{R}$, $t > 0$, the $\mathbb{F}$-stopping time

$$\tau = \inf\{t > 0 : Z_t = x\}$$

is such that $\boldsymbol{P}(\{0 \leqslant \tau < \infty\}) = 1$.

Also it's easy to verify that (c) implies (b) using the continuity of $L(\omega,t,\cdot)$.

We omit details referring the reader to Engelbert and Schmidt (1981) or to Karatzas and Shreve (1994, Proposition 6.27, p. 216). $\square$

The conditions in Remark 1.5 are consequences of the proof of Theorem 1.4: indeed, if (e) is verified, by applying Lemma 1.6 to each x, we get condition (c). If (c) is satisfied for f, then (c) is also satisfied for the function $f_x(\cdot) = f(x+\cdot)$, x being any real number. Because of Theorem 1.4, for all x, condition (b) holds with f_x in the place of f: but this is exactly condition (d). Lastly, (d) implies (e), of course.

2. Stochastic differential equations without drift. The extension of Engelbert and Schmidt condition

Consider the one-dimensional stochastic differential equation

$$X_t = x + \int_{]0,t]} b(X_s)\,\mathrm{d}Z_s, \quad t \geqslant 0 \tag{2.1}$$

where x is a real starting point, b is a Borel measurable real function and Z denotes an α-stable motion.

In the present section we investigate solution of the equation above.

Definition 2.2. A process $(X, \mathbb{F})$ defined on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$ is called *a solution of* (2.1) *starting from* x, if there exists an α-stable motion $(Z, \mathbb{F})$ such that (2.1) holds for every $t \geqslant 0$.

Usually, solutions as above are called *weak solutions*: since in this paper we do not consider other solutions, we omit this attribute.

Definition 2.3. A solution $(X, \mathbb{F})$ (2.2) of Eq. (2.1) is said to be *trivial* if

$$\boldsymbol{P}(\{X_t = X_0 \text{ for all } t \geqslant 0\}) = 1.$$

As already pointed out in the Introduction, by employing the method of random time change, Engelbert and Schmidt (1981) proved a necessary and sufficient condition for the existence of a nontrivial solution with arbitrary starting point for an equation in the class (2.1), namely an equation of the form (2.1) with Z being a Wiener process.

We shall prove below a result (cf. Theorem 2.32) which is the extension of the Engelbert and Schmidt condition to the class of α-stable motions with $1 < \alpha \leqslant 2$.

Everywhere in the following $f(t, y)$ will denote the α-stable transition density (i.e. the density of any α-stable motion) with $0 < \alpha \leqslant 2$.

Definition 2.4. Let x be a real number. We say that the coefficient b in (2.1) *satisfies condition* (H) *with respect to* x, if

$$\int_0^t \mathrm{d}s \left(\int_{|y|<L} \frac{1}{|b(x+y)|^\alpha} f(s, y)\,\mathrm{d}y \right) < \infty \quad \text{for all } t > 0,\ L > 0, \tag{H}$$

f denoting the α-stable transition density $(0 < \alpha \leqslant 2)$. (We set $|b(x)|^{-\alpha} = +\infty$ if $b(x) = 0$).

Now we state a first result which is a general sufficient condition for the existence of solutions of (2.1). From Proposition 2.29 below, it will follow that such condition extends that of Engelbert and Schmidt to stable motions with $0 < \alpha < 1$ or $1 < \alpha \leqslant 2$. Moreover it "almost" unifies the cases $0 < \alpha < 1$ and $1 < \alpha \leqslant 2$ (compare (a) and (b) of the theorem below).

Theorem 2.5. (a) *Consider Eq. (2.1) with respect to an α-stable motion Z such that $1 < \alpha \leqslant 2$. Let x be a real number and assume that function b satisfies the above condition* (H) *with respect to x. Then there exists a nontrivial solution of Eq. (2.1) starting from the point x*, i.e. *such that $X_0 = x$.*

(b) *Consider Eq. (2.1) with respect to an α-stable motion Z such that $0 < \alpha < 1$. Let x be a real number. Assume that function b satisfies the above condition* (H) *with respect to x. Suppose also that there exists a real number $U > 0$ such that $\lambda(B_U) < \infty$, where*

$$B_U = \{y \in \mathbb{R}: |b(x+y)| > U\}.$$

Then there exists a nontrivial solution of Eq. (2.1) starting from the point x.

In order to prove this theorem, begin by considering an α-stable motion $(Z, \mathbb{F})$ defined on some stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$ and such that $Z_0 = 0$. A solution of (2.1) will be constructed by random time change from $(Z, \mathbb{F})$.

Let the increasing process $(C, \mathbb{F})$ be defined by

$$C_t = \int_0^t \frac{1}{|b(x+Z_s)|^\alpha} \, \mathrm{d}s, \quad t \geqslant 0. \tag{2.6}$$

Lemma 2.7. *Under the assumptions of Theorem 2.5 the process $(C, \mathbb{F})$ verifies the following properties*:

(a) *For every $t \geqslant 0$, $\boldsymbol{P}(\{C_t < \infty\}) = 1$.*
(b) $\boldsymbol{P}(\{\lim_{t\to\infty} C_t = C_\infty = \infty\}) = 1$.

Proof. For $U > 0$ consider the $\mathbb{F}$-stopping time

$$\tau_U = \inf\{t \geqslant 0 : |Z_t| \geqslant U\}.$$

Fix any $t > 0$. For all $U > 0$ we have

$$C_t(\omega) = C_{t\wedge\tau_U}(\omega)\cdot 1_{\{t<\tau_U\}}(\omega) + C_t(\omega)\cdot 1_{\{t\geqslant\tau_U\}}(\omega).$$

Since $\tau_U > 0$, for each $U > 0$

$$\begin{aligned} \boldsymbol{E}[1_{\{t<\tau_U\}} C_{t\wedge\tau_U}] &= \boldsymbol{E}\left[1_{\{t<\tau_U\}} \int_0^{t\wedge\tau_U} \frac{1}{|b(x+Z_s)|^\alpha} \, \mathrm{d}s\right] \\ &\leqslant \boldsymbol{E}\left[\int_0^t 1_{\{|Z_s|<U\}}(s) \frac{1}{|b(x+Z_s)|^\alpha} \, \mathrm{d}s\right] \\ &= \int_0^t \mathrm{d}s \left(\int_{|y|<U} \frac{1}{|b(x+y)|^\alpha} f(s,y) \, \mathrm{d}y\right) < \infty \end{aligned}$$

because of condition (H) with respect to x.

It follows, for every U, $1_{\{t<\tau_U\}}C_{t\wedge\tau_U} = 1_{\{t<\tau_U\}}C_t < \infty$, $\boldsymbol{P}$-a.s.. Since $\tau_U \uparrow \infty$ as $U \uparrow \infty$, we get (a).

As to property (b), since $1/|b(x+y)|^\alpha > 0$ for all y, there exists $\varepsilon > 0$ such that $\lambda(B_\varepsilon) > 0$ where

$$B_\varepsilon = \left\{ y \in \mathbb{R} : \frac{1}{|b(x+y)|^\alpha} \geqslant \varepsilon \right\}.$$

So, in the case $1 < \alpha \leqslant 2$, property (b) follows from (b) of Proposition 1.3 and the fact that

$$\int_0^t \frac{1}{|b(x+Z_s)|^\alpha}\,\mathrm{d}s = \int_{\mathbb{R}} \frac{1}{|b(x+y)|^\alpha} L(t,y)\,\mathrm{d}y \geqslant \varepsilon \cdot v(t, B_\varepsilon),$$

L denoting the local time of Z.

In the case $0 < \alpha < 1$ we have for all t

$$C_t \geqslant \int_0^t \frac{1}{|b(x+Z_s)|^\alpha} 1_{\mathbb{R}\setminus B_U}(Z_s)\,\mathrm{d}s \geqslant \int_0^t \frac{1}{U^\alpha} 1_{\mathbb{R}\setminus B_U}(Z_s)\,\mathrm{d}s,$$

B_U being the set introduced in (b) of Theorem 2.5. So, using the occupation time measure v (1.1) of Z, for all t

$$C_t \geqslant U^{-\alpha}(t - v(t, B_U))$$

and in the case, property (b) follows from (a) of Proposition 1.3. □

Now we come to the proof of Theorem 2.5 (a).

In the case $\alpha = 2$ the result follows applying the same proof of the implication (ii) ⇒ (i) in Theorem 4 of Engelbert and Schmidt (1981), to the equation

$$X_t = x + \int_{]0,t]} b_1(X_{s-})\,\mathrm{d}W_s$$

with $b_1 = \sqrt{2}\cdot b$, W denoting a standard Wiener process: remark indeed that the properties (a), (b) of Lemma 2.7 are sufficient. And $\sqrt{2}\cdot W$ is a 2-stable motion.

Hence only the case $1 < \alpha < 2$ is to be considered.

In the proof we employ some results in Kallenberg (1992) such as Theorem 3.1, p. 210, i.e. *the integrability criterion with respect to stable Lévy motion* and *the predictable reduction theorem of integer-valued random measures to Poisson random measures*: below we quote the latter explicitly.

Given a measurable space (S, S), denote S_δ the augmentation $S \cup \{\delta\}$ where δ is an external point. The σ-algebra in S_δ is understood to be the one generated by S.

Theorem 2.8. (see Theorem 2.1(a) in Kallenberg). *Fix a Polish space K and a σ-finite measure space (S, S, μ), let ξ be a quasi-leftcontinuous integer-valued random measure in K (defined on some stochastic basis satisfying the usual assumptions). Let T be a predictable mapping of $\mathbb{R}_+ \times K$ into S_δ. Suppose that $T(\hat{\xi}) = \mu$ a.s. on S, $\hat{\xi}$ denoting the compensator of ξ. Then the random measure $T(\xi)$ is Poisson on S with intensity μ.*

Let τ denote the right-inverse of C (2.6), i.e.

$$\tau_t = \inf\{s \geqslant 0 : C_s > t\}, \quad t \geqslant 0. \tag{2.9}$$

Taking the above lemma into account, process τ is $\boldsymbol{P}$-a.s. finite, continuous, strictly increasing and such that $\lim_{t \to \infty} \tau_t = \tau_\infty = +\infty$. Moreover, for every t, τ_t is an $\mathbb{F}$-stopping time. Hence $\tau = (\tau_t)_{t \geqslant 0}$ defines a change of time in the sesnse of Jacod (1979, Chap. X, Section 1). Moreover, $\boldsymbol{P}$-a.s.

$$\tau_t = \int_0^t |b(x + Z_{\tau_s})|^\alpha \, \mathrm{d}s \quad \text{for every } t \geqslant 0: \tag{2.10}$$

indeed, for every ω, $|b(x + Z_{\cdot}(\omega))|^\alpha > 0$ λ-a.e. owing to (a) of the preceding lemma, hence

$$\tau_t = \int_0^{\tau_t} |b(x + Z_s)|^\alpha \, \mathrm{d}C_s = \int_0^{C_{\tau_t}} |\mathrm{b}(x + \mathrm{Z}_{\tau_s})|^\alpha \, \mathrm{d}s = \int_0^t |b(x + Z_{\tau_s})|^\alpha \, \mathrm{d}s.$$

because of Lemma 1.6 in Engelbert and Schmidt (1985).

Now consider the time changed process $(Y, \mathbb{H})$ defined, for every $t \geqslant 0$, by

$$Y_t = Z_{\tau_t}, \qquad \mathbb{H} = (\mathscr{H}_t)_{t \geqslant 0} \quad \text{where } \mathscr{H}_t = \mathscr{F}_{\tau_t}, \ t \geqslant 0. \tag{2.11}$$

We want to show that the process $(X, \mathbb{H})$ such that, for every $t \geqslant 0$

$$X_t = x + Y_t \tag{2.12}$$

is a solution of (2.1) on $(\Omega, \mathscr{F}, \mathbb{H}, \boldsymbol{P})$.

Since the basis $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$ satisfies the usual assumptions, the same holds for$(\Omega, \mathscr{F}, \mathbb{H}, \boldsymbol{P})$ (cf. 1, §a in Chap. X of Jacod, 1979).

Furthermore, in case $1 < \alpha < 2$, Z is a purely discontinuous $\mathbb{F}$-martingale. Because of Theorem 10.16, p. 316 of Jacod and taking into account the continuity of τ, we have the

Lemma 2.13. *The above defined process $(Y, \mathbb{H})$ is a purely discontinuous local martingale on the basis $(\Omega, \mathscr{F}, \mathbb{H}, \boldsymbol{P})$.*

Now let μ denote the jump-measure of the α-stable motion $(Z, \mathbb{F})$ i.e. the integer valued random measure μ biven by

$$\mu(\mathrm{d}t, \mathrm{d}x) = \sum_{s > 0} 1_{\{\Delta Z_s \neq 0\}} \varepsilon_{(s, \Delta Z_s)}(\mathrm{d}t, \mathrm{d}x), \tag{2.14}$$

$\varepsilon_{(s,x)}$ denoting Dirac measure concentrated at point (s, x). Consider the time changed random measure $\tilde{\mu} = \tau\mu$ defined as follows, for all ω

$$\tilde{\mu}(\omega, \cdot) = (\tau\mu)(\omega, \cdot) = \mu(\omega, \cdot) \circ (\tilde{C}(\omega))^{-1} \tag{2.15}$$

where

$$\tilde{C}(\omega)(t, x) = (C_t(\omega), x).$$

Then, because of Theorem 10.27(b) of Jacod (1979):

the $\mathbb{H}$-optional random measure $\tilde{\mu}$ is the jump-measure of the process Y. (2.16)

If ν denotes the $\mathbb{F}$-dual predictable projection of μ, from (e) of Theorem 10.27, taking account of Example 3.22, p. 75 in Jacod, it follows that:

the $\mathbb{H}$-dual predictable projection $\tilde{\pi}$ of $\tilde{\mu} = \tau\mu$ is given by $\tau\nu$, i.e., for all ω

$$\tilde{\pi}(\omega, \cdot) = \tau\nu(\omega, \cdot) = \nu(\omega, \cdot) \circ (\tilde{C}(\omega))^{-1}. \tag{2.17}$$

Since $\nu(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s \otimes (K/|x|^{1+\alpha})\mathrm{d}x$ where K is a constant > 0, from (2.10), (2.12) it follows that the $\mathbb{H}$-compensator $\tilde{\pi}$ of $\tilde{\mu}$ is given for all ω by

$$\tilde{\pi}(\omega, \mathrm{d}s, \mathrm{d}x) = |b(X_s(\omega))|^\alpha \mathrm{d}s \otimes \frac{K}{|x|^{1+\alpha}} \mathrm{d}x. \tag{2.18}$$

Combining (2.13), (2.16) and (2.17) we see that the process $(Y, \mathbb{H})$ can be represented as follows as a compensated sum of jumps

$$Y_t = \int_{]0,t]} \int_E x(\tilde{\mu} - \tilde{\pi})(\mathrm{d}s, \mathrm{d}x) \tag{2.19}$$

(see Jacod, 1979), where $E = \mathbb{R} - \{0\}$.

We also need the following

Lemma 2.20. *Set $N_\omega = \{s \in \mathbb{R}_+ : b(X_s(\omega)) = b(x + Y_s(\omega)) = 0\}$ where X is defined in* (2.12). *Then $\lambda(N_\omega) = 0$ for $\boldsymbol{P}$-almost all ω.*

Proof. Set $I_\omega = \{s \in \mathbb{R}_+ : b(x + Z_s(\omega)) = 0\}$. Because of Lemma 2.7(a), for $\boldsymbol{P}$-almost all ω, $\lambda(I_\omega) = 0$. On the other hand, owing to (2.11), for all ω, $\tau(N_\omega) \subset I_\omega$, $\tau(N_\omega)$ denoting the image under $\tau.(\omega)$ of N_ω (of course $\tau(\phi) = \phi$). It follows that the Borel set $\tau(N_\omega)$ (cf. §3, Chap. 15 in Royden, 1968) verifies for $\boldsymbol{P}$-almost all ω

$$\lambda(\tau(N_\omega)) = 0.$$

(Remark that, $\boldsymbol{P}$-a.s., τ is a homeomorphism of $\mathbb{R}_+$ into $\mathbb{R}_+$).

Since for all ω, $C(\tau(N_\omega)) \supset N_\omega$, it suffices to verify that, for $\boldsymbol{P}$-almost all ω

$$\lambda(C(J)) = 0 \textit{ for every Borel set } J \textit{ with } \lambda(J) = 0.$$

Taking into account (b) of Lemma 2.7, using Dellacherie (1972, Theorem 44, p. 92) we have $\boldsymbol{P}$-a.s.

$$\lambda(A) = \int_0^{C_\infty} 1_A(t)\,\mathrm{d}t = \int_0^\infty 1_A(C(t))\,\mathrm{d}C(t),$$

A denoting any Borel set in $\mathbb{R}$. Since C is strictly increasing, $1_{C(A)}(C(t)) = 1_A(t)$, hence

$$\lambda(C(A)) = \int_0^\infty 1_A(t)\,\mathrm{d}C(t)$$

and the above assertion follows from the fact that the Stieltjes measure generated by C is absolutely continuous with respect to λ. □

Now add the point $\delta = \infty$ to E and denote by E_δ the locally compact space $E \cup \{\delta\}$, by $\mathscr{E}$ (resp. $\mathscr{E}_\delta$) the Borel σ-algebra of E (resp. E_δ).

Consider the mapping $\beta : (\Omega \times \mathbb{R}_+ \times E, \mathscr{P} \otimes \mathscr{E}) \to (\Omega \times \mathbb{R}_+ \times E_\delta, \mathscr{P} \otimes \mathscr{E}_\delta)$ defined by

$$\beta(\omega, s, x) = \left(\omega, s, \frac{x}{b(X_{s-}(\omega))}\right), \tag{2.21}$$

where $\mathscr{P}$ denotes the σ-algebra of the $\mathbb{H}$-predictable sets and we adopt the convention $x/b(X_{s-}(\omega)) = \delta$ whenever $b(X_{s-}(\omega)) = 0$. This mapping is clearly measurable.

Moreover set, for each ω, $\beta_\omega(\cdot,\cdot) = \beta(\omega,\cdot,\cdot)$ and consider the $\mathbb{R}_+ \times E_\delta$-valued random measure

$$\{p(\omega,\cdot,\cdot), \omega \in \Omega\} \quad \text{where, for each } \omega,\ p(\omega,\cdot,\cdot) = \beta_\omega(\tilde{\mu}(\omega,\cdot)). \tag{2.22}$$

Lemma 2.23. *The just defined random measure p is a $\mathbb{H}$-Poisson random measure with (deterministic) $\mathbb{H}$-compensator*

$$q(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s \otimes \sigma(\mathrm{d}x),$$

where σ is the measure on $(E_\delta, \mathscr{E}_\delta)$ having density $(K/|x|^{1+\alpha})\mathrm{d}x$ on $(E, \mathscr{E})$ and such that $\sigma(\{\delta\}) = 0$.

Proof. Let us verify that, for $\boldsymbol{P}$-almost all ω

$$\text{on } \mathbb{R}_+ \times E, \quad \beta_\omega(\tilde{\pi}) = \mathrm{d}s \otimes K \frac{1}{|x|^{1+\alpha}} \mathrm{d}x. \tag{2.24}$$

Indeed, combining (2.18) and the fact that $b(X_{s-}) = b(X_s)$ for all s outside a λ-nullset, a simple computation shows that one has for all ω, $B \in \mathscr{E}$,

$$\int_{[0,t]} \int_{E_\delta} 1_B(x) \beta_\omega(\tilde{\pi})(\mathrm{d}s, \mathrm{d}x)$$

$$= \int_0^t 1_{\{b(X_{s-}(\omega)) \neq 0\}} \mathrm{d}s \left(\int_E 1_B\left(\frac{x}{b(X_{s-}(\omega))}\right) |b(X_{s-}(\omega))|^\alpha \frac{K}{|x|^{1+\alpha}} \mathrm{d}x \right).$$

From this equality, by the change of variable $x/[b(X_{s-}(\omega))] = y$ we have, for $\boldsymbol{P}$-almost all ω

$$\int_{[0,t]} \int_{E_\delta} 1_B(x) \beta_\omega(\tilde{\pi})(\mathrm{d}s, \mathrm{d}x) = \int_0^t 1_{\{b(X_{s-}(\omega)) \neq 0\}} \mathrm{d}s \left(\int_E 1_B(y) \frac{K}{|y|^{1+\alpha}} \mathrm{d}y \right)$$

$$= \int_0^t \mathrm{d}s \left(\int_E 1_B(y) \frac{K}{|y|^{1+\alpha}} \mathrm{d}y \right),$$

owing to Lemma 2.20.

From (2.24) by Theorem 2.8 we obtain the conclusion, after having noted that, for $\boldsymbol{P}$-almost all ω one has, independently of t

$$\beta_{\omega}(\tilde{\pi})([0,t]\times\{\delta\}) = \int_0^t \int_E 1_{\{\delta\}}\left(\frac{x}{b(X_{s-}(\omega))}\right)\tilde{\pi}(\omega, \mathrm{d}s, \mathrm{d}x)$$
$$= \int_0^t \int_E 1_{\{b(X_{s-}(\omega))=0\}}(\omega, s)\tilde{\pi}(\omega, \mathrm{d}s, \mathrm{d}x) = 0$$

because of (2.18).[1] □

On the basis of the last Lemma, we can choose a version of p such that, for all ω, $p(\omega, \mathbb{R}_+ \times \{\delta\}) = 0$.

Now define the process $(Z^*, \mathbb{H})$ by

$$Z_t^* = \int_{]0,t]} \int_{E_\delta} x(p-q)(\mathrm{d}s, \mathrm{d}x): \tag{2.25}$$

still because of the last lemma, $(Z^*, \mathbb{H})$ *is* $\boldsymbol{P}$*-indistinguishable from an α-stable motion.*

Combining the Definition (2.12) of X_t and (2.10), $\boldsymbol{P}$-a.s. we have for all t

$$\int_0^t |b(X_{s-})|^{\alpha}\,\mathrm{d}s < +\infty$$

and because of Theorem 3.1(a), in Kallenberg (1992), there exists the stochastic integral of the process $b(X_{s^-})$ with respect to $(Z^*, \mathbb{H})$. Recall that Z^* is a (purely discontinuous) $\mathbb{H}$-martingale and set

$$M_t = \int_{]0,t]} b(X_{s-})\,\mathrm{d}Z_s^*, \quad t \geqslant 0.$$

Lemma 2.26. *The process* $(Y, \mathbb{H})$ (2.11) *is* $\boldsymbol{P}$*-equal to the process* $(M, \mathbb{H})$.

Proof. M is a purely discontinuous $\mathbb{H}$-local martingale as well as Y (2.13). In order to verify that M, Y are indistinguishable, it suffices to show that they have the same jump processes (up to an evanescent set) (cf. Jacod and Shiryaev, 1987, Corollary 4.19, p. 43).

Indeed, the jump process of M, $\Delta M_t = b(X_{t^-})\Delta Z_t^*$ is the same as

$$\Delta Y_t 1_{\{b(X_{t-})\neq 0\}}(t)$$

because p does not charge $\mathbb{R}_+ \times \{\delta\}$. So ΔM, ΔY differ by the process

$$\Delta Y_t 1_{\{b(X_{t-})=0\}}(t)$$

which is $\boldsymbol{P}$-equal to 0: in fact, for all t

$$\sum_{s\leqslant t} (\Delta Y_s)^2 1_{\{b(X_{s-})=0\}}(s) = \int_{]0,t]}\int_E x^2 1_{\{b(X_{s-})=0\}}\tilde{\mu}(\mathrm{d}s, \mathrm{d}x) = 0$$

$\boldsymbol{P}$-a.s., owing to (2.18) and the fact that $b(X_s-) = b(X_s-)$ up to a λ-nullset. □

[1] Recall the convention $0\cdot\infty = 0$.

From the above Lemma it follows that the process $(X, \mathbb{H})$ (2.12) is a solution of Eq. (2.1) and taking (2.11) into account, it's obvious that such a solution is non-trivial.

So the proof of part (a) in Theorem 2.5 is concluded.

It remains to prove assertion (b) where the case $0 < \alpha < 1$ is considered.

One starts with an α-stable motion $(Z, \mathbb{F})$ defined on $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$, such that $Z_0 = 0$ $(0 < \alpha < 1)$. Recall that in the case, Z is of pure jump type with finite variation. Then one defines the time changed process $(Y, \mathbb{H})$ as in (2.11) and one shows that the process $(X, \mathbb{H})$ (2.12) is a solution of (2.1).

The proof follows exactly the same lines as above. It uses the properties of time-changed processes and random measures studied in N. 1, Chap. X of Jacod (1979) (in particular, the invariance of the finite-variation paths property under a time change, in the sense of Theorem 10.16 of Jacod) as well as some results in Kallenberg (1992).

So we omit going into details. □

Now we need some integrability properties of α-stable transition densities.

If Z denotes an α-stable motion $(0 < \alpha \leqslant 2)$ with transition density f, the scaling property gives, for all $y \in \mathbb{R}$, $t > 0$,

$$f(t, y) = t^{-1/\alpha} f(1, yt^{-1/\alpha}).$$

Hence f verifies, for all $y, t > 0$

$$f(t, y) \leqslant Ct^{-1/\alpha} \tag{2.27}$$

with a suitable constant C.

Moreover, since for all $y, f(1, y) \leqslant \text{const}(1 + |y|^{\alpha+1})^{-1}$ (see Zolotarev, 1986, §2.7), $f(1, \cdot)$ is in L_q i.e. $\int_{\mathbb{R}} (f(1, y))^q \, dy < \infty$, q denoting any real number $\geqslant 1$. Still because of the scaling property, for every $q \geqslant 1$ there exists a constant K (depending only on q) such that

$$\| f(t, \cdot) \|_q = K \cdot t^{(1-q)/\alpha q} \quad \text{for all} \quad t > 0, \tag{2.28}$$

$\| \cdot \|_q$ denoting the L_q-norm.

Proposition 2.29. (a) *In the case* $1 < \alpha \leqslant 2$, *assume that the following holds*:

The function $|b|^{-\alpha}$ *is locally integrable.*

(b) *In the case* $0 < \alpha < 1$, *assume that there exists a real number* $\delta > 1$ *such that the following holds*:

The function $|b|^{-\delta}$ *is locally integrable.*

Then in both cases the function b *in* (2.1) *satisfies condition* (H) (2.4) *with respect to any* x.

Proof. (a) In the case $1 < \alpha \leqslant 2$ the property follows from (2.27).

(b) In this case, because of Hölder inequality and (2.28), we have for all p, q with $p > 1$, $q > 1$, $(1/p) + (1/q) = 1$

$$\int_0^t \mathrm{d}s \left(\int_{|y|<L} \frac{1}{|b(x+y)|^\alpha} f(s,y)\,\mathrm{d}y \right) \leqslant K \cdot \left(\int_{|y|<L} |b(x+y)|^{-\alpha p}\,\mathrm{d}y \right)^{1/p} \cdot \int_0^t s^{(1-q)/\alpha q}\,\mathrm{d}s.$$

Then it suffices to take $p = \delta/\alpha > 1/\alpha > 1$ and to combine the local integrability property of $|b|^{-\delta}$ with that of $s^{(1-q)/(\alpha q)} = s^{-1/(\alpha p)}$ in order to conclude. □

Now we have the following necessary condition of existence of a nontrivial solution of (2.1). The proof is an application of Lemma 1.6 and uses *the random time change representation of stable integrals* studied in Kallenberg (1992).

Proposition 2.30. *Let* $1 < \alpha \leqslant 2$. *Assume that the process* $(X, \mathbb{F})$ *be defined on* $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$ *and constitute a nontrivial* (2.3) *solution of Eq.* (2.1) *starting from the point* x. *Then there exists a real number* $\varepsilon > 0$ *such that*

$$\int_{|y| \leqslant \varepsilon} \frac{1}{|b(x+y)|^\alpha}\,\mathrm{d}y < +\infty.$$

Proof. By assumption the process $b(X_{s^-}(\omega))$ is Z-integrable and thus $\boldsymbol{P}$-a.s. one has

$$\int_{]0,t]} |b(X_{s^-})|^\alpha\,\mathrm{d}s < +\infty \quad \text{for all } t \geqslant 0$$

owing to Theorem 3.1, (a) of Kallenberg (1992).

Now denote by A the process defined as follows, for all t

$$A_t = \int_{]0,t]} |b(X_{s^-})|^\alpha\,\mathrm{d}s.$$

Becauses of the random time change representation theorem of stable integrals (see Theorem 4.1 of Kallenberg), there exists an α-stable motion $(\tilde{Z}, \mathbb{K})$ on an extension of the underlying stochastic basis in general, so that $\int_{]0,t]} b(X_s)\,\mathrm{d}Z_s = \tilde{Z}_{A_t}$, thus

$$X_t = x + \tilde{Z}_{A_t} \quad \text{for all } t \geqslant 0, \tag{2.31}$$

holds.

We consider all random variables as defined on this extension which we also denote by $(\Omega, \mathscr{F}, \mathbb{K}, \boldsymbol{P})$. Let T denote the right-inverse of A, i.e., for all $t \geqslant 0$

$$T_t = \inf\{s \geqslant 0 \colon A_s > t\}.$$

By time change in the integral (cf. Lemma 1.6 of Engelbert and Schmidt, 1985) we have, $\boldsymbol{P}$-a.s.

$$T_t \geqslant \int_0^{T_t} |b(X_s)|^{-\alpha} \cdot |b(X_s)|^{\alpha} \, ds = \int_0^{T_t} |b(X_s)|^{-\alpha} \, dA_s$$

$$= \int_0^{t \wedge A_\infty} |b(X_{T_s})|^{-\alpha} \, ds \quad \text{for all } t \geqslant 0.$$

It follows from (2.31), $\boldsymbol{P}$-a.s.

$$T_t \geqslant \int_0^{t \wedge A_\infty} |b(x + \tilde{Z}_s)|^{-\alpha} \, ds \quad \text{for all } t \geqslant 0.$$

Because of the nontriviality of the solution we have $\boldsymbol{P}(\{A_\infty > 0\}) > 0$ (otherwise, $\boldsymbol{P}$-a.s. $\int_{]0,\cdot]} b(X_{s-}) \, dZ_s = 0$: cf. Lemma 4.3 of Kallenberg) and therefore it follows from the last relation

$$\boldsymbol{P}\left(\left\{\int_0^\tau |b(x + \tilde{Z}_s)|^{-\alpha} \, ds < \infty\right\}\right) > 0,$$

where τ is a finite and strictly positive random variable such that $\tau < A_\infty$, hence $T_\tau < \infty$, on the set $\{A_\infty > 0\}$. (The existence of such a τ is assured by Theorem 37, p. 18 of Dellacherie, 1972). The conclusion follows from Lemma 1.6. □

Now combining Theorem 2.5 (a) and Proposition 2.29 with Proposition 2.30, by a compactness argument we obtain the following extension of Engelbert and Schmidt theorem (cf. Theorem 4 of Engelbert and Schmidt, 1981 or Theorem 2.2 of Engelbert and Schmidt, 1985).

Theorem 2.32. *Consider Eq.* (2.1) *with respect to an α-stable motion Z with* $1 < \alpha \leqslant 2$. *Then the following properties are equivalent*:

(a) *For every $x \in \mathbb{R}$ there exists a nontrivial* (2.3) *solution of* (2.1) *starting from the point x.*

(b) *The function $|b|^{-\alpha}$ is locally integrable.*

3. "Local" existence conditions

Begin by giving the following

Definition 3.1. Let $I =]u, v[$ (resp. $I = [u, v]$) denote an open (resp. closed) interval of the real line with $-\infty \leqslant u < v \leqslant +\infty$ (resp. $-\infty < u < v < +\infty$) and x a real number in I (resp. in the interior of I). A process $(X, \mathbb{F})$ defined on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$, is called *a solution of* (2.1) *on the interval I starting from x*, if there exists an α-stable motion $(Z, \mathbb{F})$ such that

$$\int_{[0,t\wedge\tau]} b(X_{s^-})\,\mathrm{d}Z_s$$

exists for every $t \geqslant 0$, where

$$\tau = \inf\{s \geqslant 0 : X_s \notin I\}$$

is the first exit time of I for X and such that, up to $\boldsymbol{P}$-equality

$$X_{t\wedge\tau} = x + \int_{]0,t\wedge\tau]} b(X_{s^-})\,\mathrm{d}Z_s \quad \text{for all } t \geqslant 0.$$

Definition 3.2. A solution $(X, \mathbb{F})$ of (2.1) on $]u, v[$ (resp. $[u, v]$) starting from x in $]u, v[$ (resp. in the interior of $[u, v]$) and defined on $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$ is termed *trivial* if

$$\boldsymbol{P}(\{X_{t\wedge\tau} = x \quad \text{for all } t \geqslant 0\}) = 1,$$

τ denoting the first exit time of the interval $]u, v[$ (resp. $[u, v]$) for X.

The above definitions clearly extend Definitions 2.2 and 2.3: take $]u, v[$ with $u = -\infty, v = +\infty$.

Remark 3.3. In the case of a solution $(X, \mathbb{F})$ on $]u, v[$ (resp. on $[u, v]$) the condition of *nontriviality* is equivalent to the following one: There exists a real number $\varepsilon > 0$ such that $]x - \varepsilon, x + \varepsilon[\subset]u, v[$ and

$$\boldsymbol{P}(\{\tau < \infty\} \cup \{\tau_\varepsilon < \infty\}) > 0$$

where τ is the first exit time of $]u, v[$ (resp. $[u, v]$) and τ_ε is the first exit time of $[x - \varepsilon, x + \varepsilon]$ or, in an equivalent way, the first exit time of $]x - \varepsilon, x + \varepsilon[$ for X. This is easily checked showing that triviality (3.2) is equivalent to the contrary of the just stated condition.

Now we have a "local" version of Theorem 2.5 which is based on the just given definitions and completely unifies the cases $0 < \alpha < 1$ and $1 < \alpha \leqslant 2$.

Theorem 3.4. *Consider Eq.* (2.1) *with respect to an α-stable motion Z such that $0 < \alpha < 1$ or $1 < \alpha \leqslant 2$. Let x be a real number. Assume that there exists a real number $\varepsilon > 0$ such that*

$$\int_0^\varepsilon \mathrm{d}s \left(\int_{|y| \leqslant \varepsilon} \frac{1}{|b(x + y)|^\alpha} f(s, y)\,\mathrm{d}y \right) < \infty,$$

f denoting the α-stable transition density.

Then there exists a nontrivial solution of Eq. (2.1) *on the interval $[x - \varepsilon, x + \varepsilon]$ starting from x.*

The proof is an easy consequence of Theorem 2.5 and of the following:

Lemma 3.5. *Let $x \in \mathbb{R}$. Under the assumption of the above theorem, we have*

$$\int_0^t \mathrm{d}s\left(\int_{|y| \leqslant \varepsilon} \frac{1}{|b(x+y)|^\alpha} f(s,y)\,\mathrm{d}y\right) < \infty, \quad \textit{for all } t \geqslant 0.$$

Proof. Fix any t with $t > \varepsilon$ and set $\sigma(\cdot) = b(x + \cdot)$. We have

$$\int_0^t \mathrm{d}s\left(\int_{|y| \leqslant \varepsilon} \frac{1}{|\sigma(y)|^\alpha} f(s,y)\,\mathrm{d}y\right)$$

$$= \int_0^\delta \mathrm{d}s\left(\int_{|y| \leqslant \varepsilon} \frac{1}{|\sigma(y)|^\alpha} f(s,y)\,\mathrm{d}y\right) + \int_\delta^t \mathrm{d}s\left(\int_{|y| \leqslant \varepsilon} \frac{1}{|\sigma(y)|^\alpha} f(s,y)\,\mathrm{d}y\right), \tag{3.6}$$

where owing to the assumption, δ, $0 < \delta \leqslant \varepsilon$ has been chosen so that

$$\int_{|y| \leqslant \varepsilon} \frac{1}{|\sigma(y)|^\alpha} f(\delta,y)\,\mathrm{d}y < \infty. \tag{3.7}$$

Using the scaling property we see that $f(\cdot,\cdot)$ is continuous and strictly positive on $]0,+\infty[\times\mathbb{R}$, so there is strictly positive constant C such that

$$f(s,y) \leqslant C f(\delta,y) \tag{3.8}$$

for all couples (s,y) with $\delta \leqslant s \leqslant t$, $-\varepsilon \leqslant y \leqslant +\varepsilon$.

Combining (3.7) with (3.8) we see that the last integral in (3.6) is finite. Since $\delta \leqslant \varepsilon$ also the first integral on the right-hand side of (3.6) is finite. Conclusion follows. □

Proof of Theorem 3.4. Set $S_\varepsilon = [x-\varepsilon, x+\varepsilon]$ and define the function $\tilde{b}$ as follows:

$$\tilde{b}(y) = b(y)\cdot 1_{S_\varepsilon}(y) + 1 \cdot 1_{S_\varepsilon^c}(y), \quad y \in \mathbb{R}.$$

Owing to the last lemma, $\tilde{b}$ satisfies condition (H) (2.4) with respect to x. Since $\tilde{b}$ clearly satisfies also the second assumption in (b) of Theorem 2.5, there exists a nontrivial solution X of equation

$$\mathrm{d}X_t = \tilde{b}(X_{t-})\,\mathrm{d}Z_t \tag{3.9}$$

such that $X_0 = x$.

Now, if τ denotes the first exit time of the interval S_ε for X, for all $t < \tau$, $X_t \in S_\varepsilon$ hence $X_{t-} \in S_\varepsilon$ holds. As a consequence, on the set $\{\tau < \infty\}$, $X_{\tau-} \in S_\varepsilon$ holds and thus, for each $t > 0$, on the stochastic interval $[0, t \wedge \tau]$ we have $\tilde{b}(X_{s-}) = b(X_{s-})$. It follows, up to $\boldsymbol{P}$-equality

$$X_{t\wedge\tau} = x + \int_{]0,t\wedge\tau]} b(X_{s-})\,\mathrm{d}Z_s \quad \text{for all } t > 0,$$

$\boldsymbol{P}$ denoting the measure on the space where the solution is defined.

Thus X is a solution of Eq. (2.1) on the interval S_ε starting from x.

Moreover, since X is a nontrivial solution of Eq. (3.9) (i.e. a nontrivial solution defined for all $t \geqslant 0$), by Remark 3.3 there exists a $\rho > 0$ such that

$$P(\{\tau_\rho < \infty\}) > 0 \tag{3.9}$$

where τ_ρ is the first exit time of $[x-\rho, x+\rho]$ for X.

As a consequence, if $\rho \geqslant \varepsilon$ relation $\tau \leqslant \tau_\rho$ implies $P(\{\tau < \infty\}) > 0$; if $\rho < \varepsilon$ then $]x-\rho, x+\rho[\subset]x-\varepsilon, x+\varepsilon[$ and in both cases X is a nontrivial solution of (2.1) on S_ε because of Remark 3.3. □

Corollary 3.11. (a) *Consider Eq.* (2.1) *with respect to an α-stable motion with* $1 < \alpha \leqslant 2$. *Let* $x \in \mathbb{R}$ *be such that there exists a real number* $\varepsilon > 0$ *with*

$$\int_{|y| \leqslant \varepsilon} \frac{1}{|b(x+y)|^\alpha} \, \mathrm{d}y < \infty.$$

Then there exists a nontrivial solution of Eq. (2.1) *on the interval* $[x-\varepsilon, x+\varepsilon]$ *starting from* x.

(b) *Consider Eq.* (2.1) *with respect to an α-stable motion with* $0 < \alpha < 1$. *Let* $x \in \mathbb{R}$ *be such that there exists two real numbers* $\varepsilon > 0$ *and* $\delta > 1$ *with*

$$\int_{|y| \leqslant \varepsilon} \frac{1}{|b(x+y)|^\delta} \, \mathrm{d}y < \infty.$$

Then there exists a nontrivial solution of Eq. (2.1) *on the interval* $[x-\varepsilon, x+\varepsilon]$ *starting from* x.

Proof. As in the proof of Proposition 2.29 and using (2.27) in case (a), (2.28) in case (b), one shows that the assumption of the last theorem is verified. □

Now we prove a necessary condition for "local" existence in the case $1 < \alpha \leqslant 2$. First we state the following:

Lemma 3.12. *Let* u, v *be real numbers with* $u < v$. *Let* X *be a nontrivial* (3.2) *solution of Eq.* (2.1) *on the interval* $[u, v]$ *starting from a point* x *in the interior of* $[u, v]$. *Then* X *constitutes a nontrivial solution of* (2.1) *on the open interval* $]u, v[$.

Proof. By definition, clearly X constitutes also a solution of (2.1) on the interval $]u, v[$. So it suffices to verify that X is nontrivial as a solution on $]u, v[$.

Denote by $\bar{\tau}$ (resp. τ) the first exit time of $[u, v]$ (resp. $]u, v[$). Then $\tau \leqslant \bar{\tau}$.

Because of the assumption and Remark 3.3 there exists an $\varepsilon > 0$ such that $]x-\varepsilon, x+\varepsilon[\subset]u, v[$ and $P(\{\bar{\tau} < \infty\} \cup \{\tau_\varepsilon < \infty\}) > 0$, τ_ε being defined as in (3.3). But the relation $\tau \leqslant \bar{\tau}$ implies $\{\bar{\tau} < \infty\} \subset \{\tau < \infty\}$: it follows that $P(\{\tau < \infty\} \cup \{\tau_\varepsilon < \infty\}) > 0$ and X is a nontrivial solution on $]u, v[$ owing to the same Remark 3.3. □

Proposition 3.13. *Let $1 < \alpha \leqslant 2$. Let $I =]u, v[$ or $I = [u, v]$ denote an interval of either form considered in Definition 3.1 and x be a point in the interior of I. Assume that there exists a nontrivial solution of (2.1) on the interval I starting from x.*

Then there exists a strictly positive real number ε such that

$$\int_{|y| \leqslant \varepsilon} \frac{1}{|b(x+y)|^\alpha} \mathrm{d}y < +\infty .$$

Proof. Consider first the case of a solution on $]u, v[$. Let $(X, \mathbb{F})$ denote a solution defined on $(\Omega, \mathscr{F}, \mathbb{F}, \boldsymbol{P})$. By assumption, for all $t > 0$ we have

$$X_{t \wedge \tau} = x + \int_{]0, t \wedge \tau]} b(X_{s-}) \mathrm{d}Z_s. \tag{3.14}$$

τ denoting the first exit time of $]u, v[$ for X.

Set $I =]u, v[$ and define the function $\tilde{b}$ as follows:

$$\tilde{b}(y) = b(y) 1_I(y).$$

Since, for all $y, |\tilde{b}(y)| \leqslant |b(y)|$ it's clear that $\int_{]0, t \wedge \tau]} \tilde{b}(X_{s-}) \mathrm{d}Z_s$ exists for all $t > 0$. Moreover, for every $s < \tau(\omega)$, $X_s(\omega)$ is in I and thus $b(X_{s-}(\omega)) = \tilde{b}(X_{s-}(\omega))$ for all $s \leqslant \tau(\omega)$ up to a λ-null set. It follows $\int_{]0, t \wedge \tau]} \tilde{b}(X_{s-}) \mathrm{d}Z_s = \int_{]0, t \wedge \tau]} b(X_{s-}) \mathrm{d}Z_s$, hence, because of (3.14),

$$X_{t \wedge \tau} = x + \int_{]0, t \wedge \tau]} \tilde{b}(X_{s-}) \mathrm{d}Z_s \tag{3.15}$$

for all t, up to $\boldsymbol{P}$-equality.

Now set, for all $t \geqslant 0$

$$Y_t = X_{t \wedge \tau}.$$

When $\tau < \infty$, note that X_τ is in I^c, hence $\tilde{b}(X_\tau) = 0$, $\tilde{b}(Y_{s-}) = 0$ for all $s > \tau$ and from (3.15) we obtain

$$Y_t = x + \int_{]0, t \wedge \tau]} \tilde{b}(Y_{s-}) \mathrm{d}Z_s = \int_{]0, t]} \tilde{b}(Y_{s-}) \mathrm{d}Z_s$$

for all t, $\boldsymbol{P}$-a.s. So $(Y, \mathbb{F})$ is a solution of the same equation as (2.1) but with $\tilde{b}$ as a coefficient and this solution turns out to be nontrivial because of Definition 3.2, the assumption on X and the very definition of Y. Because of Proposition 2.30 there exists a real number $\delta > 0$ such that

$$\int_{|y| \leqslant \delta} \frac{1}{|\tilde{b}(x+y)|^\alpha} \mathrm{d}y < +\infty.$$

The conclusion follows in the present case, because of the definition of $\tilde{b}$.

In the case of a solution on the closed interval $[u, v]$ it suffices to use the above lemma. □

Remark 3.16. Owing to the definition of $\tilde{b}$, in the above proposition clearly ε cannot be greater than the distance of x from the boundary of the interval.

Combining the above proposition and Theorem 2.32, by using a compactness argument and Remark 3.3 we directly have the following theorem.

Theorem 3.17. *Consider Eq.* (2.1) *with respect to an α-stable motion Z with* $1 < \alpha \leqslant 2$. *Then the following properties are equivalent*:

(a) *For every* $x \in \mathbb{R}$ *there exists a nontrivial* (2.2) *solution starting from the point x.*

(b) *For every* $x \in \mathbb{R}$ *there exists an interval* I_x (*of either form considered in* (3.1)) *containing x in its interior and a nontrivial solution on* I_x *starting from x.*

Acknowledgements

This work originated from discussions with R. Mikulevičius and H. Pragarauskas. In particular, an actual improvement of an earlier version is due to H. Pragarauskas. The author is very grateful to them. He also thanks the referees for their helpful comments.

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