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Group gradings on superinvolution simple superalgebras ☆

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ABSTRACT

In this paper we describe all group gradings by an arbitrary finite group G on non-simple finite-dimensional superinvolution simple associative superalgebras over an algebraically closed field F of characteristic 0 or coprime to the order of G.

0. Introduction

In the paper [1], Bahturin and Giambruno described the group gradings by finite abelian groups G on the matrix algebra $M_n(F)$ over an algebraically closed field F of characteristic different from 2, which are respected by an involution. Besides, under some restrictions on the base field, they classified all G-gradings on all finite-dimensional involution simple algebras.

In this paper we deal with finite-dimensional associative superalgebras that are simple with respect to some superinvolution $*$ over an algebraically closed field of characteristic zero or coprime to the order of G. First, we give a description of such associative superalgebras. Second, we classify all group gradings on $*$-simple associative superalgebras that are not simple associative algebras.

In the same way as the description of involution gradings on involution simple associative algebras is important for the determination of group gradings on classical simple Jordan and Lie algebras [3,5] the description of superinvolution gradings on superinvolution simple associative algebras is important for

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the determination of groups gradings on simple Jordan and Lie superalgebras. The case of superalgebras that are simple algebras is due to the second and the third authors, and is to be submitted for publication shortly.

1. Definitions and introductory remarks

Let R be an associative superalgebra, or, in other words, an associative algebra with a fixed $\mathbb{Z}_2$-grading $R = R_{\bar{0}} \oplus R_{\bar{1}}$. Since all algebras and superalgebras considered in this paper are associative we will normally drop the word associative in what follows. Also, if not stated otherwise, all subalgebras, ideals and homomorphisms are $\mathbb{Z}_2$-graded. We say that R is simple if it has no non-trivial proper ($\mathbb{Z}_2$-graded) ideals. It is well-known [12] that any finite-dimensional simple (associative) superalgebra over an algebraically closed field of characteristic different from 2 is isomorphic to either $M_{k,l}(F)$, the full matrix algebra $M_n(F)$ with a $\mathbb{Z}_2$-grading completely determined by two non-negative integers $k, l,\ k + l = n$, or a subalgebra $R = Q(n)$ of $M_{2n}(F)$ consisting of all matrices of the form $\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$ with $R_{\bar{0}} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ and $R_{\bar{1}} = \begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix}$. A convenient notation for $R = Q(n)$ is $R = A \oplus tA$ where $A \cong M_n(F)$, $t^2 = 1$. Then $R_{\bar{0}} = A$ and $R_{\bar{1}} = tA$.

Definition 1. Let R be a superalgebra. A *superinvolution* on R is a $\mathbb{Z}_2$-graded linear map $* : R \to R$ such that $(x^*)^* = x$ for all $x \in R$ and $(xy)^* = (-1)^{|x||y|}y^*x^*$ for all homogeneous $x, y \in R$, of degrees $|x|$ and $|y|$, respectively. A more general notion is that of *superantiautomorphism*, that is, a linear map $\varphi : R \to R$ such that $\varphi(xy) = (-1)^{|x||y|}\varphi(y)\varphi(x)$ for all homogeneous $x, y \in R$, as above.

In this paper we will be interested in superinvolution simple superalgebras.

Definition 2. Let $(R, *)$ be a superalgebra endowed with a superinvolution $*$. We say that R is *superinvolution simple* if $R^2 \neq \{0\}$ and R has no non-trivial ideals stable under $*$.

If $*$ is a superinvolution on R, then obviously the restriction of $*$ to $R_{\bar{0}}$ is an ordinary involution. The same is true for superantiautomorphisms. Thus, there is no confusion to abbreviate the terms superinvolution and superantiautomorphism to involution and antiautomorphism, respectively. In particular, superinvolution simple superalgebras will be simply called involution simple.

The following are examples of involution simple superalgebras.

Example 1. The *orthosymplectic* involution on $R = M_{r,2s}(F)$ is given by

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_r & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_r & 0 \\ 0 & Q \end{pmatrix}$$

where t denotes the usual matrix transpose, $Q = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix}$, and I_r, I_s are the identity matrices of orders r,s, respectively.

Example 2. Let us consider $R = M_{r,r}(F)$. We will call the following involution defined on $M_{r,r}(F)$ the *transpose* involution:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix}.$$

Example 3. Let A be a superalgebra. Consider a new superalgebra A^{sop} which has the same $\mathbb{Z}_2$-graded vector space structure as A but the product of A^{sop} is given on homogeneous a, b of degrees $|a|, |b|$ by

$$a \circ b = (-1)^{|a||b|}ba.$$

Let $S = A \oplus A^{sop}$ be the direct sum of two ideals A and A^{sop}. This is a $\mathbb{Z}_2$-graded algebra with $R_{\bar{0}} = A_{\bar{0}} \oplus A_{\bar{0}}^{sop}$, $R_{\bar{1}} = A_{\bar{1}} \oplus A_{\bar{1}}^{sop}$. We denote an arbitrary element x from R as a pair of elements from A, i.e. $x = (a, b)$ where $a, b \in A$. The product in R is given by

$$\begin{aligned}(a_0 + a_1, b_0 + b_1) \cdot (a_0' + a_1', b_0' + b_1') \\ = (a_0 a_0' + a_1 a_1' + a_0 a_1' + a_1 a_0', b_0' b_0 - b_1' b_1 + b_1' b_0 + b_0' b_1),\end{aligned}$$

where $a_0, b_0, a_0', b_0' \in A_0$, $a_1, b_1, a_1', b_1' \in A_{\bar{1}}$.

A linear mapping defined by $(a, b)^{ex} = (b, a)$ is an involution called *exchange involution*. If A is simple then (S, ex) is an involution simple superalgebra.

Definition 3. Let R and S be two superalgebras endowed with involutions $*$ and $\dagger$. We say that $(R, *)$ and $(S, \dagger)$ are *isomorphic* if there exists an isomorphism of superalgebras $\varphi : R \to S$ such that $\varphi(x^*) = \varphi(x)^\dagger$ for all $x \in R$. If $R = S$ then φ is an automorphism of R and $*, \dagger$ are called *conjugate* by φ. In this case we have $\dagger = \varphi \circ * \circ \varphi^{-1}$.

If $(R, *)$ is an involution simple superalgebra then a standard argument shows that either R is a simple superalgebra or else there is a ($\mathbb{Z}_2$-graded) ideal A in R such that $R = A \oplus A^*$. In the latter case the mapping $\varphi : R \to S$ defined by $\varphi(a + b^*) = (a, b)$ where $a, b \in A$ defines an isomorphism of involution simple superalgebras between $(R, *)$ and a standard superalgebra (S, ex) of Example 3 above, where A is simple.

In [10, Propositions 13 and 14] Racine described all types of involutions on $A = M_{n,m}(F) = A_{\bar{0}} + A_{\bar{1}}$. It appears that if φ is an involution on A such that $A_{\bar{0}}$ is an involution simple algebra under φ restricted to $A_{\bar{0}}$, then $n = m$ and φ is conjugate to the transpose involution. Otherwise, φ is conjugate to the orthosymplectic involution. Also, it was shown in [8, Theorem 3.1], a superalgebra of the type $Q(n)$ has no involutions. We can summarize all the remarks above as the following.

Proposition 1. *Any finite-dimensional involution simple superalgebra over an algebraically closed field of characteristic different from* 2 *is isomorphic to one of the following*:

(1) $R = M_{n,m}(F)$ *with the orthosymplectic or transpose involution*.
(2) $R = M_{n,m}(F) \oplus M_{n,m}(F)^{sop}$ *with the ordinary exchange involution*.
(3) $R = Q(n) \oplus Q(n)^{sop}$ *with the ordinary exchange involution*.

2. Group gradings

One can define gradings of superagebras by the elements of very general sets with operations but as it turns out if the superalgebra is involution simple we can restrict ourselves to the case of abelian groups. A phenomenon of this kind was, probably, first mentioned in [9]. In the case of involutions see [3,1].

Definition 4. Given a semigroup G and a superalgebra R we say that R is *graded* by G if $R = \bigoplus_{g \in G} R_g$ where each R_g is a $\mathbb{Z}_2$-graded vector subspace and $R_g R_h \subset R_{gh}$, for any $g, h \in G$. The subset $\operatorname{Supp} R = \{g \in G \mid R_g \neq \{0\}\}$ is called the *support* of the grading.

A semigroup with 1 is called *cancellative* if each of $xg = xh, gx = hx$ implies $g = h$, for any $x, g, h \in G$.

Proposition 2. *Let R be a G-graded superalgebra, G a cancellative semigroup. Suppose R has an involution $*$ compatible with this grading, that is, $R_g^* = R_g$, for any $g \in G$, and also that R is $*$-simple. Then, given any $g, h \in \operatorname{Supp} R$ we have that $gh = hg$. If, additionally, $1 \in \operatorname{Supp} R$ then any $g \in \operatorname{Supp} R$ is invertible*.

Proof. Let $g, h \in \operatorname{Supp} R$. Suppose $R_gR_h \neq 0$. We have to show that $(R_gR_h)^* \subseteq R_{hg}$. Since we deal with a superalgebra G-grading, $R_g = R_g^{\bar{0}} + R_g^{\bar{1}}$ and $R_h = R_h^{\bar{0}} + R_h^{\bar{1}}$ where $R_g^{\bar{0}}$, $R_h^{\bar{0}}$ are even components, $R_g^{\bar{1}}$, $R_h^{\bar{1}}$ are odd components. It follows from $R_gR_h = (R_g^{\bar{0}} + R_g^{\bar{1}})(R_h^{\bar{0}} + R_h^{\bar{1}}) \subseteq R_g^{\bar{0}}R_h^{\bar{0}} + R_g^{\bar{1}}R_h^{\bar{1}} + R_g^{\bar{0}}R_h^{\bar{1}} + R_g^{\bar{1}}R_h^{\bar{0}}$ that $(R_gR_h)^* \subseteq (R_h^{\bar{0}})^*(R_g^{\bar{0}})^* + (R_h^{\bar{1}})^*(R_g^{\bar{1}})^* + (R_h^{\bar{1}})^*(R_g^{\bar{0}})^* + (R_h^{\bar{0}})^*(R_g^{\bar{1}})^* = R_h^{\bar{0}}R_g^{\bar{0}} + R_h^{\bar{1}}R_g^{\bar{1}} + R_h^{\bar{1}}R_g^{\bar{0}} + R_h^{\bar{0}}R_g^{\bar{1}} \subseteq R_{hg}$. On the other hand, $R_gR_h \subseteq R_{gh}$, $(R_gR_h)^* \subseteq R_{gh}^* = R_{gh}$. Hence, $R_{gh} = R_{hg}$, $gh = hg$.

Now, pick $g, h \in \operatorname{Supp} R$, and consider $I = R_g + RR_g + R_gR + RR_gR$. It is easily seen that I is a graded ideal. Next we want to show that $I^* = I$. Since $RR_g = \sum_l R_lR_g$, $(\sum_l R_gR_l)^* \subseteq \sum_l R_l^{\bar{0}}R_g^{\bar{0}} + R_l^{\bar{1}}R_g^{\bar{1}} + R_l^{\bar{1}}R_g^{\bar{0}} + R_l^{\bar{0}}R_g^{\bar{1}} = \sum_l (R_l^{\bar{0}} + R_l^{\bar{1}})(R_g^{\bar{0}} + R_g^{\bar{1}}) = \sum_l R_lR_g = RR_g$. In a similar manner we can show that $(RR_gR)^* = (\sum_{l,k} R_lR_gR_k)^* = \sum_{l,k} (R_lR_gR_k)^* = \sum_{k,l} R_kR_gR_l$. Therefore, I is a graded $*$-invariant non-zero ideal, hence, $I = R$. In particular, $R_h \subset R_g + RR_g + R_gR + RR_gR$. The homogeneous components on the right-hand side are of one of the forms: g, kg, gl, pgq, for some $k, l, p, q \in G$. So, h is one of these forms. It follows that one of the spaces R_g (if $g = h$), or R_kR_g, or R_gR_l, or $R_pR_gR_g$ is different from zero, with either $h = g$, or $h = kg$, or $h = gl$, or $h = pgq$. The case $h = g$ being trivial, if $R_kR_g \neq 0$ with $h = kg$ then $kg = gk$ by what was proven before and then $hg = (kg)g = g(kg) = gh$, as needed. Similarly, if $R_gR_l \neq 0$ with $gl \neq 0$. Now if $R_pR_gR_q \neq 0$ with $h = pgq$, then $R_pR_g \neq 0$ and $R_gR_q \neq 0$ so that $pg = gp$ and $gq = qg$. Again, $hg = (pgq)g = (pg)(qg) = gpgq = gh$, as required.

The invertibility claim follows in exactly the same way as in [7, Proposition 1]. □

As a result, using Proposition 1, we will assume in what follows, that we deal with abelian group gradings of finite-dimensional involution simple superalgebras. Actually, we restrict ourselves to the case where G is finite and R is not simple as a superalgebras (Cases (2) and (3) of Proposition 1). As mentioned earlier, Case (1) is to be published in a joint paper of the second and the third authors [11].

Remark 1. If A is a superalgebra graded by an *abelian* group G then the same homogeneous subspaces A_g, $g \in G$, define in A^{sop} a G-grading. We will denote these subspaces by A_g^{sop}.

The techniques we are going to use impose a further restriction on the ground field F. Namely, we are going to use the correspondence between the gradings on a (super) algebra R by a finite abelian group G and the actions on R of the dual group $\widehat{G}$ by automorphisms (see, for example, [2, Section 2]). For this to work, we need to make sure that if the order of G is d then F contains d different roots of 1 of degree d. If this condition holds then each grading $R = \bigoplus_{g\in G} R_g$ defines a homomorphism $\alpha : \widehat{G} \to \operatorname{Aut} R$ given by $\alpha(\chi)(r) = \chi(g)r$ provided that $r \in R_g$, $g \in G$. Also the grading can be recovered if we have a homomorphism α, as above.

We start with a general result (the Exchange Theorem below) obtained by the first author. An important particular case can be found in [6]. Let G be a finite abelian group and V a vector space. Suppose we have two G-gradings on V:

$$V = \oplus_{g\in G}\overline{V}_g, \quad \alpha : \widehat{G} \to \operatorname{Aut} V, \tag{1'}$$

$$V = \oplus_{g\in G}\widetilde{V}_g, \quad \beta : \widehat{G} \to \operatorname{Aut} V, \tag{2'}$$

where $\alpha, \beta : \widehat{G} \to \operatorname{Aut} V$ are homomorphisms of the dual group $\widehat{G}$ corresponding to the above gradings in the following way. Given $\chi \in \widehat{G}$ we define $\alpha(\chi)$ on an element v of $\overline{V}_g$, for each g, by $\alpha(\chi)(v) = \chi(g)v$. Similarly for (2′). Suppose $\Lambda \subset \widehat{G}$ is a subgroup such that $\alpha(\lambda) = \beta(\lambda)$, for each $\lambda \in \Lambda$. Let us denote by H the orthogonal complement $\Lambda^\perp = \{g \in G \mid \lambda(g) = 1,\ \lambda \in \Lambda\}$. Assume further that the subgroups $\alpha(\widehat{G})$ and $\beta(\hat{G})$ commute elementwise.

Let us consider a homomorphism $\gamma : \widehat{G} \to \operatorname{Aut} V$ given by $\gamma(\chi) = \alpha^{-1}(\chi)\beta(\chi)$. In this case we can define H-grading of V as follows: $V^{(h)} = \{v \mid \gamma(\chi)(v) = \chi(h)v,\ \chi \in \widehat{G}\}$.

Theorem 1 (Exchange Theorem). *The three gradings defined above are connected by the following equations*

$$\overline{V}_g = \oplus_{h\in H}(\widetilde{V}_{gh} \cap V^{(h)}), \quad \widetilde{V}_g = \oplus_{h\in H}(\overline{V}_{gh} \cap V^{(h^{-1})}) \tag{3'}$$

If V is an algebra and $(1')$ and $(2')$ *are algebra gradings, then* $(3')$ *are relations for the algebra gradings.*

Proof. Let us prove the first equality. Since all gradings are compatible, we have $\overline{V}_g = \oplus_{h\in H}(\overline{V}_g \cap V^{(h)})$. Thus it is enough to prove, for any $g \in G, h \in H$, that $\widetilde{V}_{gh} \cap V^{(h)} = \overline{V}_g \cap V^{(h)}$. If $v \in \widetilde{V}_{gh} \cap V^{(h)}$ then $\beta(\chi)(v) = \chi(gh)v$ and $\gamma(\chi)(v) = \chi(h)v$. Hence also $\gamma(\chi)^{-1}(v) = \chi(h)^{-1}v$. Now

$$\alpha(\chi)(v) = \alpha(\chi)\beta(\chi)^{-1}\beta(\chi)(v) = \gamma(\chi)^{-1}\beta(\chi)(v) = \chi(h)^{-1}\chi(gh)v$$

proving $\widetilde{V}_{gh} \cap V^{(h)} \subset \overline{V}_g \cap V^{(h)}$.

If $b \in \overline{V}_g \cap V^{(h)}$ then $\alpha(\chi)(b) = \chi(g)b,\ \gamma(\chi)(b) = \chi(h)b$. Therefore

$$\beta(\chi)(b) = \alpha(\chi)\alpha(\chi)^{-1}\beta(\chi)(a) = \alpha(\chi)\gamma(\chi)(a) = \chi(g)\chi(h)a = \chi(gh)a.$$

It follows that $\overline{V}_g \cap V^{(h)} \subset \widetilde{V}_{gh} \cap V^{(h)}$. Finally, $\overline{V}_g \cap V^{(h)} = \widetilde{V}_{gh} \cap V^{(h)}$ for any $g \in G$ and thus we have the first equality in $(3')$. The second is similar. It is easy to check that if V is an algebra and $(1')$ and $(2')$ are algebra gradings, then $(3')$ provides us with the relations between algebra gradings as well. The proof is complete. □

One of the important tools in the proof of the main results of our work is a recent result from [5], as follows.

Theorem 1. *Let $M_n(F) = A = \bigoplus_{g\in G} A_g$ be a G-grading on $M_n(F)$ over a field F of characteristic not* 2, *which contains d different roots of 1, $d = |G|$. Suppose there is a graded antiautomorphism φ whose restriction to R_e is an involution. Then there is a G-graded automorphism ψ of R such that $\varphi\psi = \psi\varphi$ and $\psi^2 = \varphi^2$.*

A consequence of this result which interests us is as follows. Let us denote by $\overline{\text{Aut}}\,(A)$ the group of automorphisms and antiautomorphisms of A. In the case $A = M_n(F)$, $[\overline{\text{Aut}}\,(A) : \text{Aut}\,(A)] = 2$.

Theorem 2. *Let P be a finite abelian subgroup in $\overline{\text{Aut}}\,(A), A = M_n(F)$ over a field F of characteristic not* 2, *which contains d different roots of* 1, $d = |G|$. *Suppose $\varphi \in P \setminus \text{Aut}\,(A)$. Then there exists $\psi \in \text{Aut}\,(A)$ commuting with all elements in P and $\psi^2 = \varphi^2$.*

Proof. Set $Q = P \cap \text{Aut}\,(A)$. Then Q is a subgroup of index 2 in P. Let G be a finite abelian group whose dual is Q. That is, the elements of Q can be viewed as multiplicative characters on G. As noted earlier, in this case A becomes G-graded if one sets $A_g = \{a \in A \mid \chi(a) = \chi(g)a \text{ for any } \chi \in Q\}$. Since φ commutes with the elements of Q, the antiautomorphism φ is a G-graded map. Also, because $\varphi^2 \in Q$, we have that the restriction of φ to R_e is an involution. Applying Theorem 1, we find a G-graded automorphism ψ such that $\varphi\psi = \psi\varphi$ and $\psi^2 = \varphi^2$. Now if χ is an arbitrary element of Q and a a homogeneous element of degree $g \in G$ then $\psi(\chi(a)) = \psi\chi(g)a = \chi(g)\psi(a) = \chi(\psi(a))$ because $\psi(a) \in A_g$. It follows that $\psi\chi = \chi\psi$ and ψ commutes with all elements of P, as required. □

3. Antiautomorphisms of graded superalgebras

Theorem 1 is no longer true in the case of (super) antiautomorphisms of matrix superalgebras. The simplest example is the trivial grading and the (super) antiautomorphism defined on $M_{n,m}$, n, m odd, by

$$\varphi(X) = \begin{pmatrix} A & -B \\ C & D \end{pmatrix}^t,$$

where $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, A and D are matrices of size $n \times n$ and $m \times m$, respectively, B and C are matrices of size $n \times m$ and $m \times n$, respectively.

Luckily, the argument of [5] can be adapted to the case of superalgebras although we have to deal with higher powers of the antiautomorphisms in question. We start with a generalization of the results of [3, Lemma 2] about fine involution gradings. Recall that a grading $R = \oplus_{g\in G} R_g$ is called *fine* if for any $g \in G$ such that $R_g \neq \{0\}$, $\dim R_g = 1$.

Theorem 3. *Let $R = M_{n,m}(F)$, $n, m \geqslant 1$, be a non-trivial matrix superalgebra with an antiautomorphism φ over an algebraically closed field F of characteristic zero or coprime to the order of G, where G is a finite abelian group. Then R admits no fine G-gradings respected by φ.*

Proof. Assume the contrary, that is $R = \oplus_{g\in G} R_g$ is a fine G-grading respected by φ, $\varphi(R_g) = R_g$. Since R is a superalgebra with a fine G-grading, according to [4], $\dim R_{\bar 0} = \dim R_{\bar 1}$, that is, $n = m$. Let $R_{\bar 0}$ be denoted by A. Then

$$A = \oplus_{g\in G} A_g,$$

where $A_g = R_{\bar 0} \cap R_g$. This grading is also fine and compatible with φ. Note that $A = J_1 \oplus J_2$, the sum of two isomorphic simple ideals.

Next let $\widehat{G}$ be the dual group of G, and $\alpha : \widehat{G} \to \operatorname{Aut} A$ the homomorphism accompanying our grading. If for each $\eta \in \widehat{G}$, $\alpha(\eta)(J_i) = J_i$, then a fine G-grading of A induces G-gradings on both ideals such that $A_g = (J_1)_g \oplus (J_2)_g$. In particular, $A_e = (J_1)_e \oplus (J_2)_e$, where $(J_i)_e \neq \{0\}$. This contradicts the fact that our G-grading is fine. Therefore, there exists $\xi \in \widehat{G}$ such that $\alpha(\xi)(J_1) = J_2$. Hence, $\widehat{G} = \Lambda \cup \Lambda\xi$ where $\Lambda = \{\eta \in \widehat{G} | \alpha(\eta)(J_i) = J_i\}$ and $\xi^2 \in \Lambda$. Then $H = \Lambda^{\perp}$ is a subgroup of G of order 2 and $\widehat{G/H} \cong \Lambda$. Let $H = \{e, h\}$ where $h^2 = e$. Next we can consider the induced $\overline{G} = G/H$-grading of A. Let $\bar g = gH$ for any $g \in G$. Then $A_{\bar g} = A_g + A_{gh}$. Since $\widehat{G/H} * J_i = \Lambda * J_i =_i$ where $i \in 1, 2$, J_i is a G/H-graded ideal. It follows from $A_{\bar e} = (J_1)_{\bar e} \oplus (J_2)_{\bar e}$, $(J_i)_{\bar e} \neq \{0\}$, and $\dim A_{\bar e} = 2$ that $\dim (J_i)_{\bar e} = 1$. Therefore, both G/H-gradings on J_1 and J_2 are fine.

The following two cases may occur.

Case 1. Let $\varphi(J_1) = J_2$. Note that $A_{\bar g} = (J_1)_{\bar g} \oplus (J_2)_{\bar g}$ for each $\bar g \in \overline{G}$. By Proposition 2 [6], we can recover our original G-grading. In fact,

$$A_g = \{X + \xi(g)^{-1}(\xi * X) | X \in A_{\bar g}\}. \tag{1}$$

For example, let us take $X = \begin{pmatrix} X_{\bar g} & 0 \\ 0 & 0 \end{pmatrix}$, $X_{\bar g} \in (J_1)_{\bar g}$. Then, by (1),

$$0 \neq X + \xi(g)^{-1}(\xi * X) = \begin{pmatrix} X_{\bar g} & 0 \\ 0 & \xi(g)^{-1}(\xi * X_{\bar g}) \end{pmatrix} \in A_g.$$

Since $\dim A_g = 1$, $A_g = \operatorname{span}\left\{\begin{pmatrix} X_{\bar g} & 0 \\ 0 & \xi(g)^{-1}(\xi * X_{\bar g}) \end{pmatrix}\right\}$. Recall that $\varphi(A_g) = A_g$ where φ can be represented as follows:

$$\varphi * \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} \varphi_0(Y) & 0 \\ 0 & \varphi_1(X) \end{pmatrix},$$

where φ_0 and φ_1 are antiautomorphisms. Hence

$$\varphi * \begin{pmatrix} X_{\bar g} & 0 \\ 0 & \xi(g)^{-1}(\xi * X_{\bar g}) \end{pmatrix} = \begin{pmatrix} \xi(g)^{-1}(\varphi_0\xi) * (X_{\bar g}) & 0 \\ 0 & \varphi_1(X_{\bar g}) \end{pmatrix}$$
$$= \lambda_g \begin{pmatrix} X_{\bar g} & 0 \\ 0 & \xi(g)^{-1}(\xi * X_{\bar g}) \end{pmatrix}$$

for some non-zero scalar λ_g. Therefore, for each $\bar{g} \in \overline{G}, X_{\bar{g}} = (\lambda_g \xi(g))^{-1})(\varphi_0 \xi) * (X_{\bar{g}})$ where $\varphi_0 \xi$ is also an antiautomorphism. In other words a fine $\overline{G}$-grading on J_1 is respected by antiautomorphism $\varphi_0 \xi$. Then, by [3, Lemma 2], $G/H \cong N_1 \times \cdots \times N_k$ where $N_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. Let $\varphi(J_i) = J_i$. Then a fine $\overline{G}$-grading on each J_i is also compatible with φ. Hence, according to [3, Lemma 2], $G/H \cong N_1 \times \cdots \times N_k$ where $N_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Therefore, $|G| = 2 \cdot 2^{2l} = 2^{2l+1}$, for some natural number l. Moreover, we have that for each $g \in G$, either $g^2 = e$ or $g^4 = e$. On the other hand, according to Theorem 5 [2], $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \times \mathbb{Z}_{n_k}$, $n_i \in \mathbb{N}$. Moreover, either $n_i = 2$ or $n_i = 4$. Therefore, $|G| = 2^{2r} \cdot 4^{2s} = 2^{2r+4s}$, for some natural numbers r and s, which is contradiction. □

In what follows, let τ denote an antiautomorphism of $M_{n,m}(F)$ defined by the formula:

$$X^\tau = \begin{pmatrix} A & -B \\ C & D \end{pmatrix}^t,$$

where $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, A and D are matrices of size $n \times n$ and $m \times m$, respectively, B and C are matrices of size $n \times m$ and $m \times n$, respectively.

Recall that a grading $R = \oplus_{g \in G} R_g$ on the matrix algebra $R = M_n(F)$ is called *elementary* if there exists an n-tuple $\theta = (g_1, \ldots, g_n) \in G^n$ such that the matrix units E_{ij}, $1 \leqslant i, j \leqslant n$ are homogeneous and $E_{ij} \in R_g$ if and only if $g = g_i^{-1} g_j$.

Lemma 1. *Let $R = M_n = \oplus_{g \in G} R_g$ be a matrix algebra with the elementary G-grading. If $R_e = A_1 \oplus A_2$ is the sum of two simple subalgebras, then there exists $g \in G, g \neq e$, such that $A_1 R A_2 \subseteq R_g$.*

Lemma 2. *If $R = R_{\bar{0}} + R_{\bar{1}}$ is a superalgebra with an antiautomorphism φ, then for any $x, y \in R$,*

$$\varphi(xRy) \subseteq \varphi(y) R \varphi(x). \tag{2}$$

Lemma 3. *Let $R = C \otimes D = \oplus_{g \in G} R_g$ be a G-graded matrix superalgebra with an elementary grading on C, and a fine grading on D over an algebraically closed field F of characteristic not 2. Let $\varphi : R \to R$ be an antiautomorphism on R preserving G-grading and $\sigma : R \to R$ be an automorphism of order 2 of R that defines a superalgebra structure on R. Let also φ act as a superinvolution on R_e. Then*

(1) *$C_e \otimes I$ is φ-stable and σ-stable where I is the unit of D and hence σ induces a $\mathbb{Z}_2$-grading on C_e and φ induces a superinvolution $*$ on C_e compatible with the $\mathbb{Z}_2$-grading.*
(2) *there are $*$-subsuperalgebras $B_1, \ldots, B_k \subseteq C_e$ such that $C_e = B_1 \oplus \cdots \oplus B_k$, and $B_1 \otimes I, \ldots, B_k \otimes I$ are φ-stable and σ-stable.*
(3) *φ acts on $R_e = C_e \otimes I$ as $\varphi * X = S^{-1} X^\tau S$ where $S = S_1 \otimes I + \cdots + S_k \otimes I, S_i \in B_i C B_i$ and $S_i = \begin{pmatrix} I_{s_i} & 0 \\ 0 & Q_{r_i} \end{pmatrix}$ if B_i is of type $M_{s_i,2r_i}(F)$ with orthosymplectic superinvolution; $S_i = \begin{pmatrix} 0 & I_{s_i} \\ I_{s_i} & 0 \end{pmatrix}$ if B_i is of type $M_{s_i,s_i}(F)$ with transpose superinvolution; $S_i = \begin{pmatrix} 0 & I_{s_i+r_i} \\ I_{s_i+r_i} & 0 \end{pmatrix}$ if B_i is of type $M_{s_i,r_i}(F) \oplus M^{sop}_{s_i,r_i}(F)$ with exchange superinvolution; $S_i = \begin{pmatrix} 0 & I_{2s_i} \\ I_{2s_i} & 0 \end{pmatrix}$ if B_i is of type $Q(s_i) \oplus Q(s_i)^{sop}$ with exchange superinvolution.*
(4) *if e_i is the identity of B_i, then $D_i = e_i \otimes D$ is φ-stable and σ-stable.*
(5) *the centralizer of $R_e = C_e \otimes I$ in R can be decomposed as $Z_1 D_1 \oplus \cdots \oplus Z_k D_k$ where $Z_i = Z'_i \otimes I, Z'_i$ is the center of B_i.*

Proof. It follows from [2, Theorem 5] that the identity component R_e equals to $C_e \otimes I$. Since R_e is φ- and σ-stable, both φ and σ induce a superinvolution $*$ and a superalgebra structure on C_e. Both structures are compatible with each other.

Since C_e is semisimple, it is the direct sum of simple subalgebras,

$$C_e = A_1 \oplus \cdots \oplus A_l.$$

If for some i, $1 \leqslant i \leqslant l$, $\sigma(A_i) = A_j$ where $i \neq j$, then it is easily seen that $A_i' = A_i + A_j$ is σ-stable. Therefore, C_e can be written as a direct sum of σ-stable superalgebras,

$$C_e = A_1' \oplus \cdots \oplus A_s'.$$

Next, if for some i, $1 \leqslant i \leqslant s$, $(A_i')^* = A_j'$ where $i \neq j$, then $B_i = A_i' + A_j'$ is $*$-stable. Finally, C_e can be written as a direct sum of $*$-simple superalgebras.

$$C_e = B_1 \oplus \cdots \oplus B_k.$$

Now (1), (2) and (3) follows from the classification of involution simple superalgebras (see Proposition 1).

Next we fix $1 \leqslant i \leqslant k$, and consider $R' = (e_i \otimes I)(C \otimes D)(e_i \otimes I) = e_iCe_i \otimes D$ where e_i is the identity of B_i. Since $\varphi(e_i \otimes I) = e_i \otimes I$ and $\sigma(e_i \otimes I) = e_i \otimes I$, R' is φ- and σ-stable.

To prove (4), we consider the following three cases.

Case 1. Let B_i be of the type $M_{r,s}(F)$. Then

$$e_iCe_i = B_i, \tag{3}$$

and $e_iCe_i \otimes I = B_i \otimes I$. Hence, $e_iCe_i \otimes I$ is φ- and σ-stable. Since $e_i \otimes D$ is a centralizer of $e_iCe_i \otimes I$, it is also φ- and σ-stable.

Case 2. Let $B_i = A \oplus A^{sop}$ where $A = M_{r,s}(F)$. Denote the identity of A by ε_i. Then, ε_i^* is the identity of A^{sop}, and $e_i = \varepsilon_i + \varepsilon_i^*$. Notice that

$$e_iCe_i \otimes I = \varepsilon_iC\varepsilon_i \otimes I + \varepsilon_iC\varepsilon_i^* \otimes I + \varepsilon_i^*C\varepsilon_i \otimes I + \varepsilon_i^*C\varepsilon_i^* \otimes I. \tag{4}$$

Next we want to prove that both φ and σ permute the terms of (4) leaving $e_iCe_i \otimes I$ invariant. Without any loss of generality we consider just one term of the form $\varepsilon_iC\varepsilon_i^* \otimes I$. Since $\varepsilon_iC\varepsilon_i^* \otimes I = (\varepsilon_i \otimes I)(C \otimes I)(\varepsilon^* \otimes I)$, by (2), $\varphi(\varepsilon_iC\varepsilon_i^* \otimes I) \subseteq (\varepsilon_i \otimes I)(C \otimes D)(\varepsilon^* \otimes I) = \varepsilon_iC\varepsilon_i^* \otimes D$ and $\sigma(\varepsilon_iC\varepsilon_i^* \otimes I) \subseteq (\varepsilon_i \otimes I)(C \otimes D)(\varepsilon^* \otimes I) = \varepsilon_iC\varepsilon_i^* \otimes D$.

By Lemma 2, there exists a $g \in G, g \neq e$ such that $\varepsilon_iC\varepsilon_i^* \subseteq C_g$. Hence, $\varepsilon_iC\varepsilon_i^* \otimes I \subseteq R_g$. Consequently, $\varphi(\varepsilon_iC\varepsilon_i^* \otimes I) \subseteq R_g$ and $\sigma(\varepsilon_iC\varepsilon_i^* \otimes I) \subseteq R_g$. Next we take a homogeneous $x \in \varepsilon_iC\varepsilon_i^*$ of degree g and a homogeneous $y \in D$ of degree h such that $x \otimes y \in R_g$. Then $\deg(x \otimes y) = gh = g, h = e$. This implies $y = \lambda I, \lambda \in F$ for any $x \otimes y \in R_g \cap \varepsilon_iC\varepsilon_i^* \otimes D$. It follows that $R_g \cap \varepsilon_iC\varepsilon_i^* \otimes D \subseteq \varepsilon_iC\varepsilon_i^* \otimes I$.

As a consequence, $\varphi(e_iCe_i \otimes I) = e_iCe_i \otimes I$ and $\sigma(e_iCe_i \otimes I) = e_iCe_i \otimes I$, that is, $e_iCe_i \otimes I$ is φ- and σ-stable. From the decomposition $R' = e_iCe_i \otimes D$ it follows that $e_i \otimes D$, the centralizer of $e_iCe_i \otimes I$ in R', is φ- and σ-stable.

Case 3. Let $B_i = Q(s_i) \oplus Q(s_i)^{sop}$. Since $Q(s_i) = I_1 \oplus I_2$ where I_1, I_2 are simple ideals isomorphic to $M_{s_i}(F)$, $B_i = (I_1 \oplus I_2) \oplus (I_1^* \oplus I_2^*)$. Let $\varepsilon_i, \hat{\varepsilon}_i, \varepsilon_i^*, \hat{\varepsilon}_i^*$ be the identities of I_1, I_2, I_1^*, I_2^*, respectively. Then we notice that $\sigma(\varepsilon_i \otimes I) = \hat{\varepsilon}_i \otimes I$. We have that

$$e_i = \varepsilon_i + \varepsilon_i^* + \hat{\varepsilon}_i + \hat{\varepsilon}_i^*. \tag{5}$$

Therefore, $e_iCe_i \otimes I = N_1 \otimes I + N_2 \otimes I + N_3 \otimes I + N_4 \otimes I$ where $N_1 = \varepsilon_iC\varepsilon_i + \varepsilon_i^*C\varepsilon_i + \varepsilon_iC\varepsilon_i^* + \varepsilon_i^*C\varepsilon_i^*$, $N_2 = \varepsilon_iC\hat{\varepsilon}_i + \varepsilon_i^*C\hat{\varepsilon}_i + \varepsilon_iC\hat{\varepsilon}_i^* + \varepsilon_i^*C\hat{\varepsilon}_i^*$, $N_3 = \hat{\varepsilon}_iC\varepsilon_i + \hat{\varepsilon}_i^*C\varepsilon_i + \hat{\varepsilon}_iC\varepsilon_i^* + \hat{\varepsilon}_i^*C\varepsilon_i^*$, and $N_4 = \hat{\varepsilon}_iC\hat{\varepsilon}_i + \hat{\varepsilon}_i^*C\hat{\varepsilon}_i + \hat{\varepsilon}_iC\hat{\varepsilon}_i^* + \hat{\varepsilon}_i^*C\hat{\varepsilon}_i^*$.

Arguing in the same way as in the second case, we can prove that $\varphi(N_1 \otimes I) = N_1 \otimes I$ and $\varphi(N_4 \otimes I) = N_4 \otimes I$. Now we consider N_2 and N_3. Suppose that the elementary grading on B_iCB_i induced from C is defined by (g_1, g_2, g_3, g_4). It is easy to see that $\deg(\varepsilon_iC\hat{\varepsilon}_i) = g_1^{-1}g_3$, $\deg(\varepsilon_iC\hat{\varepsilon}_i^*) = g_1^{-1}g_4$, $\deg(\varepsilon_i^*C\hat{\varepsilon}_i) = g_2^{-1}g_3$, $\deg(\varepsilon_i^*C\hat{\varepsilon}_i^*) = g_2^{-1}g_4$, $\deg(\hat{\varepsilon}_iC\varepsilon_i) = g_3^{-1}g_1$, $\deg(\hat{\varepsilon}_i^*C\varepsilon_i) = g_4^{-1}g_1$, $\deg(\hat{\varepsilon}_i^*C\varepsilon_i^*) = g_4^{-1}g_2$, $\deg(\hat{\varepsilon}_iC\varepsilon_i^*) = g_3^{-1}g_2$.

Let us take, for example, the first term $\varepsilon_iC\hat{\varepsilon}_i$ of N_2. Then $\varphi(\varepsilon_iC\hat{\varepsilon}_i \otimes I) \subseteq R_{g_1^{-1}g_3}$. On the other hand, by (2), $\varphi(\varepsilon_iC\hat{\varepsilon}_i \otimes I) \subseteq \hat{\varepsilon}_i^*C\varepsilon_i^* \otimes D \subseteq C_{g_4^{-1}g_2} \otimes D$. If we take $x \in C_{g_4^{-1}g_2}$ and a homogeneous $y \in D$ of degree h such that $\deg(x \otimes y) = g_1^{-1}g_3$, then $g_1^{-1}g_3 = g_4^{-1}g_2h$, that is, $h = g_1^{-1}g_4g_2^{-1}g_3$. This implies that

$$\varphi(\varepsilon_iC\hat{\varepsilon}_i \otimes I) \subseteq \hat{\varepsilon}_i^*C\varepsilon_i^* \otimes D_h. \tag{6}$$

Similarly, we can show that for each term of N_2 there should be D_h on the right-hand side of (6). Hence $\varphi(N_2 \otimes I) = N_3 \otimes D_h$.

Next we take the first term $\hat{\varepsilon}_i C \varepsilon_i$ of N_3. Likewise we can show that $\varphi(\hat{\varepsilon}_i C\varepsilon_i \otimes I) \subseteq \varepsilon_i^* C \hat{\varepsilon}_i^* \otimes D_{h^{-1}}$. Hence for each term of N_3 we have $D_{h^{-1}}$ on the right-hand side, and $\varphi(N_3 \otimes I) = N_2 \otimes D_{h^{-1}}$.

Note that the centralizer of $(N_2 + N_3) \otimes I$ in R' is $e_i \otimes D$. Hence, the centralizer of $\varphi((N_2 + N_3) \otimes I) = N_3 \otimes D_{h^{-1}} + N_2 \otimes D_h$ in R' is $\varphi(e_i \otimes D)$. Next we take any $x \otimes y$ in $\varphi(e_i \otimes D)$. Since $x \otimes y$ lies in the centralizer of $N_3 \otimes D_{h^{-1}} + N_2 \otimes D_h$, $x \otimes y$ commutes with each element in $N_3 \otimes D_{h^{-1}}$ and $N_2 \otimes D_h$. Therefore, x commutes with any matrix in N_3 and N_2. Direct computations show that $x = e_i$, and $\varphi(e_i \otimes D) = e_i \otimes K$ where K is a subspace of D. By dimension arguments, $K = D$, and $\varphi(e_i \otimes D) = e_i \otimes D$.

To prove that $e_i \otimes D$ is σ-stable, we represent $e_i C e_i \otimes I$ as follows: $e_i C e_i \otimes I = N_1' \otimes I + N_2' \otimes I + N_3' \otimes I + N_4' \otimes I$ where $N_1' = \varepsilon_i C \varepsilon_i + \hat{\varepsilon}_i C \varepsilon_i + \varepsilon_i C \hat{\varepsilon}_i + \hat{\varepsilon}_i C \hat{\varepsilon}_i$, $N_2' = \varepsilon_i C \varepsilon_i^* + \hat{\varepsilon}_i C \varepsilon_i^* + \varepsilon_i C \hat{\varepsilon}_i^* + \hat{\varepsilon}_i C \hat{\varepsilon}_i^*$, $N_3' = \varepsilon_i^* C \varepsilon_i + \hat{\varepsilon}_i^* C \varepsilon_i + \varepsilon_i^* C \hat{\varepsilon}_i + \hat{\varepsilon}_i^* C \hat{\varepsilon}_i$, and $N_4' = \varepsilon_i^* C \varepsilon_i^* + \hat{\varepsilon}_i^* C \varepsilon_i^* + \varepsilon_i^* C \hat{\varepsilon}_i^* + \hat{\varepsilon}_i^* C \hat{\varepsilon}_i^*$. In the same way as above, we can show that $e_i \otimes I$ is σ-stable.

Hence (4) is proved.

To prove (5) we note that the centralizer Z of C_e in C is equal to $Z_1' \oplus \cdots \oplus Z_k'$ where Z_i' is the center of B_i and the centralizer of R_e in R coincides with $Z \otimes D = Z_1 D_1 \oplus \cdots \oplus Z_k D_k$ where $Z_i = Z_i' \otimes I$ and $D_i = e_i \otimes D$.

Our proof is complete. □

Proposition 3. *Let G be a finite abelian group, and R a superalgebra of type $M_{n,m}(F)$ over an algebraically closed field F of characteristic not 2. Suppose that φ is an antiautomorphism of R that preserves a G-grading of R. Then there exists an automorphism ψ of R preserving the G-grading of R such that ψ commutes with φ and $\psi^4 = \varphi^4$.*

Proof. It is easy to check that the φ-action on R is defined by

$$\varphi * X = \Phi^{-1} X^\tau \Phi$$

for some matrix Φ, and $X^\tau = \begin{pmatrix} A & -B \\ C & D \end{pmatrix}^t$. First let $X \in R_e$. Consider the decomposition $C_e = B_1 \oplus \cdots \oplus B_k$ found in the previous lemma. Then $X = X_1 \otimes I + \cdots + X_k \otimes I$ with $X_i \in B_i$, $1 \leqslant i \leqslant k$. Then φ acts on X as

$$\varphi * X = S^{-1} X^\tau S,$$

where S as in (3) of Lemma 4. Hence the matrix ΦS^{-1} commutes with X^τ for any $X \in R_e$, that is ΦS^{-1} is an element of the centralizer of R_e in R. Hence, we obtain

$$\Phi = S_1 Y_1 \otimes Q_1 + \cdots + S_k Y_k \otimes Q_k \tag{7}$$

where $Q_i \in D$, $Y_i \in Z_i'$, $1 \leqslant i \leqslant k$. Compute now the action of φ^4 on an arbitrary $X \in R$:

$$\varphi^4 * X = ((\Phi^{-1})^t \Phi)^{-1})^2 X ((\Phi^{-1})^t \Phi)^2$$

Set $P = ((\Phi^{-1})^t \Phi)^2$. We need to show that there exists an inner automorphism ψ such that $\psi^4 * X = P^{-1} X P$ for all $X \in R$. Note that for any $T_i, T_i' \in B_i C B_i$ and $Q_i, Q_i' \in D$, $i = 1, \ldots, k$, the relation

$$\left(\sum_i T_i \otimes Q_i\right)\left(\sum_i T_i' \otimes Q_i'\right) = \sum_i T_i T_i' \otimes Q_i Q_i'$$

holds.

We compute the value of P:

$$\begin{aligned} P = ((\Phi^{-1})^t \Phi)^2 &= \sum_{i=1}^k ((Y_i^t S_i^t)^{-1} S_i Y_i)^2 \otimes ((Q_i^t)^{-1} Q_i)^2 \\ &= \sum_i ((S_i^t)^{-1} (Y_i^t)^{-1} S_i Y_i)^2 \otimes ((Q_i^t)^{-1} Q_i)^2. \end{aligned} \tag{8}$$

Lemma 4. *All Q_i satisfy $(Q_i^t)^{-1}Q_i = \pm I$.*

Obviously it is sufficient to prove the relation

$$e_i \otimes (Q_i^t)^{-1}Q_i = \pm e_i \otimes I$$

in $D_i = e_i \otimes D$. Recall that D_i is φ- and σ-stable. Moreover, D_i is G-graded algebra with a fine G-grading compatible with φ and σ. Therefore, this is G-graded superalgebra with a fine G-grading respected by φ. According to Theorem 3, D_i cannot be non-trivial. Therefore, D_i is a trivial superalgebra, that is, $D_i \subseteq R_{\bar{0}}$, and τ acts on D_i as a usual transpose. For any $X \in D$ we have

$$\varphi * (e_i \otimes X) = \Phi^{-1}(e_i \otimes X)^{\tau}\Phi = (S_iY_i)^{-1}(e_i)(S_iY_i) \otimes (Q_i^{-1}X^tQ_i) = e_i \otimes Q_i^{-1}X^tQ_i$$

i.e. action by φ induces an antiautomorphism $e_i \otimes X \to e_i \otimes Q_i^{-1}X^tQ_i$ on D_i. Arguing in the same way as in Lemma 6.5 (see [6]) we can conclude that $e_i \otimes (Q_i^t)^{-1}Q_i = \pm e_i \otimes I$. $\square$

Now we compute $((Y_i^tS_i^t)^{-1}S_iY_i)^2$. If B_i is simple then Y_i is a scalar matrix and $((S_i^t)^{-1}S_i)^2 = I$ by Lemma 4. If B_i is of type $M_{s_i,r_i}(F) \oplus M_{s_i,r_i}^{sop}(F)$, then

$$Y_i = \begin{pmatrix} \lambda I & 0 \\ 0 & \mu I \end{pmatrix}, \quad S_i = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and $((Y_i^tS_i^t)^{-1}S_iY_i)^2 = \begin{pmatrix} \left(\frac{\lambda}{\mu}\right)^2 I & 0 \\ 0 & \left(\frac{\mu}{\lambda}\right)^2 I \end{pmatrix} = \begin{pmatrix} (\gamma)^2 I & 0 \\ 0 & (\gamma^{-1})^2 I \end{pmatrix}$ where $\gamma = \frac{\lambda}{\mu}$. If B_i if of type $Q(s_i) \oplus Q(s_i)^{sop}$, then

$$Y_i = \begin{pmatrix} \alpha I & 0 & 0 & 0 \\ 0 & \alpha_1 I & 0 & 0 \\ 0 & 0 & \beta I & 0 \\ 0 & 0 & 0 & \beta_1 I \end{pmatrix}, \quad S_i = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix},$$

and

$$((Y_i^tS_i^t)^{-1}S_iY_i)^2 = \begin{pmatrix} (\beta^{-1}\alpha)^2 I & 0 & 0 & 0 \\ 0 & (\alpha_1\beta_1^{-1})^2 I & 0 & 0 \\ 0 & 0 & (\alpha^{-1}\beta)^2 I & 0 \\ 0 & 0 & 0 & (\alpha_1^{-1}\beta_1)^2 I \end{pmatrix}$$
$$= \begin{pmatrix} \gamma^2 I & 0 & 0 & 0 \\ 0 & \mu^2 I & 0 & 0 \\ 0 & 0 & (\gamma^{-1})^2 I & 0 \\ 0 & 0 & 0 & (\mu^{-1})^2 I \end{pmatrix}$$

where $\gamma = \beta^{-1}\alpha$, and $\mu = \alpha_1\beta_1^{-1}$.

We have proved that $P = (P_1 + \cdots + P_k) \otimes I$ where $P_1 \in B_1, \ldots, P_k \in B_k$ and P_i has one of the forms: $P_i = I, P_i = \begin{pmatrix} (\gamma)^2 I & 0 \\ 0 & (\gamma^{-1})^2 I \end{pmatrix}$, and $P_i = \begin{pmatrix} \gamma^2 I & 0 & 0 & 0 \\ 0 & \mu^2 I & 0 & 0 \\ 0 & 0 & (\gamma^{-1})^2 I & 0 \\ 0 & 0 & 0 & (\mu^{-1})^2 I \end{pmatrix}$. Now we present a matrix $T = (T_1 + \cdots + T_k) \otimes I, T_1 \in B_1, \ldots, T_k \in B_k$ such that $T_i^4 = P_i$ for all i and hence $T^4 = P$. In case $P_i = I$ we take $T_i = I$. If $P_i = \begin{pmatrix} (\gamma)^2 I & 0 \\ 0 & (\gamma^{-1})^2 I \end{pmatrix}$, then we take $T_i = \begin{pmatrix} \gamma_1 I & 0 \\ 0 & \gamma_1^{-1} I \end{pmatrix}$ where

$\gamma^2=\gamma_1^4$. If $P_i=\begin{pmatrix}\gamma^2 I & 0 & 0 & 0\\ 0 & \mu^2 I & 0 & 0\\ 0 & 0 & (\gamma^{-1})^2 I & 0\\ 0 & 0 & 0 & (\mu^{-1})^2 I\end{pmatrix}$, we take $T_i=\begin{pmatrix}\gamma_1 I & 0 & 0 & 0\\ 0 & \mu_1 I & 0 & 0\\ 0 & 0 & \gamma_1^{-1} I & 0\\ 0 & 0 & 0 & \mu_1^{-1} I\end{pmatrix}$ where $\gamma^2=\gamma_1^4$, and $\mu^2=\mu_1^4$. Note that $T\in R_e$, hence the map $\psi: X\to T^{-1}XT$ is an inner automorphism preserving G-grading. Moreover, since $T^4=P$, $\psi^4=\varphi^4$.

Now we need to check that ψ and φ commute. Direct computations show that $\varphi\psi=\psi\varphi$ if and only if

$$T^\tau \Phi T=\lambda\Phi,\quad (\Phi^{-1}T^\tau\Phi)T=\lambda I, \tag{9}$$

for some scalar λ. Since $T=T_1\otimes I+\cdots+T_k\otimes I$ where $T_i\in B_i$, $\Phi^{-1}T^\tau\Phi=\varphi * T=S^{-1}T^\tau S$ (see Lemma 4). If B_i is simple, then $T_i=I$ and $S_i^{-1}T_i^\tau S_i=T_i=I$. If $B_i=A\oplus A^{sop}$, then the restriction of φ to B_i acts as the exchange superinvolution, and

$$S_i^{-1}T_i^\tau S_i=\begin{pmatrix}\gamma_1^{-1} I & 0 & 0 & 0\\ 0 & \mu_1^{-1} I & 0 & 0\\ 0 & 0 & \gamma_1 I & 0\\ 0 & 0 & 0 & \mu_1 I\end{pmatrix}$$

for $T_i=\begin{pmatrix}\gamma_1 I & 0 & 0 & 0\\ 0 & \mu_1 I & 0 & 0\\ 0 & 0 & \gamma_1^{-1} I & 0\\ 0 & 0 & 0 & \mu_1^{-1} I\end{pmatrix}$, or

$$S_i^{-1}T_i^\tau S_i=\begin{pmatrix}\gamma_1^{-1} I & 0\\ 0 & \gamma_1 I\end{pmatrix}$$

for $T_i=\begin{pmatrix}\gamma_1 I & 0\\ 0 & \gamma_1^{-1} I\end{pmatrix}$. In both cases (9) holds with $\lambda=1$ and thus the proof is complete.

4. Main results

In this section we describe group gradings compatible with superinvolution of involution simple superalgebras which are not simple as superalgebras. Notice that these results depend on the classification of gradings by a finite abelian group on matrix algebras [2], involution gradings on matrix algebras [3], [5], involution gradings on involution simple algebras [1], and group gradings on simple superalgebras [4]. Finally, the superinvolution gradings on $M_{n,m}(F)$ have been described in [11].

We start the following general result.

Lemma 5. *Let R be a simple superalgebra as in Example 3, that is, $R=A\oplus A^{sop}$ where A is a simple superalgebra, and $*$ denote the ordinary exchange involution. If φ is an automorphism of R that commutes with $*$, then there exists a linear mapping $\varphi_0: A\to A$ such that one of the following cases holds:*

Type 1: $\varphi((x,y))=(\varphi_0(x),\varphi_0(y))$, and φ_0 is an automorphism of A.
Type 2: $\varphi((x,y))=(\varphi_0(y),\varphi_0(x))$, and φ_0 is an antiautomorphism of A.

Proof. Since $R=A\oplus A^{sop}$, we will represent an arbitrary element of a superalgebra R as a pair of elements from A, i.e. (x,y) where $x,y\in A$. We also recall that $A=A_{\bar 0}+A_{\bar 1}$. If φ is an automorphism of R that commutes with $*$, then the following two cases may occur:

1. $\varphi(A)=A$, $\varphi(A^{sop})=A^{sop}$. Then, there exist two linear mappings $\varphi_0,\varphi_1: A\to A$ such that $\varphi((x,y))=(\varphi_0(x),\varphi_1(y))$. Now φ commutes with the involution $*$. Hence

$$\begin{aligned}(\varphi_1(y),\varphi_0(x))&=(\varphi_0(x),\varphi_1(y))^*=(\varphi((x,y)))^*\\ &=\varphi((x,y))^*=\varphi((y,x))=(\varphi_0(y),\varphi_1(x)).\end{aligned}$$

Hence, $\varphi_0 = \varphi_1$. Thus φ is completely defined by $\varphi_0 : A \to A, \varphi((x,y)) = (\varphi_0(x), \varphi_0(y))$. Next, for any homogeneous $x, y \in A$,

$$\begin{aligned}\varphi((xy,0)) &= (\varphi((0,xy)))^* = ((-1)^{|x||y|}\varphi((0,y))\varphi((0,x)))^* \\ &= (-1)^{|x||y|}((0,\varphi_0(y)) \cdot (0,\varphi_0(x)))^* \\ &= (-1)^{|x||y|}(-1)^{|\varphi_0(x)||\varphi_0(y)|}(0,\varphi_0(x)\varphi_0(y))^* \\ &= (\varphi_0(x)\varphi_0(y),0).\end{aligned}$$

Hence φ_0 is indeed an automorphism of A, and we have a Type 1 automorphism.

2. $\varphi(A) = A^{sop}$, $\varphi(A^{sop}) = A$. Again there exist two linear mappings $\varphi_0, \varphi_1 : A \to A$ such that $\varphi((x,y)) = (\varphi_0(y), \varphi_1(x))$. Since φ commutes with the involution, we must have

$$(\varphi_1(x), \varphi_0(y)) = \varphi((x,y))^* = \varphi((y,x)) = (\varphi_0(x), \varphi_1(y)).$$

Again, as before $\varphi_0 = \varphi_1$. Now let x, y be homogeneous elements from A. Therefore, $(\varphi_0(xy), 0) = \varphi((0,xy)) = \varphi(((-1)^{|x||y|}(0,y) \cdot (0,x))) = (-1)^{|x||y|}\varphi((0,y)) \cdot \varphi((0,x)) = (-1)^{|x||y|}(\varphi_0(y),0) \cdot (\varphi_0(x),0)$. It follows that

$$\varphi_0(xy) = (-1)^{|x||y|}\varphi_0(y)\varphi_0(x),$$

and we have a Type 2 automorphism. □

By a *Type I* involution grading of a superalgebra $R_g = A \oplus A^{sop}$, as above, we understand a grading in which A is a graded subspace, that is, $A = \bigoplus_{g\in G}(R_g \cap A)$. In this case also

$$A^{sop} = A^* = \bigoplus_{g\in G}(R_g^* \cap A^*) = \bigoplus_{g\in G}(R_g \cap A^{sop}),$$

so that A^{sop} is also graded. Then there is a G-grading on A, hence on A^{sop}, as in Remark 1, such that $R_g = A_g \oplus A_g^{sop}$.

Theorem 4. *Let G be a finite abelian group and F an algebraically closed field of characteristic* 0 *or coprime to the order of G. Then any G-grading of $R = A \oplus A^{sop}$ where $A = M_{n,m}(F)$, with the ordinary exchange superinvolution $*$ compatible with G-grading has one of the following types*:

Type I: $R_g = A_g \oplus A_g^{sop}$, *for a G-grading of* $A = \oplus_{g\in G}A_g$,

Type II: $R_g = \{(x,x^\dagger) \mid x \in A_g\} \oplus \{(x,-x^\dagger) \mid x \in A_{gh}\}$, *for a $\dagger$-involution grading* $A = \oplus_{g\in G}A_g$ *where $\dagger$ is a graded superinvolution on A*, $h \in G, o(h) = 2$.

Type III: $R_g = \{(x,x^\dagger) \mid x \in A_g \cap A_+\} \oplus \{(x,-ix^\dagger) \mid x \in A_{gh} \cap A_-\} \oplus \{(x,-x^\dagger) \mid x \in A_{gh^2} \cap A_+\} \oplus \{(x,ix^\dagger) \mid x \in A_{gh^3} \cap A_-\}$ *where h is an element of order* 4 *in G, $\dagger$ is an antiautomorphism of order* 4 *on A, $A = \bigoplus_{g\in G} A_g$ is a $\dagger$-grading on A, A_+, A_- are symmetric and skew-symmetric elements of A with respect to* $\dagger^2$.

Proof. If $\widehat{G}$ acts on R by automorphisms of Type 1 only, we arrive at Type I gradings described just before the statement of this theorem. Now let $\widehat{G}$ act on R by automorphisms of both Type 1 and Type 2, and $\alpha : \widehat{G} \to \operatorname{Aut} R$ the homomorphism accompanying our grading. Let Λ stand for the set of all $\chi \in \widehat{G}$ that act on R by automorphisms of Type 1. As earlier, Λ is a subgroup of index 2 in $\widehat{G}$. Choose $\xi \in \widehat{G}$, such that $\alpha(\xi) = \varphi$ is an automorphism of Type 2, $\widehat{G} = \Lambda \cup \Lambda\xi$.

Next we assume that there exists an automorphism ψ of Type 1 such that $\psi^2 = \varphi^2$, and ψ commutes with $\alpha(\widehat{G})$. Then we can apply the Exchange Theorem. For this, we consider two gradings of R. The first is our original one defined by α. The second one is defined by a new homomorphism β such that $\beta|_\Lambda = \alpha|_\Lambda$, $\beta(\xi) = \psi$. It is easily seen that β is indeed a homomorphism. Now by the Exchange Theorem there exists a grading by a subgroup $H = \Lambda^\perp = \{e,h\}$, corresponding to the action of $\gamma = \alpha\beta^{-1}$. Moreover, $\gamma(\widehat{G}) = \{id, \varphi\psi^{-1}\}$. Denote $\omega = \varphi\psi^{-1}$. Then, $\omega((x,y)) = (\omega_0(y), \omega_0(x))$, $\omega_0^2 = id$ and $\omega_0(xy) = (-1)^{|x||y|}\omega_0(y)\omega_0(x)$. Therefore, ω_0 is an involution on $M_{n,m}(F)$ which we denote by $\omega_0(x) = x^\dagger$. Since the grading defined by β is a grading of the Type I, it follows from the first part of the proof of this theorem that $\overline{R}_g = A_g \oplus A_g^{sop}$ where $A = \oplus_{g\in G} A_g$ is a †-grading of R. By the Exchange Theorem there exists an element h of G of order 2 such that $R_g = \overline{R}_g \cap R^{(e)} + \overline{R}_{gh} \cap R^{(h)}$. Here,

$$\begin{aligned} R^{(e)} &= \{(x,y) | \omega((x,y)) = (x,y)\} \\ &= \{(x,y) | (\omega_0(y), \omega_0(x)) = (x,y)\} = \{(x, x^\dagger) | \, x \in A\}. \end{aligned}$$

Also

$$R^{(h)} = \{(x,y) | \omega((x,y)) = -(x,y)\} = \{(x, -x^\dagger) | x \in A\}.$$

This allows us to write $R_g = \{(x,x^\dagger) | \, x \in A_g\} \cup \{(x, -x^\dagger) | \, x \in A_{gh}\}$.

Now we consider the remaining case when there is no automorphism ψ of R of Type 1 such that $\psi^2 = \varphi^2$ and ψ commutes with $\alpha(\widehat{G})$. Let Λ_1 denote the set of all $\eta \in \widehat{G}$ for which there exists an automorphism τ of Type 1 such that $\alpha(\eta) = \tau^2$ and τ commutes with $\alpha(\widehat{G})$. Clearly, Λ_1 is a subgroup of Λ. Moreover, since $\eta^2 \in \Lambda_1$ for each $\eta \in \Lambda$, this subgroup has index 2 in Λ, and therefore, has index 4 in $\widehat{G}$. By our assumption $\xi^2 \notin \Lambda_1$. Hence $\widehat{G} = \Lambda_1 \cup \Lambda_1\xi \cup \Lambda_1\xi^2 \cup \Lambda_1\xi^3$. Next we can write $\varphi((x,y)) = (\varphi_0(y), \varphi_0(x))$ where φ_0 is an antiautomorphism of A that commutes with $\alpha(\Lambda_1)$. By Proposition 3, there exists an automorphism ψ_0 of A such that $\psi_0^4 = \varphi_0^4$, $\psi_0\varphi_0 = \varphi_0\psi_0$, and ψ_0 commutes with $\alpha(\Lambda_1)$. Set $\psi((x,y)) = (\psi_0(x), \psi_0(y))$. Obviously, $\psi^4((x,y)) = (\psi_0^4(x), \psi_0^4(y)) = (\varphi_0^4(x), \varphi_0^4(y)) = \varphi^4((x,y))$. Besides,

$$\begin{aligned} &\psi\varphi((x,y)) = \psi((\varphi_0(y), \varphi_0(x))) = (\psi_0\varphi_0(y), \psi_0\varphi_0(x)) \\ &= (\varphi_0\psi_0(y), \varphi_0\psi_0(x)) = \varphi((\psi_0(x), \psi_0(y))) = \varphi\psi((x,y)). \end{aligned}$$

This implies that $\psi\varphi = \varphi\psi$. Moreover, ψ commutes with $\alpha(\Lambda_1)$. Next we consider a new homomorphism $\beta : \widehat{G} \to \mathrm{Aut}\, R$ defined as follows: $\beta(\xi^k\eta) = \psi^k\alpha(\eta)$ for $k = 0, 1, 2, 3$. Let also $R = \bigoplus_{g\in G} \overline{R}_g$ be the G-grading defined by β. Since this is a Type I grading, $\overline{R}_g = A_g \oplus A_g^{sop}$ for some G-grading of $A = \oplus_g A_g$.

This allows us to apply the Exchange Theorem, in which $\Lambda^\perp = H = \{e, h, h^2, h^3\}$. The homomorphism $\gamma : \widehat{G} \to \mathrm{Aut}\,(R)$ defined by $\gamma(\chi) = \alpha^{-1}(\chi)\beta(\chi)$ defines a grading $R = R^{(e)} \oplus R^{(h)} \oplus R^{(h^2)} \oplus R^{(h^3)}$ and

$$R_g = \overline{R}_g \cap R^{(e)} \oplus \overline{R}_{gh} \cap R^{(h)} \oplus \overline{R}_{gh^2} \cap R^{(h^2)} \oplus \overline{R}_{gh^3} \cap R^{(h^3)}.$$

Let us consider $\theta = \gamma(\xi)$. Then the respective θ_0 is an antiautomorphism of A of order 4 which we denote by †. Notice that θ^2 is an automorphism of order 2. Let $A_+ = \{x \in A | \; \theta_0^2(x) = x\}$ and $A_- = \{x \in A | \; \theta_0^2(x) = -x\}$. Direct computations show that $\overline{R}_g \cap R^{(e)} = \{(x, x^\dagger) \,|\, x \in A_g \cap A_+\}$, $\overline{R}_{gh} \cap R^{(h)} = \{(x, -ix^\dagger) \,|\, x \in A_{gh} \cap A_-\}$, $\overline{R}_{gh^2} \cap R^{(h^2)} = \{(x, -x^\dagger) \,|\, x \in A_{gh^2} \cap A_+\}$ and $\overline{R}_{gh^3} \cap R^{(h^3)} = \{(x, ix^\dagger) \,|\, x \in A_{gh^3} \cap A_-\}$.

The proof is now complete. □

Example. Let us consider the $\mathbb{Z}_4 = \{\pm 1, \pm i\}$-grading of $R = A \oplus A^{sop}$, $A \cong M_{n,m}(F)$ induced by $\varphi * (X, Y) = (Y^\tau, X^\tau)$. Clearly, φ is of order 4. Direct computations show that

$$R_1 = \left\{ \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \begin{bmatrix} A^t & 0 \\ 0 & D^t \end{bmatrix} \right) \right\}, \quad R_{-1} = \left\{ \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \begin{bmatrix} -A^t & 0 \\ 0 & -D^t \end{bmatrix} \right) \right\},$$

$$R_i = \left\{ \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}, \begin{bmatrix} 0 & -iC^t \\ iB^t & 0 \end{bmatrix} \right) \right\}, \; R_{-i} = \left\{ \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}, \begin{bmatrix} 0 & iC^t \\ -iB^t & 0 \end{bmatrix} \right) \right\},$$

where A, B, C, D are any matrices of appropriate orders. This is in fact a grading of Type III for $A = A_e$ (a trivial grading) and $h = -i$.

Lemma 6. *Let $A = B + tB$ where $B \cong M_n(F)$ be an associative superalgebra of type $Q(n)$, and ψ an automorphism of A. Then there exists an automorphism ψ_0 of B such that for any $X + tY \in R$, either $\psi(X + tY) = \psi_0(X) + t\psi_0(Y)$ or $\psi(X + tY) = \psi_0(X) - t\psi_0(Y)$.*

Proof. Let us consider $A = B + tB$ with a $\mathbb{Z}_2$-grading $A_{\bar{0}} = B$ and $A_{\bar{1}} = tB$. Then both B and tB are invariant subspaces under the action of ψ. Namely, there exists two linear mappings $\psi_0, \psi_1 : B \to B$ such that for any $X + tY \in R$, $\psi(X + tY) = \psi_0(X) + t\psi_1(Y)$. If we use that ψ is an automorphism, we can easily derive the following relations:

$$\psi_0(X_1X_2) = \psi_0(X_1)\psi_0(X_2), \tag{10}$$
$$\psi_1(XY) = \psi_0(X)\psi_1(Y), \tag{11}$$
$$\psi_1(YX) = \psi_1(Y)\psi_0(X), \tag{12}$$
$$\psi_0(Y_1Y_2) = \psi_1(Y_1)\psi_1(Y_2), \tag{13}$$

where all $X_1, X_2, X, Y_1, Y_2, Y \in B$. It follows from (10) that ψ_0 is an automorphism of B. Now in (11) and (12) we set $Y = I$, the identity matrix, then we obtain $\psi_0(X)\psi_1(I) = \psi_1(I)\psi_0(X)$ for all $X \in B$. It follows then that $\psi_1(I)$ is a scalar matrix, $\psi_1(I) = \lambda I$, and $\psi_1 = \lambda\psi_0$. Now if we apply (13) we will obtain $I = \psi_0(I \cdot I) = \psi_1(I)\psi_1(I) = \lambda^2 I$. In this case $\lambda = \pm 1$. This argument allows us to conclude that for each automorphism ψ there is an automorphism ψ_0 of B such that either $\psi(X + tY) = \psi_0(X) + t\psi_0(Y)$ or $\psi(X + tY) = \psi_0(X) - t\psi_0(Y)$. The proof is complete. □

Lemma 7. *Let $A = B + tB$ where $B \cong M_n(F)$ be an associative superalgebra of type $Q(n)$, and ψ be an antiautomorphism of A. Then there exists an antiautomorphism ψ_0 of B such that for any $X + tY \in A$, either $\psi(X + tY) = \psi_0(X) + it\psi_0(Y)$ or $\psi(X + tY) = \psi_0(X) - it\psi_0(Y)$ where $i^2 = -1$.*

Proof. The proof of this lemma is similar to the previous one except that in the case where ψ is a (super!)antiautomorphism the Eqs. (10)–(13) are replaced by

$$\psi_0(X_1X_2) = \psi_0(X_2)\psi_0(X_1), \tag{14}$$
$$\psi_1(XY) = \psi_1(Y)\psi_0(X), \tag{15}$$
$$\psi_1(YX) = \psi_0(X)\psi_1(Y), \tag{16}$$
$$\psi_0(Y_1Y_2) = -\psi_1(Y_2)\psi_1(Y_1), \tag{17}$$

where all $X_1, X_2, X, Y_1, Y_2, Y \in B$. Now (14) implies ψ_0 being an antiautomorphism. Also (15) and (16) imply $\psi_1(I) = \lambda I$ and $\psi_1 = \lambda\psi_0$. Using (17), we now derive that $\lambda = \pm i$. □

Now we are ready to prove the second main result of this paper.

Theorem 5. *Let G be a finite abelian group and $R = A \oplus A^{sop}$ an involution simple superalgebra with A of type $Q(n)$, as in item(3) of Proposition 1. Suppose the base field F is algebraically closed of characteristic 0 or coprime to the order of G. Then any G-grading of $A = B + tB \cong Q(n)$ with the ordinary exchange involution $*$ compatible with G-grading has one of the following forms:*

Type I: $R_g = A_g \oplus A_g^{sop}$, for a grading of $A = \oplus_{g\in G} A_g$,

Type II: $R_g = \{(x, x^\dagger) \mid x \in B_g\} \oplus \{(tx, -tx^\dagger) \mid x \in B_{gh}\} \oplus \{(x, -x^\dagger) \mid x \in B_{gh^2}\} \oplus \{(tx, tx^\dagger) \mid x \in B_{gh^3}\}$, where h is an element of order 4 in G, $\dagger$ is an involution on $B \cong M_n(F)$, $B = \bigoplus_{g\in G} B_g$ is an involution grading on B with respect to involution $\dagger$.

Proof. If $\widehat{G}$ acts on R by automorphisms of Type 1 only, we arrive at Type I gradings described just before the statement of Theorem 4. Otherwise, let $\alpha(\widehat{G})$ contain all possible automorphisms, where, as before, $\alpha : \widehat{G} \to \mathrm{Aut}\,(R)$ is the homomorphism corresponding to our grading. Let $\varphi \in \alpha(\widehat{G})$ be such that $\varphi((x, y)) = (\varphi_0(y), \varphi_0(x))$ where φ_0 is an antiautomorphism of A, $x, y \in A$. Then, according to

Lemma 7, φ_0 has one of two forms $\varphi_0(u+tv)=\varphi_1(u)\pm it\varphi_1(v)$, where φ_1 is an antiautomorphism of $B=M_n(F)$, and $x=u+tv$ with $u,v\in B$. If we compute the powers of φ on $(x,y)=(u+tv,p+tq)$ where also $p,q\in B$ then we obtain the following:

$$\varphi^2((u+tv,p+tq))=(\varphi_1^2(u)-t\varphi_1^2(v),\varphi_1^2(p)-t\varphi_1^2(q)), \tag{18}$$

$$\varphi^3((u+tv,p+tq))=(\varphi_1^3(p)\mp t\varphi_1^3(q),\varphi_1^3(u)\mp t\varphi_1^3(v)), \tag{19}$$

$$\varphi^4((u+tv,p+tq))=(\varphi_1^4(u)+t\varphi_1^4(v),\varphi_1^4(p)+t\varphi_1^4(q)). \tag{20}$$

Clearly, if we replace φ by φ^3 we may assume from the very beginning that $\varphi_0(u+tv)=\varphi_1(u)+it\varphi_1(v)$, for an antiautomorphism φ_1 of B. Let $\zeta\in\widehat{G}$ be such that $\alpha(\zeta)=\varphi$ and let

$$\Lambda=\{\chi\in\widehat{G}\mid\alpha(\chi)((u+tv,p+tq))=(\pi_1(u)+t\pi_1(v),\pi_1(p)+t\pi_1(q))\}$$

for any $u,v,p,q\in B$, $\pi_1\in\mathrm{Aut}\,(B)$. Then $\widehat{G}=\Lambda\cup\Lambda\zeta\cup\Lambda\zeta^2\cup\Lambda\zeta^3$. Indeed, choose any $\chi\in\widehat{G}$ and consider $\pi=\alpha(\chi)$. Then $\pi((x,y))$ is described by Lemmas 5 and then 6 or 7. Direct calculations using Eqs. (18)–(20) show that if π is one of the cases of Lemma 6 then either $\chi\in\Lambda$ or $\varphi^2\chi\in\Lambda$. If π is one of the cases of Lemma 7 then either $\varphi\chi\in\Lambda$ or $\varphi^3\chi\in\Lambda$.

Let us define a mapping $\alpha_1:\widehat{G}\to\overline{\mathrm{Aut}}\,(B)$ by associating with each $\chi\in\widehat{G}$ the mapping π_1 as in the previous paragraph. Obviously, this is a homomorphism of groups and the image φ_1 of ζ is an antiautomorphism. In this case Theorem 2 applies and there exists an automorphism ψ_1 of B such that $\psi_1^2=\varphi_1^2$ and ψ_1 commutes with every $\pi_1\in\alpha_1(\widehat{G})$. Let use our previous notation to define an automorphism ψ of R by setting $\psi((x,y))=(\psi_0(x),\psi_0(y))$ where $\psi_0(u+tv)=\psi_1(u)+t\psi_1(v)$. Immediate calculations using different cases of Lemmas 6 or 7 show that ψ commutes with any element of $\alpha(\widehat{G})$. For example, if $\pi\in\alpha(\widehat{G})$ has the form $\pi((u+tv,p+tq))=(\pi_1(p)-it\pi_1(q),\pi_1(u)-it\pi_1(v))$ then using that $\psi_1\pi_1=\pi_1\psi_1$, we easily find both $\psi\pi$ and $\pi\psi$ acting on $(u+tv,p+tq)$ produce the same $(\psi_1\pi_1(p)-it\psi_1\pi_1(q),\psi_1\pi_1(u)-it\psi_1\pi_1(v))$.

In order to apply Exchange Theorem, we define another mapping $\beta:\widehat{G}\to\mathrm{Aut}\,(R)$ by setting $\beta(\zeta^k\lambda)=\psi^k\alpha(\lambda)$ for $k=0,1,2,3$. By Eq. (19), $\varphi^4=\psi^4$ and so this mapping is well defined and is a homomorphism of groups coinciding with α on Λ. Let also $R=\bigoplus_{g\in G}\overline{R}_g$ be a G-grading defined by β. This allows to apply Exchange Theorem, in which $\Lambda^\perp=H=\{e,h,h^2,h^3\}$. The homomorphism $\gamma:\widehat{G}\to\mathrm{Aut}\,(R)$ defined by $\gamma(\chi)=\alpha^{-1}(\chi)\beta(\chi)$ defines a grading $R=R^{(e)}\oplus R^{(h)}\oplus R^{(h^2)}\oplus R^{(h^3)}$ and

$$R_g=\overline{R}_g\cap R^{(e)}\oplus\overline{R}_{gh}\cap R^{(h)}\oplus\overline{R}_{gh^2}\cap R^{(h^2)}\oplus\overline{R}_{gh^3}\cap R^{(h^3)}. \tag{21}$$

Let us consider $\theta=\gamma(\zeta)$. Then the respective $\theta_1\in\overline{\mathrm{Aut}}\,(B)$ is an involution, which we denote by †. The grading of R defined by β induces a grading $B=\bigoplus_{g\in G}B_g$ on B, which permutes with †, hence is an involution grading on the matrix algebra $B=M_n(F)$. We have $\overline{R}_g=\{(u+tv,p+tq)\mid u,v,p,q\in B_g\}$. To finally compute the homogeneous components of our original grading by Eq. (21), we need to compute the components of the H-grading $R^{(t)}$, $t\in H$. We have $(x,y)\in R^{(e)}$ if $\theta((x,y))=(x,y)$. Using the same notation for $x,y\in A$, as before, we get

$$\theta((u+tv,p+tq))=(\theta_1(p)-it\theta_1(q),\theta_1(u)-it\theta_1(v))=(p^\dagger-itq^\dagger,u^\dagger-itv^\dagger).$$

If $(x,y)\in R^{(e)}$ we must have $p^\dagger-itq^\dagger=u+tv$, $u^\dagger-itv^\dagger=p+tq$ and so $p^\dagger=u$, $-iq^\dagger=v$, $u^\dagger=p$, and $-iv^\dagger=q$. It follows that $p=u^\dagger$, $v=q=0$. Finally, $R^{(e)}=\{(u,u^\dagger)\mid u\in B\}$. Now since $\overline{R}_g=\{(u+tv,p+tq)\mid u,v,p,q\in B_g\}$, we finally obtain $\overline{R}_g\cap R^{(e)}=\{(u,u^\dagger)\mid u\in B\}$.

A similar computation gives us also $\overline{R}_{gh}\cap R^{(h)}=\{(tv,-tv^\dagger)\mid v\in B\}$, $\overline{R}_{gh^2}\cap R^{(h^2)}=\{(u,-u^\dagger)\mid u\in B\}$, and $\overline{R}_{gh^3}\cap R^{(h^3)}=\{(tv,tv^\dagger)\mid v\in B\}$.

Now the proof of our theorem is complete. □

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