Integral mean values of modular $L$-functions

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Abstract

We consider the mean squares of $L$-functions associated to modular forms with respect to Hecke congruence subgroups, expressing the mean value as an inner product. This avoids the discussion of generalized additive divisor problems. As applications, we obtain asymptotic formulas for both weighted and unweighted mean squares.

Keywords: Mean values; $L$-function; Modular form; Hecke congruence subgroup

1. Introduction

Let $f$ be an automorphic form over $GL(n)$ and $L(s, f)$ its associated $L$-function, normalized so that the central point is at $s = 1/2$. An important problem in analytic number theory is to estimate integral mean values

$$
\int_0^T \left| L \left( \frac{1}{2} + it, f \right) \right|^{2m} \, dt.
$$

(1.1)

For the Riemann zeta function, a first culmination was reached by Hardy and Littlewood in 1918 [5] who considered the mean square case by using the Mellin transform, and again in 1923 [6] by introducing the method of approximate functional equation. This method dominated the next 60 years of the mean value theory. In 1926, Ingham [9] obtained the asymptotic formula for the fourth moment, and in 1979 Heath-Brown [7]...
substantially improved the error term estimates. The first successful application of the Kuznetsov trace formula to the mean value theory of $\zeta(s)$ was due to Iwaniec [10] in 1979 for the fourth moment over short intervals. Recent progress on the fourth moment of $\zeta(s)$ is largely due to a fundamental explicit formula of Motohashi [14,15] via the spectral decomposition with respect to $\text{SL}(2, \mathbb{Z})$. In a series of papers authored by Motohashi, Ivić, or both, various properties of the error term for the fourth moment have been extensively studied.

The study of mean squares for $L$-functions over $\text{GL}(2)$ was mainly pioneered by Good in the middle 1970s who studied asymptotic formulas for the mean squares of $L$-functions associated to cusp forms with respect to $\text{SL}(2, \mathbb{Z})$. Motohashi [16] also considered this case and established an explicit formula for cusp forms with respect to $\text{SL}(2, \mathbb{Z})$ analogous to that for $\zeta(s)^4$, thus giving a complete spectral expansion of the mean squares. On the other hand, Kuznetsov [13] studied the mean squares for Maass forms with respect to $\text{SL}(2, \mathbb{Z})$. In 1997, Jutila [12] used a Laplace transform method to study in a unified way the fourth moment of $\zeta(s)$ and the mean squares of $L$-functions associated to both cusp forms and Maass forms with respect to $\text{SL}(2, \mathbb{Z})$.

The present paper is devoted to the study of mean squares of $L$-functions through a different approach. With complex Tauberian theorems in mind, we consider the Dirichlet integral

$$Z_f(w) = \int_1^\infty \left| L \left( \frac{1}{2} + it, f \right) \right|^2 t^{-w} \, dt \quad (\Re w \gg 1) \quad (1.2)$$

and study its analytic properties, especially its meromorphic continuation beyond $\Re w = 1$ and its polar behavior. Our goal is to express $Z_f(w)$ as an inner product of $f$ with a certain kernel function (cf. (4.10) and (5.4)), by directly exploring the symmetries satisfied by $f$ itself, so that it suffices to study the analytic properties of this kernel function alone. This realizes and generalizes the ideas suggested by Good [4]. One may observe some similarities in this process with the above-mentioned methods, but a distinctive difference here is that we have avoided delicate discussions of the arithmetic nature of those Fourier coefficients, such as estimates of the “generalized additive divisor problems” $\sum_{n \leq x} \lambda_f(n)\lambda_f(n + r)$. Instead, we reduce the problem to more routine analytic manipulations. Hence this approach is applicable to cases where our knowledge of the arithmetic nature of $f$ is still limited. This is the main motivation of the present work.

To carry out this approach, in this paper we consider modular forms with respect to general Hecke congruence subgroups, and study the mean square of their $L$-functions. The main results are the following theorems.

**Theorem 1.** Let $f \in S_k(N, \chi)$ be a cusp form for $\Gamma_0(N)$ of weight $k$ and character $\chi \pmod{N}$. Then asymptotically we have

$$\int_0^T \left| L \left( \frac{1}{2} + it, f \right) \right|^2 \, dt \sim \frac{2(4\pi)^k}{\Gamma(k)} \frac{\| f \|^2}{\text{vol}(\Gamma_0(N) \backslash \mathbb{H})} T \log T, \quad (1.3)$$
where the norm on $S_k(N, \chi)$ is induced from the inner product

$$\langle f_1, f_2 \rangle = \int \int_{\Gamma_0(N) \backslash \mathfrak{H}} f_1(z) f_2(z)y^k \frac{dx \, dy}{y^2}. \quad (1.4)$$

More precisely, write

$$G_T(t) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(w + k - 1)}{2^{w-1}(4\pi)^k} \frac{|\Gamma(\frac{k}{2} + it)|^2 T^w}{\Gamma(\frac{w+k}{2} + it)\Gamma(\frac{w+k}{2} - it)} \frac{dw}{w^3}, \quad (1.5)$$

where $\int_{(c)}$ denotes the integral $\int_{c-i\infty}^{c+i\infty}$, then we can derive a much more accurate asymptotic formula for a weighted mean value problem.

**Theorem 2.** Let $f \in S_k(N, \chi)$. Then we have

$$\int_{-\infty}^{\infty} G_T(t) \left| L \left( \frac{1}{2} + it, f \right) \right|^2 dt = \frac{4\|f\|^2}{\text{vol}(\Gamma_0(N) \backslash \mathfrak{H})} T \log T + c_0 T + \sum_{j=1}^{m} c_j T^{x_j} + O \left( T^{\frac{1}{2} + \varepsilon} \right), \quad (1.6)$$

where $\frac{1}{2} < x_1 < \cdots < x_m < 1$ such that $x_1(1-x_1), \ldots, x_m(1-x_m)$ are the exceptional eigenvalues for $\Gamma_0(N)$, and $c_0, c_1, \ldots, c_m$ are computable constants.

**Remark 3.** The $O$-constant in (1.6) depends polynomially on both $\|f\|$ and $N$, but for simplicity we do not describe this dependence explicitly.

The same argument can be also applied to Maass forms with respect to Hecke congruence subgroups, but with additional difficulties. The main obstacle is that in this case it is not obvious how to relate the Dirichlet integral (1.2) to an inner product representation, like in (5.4) for cusp forms. Similar difficulties appear when recently Beineke and Bump [2] took yet another approach to study the mean squares of $L$-functions for Maass forms but to get only

$$\int_{0}^{T} \left| L \left( \frac{1}{2} + it, f \right) \right|^2 dt \ll T \log T, \quad (1.7)$$

where $f$ is an even Maass form with respect to $\text{SL}(2, \mathbb{Z})$. We wish to return to this topic in a separate paper.
2. $L$-functions associated to Eisenstein series

Let $N \geq 1$ and write $\Gamma = \Gamma_0(N)$. Every cusp $\alpha$ of $\Gamma$ is associated with an Eisenstein series

$$E_\alpha(z, s) = \sum_{\gamma \in \Gamma \setminus \Gamma} \left( \zeta(\sigma_\alpha^{-1} \gamma z) \right)^s = \delta_{\alpha \infty} y^s + \phi_{\alpha \infty}(s) y^{1-s} + \sum_{n=\infty}^{\infty} \frac{\phi_{\alpha \infty}(s, n)}{\sqrt{|n|}} W_s(nz),$$

where $\sigma_\alpha \in \text{SL}(2, \mathbb{R})$ is the scaling matrix for $\alpha$,

$$W_s(z) = \sqrt{|y|} K_{s-\frac{1}{2}}(2\pi |y|) e^{2\pi i x}$$

is the Whittaker function for $\text{GL}(2)$, and $K_s$ is the modified Bessel function of the second kind. In fact, write $\alpha = \frac{u}{v}$ with $v|N$ and $(u, v) = 1$, and let $\chi_0^0$ be the principal character modulo $N$, then we have [3,11]

$$\phi_{\alpha \infty}(s) = \frac{\phi(v)}{\phi\left((v, \frac{N}{v})\right)} \frac{(v, \frac{N}{v})^s}{v^s N^s} \frac{L(2s - 1, \chi_0^0/v)}{L(2s, \chi_0^0)},$$

$$\phi_{ab}(s, n) = \frac{2\pi^s |n|^{s-\frac{1}{2}}}{\Gamma(s)} \sum_{c>0, (c, \sigma_\alpha^{-1} \sigma_b)} \frac{S_{ab}(0, n; c)}{c^{2s}}.$$

For our future discussions, we will study the Dirichlet series formed by the Fourier coefficients of $E_\alpha(z, s)$, i.e.

$$L_\alpha(v, s) = \frac{1}{2} \sum_{n=\infty}^{\infty} \frac{\phi_{\alpha \infty}(s, n)}{|n|^v},$$

called the $L$-function associated to $E_\alpha(z, s)$. We also write

$$A_\alpha(v, s) = \left( \frac{\sqrt{N}}{\pi} \right)^v \Gamma\left( \frac{v + s - \frac{1}{2}}{2} \right) \Gamma\left( \frac{v - s + \frac{1}{2}}{2} \right) L_\alpha(v, s)$$

and call it the complete $L$-function associated to $E_\alpha(z, s)$.

**Proposition 4.** The complete $L$-function $A_\alpha(v, s)$ has a meromorphic continuation, in both $v$ and $s$, to the complex plane, with simple poles along $v - s = \pm \frac{1}{2}$, $s + v = \frac{3}{2}$, $s + v = \frac{1}{2}$ and $s = \frac{p}{2}$, where $p$ runs through the nontrivial zeros of $L(s, \chi_0^0)$. The
residues of $A_a(v, s)$, when regarded as a function in $s$, are given as follows, where $w_N = \begin{pmatrix} N & -1 \\ 1 & 0 \end{pmatrix}$ and $b = w_N a$. 

\[
Res_{s=v-\frac{1}{2}} A_a(v, s) = -2\delta_{b\infty} N^{\frac{1-v}{2}} , \tag{2.6}
\]

\[
Res_{s=v+\frac{1}{2}} A_a(v, s) = 2\phi_{a\infty} \left( \frac{v}{2} + \frac{1}{2} \right) N^{\frac{v}{2}} , \tag{2.7}
\]

\[
Res_{s=\frac{1}{2}-v} A_a(v, s) = -2\delta_{a\infty} N^{\frac{v}{2}} , \tag{2.8}
\]

\[
Res_{s=\frac{1}{2}-v} A_a(v, s) = 2\phi_{b\infty} \left( \frac{3}{2} - v \right) N^{\frac{1-v}{2}} . \tag{2.9}
\]

**Proof.** The function 

\[
\tilde{E}_a(z, s) = E_a(z, s) - \delta_{a\infty} y^s - \phi_{a\infty}(s) y^{1-s} = \sum_{n=-\infty}^{\infty} \frac{\phi_{a\infty}(s, n)}{\sqrt{|n|}} W_s(nz)
\]

decays exponentially as $y \to \infty$, and for $\Re v \gg 1$ we have 

\[
A_a(v, s) = 2N^{\frac{v}{2}} \int_0^{\infty} \tilde{E}_a(iy, s) y^{\frac{v}{2} - \frac{3}{2}} dy .
\]

Now write $b = w_N a$, then $E_a(w_N z, s) = E_b(z, s)$, so in particular 

\[
E_b(iy, s) = E_a(w_N iy, s) = E_a \left( \frac{i}{N y} , s \right) ,
\]

\[
\int_0^{\frac{1}{\sqrt{N}}} \tilde{E}_a(iy, s) y^{v - \frac{1}{2}} \frac{dy}{y} = \int_0^{\frac{1}{\sqrt{N}}} E_a(iy, s) y^{v - \frac{1}{2}} \frac{dy}{y} - \frac{\delta_{a\infty} N^{-\frac{v+s-1}{2}}}{v+s-\frac{1}{2}} - \frac{\phi_{a\infty}(s) N^{-\frac{v+s-1}{2}}}{v-s+\frac{1}{2}}
\]

\[
= \int_0^{\frac{1}{\sqrt{N}}} E_b(iy, s) (Ny)^{v-\frac{1}{2}} dy \frac{dy}{y} - \frac{\delta_{a\infty} N^{-\frac{v+s-1}{2}}}{v+s-\frac{1}{2}} - \frac{\phi_{a\infty}(s) N^{-\frac{v+s-1}{2}}}{v-s+\frac{1}{2}}
\]
\[ = \int_{1/\sqrt{N}}^{\infty} \tilde{E}_b(iy, s) (Ny)^{\frac{1}{2} - v} \frac{dy}{y} - \frac{\delta_{a\infty}(s) N^{\frac{v-s}{2}}}{v-s+\frac{1}{2}} - \frac{\delta_{b\infty}(s) N^{\frac{v-s}{2}}}{\frac{1}{2} - s} - \frac{\phi_{a\infty}(s) N^{\frac{v-s}{2}}}{\frac{3}{2} - v-s}. \]

Hence

\[ A_a(v, s) = 2N^\frac{1}{2} \int_{1/\sqrt{N}}^{\infty} \tilde{E}_a(iy, s)y^{\frac{1}{2} - \frac{1}{2}} \frac{dy}{y} + \frac{2N^{\frac{1}{2}}}{\sqrt{N}} \int_{1/\sqrt{N}}^{\infty} \tilde{E}_b(iy, s)y^{\frac{1}{2} - v} \frac{dy}{y} \]

\[ - \frac{2\delta_{a\infty}(s) N^{\frac{1}{2} - \frac{1}{2}}}{v-s+\frac{1}{2}} - \frac{2\phi_{a\infty}(s) N^{\frac{1}{2} - \frac{1}{2}}}{\frac{1}{2} - s} - \frac{2\delta_{b\infty}(s) N^{\frac{1}{2} - \frac{1}{2}}}{\frac{3}{2} - v-s} - \frac{2\phi_{b\infty}(s) N^{\frac{1}{2} - \frac{1}{2}}}{\frac{3}{2} - v-s}. \]

(2.10)

This gives the meromorphic continuation of \( A_a(v, s) \), with the prescribed poles and residues. Note that the pole of \( A_a(v, s) \) at \( s = \frac{\rho}{2} \) comes from the corresponding pole for \( \phi_{a\infty}(s) \) and \( \phi_{b\infty}(s) \), as can be observed from (2.2). \( \square \)

**Proposition 5.** Write \( s = \sigma + it \). Assume that \( \sigma_1 \leq \sigma \leq \sigma_2 \). Then as \( |t| \to \infty \) we have

\[ \zeta(2s) A_a \left( \frac{1}{2}, s \right) \ll |t|^c \]  

(2.11)

for some constant \( c > 0 \).

**Proof.** By definition (2.10), we have

\[ A_a \left( \frac{1}{2}, s \right) = 2N^\frac{1}{2} \int_{1/\sqrt{N}}^{\infty} \left( \tilde{E}_a(iy, s) + \tilde{E}_b(iy, s) \right) \frac{dy}{y} - \frac{2N^{\frac{1}{2}}}{\sqrt{N}} \left( \frac{\delta_{a\infty} + \delta_{b\infty}}{s} \right) \]

\[ - \frac{2N^{\frac{1}{2}}}{1-s} \left( \phi_{a\infty}(s) + \phi_{b\infty}(s) \right) \]

\[ = 2N^\frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \phi_{a\infty}(s, n) + \phi_{b\infty}(s, n) \right) \int_{1/\sqrt{N}}^{\infty} K_{s-\frac{1}{2}}(2\pi|n|y) \frac{dy}{\sqrt{y}} \]

\[ + O \left( \frac{|\phi_{a\infty}(s)| + |\phi_{b\infty}(s)| + 1}{|t|} \right). \]
Recall that \[1\]

\[K_{s-\frac{1}{2}}(z) = \left(\frac{2}{z}\right)^{s-\frac{1}{2}} \Gamma(s) \int_0^\infty \frac{\cos zx}{(x^2 + 1)^s} \, dx.\]

If \(\sigma > \frac{3}{4}\), then we have

\[
\int_{\sqrt{N}}^\infty K_{s-\frac{1}{2}}(2\pi n|y) \frac{dy}{\sqrt{y}} = \frac{\Gamma(s)}{\pi^s |n|^s} \int_{\sqrt{N}}^\infty \left(\int_0^\infty \frac{\cos(2\pi n|y|x)}{(x^2 + 1)^s} \, dx\right) \frac{dy}{y^\sigma} = \frac{s \Gamma(s)}{2\pi^s |n|^s} \int_{\sqrt{N}}^\infty \left(\int_0^\infty \frac{(1 - (2s + 1)x^2) \cos(2\pi n|y|x)}{(x^2 + 1)^{s+2}} \, dx\right)
\]

by applying partial integration twice to the inner integral. By (2.3), Weil's estimate for Kloosterman sums implies \(\zeta(s) \phi_{a\infty}(s,n) \ll |n|^{-\frac{1}{2}}\) for \(\sigma > \frac{3}{4}\). Hence we have

\[
\zeta(2s) A_a \left(\frac{1}{2}, s\right) \ll |t|^2 \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{|\zeta(2s)\Gamma(s)|}{|n|^\sigma} \left(\frac{|\phi_{a\infty}(s,n)| + |\phi_{b\infty}(s,n)|}{|n|^{-\frac{1}{2}}\sigma} + 1\right) \ll |t|^2.
\]

On the other hand, if \(\sigma < \frac{1}{4}\), as \(K_{s-\frac{1}{2}}(z) = K_{\frac{1}{2}-s}(z)\), then similarly we have

\[
\int_{\sqrt{N}}^\infty K_{\frac{1}{2}-s}(2\pi n|y) \frac{dy}{\sqrt{y}} \ll \frac{|t|^2 |\Gamma(1-s)|}{|n|^{\frac{3}{2}-\sigma}}, \quad \zeta(2s) A_a \left(\frac{1}{2}, s\right) \ll |t|^{3-2\sigma}.
\]

Since \(\zeta(2s) A_a \left(\frac{1}{2}, s\right)\) has only finitely many poles for \(\frac{1}{4} < \sigma < \frac{3}{4}\), we may apply the Phragmen–Lindelöf principle to complete the proof. \(\square\)

**Proposition 6** *(Functional equation).* We have

\[A_a(v, s) = \sum_b \phi_{ab}(s) A_b(v, 1 - s).\]
Proof. By the functional equation of the Eisenstein series, we have

\[ \phi_{a\infty}(s, n) = \sum_{b} \phi_{ab}(s) \phi_{b\infty}(1 - s, n), \]

then (2.12) follows directly by definition. □

Corollary 7. We have

\[ \sum_{a} A_{a}(v, 1 - s) E_{a}(z, s) = \sum_{a} A_{a}(v, s) E_{a}(z, 1 - s). \] (2.13)

Proof. The functional equations of Eisenstein series and (2.12) imply that

\[ \sum_{a} A_{a}(v, 1 - s) E_{a}(z, s) = \sum_{a} A_{a}(v, 1 - s) \sum_{b} \phi_{ab}(s) E_{b}(z, 1 - s) \]
\[ = \sum_{b} E_{b}(z, 1 - s) \sum_{a} \phi_{ab}(s) A_{a}(v, 1 - s) \]
\[ = \sum_{b} A_{b}(v, s) E_{b}(z, 1 - s). \]

Here we have implicitly used the fact that \( \phi_{ab}(s) = \phi_{ba}(s) \). □

3. Nonholomorphic Poincaré series

Let \( n \in \mathbb{Z} \) be a nonzero integer, \( N \geq 1 \), and write \( \Gamma = \Gamma_0(N) \). Then

\[ P_{n; w, \tau}(z) = \frac{1}{\sqrt{|n|}} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \Im(\gamma z)^{\tau} W_{w + \frac{1}{2}}(n \cdot \gamma z) \] (3.1)

is called the nonholomorphic Poincaré series of level \( N \) and weight \( n \), where \( W_{\tau}(z) \) is the Whittaker function as defined in (2.1). By the estimate [1]

\[ K_{\tau}(y) \ll \begin{cases} y^{-|\Re \tau|}, & y \to 0, \\ y^{-\frac{1}{2}} e^{-y}, & y \to \infty, \end{cases} \] (3.2)

we see that \( P_{n; w, \tau}(z) \) is absolutely convergent for \( \Re \tau > |\Re w| + \frac{1}{2} \) and so defines an analytic function over \( \mathcal{H} \) in this region.
Proposition 8 (Petersson formula). Let
\[ u(z) = a_+ y^{1+i\nu} + a_- y^{1-i\nu} + \sum_{n=\infty}^{\infty} \frac{a(n)}{\sqrt{|n|}} W_{\frac{1}{2}+i\nu}(nz) \]
be an automorphic form for \( \Gamma \). Assume that \( \Re \tau > |\Re w| + \frac{1}{2} \), then
\[ \langle P_{n;\nu}, u \rangle = \frac{a(n)}{8|n|^\tau \pi^\tau} \frac{\Gamma\left(\frac{\tau+w+i\nu}{2}\right)\Gamma\left(\frac{\tau+w-i\nu}{2}\right)\Gamma\left(\frac{\tau-w+i\nu}{2}\right)\Gamma\left(\frac{\tau-w-i\nu}{2}\right)}{\Gamma(\tau)}. \] (3.2)

Proof. This follows easily using the unfolding technique and the well-known formula [1]
\[ \int_0^\infty K_\mu(y)K_\nu(y)y^s \frac{dy}{y} = \frac{\Gamma\left(\frac{s+\mu+\nu}{2}\right)\Gamma\left(\frac{s+\mu-\nu}{2}\right)\Gamma\left(\frac{s-\mu+\nu}{2}\right)\Gamma\left(\frac{s-\mu-\nu}{2}\right)}{2^{3-s}\Gamma(s)}, \]
where \( \Re s > |\Re \mu| + |\Re \nu| \). □

Proposition 9. Let \( \Re \tau > |\Re w| + \frac{1}{2} \). Then \( P_{n;\nu} \in L^2(\Gamma \backslash \mathfrak{H}) \).

Proof. Let
\[ \mathcal{D} = \left\{ z \in \mathfrak{H} \mid -\frac{1}{2} < \Re z < \frac{1}{2}, |cz+d| > 1 \text{ for every } (c d)^* \in \Gamma - \Gamma_\infty \right\} \]
be the standard polygon for \( \Gamma \) and so a fundamental domain, then
\[ \|P_{n;\nu}(z)\|_{L^2(\Gamma \backslash \mathfrak{H})}^2 = \int_\mathcal{D} \int |P_{n;\nu}(z)|^2 \frac{dx \, dy}{y^2}. \]
Let \( z \in \mathcal{D} \), then for every \( \left(\begin{smallmatrix} c & * \\ d & \end{smallmatrix}\right) \in \Gamma \), since
\[ |cz+d| = \sqrt{(cx+d)^2 + c^2 y^2} \geq 1, \]
the condition \( |cx+d| \leq \frac{\sqrt{3}}{2} \) would imply that \(|y| \geq \frac{1}{2}, \text{ i.e. } |c| \geq \max\{1, \frac{1}{2x}\}\). Furthermore, for every such a \( c \) there are at most two choices for \( d \), so by (3.2) we have
\[ P_{n;\nu}(z) \ll 1 + \sum_{c=1}^{\infty} \sum_{d=-\infty}^{\infty} \left( \frac{y}{(cx+d)^2 + c^2 y^2} \right)^{\Re \tau - |\Re w| + \frac{1}{2}}. \]
\[
\ll 1 + \sum_{c=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{y}{m^2 + c^2 y^2} \right)^{\Re \tau - |\Re w| + \frac{1}{2}} + \sum_{c \geq \max\{1, \frac{1}{c^2 y}\}} \left( \frac{y}{c^2 y^2} \right)^{\Re \tau - |\Re w| + \frac{1}{2}} \\
\ll 1 + \sum_{c=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{1}{m c} \right)^{\Re \tau - |\Re w| + \frac{1}{2}} + \sum_{c \geq \max\{1, \frac{1}{c^2 y}\}} \left( \frac{1}{c^2 y} \right)^{\Re \tau - |\Re w| + \frac{1}{2}}.
\]

As we have assumed that \(\Re \tau > |\Re w| + \frac{1}{2}\), the summations over \(c\) and \(m\) are both convergent. On the other hand, if \(y \geq \frac{1}{2}\), then

\[
\sum_{c \geq \max\{1, \frac{1}{c^2 y}\}} \left( \frac{1}{c^2 y} \right)^{\Re \tau - |\Re w| + \frac{1}{2}} \ll \frac{1}{2\Re \tau - |\Re w| + 1} \ll 1
\]

and if \(y \leq \frac{1}{2}\), then

\[
\sum_{c \geq \max\{1, \frac{1}{c^2 y}\}} \left( \frac{1}{c^2 y} \right)^{\Re \tau - |\Re w| + \frac{1}{2}} \ll \frac{1}{2\Re \tau - |\Re w| + 1} \ll 1.
\]

Hence \(P_{n; w, \tau}(z)\) is bounded over \(D\), and

\[
\iint_{D} |P_{n; w, \tau}(z)|^2 \frac{dx \, dy}{y^2} \ll \iint_{D} \frac{dx \, dy}{y^2} \ll 1.
\]

Therefore \(P_{n; w, \tau}\) is square-integrable over \(\Gamma \setminus \mathcal{S}\). □

**Proposition 10.** The function \(P_{n; w, \tau}(z)\) has a meromorphic continuation to the whole complex space \((w, \tau) \in \mathbb{C}^2\). Hence the Petersson formula (3.3) holds valid except at the poles of the right-hand side.

**Proof.** To begin with, we assume that \(\Re \tau > |\Re w| + \frac{1}{2}\), in which case \(P_{n; w, \tau} \in L^2(\Gamma \setminus \mathcal{S})\) and so has the spectral decomposition

\[
P_{n; w, \tau}(z) = \sum_{a} \frac{1}{4\pi i} \int_{(\frac{1}{2})} (P_{n; w, \tau}, E_a(\cdot, s)) E_a(z, s) \, ds + \sum_{j=1}^{\infty} (P_{n; w, \tau}, u_j) u_j(z),
\]
where $\alpha$ runs through inequivalent cusps for $\Gamma$, and $\{u_1, u_2, \ldots\}$ is an orthonormal basis for $L_0^2(\Gamma \backslash \mathcal{S})$ consisting of Hecke eigenforms with $\Lambda$-eigenvalues $\{\frac{1}{4} + v_1^2, \frac{1}{4} + v_2^2, \ldots\}$, respectively. Note that $\langle P_n; w, \tau, 1 \rangle = 0$ as $n \neq 0$.

By the Petersson formula (3.3), if $s = \frac{1}{2} + it$, then the Stirling formula gives

$$
\langle P_n; w, \tau, E_a(\cdot, s) \rangle = e^{-\pi|t|} \phi_{a\infty} \left( \frac{1}{2} - it, n \right) \ll e(-\frac{\pi}{2}|t|),
$$

where we have used the well-known estimate

$$
\zeta(2s) \Gamma(s) \phi_{a\infty} (s, n) \ll (|t| + 2)^c
$$

for some constant $c$ depending on $\Re s$ only, so the integrals in (3.4) are absolutely convergent. On the other hand, write

$$
u_j(z) = \sum_{n=-\infty}^{\infty} \frac{a_j(n)}{\sqrt{|n|}} W_{\frac{1}{2} + iv_j}(nz),$$

then by Hoffstein and Lockhart [8] we have

$$a_j(n) \ll (|v_j| + 2)^c e^{\frac{\pi}{2}|v_j|} \ll e^{\left(\frac{\pi}{2}+c\right)|v_j|}$$

as $j \to \infty$, so

$$
\langle P_n; w, \tau, u_j \rangle = e^{(\frac{\pi}{2}-\frac{\pi}{2})|v_j|} \ll e^{-\frac{\pi}{2} \sqrt{j}},
$$

therefore the summation over $j$ also converges absolutely.

In conclusion, the right-hand side of the spectral decomposition (3.4), when applied with the Petersson formula (3.3), has a meromorphic continuation in $w, \tau$ everywhere, thus yielding a meromorphic continuation for $P_n; w, \tau(z)$ itself. Furthermore, by the meromorphic continuation, the Petersson formula (3.3) is valid wherever it is meaningful. \qed
4. Nonholomorphic Kernel function

In this section, we will discuss the analytic properties of the nonholomorphic kernel function

\[ P_{w, \tau}(z) = \sum_{\gamma \in \Gamma} (\Im \gamma z)^w \left( \frac{\Im \gamma z}{|\gamma|} \right)^w. \]  

(4.1)

As always, we write \( \Gamma = \Gamma_0(N) \).

Obviously, \( P_{w, \tau}(z) \) is absolutely convergent for \( \Re \tau > 1, \Re w > 0 \) and so defines an analytic function over \( \mathfrak{H} \). By the Poisson summation formula, we have

\[
P_{w, \tau}(z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} (\Im \gamma z)^{\tau + w} \left( \sum_{n=-\infty}^{\infty} \frac{1}{|n + \gamma z|^w} \right)
= \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} (\Im \gamma z)^{\tau + w} \left[ \sum_{n=-\infty}^{\infty} \left( e^{2\pi n i \Im \gamma z} \int_{-\infty}^{\infty} e^{2\pi n u i} \frac{du}{u^2 + (\Im \gamma z)^2} \right) \right].
\]

Recall that

\[
\int_{-\infty}^{\infty} e^{2\pi n i u} \frac{du}{(u^2 + y^2)^{s}} = \begin{cases} 
2\pi |n|^{-1-\frac{s}{2}} y^{1-s} K_{s-\frac{1}{2}}(2\pi |n| y) & \text{if } n \neq 0; \\
\sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} & \text{if } n = 0,
\end{cases}
\]

so by definition (3.1) this gives

\[
P_{w, \tau}(z) = \sqrt{\pi} \frac{\Gamma\left(\frac{w-1}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} E_{\infty}(z, \tau + 1) + \frac{2\pi^w}{\Gamma\left(\frac{w}{2}\right)} \sum_{n=-\infty}^{\infty} |n|^{-\frac{w}{2}} P_{n, \frac{w-1}{2}, \tau + \frac{w}{2}}(z).
\]

We write

\[
P_{w, \tau}^*(z) = \frac{2\pi^w}{\Gamma\left(\frac{w}{2}\right)} \sum_{n=-\infty}^{\infty} |n|^{-\frac{w}{2}} P_{n, \frac{w-1}{2}, \tau + \frac{w}{2}}(z),
\]

(4.2)

which turns out to be the “cuspidal part” of \( P_{w, \tau}(z) \), then

\[
P_{w, \tau}(z) = \sqrt{\pi} \frac{\Gamma\left(\frac{w-1}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} E_{\infty}(z, \tau + 1) + P_{w, \tau}^*(z).
\]

(4.3)
Proposition 11 (Petersson formula). Assume that \( \Re \tau > 1, \Re w > 0 \).

1. Let

\[
u(z) = \sum_{n=\infty}^{\infty} \frac{a(n)}{\sqrt{|n|}} W_{\frac{1}{2} + i\nu(nz)}
\]

be a Maass form for \( \Gamma \) with conductor \( D \), then

\[
\langle P_{w,\tau}^*, \nu \rangle = \sqrt{\frac{\sqrt{D}}{\pi}} \frac{\Gamma(\frac{\tau+w-1+i\nu}{2})\Gamma(\frac{\tau+w-1-i\nu}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} \Lambda \left( \tau + \frac{1}{2} + \overline{\nu} \right),
\]

where

\[
\Lambda(s, \overline{\nu}) = \left( \frac{\sqrt{D}}{\pi} \right)^s \Gamma \left( \frac{s + i\nu}{2} \right) \Gamma \left( \frac{s - i\nu}{2} \right) \cdot \frac{1}{2} \sum_{n=\infty}^{\infty} \frac{a(n)}{n^s} \sum_{n=\infty}^{\infty} \frac{a(n)}{|n|^s}
\]

is the complete \( L \)-function for \( \overline{\nu} \).

2. Let \( \alpha \) be a cusp for \( \Gamma \), then

\[
\langle P_{w,\tau}^*, E_\alpha(\cdot, s) \rangle = \sqrt{\frac{\sqrt{D}}{\pi}} \frac{\Gamma(\frac{\tau+w-s}{2})\Gamma(\frac{\tau+w-1+s}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} \Lambda_\alpha \left( \tau + \frac{1}{2} + \overline{s} \right),
\]

where \( \Lambda_\alpha(\nu, s) \) is as defined in (2.5).

Proof. This follows directly from the definition (4.2) of \( P_{w,\tau}^* \) and the Petersson formula (3.3) of nonholomorphic Poincaré series.

Proposition 12. The function \( P_{w,\tau}(z) \) has a meromorphic continuation to \( \Re \tau > -\varepsilon, \Re w > \frac{1}{2} \). In particular, \( P_{w,0}(z) \) is holomorphic over \( \Re w > \frac{1}{2} \) except for the simple poles at \( w = \varepsilon_1, \ldots, \varepsilon_m \), as defined in Theorem 2, and the double pole at \( w = 1 \) with Laurent expansion

\[
P_{w,0}(z) = \frac{4}{\text{vol}(\Gamma \backslash \mathcal{S})} (w - 1)^{-2} + O(|w - 1|^{-1}).
\]
Proof. We start with the assumption that $1 < \Re \tau < \frac{4}{3}$ and $\frac{1}{2} < \Re w < 1$. In this case, the spectral decomposition for $P_{w, \tau}^*(z)$ gives

$$P_{w, \tau}(z) = \sqrt{\pi} \frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})} E_{\infty}(z, \tau + 1) + \sum_{a} \frac{1}{4\pi i} \int_{\frac{1}{2}}^{\infty} \langle P_{w, \tau}^*, E_a(\cdot, s) \rangle E_a(z, s) \, ds$$

$$+ \sum_{j=1}^{\infty} \langle P_{w, \tau}^*, u_j \rangle u_j(z).$$

By the Petersson formula (4.4), we have

$$\sum_{j=1}^{\infty} \langle P_{w, \tau}^*, u_j \rangle u_j(z) = \frac{\sqrt{\pi}}{2} \sum_{j=1}^{\infty} \frac{\Gamma(\frac{\tau + w - \frac{1}{2} + i\nu_j}{2})\Gamma(\frac{\tau + w - \frac{1}{2} - i\nu_j}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} A(\tau + \frac{1}{2}, u_j) D_{\frac{\tau + w}{2} + \frac{1}{2}} u_j(z).$$

It has an analytic continuation to the region $\Re \tau > -\varepsilon$, $\Re w > \frac{1}{2}$. In case $\tau = 0$, it has simple poles at $w = z_1, \ldots, z_m$ but is holomorphic at $w = 1$.

For the integrals against the Eisenstein series, we have

$$\sum_{a} \frac{1}{4\pi i} \int_{\frac{1}{2}}^{\infty} \langle P_{w, \tau}^*, E_a(\cdot, s) \rangle E_a(z, s) \, ds$$

$$= \sum_{a} \frac{\sqrt{\pi}}{4\pi i} \int_{\frac{1}{2}}^{\infty} \frac{\Gamma(\frac{\tau + w - \frac{1}{2} + s}{2})\Gamma(\frac{\tau + w - \frac{1}{2} - s}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} \frac{A_a(\tau + \frac{1}{2}, s)}{2N^{\frac{\tau + w}{2} + \frac{1}{2}}} E_a(z, s) \, ds$$

$$= \sum_{a} \frac{\sqrt{\pi}}{4\pi i} \int_{\frac{1}{2}}^{\infty} \frac{\Gamma(\frac{\tau + w - \frac{1}{2} + s}{2})\Gamma(\frac{\tau + w - \frac{1}{2} - s}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} \frac{A_a(\tau + \frac{1}{2}, 1 - s)}{2N^{\frac{\tau + w}{2} + \frac{1}{2}}} E_a(z, s) \, ds.$$

Shift the integration line from $\Re s = \frac{1}{2}$ to $\Re s = -\frac{4}{3}$, then

$$\sum_{a} \frac{1}{4\pi i} \int_{\frac{1}{2}}^{\infty} \langle P_{w, \tau}^*, E_a(\cdot, s) \rangle E_a(z, s) \, ds$$

$$= \sum_{a} \frac{\sqrt{\pi}}{4\pi i} \int_{-\frac{4}{3}}^{\frac{1}{2}} \frac{\Gamma(\frac{\tau + w - \frac{1}{2} + s}{2})\Gamma(\frac{\tau + w - \frac{1}{2} - s}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} \frac{A_a(\tau + \frac{1}{2}, 1 - s)}{2N^{\frac{\tau + w}{2} + \frac{1}{2}}} E_a(z, s) \, ds$$

$$+ R_{0; w, \tau}(z) + R_{1; w, \tau}(z) + R_{2; w, \tau}(z) + \sum_{\rho} R_{\rho; w, \tau}(z),$$

where the terms

$$R_{0; w, \tau}(z) = \sqrt{\pi} \frac{\Gamma(\tau + w - \frac{1}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} \sum_{a} \frac{A_a(\tau + \frac{1}{2}, \tau + w)}{2N^{\frac{\tau + w}{2} + \frac{1}{2}}} E_a(z, 1 - \tau - w),$$
\[ R_{1; w, \tau}(z) = \frac{\sqrt{\pi}}{2N^\tau} \frac{\Gamma(\tau + \frac{w-1}{2})}{\Gamma(\tau + \frac{w}{2})} \sum_a \delta_{b\infty} E_a(z, 1 - \tau), \]

\[ R_{2; w, \tau}(z) = -\frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})} \sum_a \phi_{a\infty}(1 + \tau) E_a(z, -\tau), \]

\[ R_{\rho; w, \tau}(z) = \sqrt{\pi} \text{Res}_{s=\rho} \left( \frac{\Gamma(\frac{\tau+w-s}{2})\Gamma(\frac{\tau+w-1+s}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} \sum_a A_a(\tau + \frac{1}{2}, 1-s) \frac{1}{4N^{\frac{\tau}{2}+\frac{1}{4}}} \right) E_a(z, s) \]

come from the residues at \( s = 1 - \tau - w, 1 - \tau, -\tau, \frac{\rho}{2} \), respectively, and \( \rho \) runs through the nontrivial zeros of \( L(s, \chi_N^0) \). Recall that \( \chi_N^0 \) is the principal character modulo \( N \). So

\[
\sqrt{\pi} \Gamma(\frac{w-1}{2}) \frac{E_{\infty}(z, \tau + 1)}{\Gamma(\frac{w}{2})} + \sum_a \frac{1}{4\pi i} \int_{\frac{1}{2}}^1 \langle P_{w, \tau}, E_a(\cdot, s) \rangle E_a(z, s) ds
\]

\[
= \sum_a \frac{\sqrt{\pi}}{4\pi i} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\Gamma(\frac{\tau+w-s}{2})\Gamma(\frac{\tau+w-1+s}{2})}{\Gamma(\frac{w}{2})\Gamma(\tau + \frac{w}{2})} A_a(\tau + \frac{1}{2}, 1-s) \frac{1}{2N^{\frac{\tau}{2}+\frac{1}{4}}} E_a(z, s) ds
\]

\[ + \left( R_{0; w, \tau}(z) + \sum_\rho R_{\rho; w, \tau}(z) \right) + R_{w, \tau}(z), \tag{4.8} \]

where

\[ R_{w, \tau}(z) = \sqrt{\pi} \Gamma(\frac{w-1}{2}) \frac{E_{\infty}(z, \tau + 1)}{\Gamma(\frac{w}{2})} R_{1; w, \tau}(z) + R_{2; w, \tau}(z). \]

By our previous discussion of \( A_a(v, s) \), every term in (4.8) has a meromorphic continuation to \( \Re \tau > -\epsilon, \Re w > \frac{1}{2} \). Furthermore, the integral along \( \Re s = -\frac{4}{3} \), in case \( \tau = 0 \), is holomorphic over \( \Re w > \frac{1}{2} \).

Over the region \( \Re w > \frac{1}{2} \), the function \( R_{\rho; w, 0}(z) \) has a unique pole at \( w = 1 - \frac{\rho}{2} \). On the other hand,

\[ R_{0; w, 0}(z) = \frac{\sqrt{\pi}}{2N^{\frac{\tau}{2}}} \frac{\Gamma(w - \frac{1}{2})}{\Gamma(\frac{w}{2})^2} \sum_a A_a \left( \frac{1}{2}, w \right) E_a(z, 1 - w), \]

so over the region \( \Re w > \frac{1}{2} \) it has poles at \( w = 1 \) and at \( 1 - \frac{\rho}{2} \) with

\[ \text{Res}_{w=1-\frac{\rho}{2}} R_{0; w, 0}(z) = - \text{Res}_{w=1-\frac{\rho}{2}} R_{\rho; w, 0}(z). \]
Hence $R_{0;w,0}(z) + \sum_\rho R_{\rho;w,0}(z)$ is holomorphic over the region $\Re w > \frac{1}{2}$ except for the only pole $w = 1$, and residues (2.7), (2.9) imply

$$A_a\left(\frac{1}{2}, w\right) = \frac{2\phi_{a\infty}(w) + 2\phi_{b\infty}(w)}{w - 1} N \frac{w - \frac{1}{2}}{w - \frac{1}{4}} + O(1),$$

where $b = wN$, so

$$R_{0;w,0}(z) + \sum_\rho R_{\rho;w,0}(z)$$

$$= \frac{\sqrt{\pi}}{N^{1 - \frac{w}{2}}} \frac{1}{\Gamma\left(\frac{w}{2}\right)^2} \left(\sum_a \phi_{a\infty}(w) E_a(z, 1 - w) + \sum_a \phi_{b\infty}(w) E_a(z, 1 - w)\right) + O(|w - 1|^{-1})$$

$$= \frac{\sqrt{\pi}}{N^{1 - \frac{w}{2}}} \frac{1}{\Gamma\left(\frac{w}{2}\right)^2} \frac{E_\infty(z, w) + E_\infty(wNz, w)}{w - 1} + O(|w - 1|^{-1})$$

$$= \frac{2}{\text{vol}(\Gamma\backslash\mathcal{D})} \frac{1}{(w - 1)^2} + O(|w - 1|^{-1}).$$

On the other hand,

$$R_{2;w,\tau}(z) = -\frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{w - 1}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} E_\infty(z, 1 + \tau)$$

and, by writing

$$E_\infty(wNz, s) = \frac{1}{\text{vol}(\Gamma\backslash\mathcal{D})} \frac{1}{s - 1} + u(z) + (s - 1)v(z),$$

as $\tau \to 0$ we have

$$R_{1;w,\tau}(z) = \frac{\sqrt{\pi}}{2N^\tau} \frac{\Gamma(\tau + \frac{w - 1}{2})}{\Gamma(\tau + \frac{w}{2})} E_\infty(wNz, 1 - \tau)$$

$$= -\frac{\sqrt{\pi}}{2\text{vol}(\Gamma\backslash\mathcal{D})} \left(\frac{\Gamma\left(\frac{w - 1}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} \frac{1}{\tau} + \frac{\Gamma'(\frac{w - 1}{2})\Gamma\left(\frac{w}{2}\right) - \Gamma\left(\frac{w - 1}{2}\right)\Gamma'(\frac{w}{2})}{\Gamma\left(\frac{w}{2}\right)^2}\right)$$

$$+ \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{w - 1}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} \left(\frac{\log N}{\text{vol}(\Gamma\backslash\mathcal{D})} + u(z)\right) + O(\tau).$$
so by definition $R_{w,0}^*(z)$ equals to

$$\frac{-\sqrt{\pi}}{2} \left( \frac{\Gamma'(\frac{w-1}{2}) \Gamma(\frac{w}{2}) - \Gamma'(\frac{w-1}{2}) \Gamma'(\frac{w}{2})}{\text{vol}(\Gamma \backslash \mathfrak{H}) \Gamma(\frac{w}{2})^2} - \frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})} \left( \frac{\log N}{\text{vol}(\Gamma \backslash \mathfrak{H})} + u(z) \right) \right),$$

which is holomorphic over $\Re w > \frac{1}{2}$ except the pole at $w = 1$, and

$$R_{w,0}^*(z) = \frac{2}{\text{vol}(\Gamma \backslash \mathfrak{H})} \frac{1}{(w-1)^2} + O(|w-1|^{-1}).$$

In conclusion, the expression

$$P_{w,\tau}(z) = \sum_a \frac{\sqrt{\pi}}{4\pi i} \int_{(-\frac{1}{2})} \frac{\Gamma(\frac{\tau+w-1-s}{2}) \Gamma'(\frac{\tau+w-1+s}{2})}{\Gamma(\frac{w}{2}) \Gamma(\frac{\tau}{2} + \frac{w}{2})} \frac{A_a(\tau + \frac{1}{2}, 1 - s)}{2N^{\frac{1}{2} + \frac{s}{2}}} E_a(z, s) \ ds$$

$$+ \frac{-\sqrt{\pi}}{2} \sum_{j=1}^{\infty} \frac{\Gamma(\frac{\tau+w-1+iv_j}{2}) \Gamma'(\frac{\tau+w-1-iv_j}{2})}{\Gamma(\frac{w}{2}) \Gamma(\frac{\tau}{2} + \frac{w}{2})} \frac{A(\tau + \frac{1}{2}, v_j)}{D_j^{\frac{1}{2} + \frac{1}{2}}} u_j(z) + R_{w,\tau}(z)$$

$$+ R_{0,1}(z) + \sum_{\rho} R_{\rho, w, \tau}(z)$$

(4.9)

yields the meromorphic continuation of $P_{w,\tau}(z)$ up to $\Re \tau > -\varepsilon, \Re w > \frac{1}{2}$, in particular, $P_{w,0}(z)$ is holomorphic up to $\Re w > \frac{1}{2}$ except for simple poles at $\alpha_1, \ldots, \alpha_m$ and a double pole at $w = 1$ with leading term

$$\frac{4}{\text{vol}(\Gamma \backslash \mathfrak{H})} \frac{1}{(w-1)^2}.$$

This completes the proof of the proposition. □

**Corollary 13.** Let $f \in S_k(N, \gamma)$. Write $F(z) = |f(z)|^2 y^k$, then the function

$$Z_f(w, \tau) = \langle P_{w,\tau}, F \rangle = \int \int_{\Gamma \backslash \mathfrak{H}} P_{w,\tau}(z) |f(z)|^2 y^k \frac{dx \, dy}{y^2}$$

(4.10)

has an analytic continuation to $\Re \tau > -\varepsilon, \Re w > \frac{1}{2}$, and $Z_f(w, 0)$ has simple poles at $w = \alpha_1, \ldots, \alpha_m$ and a double pole at $w = 1$ with

$$Z_f(w, 0) = \frac{4 \| f \|^2}{\text{vol}(\Gamma \backslash \mathfrak{H})} \frac{1}{(w-1)^2} + O(|w-1|^{-1}).$$

(4.11)
The last ingredient is the following growth estimate of \( Z_f(w, 0) \), which will be used in the proof of Theorem 2.

**Proposition 14.** Assume \( \frac{1}{2} < \Re w < \frac{3}{2} \). Then as \( |\Im w| \to \infty \) we have

\[
Z_f(w, 0) \ll |\Im w|^{\frac{9}{4} - 2\Re w + \varepsilon}.
\]

**Proof.** To begin with, write \( j \left( \begin{array}{cc} a & c \\ b & d \end{array} \right), z \) \( = cz + d \), then the function \( f_a(z) = j(\sigma_a, z)^{-k} f(\sigma_0 z) \) is a cusp form of weight \( k \) over with respect to a congruence subgroup, where \( \sigma_a \) is the scaling matrix of \( a \), so for \( \Re s > 1 \) we have

\[
\zeta(2s)(E_a(\cdot, s), F) = \frac{\Gamma(s + k - 1)}{(4\pi)^{s+k-1}} L(s, f_a \times \overline{f_a}) \ll e(\frac{s-\frac{k}{2}}{2})|t|.
\]

By definition (4.9) of \( P_{w, r}(z) \), we have

\[
Z_f(w, 0) = \frac{\sqrt{\pi}}{4\pi i} \int_{(-\frac{1}{2})} \frac{\Gamma(\frac{w-s}{2}) \Gamma(\frac{w-1+s}{2})}{\Gamma(\frac{w}{2})^2} \sum_a A_a(\frac{1}{2}, 1-s) \langle E_a(\cdot, s), F \rangle \frac{1}{2N_{\frac{1}{2}}} \, ds
\]

\[
+ \frac{\sqrt{\pi}}{2} \sum_{j=1}^\infty \frac{\Gamma(\frac{w-1/2+iv_j}{2}) \Gamma(\frac{w-1/2-iv_j}{2})}{\Gamma(\frac{w}{2})^2} A\left(\frac{1}{2}, u_j\right) \langle u_j, F \rangle \frac{D_j^{1/2}}{D_j}
\]

\[
+ \{ R_{\rho, w, 0} + \sum_{\rho} R_{\rho; w, 0}, F \}
\]

\[
= P_1(w) + P_2(w) + P_3(w) + P_4(w),
\]

say. For simplicity, write \( w = u + iv \) and assume \( \frac{1}{2} < u < \frac{3}{2}, v \gg 1 \).

(1) By the functional equation (2.13), we have

\[
P_1(w) = \frac{\sqrt{\pi}}{4\pi i} \int_{(-\frac{1}{2})} \frac{\Gamma(\frac{w-s}{2}) \Gamma(\frac{w-1+s}{2})}{\Gamma(\frac{w}{2})^2} \sum_a A_a(\frac{1}{2}, s) \langle E_a(\cdot, 1-s), F \rangle \frac{1}{2N_{\frac{1}{2}}} \, ds,
\]

so the growth conditions (2.11) and (4.13) give

\[
P_1(w) \ll \int_{-\infty}^{\infty} \left| \frac{\Gamma(\frac{w+\frac{4}{2}-it}{2}) \Gamma(\frac{w-\frac{7}{2}+it}{2})}{\Gamma(\frac{w}{2})^2} \right| e(\frac{s-\frac{k}{2}}{2})|t| \, dt \ll 1.
\]

(4.14)
(2) Since \( \|u_j\| = 1 \), we have \( \langle u_j, F \rangle \ll 1 \), so

\[
P_2(w) \ll \sum_{j=1}^{\infty} \left| \frac{\Gamma(\frac{w-\frac{1}{2}+iv_j}{2})\Gamma(\frac{w-\frac{1}{2}-iv_j}{2})}{\Gamma(\frac{w}{2})^2} \frac{A(\frac{1}{2}, u_j)}{D_j^{\frac{1}{2}}} \right|
\]

\[
= \sum_{j=1}^{\infty} \left| \frac{\Gamma(\frac{w-\frac{1}{2}+iv_j}{2})\Gamma(\frac{w-\frac{1}{2}-iv_j}{2})\Gamma(\frac{1}{2}+iv_j)\Gamma(\frac{1}{2}-iv_j)}{\Gamma(\frac{w}{2})^2} \frac{A(\frac{1}{2}, u_j)}{D_j^{\frac{1}{2}}} \right| \cdot L\left(\frac{1}{2}, \overline{u_j}\right).
\]

With at most finitely many exceptions, we may assume that \( v_j > 0 \), then the Stirling formula and Weyl’s law give

\[
P_2(w) \ll \sum_{j=1}^{\infty} \left| \frac{e^{-\frac{3}{4}v_j v_j^{-\frac{1}{2}}(v + v_j)^{\frac{w}{2} - \frac{3}{4}}}}{e^{-\frac{3}{4}v_j v_j^{\frac{w}{2} - \frac{3}{4}}} L\left(\frac{w - \frac{1}{2} - iv_j}{2}\right)} \right|
\]

\[
\ll e^{3v_j v_j^{-\frac{1}{2}}(v + v_j)^{\frac{w}{2} - \frac{3}{4}}} \sum_{j=1}^{\infty} \left| \frac{e^{-\frac{3}{4}v_j v_j^{-\frac{1}{2}}(v + v_j)^{\frac{w}{2} - \frac{3}{4}}}}{e^{-\frac{3}{4}v_j v_j^{\frac{w}{2} - \frac{3}{4}}} L\left(\frac{w - \frac{1}{2} - iv_j}{2}\right)} \right|
\]

\[
\ll e^{3v_j v_j^{-\frac{1}{2}}(v + v_j)^{\frac{w}{2} - \frac{3}{4}}} \sum_{j=1}^{\infty} \left( e^{-\frac{3}{4}v_j v_j^{\frac{w}{2} - \frac{3}{4}}} + v_j^{\frac{w}{2} - \frac{3}{4}} \right) \sum_{v_j < v-1} v_j^{\frac{w}{2} - \frac{3}{4} + \varepsilon}.
\]

(4.15)

(3) By definition and (4.13) we have

\[
P_3(w) = \frac{3\Gamma(k)\Gamma(\frac{w-1}{2})}{\pi^{\frac{3}{2}}(4\pi)^k \Gamma(\frac{w}{2})} \lim_{\tau \to 0} \left( L(1 + \tau, f \otimes \overline{f}) + L(1 - \tau, f_{w-N} \otimes \overline{f}_{w-N}) \right),
\]

so (4.13) gives

\[
P_3(w) \ll \left| \frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})} \right| \ll 1.
\]

(4.16)

(4) By definition, we have

\[
P_4(w) = \frac{\sqrt{\pi}}{2N^{\frac{1}{2}}} \frac{\Gamma(\frac{w-\frac{1}{2}}{2})}{\Gamma(\frac{w}{2})^2} \sum_{a} A_a\left(\frac{1}{2}, 1 - w\right) \langle E_a(\cdot, w), F \rangle
\]

\[
+ \sum_{\rho} \operatorname{Res}_{\rho = \frac{w}{2}} \left( \frac{\sqrt{\pi}}{4N^{\frac{1}{2}}} \frac{\Gamma(\frac{w-\frac{1}{2}}{2})\Gamma(\frac{w-\frac{1}{2}+s}{2})}{\Gamma(\frac{w}{2})^2} \sum_{a} A_a\left(\frac{1}{2}, s\right) \langle E_a(\cdot, 1-s), F \rangle \right).
\]
Now we consider two cases $u = \frac{3}{2}$ and $u = -\frac{1}{2}$. In both cases, by (2.11) and (4.13) we have that
\[
\frac{\Gamma(w - \frac{1}{2})}{\Gamma(\frac{w}{2})^2} \sum_a A_a \left(\frac{1}{2}, 1 - w\right) \langle E_a(\cdot, w), F \rangle \ll \sum_a \left| A_a \left(\frac{1}{2}, 1 - w\right) \langle E_a(\cdot, w), F \rangle \right|
\]
is always bounded. On the other hand,
\[
\sum \text{Res} \left( \sum_a A_a \left(\frac{1}{2}, s\right) \langle E_a(\cdot, 1 - s), F \rangle \right) 
\]
\[
= \sum_a \frac{\sqrt{\pi}}{2\pi i} \left( \int_{(1+i\epsilon)} - \int_{(-i\epsilon)} \right) \frac{\Gamma(\frac{w-s}{2})\Gamma(\frac{w-1+s}{2})}{\Gamma(\frac{w}{2})^2} \frac{A_a \left(\frac{1}{2}, 1 - s\right) \langle E_a(\cdot, s), F \rangle}{4N^{\frac{1}{4}}} ds 
\]
\[
+ \frac{\sqrt{\pi}}{2N^{\frac{1}{4}}} \left| f \right|^2 \text{ vol}(\Gamma \setminus \mathbb{H}) \sum_a A_a \left(\frac{1}{2}, 0\right), 
\]
where the last term comes from the residues at $s = 0$ and 1, so in both cases $u = -\frac{1}{2}$ and $\frac{3}{2}$ we also have, by (2.11) and (4.13), that
\[
\sum \text{Res} \left( \sum_a A_a \left(\frac{1}{2}, s\right) \langle E_a(\cdot, 1 - s), F \rangle \right) \ll 1.
\]
Hence $P_4(w) \ll 1$ for both $u = \frac{3}{2}$ and $-\frac{1}{2}$. We can show, exactly as before, that $P_4(w)$ is holomorphic over the region $-\frac{1}{4} < u < \frac{3}{4}$ except the pole at $w = 1$, so the Phragmen–Lindelöf principle implies that $P_4(w) \ll 1$. Combining this with (4.14)–(4.16), we have completed our proof. □

5. Proofs of theorems

Now we are ready to prove our main theorems.

**Proof of Theorem 1.** Let $D$ be the conductor of $f$ and
\[
A(s, f) = \left(\frac{\sqrt{D}}{2\pi}\right)^{\frac{k-1}{2}} \Gamma \left(s + \frac{k-1}{2}\right) L(s, f) = \int_0^\infty f(iy)(y\sqrt{D})^{s-k+\frac{1}{2}} \frac{dy}{y}
\]
its complete $L$-function, then the inverse Mellin transformation gives
\[
f(iy) = \frac{1}{2\pi i} \int_{(2)} A(s, f)(y\sqrt{D})^{-s-k+\frac{1}{2}} \, ds.
\]
Since \( f(z) \) is an analytic function in \( z \), this implies that

\[
f(z) = \frac{1}{2\pi i} \int_{(2)} A(s, f) \left( \frac{i}{z\sqrt{D}} \right)^{s+k-1} ds.
\]  

(5.1)

Now assume that \( \Re w > 1 \), then by (4.10) the standard unfolding technique shows that

\[
Z_f(w, \tau) = \int \int_{\mathcal{D}} y^{\tau} \left( \frac{y}{|z|} \right)^w |f(z)|^2 y^k \frac{dx \, dy}{y^2}
= \frac{1}{2\pi i} \int_{(2)} \frac{i^{s+k-1}}{D^{s+k-1}} A(s, f) \left( \int \int_{\mathcal{D}} \frac{y^{\tau+k+w-2}}{z^{s+k-1}|z|^w} f(z) \, dx \, dy \right) ds.
\]

Using the polar coordinates, we have

\[
\int \int_{\mathcal{D}} \frac{y^{\tau+k+w-2}}{e^{s+k-1}|z|^w} f(z) \, dx \, dy
= \int_0^{\pi} (\sin \theta)^w e^{i\theta} \left( \int_0^\infty r^{\tau+k-1} f(r e^{i\theta}) \, dr \right) d\theta
= \frac{i^{\tau-k+1}}{D^{s+k-1}} A(\tau + 1 - s, f) \int_0^{\pi} (\sin \theta)^w e^{i\theta(\tau+1-2s)} d\theta.
\]

Recall that [1]

\[
\int_0^{\pi} (\sin \theta)^x e^{i\theta} d\theta = \frac{\pi i^x}{2^x} \frac{\Gamma(1+z)}{\Gamma(1+\frac{x+\beta}{2})\Gamma(1+\frac{z-\beta}{2})},
\]

(5.2)

so a shift of the integration line gives

\[
Z_f(w, \tau) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(w + \tau + k - 1)\Gamma(s + k-1)\Gamma(\tau + k+1 - s)}{2^w (4\pi)^{\tau+k-1} \Gamma(w+k-1+s)\Gamma(\tau + w+k+1 - s)} \times L(s, f) L(\tau + 1 - s, f) \, ds.
\]

In particular, we have

\[
Z_f(w, 0) = \frac{\Gamma(w + k - 1)}{2^{w-1}(4\pi)^k} \int_{-\infty}^{\infty} \frac{|\Gamma(\frac{k}{2} + it)|^2 |L(\frac{1}{2} + it, f)|^2}{\Gamma(\frac{w+k}{2} + it)\Gamma(\frac{w+k}{2} - it)} \, dt.
\]  

(5.3)
Now we assume that \( w > 1 \) is real, then the Stirling formula gives
\[
Z_f(w, 0) = \frac{\Gamma(w + k - 1)}{2^{w - 2} (4\pi)^k} \int_1^\infty \left| L \left( \frac{1}{2} + it, f \right) \right|^2 t^{-w} \, dt + G(w)
\]
(5.4)
for some function \( G(w) \) holomorphic at \( w = 1 \). Combing (5.4) and (4.11), we have
\[
\int_1^\infty \left| L \left( \frac{1}{2} + it, f \right) \right|^2 t^{-w} \, dt = \frac{2(4\pi)^k}{\Gamma(k)} \frac{\| f \|^2}{\text{vol}(\Gamma \backslash \mathcal{H})} (w - 1)^{-2} + O(|w - 1|^{-1}).
\]
Hence a Wiener–Ikehara Tauberian argument (see, for example, Theorem 1.3 in [17]) readily implies that
\[
\int_1^T \left| L \left( \frac{1}{2} + it, f \right) \right|^2 \, dt \sim \frac{2(4\pi)^k}{\Gamma(k)} \frac{\| f \|^2}{\text{vol}(\Gamma \backslash \mathcal{H})} T \log T.
\]
This completes the proof of Theorem 1. \( \square \)

**Proof to Theorem 2.** Consider the integral
\[
I_f(T) = \frac{1}{2\pi i} \int_{(2)} \frac{Z_f(w, 0) T^w}{w^3} \, dw,
\]
(5.5)
where the growth condition (4.12) guarantees its absolute convergence. By (5.3), we may exchange the order of integrations to get
\[
I_f(T) = \int_1^\infty G_T(t) \left| L \left( \frac{1}{2} + it, f \right) \right|^2 \, dt,
\]
(5.6)
where \( G_T(t) \) is as defined in (1.5). On the other hand, by the growth condition (4.12) we can shift the integration line for \( I_f(T) \) from \( \Re s = 2 \) to \( \Re s = \frac{1}{2} + \varepsilon \), then by (4.11) we have
\[
I_f(T) = \left( \text{Res}_{w=1} + \sum_{j=1}^m \text{Res}_{w=\eta_j} \right) \frac{Z_f(w, 0) T^w}{w^3} + \frac{1}{2\pi i} \int_{(\frac{1}{2}+\varepsilon)} \frac{Z_f(w, 0) T^w}{w^3} \, dw
\]
\[
= \frac{4\| f \|^2}{\text{vol}(\Gamma \backslash \mathcal{H})} T \log T + c_0 T + \sum_{j=1}^m c_j T^{\eta_j} + O \left( T^{\frac{1}{2} + \varepsilon} \right)
\]
for some constants \( c_0, c_1, \ldots, c_m \). This completes the proof. \( \square \)
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