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Transition Layers in Singular Perturbation Problems*

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1. INTRODUCTION

The phenomenon of interior nonuniformities in solutions of singular perturbation problems has been extensively studied within the context of second order differential equations whose degenerate form is a first order equation (see, for example, O'Malley [5] and Wasow [6] and the many references contained therein). In this paper, on the other hand, we shall study equations which reduce to the simple form $h(x, u) = 0$ when the perturbation parameter ϵ is zero. The type of nonuniformity we study is that of an abrupt transition at some interior point x_0 between distinct solutions of the reduced equation. In other words, we assume that the reduced equation has (at least) two solutions $u = g^1(x)$ and $u = g^2(x)$. We then look for families $u(x, \epsilon)$ of solutions of the original problem for which $u(x, \epsilon) \to g^1(x)$ as $\epsilon \to 0$ uniformly for $x < x_0 - \delta$, and $u(x, \epsilon) \to g^2(x)$ uniformly for $x > x_0 + \delta$, where δ is any positive number. For simplicity, our solutions are defined for all values of x. They are not conceived as being solutions of any particular boundary value problem. Transition layers in the context of boundary value problems will be the subject of a later paper.

It is not unusual for a second order equation to have a solution family with a transition layer, but the location x_0 of such a layer is subject to restrictive conditions. Consider the example

$$\epsilon^2(p(u, x)u')' - h(u, x) = 0,$$

where $p > 0$. Sufficient conditions for a transition layer at $x = x_0$ to exist, in the case $g^2(x_0) > g^1(x_0)$, are

(i) $h_u(g^i(x), x) \geqslant \alpha > 0$,

(ii) $$\int_{g^1(x_0)}^{k} p(v, x_0)\, h(v, x_0)\, dv \begin{cases} > 0 & \text{for } k \in (g^1(x_0), g^2(x_0)), \\ = 0 & \text{for } k = g^2(x_0), \end{cases}$$

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and

(iii) $$\int_{g^1(x_0)}^{g^2(x_0)} [p_x(v, x_0)\, h(v, x_0) + p(v, x_0)\, h_x(v, x_0)]\, dv \neq 0.$$

This example is treated in detail in Section 6. In the general case

$$F(\epsilon^2 u'', \epsilon u', x, \epsilon) = 0$$

treated in Sections 2–5, conditions analogous to these three are given. The analog of (ii) is simply

(ii)′: *the equation*

$$F(y'', y', y, 0, 0) = 0 \tag{1.1}$$

has a solution satisfying $y(-\infty) = g_1(x_0)$, $y(\infty) = g_2(x_0)$.

In the example mentioned above, this condition is equivalent to (ii).

The simple case when F does not depend on x or ϵ should be considered separately; in fact, our Hypothesis 3 explicitly excludes it. This case is easily treated, because (ii)′ alone is seen to be sufficient for a transitional family to exist. In fact, necessarily $g^i(x)$ are constant, so the family $u(x, \epsilon) = y(x/\epsilon)$, where $y(\eta)$ is a solution of $F(y'', y', y) = 0$, $y(\infty) = g^2$, $y(-\infty) = g^1$, is transitional in nature.

The results in this paper are mainly concerned with existence and uniqueness, although methods for constructing asymptotic expansions for our solutions are given in Section 7. We first establish, in Section 2, the existence of regular families $w^i(x, \epsilon)$ approaching $g^i(x)$ uniformly in x, as $\epsilon \to 0$. These families are analogous to those obtained in other papers by the usual outer expansion techniques. Inner expansions correspond to the construction of our transition function $y(\eta, \epsilon)$ discussed in Section 4.

In past approaches to singular perturbation problems, stretched variable techniques have been used to obtain boundary layer "corrections," or boundary layer "matched expansions." In the former approach, the solution is obtained as the sum of inner and outer expansions; in the second, it is pieced together from expansions of these types. Our approach is different from either of these, in that we use the *product* of an inner expansion and an outer expansion. In fact, our family of solutions is of the form

$$u = w^1 + y(w^2 - w^1), \tag{1.2}$$

where the w^i are the families mentioned above, and y is a function of a stretched variable η and ϵ which approaches 0 as $\eta \to -\infty$ and 1 as $\eta \to \infty$.

The proofs for the existence of appropriate families w^i in Section 2, and of the family y in Section 4, are both based on the implicit function theorem, but they use that theorem in completely different ways. In both cases, difficulties force the problem to be reformulated in order for the theorem to apply. In Section 2, the difficulty is that the essential features of the problem when $\epsilon = 0$ are radically different from those when $\epsilon \neq 0$; thus, the problem as it stands is out of the scope of the implicit function theorem. This difficulty is surmounted by forming an equivalent problem which does not have this liability. In Section 4, however, the main difficulty in using the implicit function theorem is that the operator involved does not have an invertible Frechet derivative at the origin, as is required; in fact the derivative has a one-dimensional nullspace. This difficulty is overcome by introducing an extra parameter λ into the problem, thus recasting the problem as one for which not only a solution $v(\eta, \epsilon)$ of the appropriate differential equation is sought, but also a function $\lambda(\epsilon)$. This new problem in a space of one higher dimension is now amenable to the implicit function theorem. The technique for such a recasting is given in general terms in Section 3. The ideas involved here are sometimes used in bifurcation problems (see [3], for example). As applied in Section 4, the extra parameter λ appears in the particular form of the stretched variable η.

Another difficulty in Section 4 is the fact that the Frechet derivative L (in fact, the differential equation itself) is on an infinite interval. Although one knows that 0 is an eigenvalue, it is necessary to prove that the continuous spectrum of L is bounded away from 0. This is done with a result of Weyl dating from 1909.

As regards uniqueness for the families constructed herein, it turns out that the regular families are unique; this is proved in Section 2. When it comes to transitional families, we focus attention on families of the form (1.2), where $y((x/\epsilon), \epsilon)$ is such that $\lim_{\zeta\to\infty} y(\zeta, \epsilon) = 1$; $\lim_{\zeta\to-\infty} y(\zeta, \epsilon) = 0$, uniformly in ϵ. Under our hypotheses, in general there can be only one transitional family of this type, although in exceptional circumstances, there exist two. The proof, given in Section 5, surprisingly reduces to the question of uniqueness of the initial value problem for a certain non-Lipschitzian differential equation. Whether there exist transitional families which are not of the type (1.2) with y having the properties indicated, is an open question.

One notational convention should be made clear at the outset. Partial derivatives of functions of several variables will usually be denoted by numerical subscripts, the number denoting the argument with respect to which the derivative is taken.

2. Existence of Regular Families of Solutions

Consider the differential equation

$$F(\epsilon^2 u'', \epsilon u', u, x, \epsilon) = 0 \tag{2.1}$$

for a function $u(x)$, $-\infty < x < \infty$. Our basic assumption follows.

HYPOTHESIS 1. *The degenerate equation* $F(0, 0, u, x, 0) = 0$ *has two bounded solutions* $u = g^i(x)$, $i = 1, 2$. *For some* $\kappa \neq 0$, $i = 1, 2$, *and* $-\infty < x < \infty$,

$$F_1(0, 0, g^i(x), x, 0) \geqslant \kappa^2, \tag{2.2a}$$

$$F_3(0, 0, g^i(x), x, 0) \leqslant -\kappa^2. \tag{2.2b}$$

In addition, it will be assumed throughout the paper, but not stated again, that F and g^i have a sufficient number of derivatives bounded uniformly in x, for the steps indicated in this and the succeeding sections to be meaningful.

THEOREM 2.1. *Under Hypothesis* 1, *there exist unique families* $u = w^1(x, \epsilon)$ *and* $u = w^2(x, \epsilon)$ *of solutions of* (2.1), *defined for* $x \in (-\infty, \infty)$, *and* $|\epsilon| < \epsilon_0$ *for some* $\epsilon_0 > 0$, *such that the limit relation*

$$\lim_{\epsilon\to 0} w^i(x, \epsilon) = g^i(x) \tag{2.3}$$

holds uniformly in x, *and such that* w_x^i *and* w_{xx}^i *are bounded for* x *and* ϵ *in the above range.*

Proof. Since the argument is the same for w^1 and w^2, we suppress the superscripts on w and g.

Let C^k denote the space of functions $u(x)$ with k bounded continuous derivatives on the real line $(-\infty, \infty)$. The associated norm is

$$|u|_k \equiv \sum_{j=0}^{k} \sup |u^{(j)}(x)|.$$

Let $\mathbf{F}$ denote the mapping from $C^2 \times R^1$ to C^0 defined by

$$\mathbf{F}(u, \epsilon) = F(\epsilon^2 u'', \epsilon u', u, x, \epsilon). \tag{2.4}$$

It can be checked that $\mathbf{F}$ has continuous Frechet derivatives to any desired order, provided that F is uniformly continuously differentiable to a sufficiently high order in all its arguments.

For some $m > 0$, we construct a function $U(x, \epsilon) = \sum_{n=0}^{m} \epsilon^n u_n(x)$ satisfying

$$\partial_\epsilon^k \mathbf{F}(U(\cdot, \epsilon), \epsilon)|_{\epsilon=0} = 0, \qquad k = 0, \dots, m. \tag{2.5}$$

For $k = 0$, (2.5) is $F(0, 0, u_0, x, 0) = 0$, with solution $u_0(x) = g(x)$. For $k > 0$, the equation is

$$F_3(0, 0, g(x), x, 0) u_k(x) = h_k(x),$$

where the function $h_k(x)$ is determinable from the previously found functions

u_j, $j < k$. Thus, by virtue of (2.2b), all the terms u_k, $k \geqslant 1$, are uniquely determined.

From (2.5) and Taylor's formula, we have that

$$\mathbf{F}(U, \epsilon) = \epsilon^{m+1} q(x, \epsilon), \tag{2.6}$$

where $q(x, \epsilon) = (1/(m+1)!)\, \partial_\epsilon^{m+1} \mathbf{F}(U(\cdot, \theta\epsilon), \theta\epsilon)$. The derivative on the right, being expressible in terms of the derivatives of F, is bounded uniformly in x, for $|\epsilon| < 1$:

$$|q|_0 < K_1. \tag{2.7}$$

In addition to the usual norm $|u|_2$, we shall use the following family of norms on C^2 for $\epsilon \neq 0$:

$$|u|_2^\epsilon = \epsilon^2 \sup |u''(x)| + |\epsilon| \sup |u'(x)| + \sup |u(x)|.$$

LEMMA 2.2. *For ϵ small enough but nonzero, the derivative $\mathbf{F}_1(U, \epsilon)$ is a homeomorphism between C^2 and C^0. Furthermore there exists a K_2 independent of ϵ such that*

$$K_2^{-1} |v|_2^\epsilon \leqslant |F_1(U, \epsilon; v)|_0 \leqslant K_2 |v|_2^\epsilon. \tag{2.8}$$

Proof. We denote by $\mathbf{L}^\epsilon$ the above indicated derivative:

$$\mathbf{L}^\epsilon v \equiv \mathbf{F}_1(U, \epsilon; v) \equiv \epsilon^2 a(x, \epsilon) v'' + \epsilon b(x, \epsilon) v' + c(x, \epsilon) v,$$

where $a(x, \epsilon) = F_1(\epsilon^2 U_{xx}(x, \epsilon), \ \epsilon U_x(x, \epsilon), \ U(x, \epsilon), x, \epsilon)$, $b = F_2(\cdots)$, and $c = F_3(\cdots)$. Clearly $\mathbf{L}^\epsilon$ is bounded. We consider its invertibility.

By virtue of (2.2), and the fact that $U(x, 0) = g(x)$, we have, for some constant $\epsilon_1 > 0$,

$$a \geqslant \tfrac{1}{2}\kappa^2, \qquad c \leqslant -\tfrac{1}{2}\kappa^2 \qquad \text{for} \quad |\epsilon| < \epsilon_1. \tag{2.9}$$

Let $f \in C^0$ and consider the equation

$$\mathbf{L}^\epsilon v = f. \tag{2.10}$$

From (2.9) and the maximum principle, we have that any bounded solution v must satisfy

$$|v|_0 \leqslant 2\kappa^{-2} |f|_0.$$

In particular, $f = 0$ implies $v = 0$, so $\mathbf{L}^\epsilon$ is one-to-one.

For the existence of a solution $v \in C^2$ for arbitrary $f \in C^0$, we use a comparison argument with the constant function $\alpha \equiv 2\kappa^{-2} |f|_0$. Let v_n be the solution of the boundary value problem $\mathbf{L}^\epsilon v_n = f$ in $(-n, n)$, $v_n(-n) =$

$v_n(n) = 0$. Since, for $|\epsilon| < \epsilon_1$, $\mathbf{L}^\epsilon\alpha - f = c\alpha - f \leqslant 0$ and $\mathbf{L}^\epsilon(-\alpha) - f = -c\alpha - f \geqslant 0$, the maximum principle insures that any solution of $\mathbf{L}^\epsilon v - f = 0$ in any interval (x_0, x_1) lies between $-\alpha$ and α in that interval, if it does so at the endpoints. In particular, $|v_n|_0 < \alpha$ for all n. The equation $\mathbf{L}^\epsilon v_n = f$, together with (2.9), imply that

$$\begin{aligned}\epsilon^2 |v_n''|_0 &\leqslant 2\kappa^{-2}(|f|_0 + |\epsilon||b|_0|v_n'|_0 + |c|_0|v_n|_0)\\ &\leqslant C(|f|_0 + |\epsilon||v_n'|_0),\end{aligned}$$

where C is independent of ϵ. This, together with the interpolation inequality (see [4, p. 114])

$$|\epsilon||v_n'|_0 \leqslant \delta\epsilon^2|v_n''|_0 + 2|v_n|_0/\delta \qquad (\delta > 0, \text{ arbitrary})$$

yields the result that $|v_n|_2^\epsilon$ is bounded uniformly in n

$$|v_n|_2^\epsilon \leqslant K_3|f|_0. \tag{2.11}$$

The differential equation itself again provides the equicontinuity of the sequence $\{v_n''\}$, so a subsequence of $\{v_n\}$ converges to a solution $v(x)$ of $\mathbf{L}^\epsilon v = f$ for all x. Since (2.11) continues to hold in the limit, we have the invertibility of $\mathbf{L}^\epsilon$ with the left side of (2.8) holding. But the right side is immediate; the lemma is proved.

Continuing with the proof of Theorem 2.1, we set $m = 5$, and let I be an open interval on the real axis containing 0, such that the conclusion of Lemma 2.2 holds for $\epsilon \in I\backslash\{0\}$. We then define the operator $\mathbf{H}$: $C^2 \times I \to C^2$ by

$$\mathbf{H}(s, \epsilon) = \begin{cases}\epsilon^{-3}(\mathbf{L}^\epsilon)^{-1}\,\mathbf{F}(U + \epsilon^3 s, \epsilon), & \epsilon \neq 0,\\ s, & \epsilon = 0.\end{cases} \tag{2.12}$$

Clearly $\mathbf{H}$ is continuous differentiable for $\epsilon \neq 0$. We shall show that it is also for $\epsilon = 0$. Using a Taylor series expansion with remainder of order 2, we have

$$\begin{aligned}\mathbf{F}(U + \epsilon^3 s, \epsilon) &= \mathbf{F}(U, \epsilon) + \epsilon^3\mathbf{F}_1(U, \epsilon; s) + \epsilon^6\Psi(s, \epsilon)\\ &= \epsilon^6 q + \epsilon^3\mathbf{L}^\epsilon s + \epsilon^6\Psi(s, \epsilon),\end{aligned}$$

where $|\Psi(s, \epsilon)|_0 \leqslant K_4$ for $|s|_2 < 1$ and $|\epsilon| < 1$. Thus, $\mathbf{H}(s, \epsilon) - s = \epsilon^3(\mathbf{L}^\epsilon)^{-1}q + \epsilon^3(\mathbf{L}^\epsilon)^{-1}\Psi$, and from (2.7) and (2.8)

$$\begin{aligned}|\mathbf{H}(s, \epsilon) - s|_2 &\leqslant \epsilon^{-2}|\mathbf{H}(s, \epsilon) - s|_2^\epsilon\\ &\leqslant K_5\epsilon(|q|_0 + |\Psi|_0) \leqslant K_6\epsilon\end{aligned} \tag{2.13}$$

for $|s|_2 < 1$ and $|\epsilon| < 1$. Thus $\mathbf{H}$ is continuous at $\epsilon = 0$.

For $\epsilon \neq 0$ we have

$$\begin{aligned}\mathbf{H}_1(s, \epsilon) &= (\mathbf{L}^\epsilon)^{-1}\mathbf{F}_1(U + \epsilon^3 s, s)\\ &= \mathbf{I} + (\mathbf{L}^\epsilon)^{-1}(\mathbf{F}_1(U + \epsilon^3 s, \epsilon) - \mathbf{F}_1(U, \epsilon)) \equiv \mathbf{I} + \mathbf{T}(s, \epsilon).\end{aligned}$$

But $\mathbf{F}_1(u, \epsilon)$ is a linear second order differential operator with ϵ multiplying derivatives as in $\mathbf{L}^\epsilon$, and coefficients depending differentiably on u, $\epsilon u'$, and $\epsilon^2 u''$. Thus, by the mean value theorem, $\mathbf{G}(s, \epsilon) \equiv \mathbf{F}_1(U + \epsilon^3 s, \epsilon) - \mathbf{F}_1(U, s)$ is also such an operator, with coefficients bounded in C^0 by $K_7\epsilon^3 | s |_2^\epsilon$ for $\epsilon^3 | s |_2^\epsilon < 1$. Thus, $| \mathbf{G}(s, \epsilon; v)|_0 \leqslant K_8\epsilon^3 | s |_2^\epsilon | v |_2^\epsilon \leqslant K_8\epsilon^3 | s |_2 | v |_2$, so by (2.8), $| \mathbf{T}(s, \epsilon; v)|_2 \leqslant \epsilon^{-2} | \mathbf{T}(s, \epsilon, v)|_2^\epsilon = \epsilon^{-2} |(\mathbf{L}^\epsilon)^{-1}\mathbf{G} |_2^\epsilon \leqslant \epsilon^{-2} K_2 | \mathbf{G} |_0 \leqslant K_9 \epsilon | s |_2 | v |_2$. Thus, in operator norm, $| \mathbf{T}(s, \epsilon)|_{B(C^2, C^2)} \leqslant K_9\epsilon | s |_2$. This implies that $\mathbf{H}_1(s, \epsilon)$ is continuous at $\epsilon = 0$, and, of course, $\mathbf{H}_1(s, 0) = \mathbf{I}$.

By the implicit function theorem, there is a function $s(\epsilon) \in C^2$ defined for $| \epsilon | < \epsilon_0$, satisfying $| s(\epsilon)|_2 \leqslant K_{10}\epsilon$ and $\mathbf{H}(s(\epsilon), \epsilon) = 0$. From (2.12) we, therefore, have $\mathbf{F}(w(\epsilon), \epsilon) = 0$, where $w(\epsilon) = U(\epsilon) + \epsilon^3 s(\epsilon)$. The existence of the required families is thereby proved.

We consider the question of uniqueness. Suppose that $\bar{w}(\epsilon) \in C^2$ is another family approaching g, with $| \bar{w}(\epsilon)|_2$ bounded for $| \epsilon | < 1$. Then clearly $| \bar{w}(\epsilon) - g |_2^\epsilon \to 0$ as $\epsilon \to 0$. Let $v(\epsilon) = \bar{w}(\epsilon) - w(\epsilon)$. Since both w and $\bar{w}$ satisfy (2.1), we have by the mean value theorem,

$$0 = \mathbf{F}(\bar{w}, \epsilon) - \mathbf{F}(w, \epsilon) = \mathbf{F}_1(w + \Theta v, \epsilon; v)$$

for $0 < \theta < 1$. The right side can be thought of as a linear differential operator in v, with coefficients depending also on v. By (2.2), this operator is subject to the maximum principle for small enough ϵ and small enough $| v |_2^\epsilon$. But $| v |_2^\epsilon$ can be guaranteed small by making ϵ small, because by construction $| w - g |_2^\epsilon \to 0$, and by assumption $| \bar{w} - g |_2^\epsilon \to 0$. Thus, for small enough ϵ, the maximum principle applies, and we obtain that $v \equiv 0$, so $\bar{w} = w$. This completes the proof.

3. A Lemma of Implicit Function Type

The proof of existence of a transition layer given in Section 4 will be based on a variant of the implicit function theorem. This section is devoted to proving the needed variant.

In the following, X and Y denote Banach spaces, D a neighborhood of the origin in X, and I an open interval on the real line containing the origin. Let $M\colon D \times I \times I \to Y$ and $m\colon D \times I \times I \to R^1$ be continuous mappings, continuously differentiable in their first two arguments, satisfying the following hypotheses (here subscripts denote partial Frechet derivatives):

(i) $M(0, 0, 0) = m(0, 0, 0) = 0$;

(ii) $M_1(0, 0, 0)$ is a linear operator with one-dimensional nullspace spanned by $\phi \in X$ and range characterized by

$$\mathscr{R}(M_1(0, 0, 0)) = \{v \in Y: \langle \phi^*, v \rangle = 0\}$$

for some $\phi^* \in Y^*$;

(iii) $\langle \phi^*, M_2(0, 0, 0; 1) \rangle \neq 0$;

(iv) $m_1(0, 0, 0; \phi) \neq 0$.

LEMMA 3.1. *Let M and m be as described above. Then there exist unique continuous functions $u(\epsilon)$, $\lambda(\epsilon)$, defined for $|\epsilon| < \epsilon_1$ for some $\epsilon_1 > 0$, satisfying*

$$u(0) = 0, \qquad \lambda(0) = 0, \qquad M(u(\epsilon), \lambda(\epsilon), \epsilon) = 0; \tag{3.1}$$

$$m(u(\epsilon), \lambda(\epsilon), \epsilon) = 0. \tag{3.2}$$

Proof. Let P_1 denote any projection of X onto the nullspace $\mathscr{N}(M_1(0, 0, 0))$ and $Q_1 = I_X - P_1$. Let Q_2 be any projection of Y onto $\mathscr{R}(M_1(0, 0, 0))$, and $P_2 = I_Y - Q_2$, where I_X, I_Y are the identity mappings in the respective spaces. Let L denote the restriction of $M_1(0, 0, 0)$ to Q_1X. It is one-to-one since $Lu = 0$ implies $u = a\phi$, which implies $u = 0$. Also it maps onto Q_2Y by (ii), so by the closed graph theorem, it is a bicontinuous map between Q_1X and Q_2Y.

Let $I_1 \subset I$ be an open interval containing the origin, and D_1 an open neighborhood of the origin in Q_1X, such that $D \supset \{z + \alpha\phi: z \in D_1, \alpha \in I_1\}$. Let $W_1 = D_1 \times I_1 \times I_1$, $W_2 = Q_2Y \times P_2Y \times R^1$, and $N: W_1 \times I \to W_2$ be the mapping defined by

$$N(w, \epsilon) = (A(z, \alpha, \lambda, \epsilon), B(z, \alpha, \lambda, \epsilon), C(z, \alpha, \lambda, \epsilon)),$$

where $w = (z, \alpha, \lambda) \in W_1$ and the operators on the right are defined by

$$A(z, \alpha, \lambda, \epsilon) = Q_2M(z + \alpha\phi, \lambda, \epsilon),$$
$$B(z, \alpha, \lambda, \epsilon) = P_2M(z + \alpha\phi, \lambda, \epsilon),$$

and

$$C(z, \alpha, \lambda, \epsilon) = m(z + \alpha\phi, \lambda, \epsilon).$$

It suffices to prove that $N(w, \epsilon) = 0$ has a unique continuous solution $w(\epsilon)$

with $w(0) = 0$, since this equation is equivalent to (3.1), (3.2) with $u(\epsilon) \equiv z(\epsilon) + \alpha(\epsilon)\phi$.

Clearly $N(0, 0) = 0$. Also N is continuously differentiable in w and continuous in ϵ. If the derivative $N_1(0, 0)$ is a homeomorphism from W_1 to W_2, the implicit function theorem will then yield the desired result. This derivative can be expressed as the following Jacobian matrix, where for simplicity we use such notation as $A_1(0)$ in place of $A_1(0, 0, 0, 0)$:

$$N_1(0, 0) = \begin{pmatrix} A_1(0) & A_2(0) & A_3(0) \\ B_1(0) & B_2(0) & B_3(0) \\ C_1(0) & C_2(0) & C_3(0) \end{pmatrix}.$$

We proceed to evaluate some of the elements:

$A_1(0) = Q_2M_1(0)|_{Q_1X} = L$;

$A_2(0)\colon R^1 \to Q_2Y$ is the operator $A_2(0; \alpha) = \alpha Q_2M_1(0; \phi) = 0$, since ϕ is in the nullspace of $M_1(0)$;

$B_1(0) = P_2M_1(0) = 0$, by the definition of P_2;

$B_2(0)\colon R^1 \to P_2Y$ is the operator $B_2(0; \alpha) = \alpha P_2M_1(0; \phi) = 0$;

$B_3(0)$ is given by $B_3(0; \alpha) = P_2M_2(0; \alpha) = \alpha P_2\psi$, where $\psi = M_2(0; 1)$, so that by (iii), $P_2\psi \neq 0$;

$C_2(0)\colon R^1 \to R^1$ is the operation of multiplication by $m_1(0; \phi) = \beta$, which is different from 0, by (iv).

Thus,

$$N_1(0, 0) = \begin{pmatrix} L & 0 & A_3(0) \\ 0 & 0 & S_{P_2\psi} \\ C_1(0) & S_\beta & C_3(0) \end{pmatrix},$$

where we use the symbols $S_{P_2\psi}$ and S_β to denote the linear operators with domain R^1 consisting of multiplication by $P_2\psi$ and β, respectively. It is now easily seen that $N_1(0, 0)$ has a bounded inverse, so is a homeomorphism of W_1 onto W_2. In fact, for any $v = (\bar{z}, \bar{y}, \gamma) \in W_2$, the equation

$$N_1(0, 0; w) = v \tag{3.3}$$

may be solved for $w = (z, \alpha, \lambda)$ as follows. Since P_2X is one-dimensional, it is spanned by any nonzero element such as $P_2\psi$; therefore, we may write $\bar{y} = \nu P_2\psi$ for some $\nu \in R^1$. The second equation in the system (3.3) reads $\lambda P_2\psi = \nu P_2\psi$, which has the unique solution $\lambda = \nu$. The first equation then reads $Lz + A_3(0; \lambda) = \bar{z}$, which has a unique solution z depending continuously on $\nu = \lambda$ and $\bar{z}$, by virtue of the bounded invertibility of L. Finally, the third equation can be solved uniquely for α since by (iv), $\beta \neq 0$.

The implicit function theorem is now applicable, and the lemma is proved.

4. Existence of a Transitional Family

We now seek solutions $u(x, \epsilon)$ of (2.1) with the property that for any $\delta > 0$,

$$\lim_{\epsilon \to 0} u(x, \epsilon) = \begin{cases} g^1(x) & \text{uniformly in } (-\infty, -\delta), \\ g^2(x) & \text{uniformly in } (\delta, \infty). \end{cases} \tag{4.1}$$

Our goal will be to obtain such a family in the form

$$u(x, \epsilon) = w^1(x, \epsilon) + \tilde{y}(x/\epsilon, \epsilon)(w^2(x, \epsilon) - w^1(x, \epsilon)),$$

where w^i are the regular families constructed in Theorem 2.1. Thus, a sharp transition at the origin is brought about by prescribing $\tilde{y}$ to depend on the stretched variable (x/ϵ). However, it proves to be more convenient to use a shifted stretched variable $\eta(x, \epsilon) = (x - \lambda(\epsilon)/\epsilon)$, and a transition function $y(\eta, \epsilon) = \tilde{y}(x/\epsilon, \epsilon) = \tilde{y}(\eta + (\lambda/\epsilon), \epsilon)$, where λ is a regular function of ϵ satisfying $\lambda(0) = 0$, and adjusted so that $y(0, \epsilon)$ is independent of ϵ. If $\tilde{y}((x/\epsilon), \epsilon)$ were known, and a number a were given such that $\tilde{y}_1(a, 0) \neq 0$ (and there must be such a number, since $\tilde{y}$ is not constant), then the equation $\tilde{y}(\mu(\epsilon), \epsilon) = \tilde{y}(a, 0)$ could be solved for an appropriate $\mu(\epsilon)$ satisfying $\mu(0) = a$. Setting $\lambda(\epsilon) = \epsilon\mu(\epsilon)$, we would then indeed have that $y(0, \epsilon) = \tilde{y}(\mu(\epsilon), \epsilon)$ is independent of ϵ. However, we shall work from the other direction and attempt to find a family of solutions of (2.1) of the form

$$\begin{aligned} u(x, \epsilon) &= w^1(x, \epsilon) + y(\eta(x, \epsilon), \epsilon)(w^2(x, \epsilon) - w^1(x, \epsilon)) \\ &\equiv w^1 + y\,\Delta w, \end{aligned} \tag{4.2}$$

where y satisfies

$$\lim_{\eta \to -\infty} y(\eta, \epsilon) = 0; \qquad \lim_{\eta \to \infty} y(\eta, \epsilon) = 1, \tag{4.3}$$

these convergence processes being uniform in ϵ for ϵ in some open interval containing the origin. As mentioned above, at the same time we adjust $\lambda(\epsilon)$ so that $y(0, \epsilon)$ is independent of ϵ.

To see that (4.2) will then be a transitional family, we note that

$$| u(x, \epsilon) - g^1(x)| \leqslant | w^1(x, \epsilon) - g^1(x)| + | y(\eta(x, \epsilon), \epsilon)| \; | \Delta w(x, \epsilon)|.$$

By (2.3), the first term on the right converges to 0 as $\epsilon \to 0$ uniformly in x.

By virtue of the uniformity in (4.3), the factor $y(\eta(x, \epsilon), \epsilon) \to 0$ as $\epsilon \to 0$ uniformly for $x \leqslant -\delta < 0$. Finally, the factor $| \Delta w |$ is bounded. Thus, u satisfies the first part of (4.1). The second part is established analogously. It will be proved in Section 5 that in general there exists only one transitional family of the type (4.2).

Accordingly, we define a differential operator G for a function $y(\eta, \epsilon)$ by substituting (4.2) into the differential operator F and writing the result in terms of η, λ, and ϵ (noting that $x = \epsilon\eta + \lambda$):

$$G(y_{\eta\eta}, y_\eta, y, \epsilon\eta + \lambda, \epsilon) \equiv F(\Delta w(\epsilon\eta + \lambda, \epsilon) y_{\eta\eta} + \epsilon^2 w^1_{xx}(\epsilon\eta + \lambda, \epsilon) + 2\epsilon \Delta w_x y_\eta + \epsilon^2 \Delta w_{xx} y, \Delta w y_\eta + \epsilon w_x{}^1 + \epsilon \Delta w_x y, \ldots).$$

We then seek a solution pair $y(\eta, \epsilon)$, $\lambda(\epsilon)$ of $G = 0$, satisfying (4.3) and $\lambda(0) = 0$. In particular for $\epsilon = 0$, we obtain the following equation for $y_0(\eta) \equiv y(\eta, 0)$:

$$G(y_0'', y_0', y_0, 0, 0) \equiv F(\Delta g(0) y_0'', \Delta g(0) y_0', \Delta g(0) y_0 + g^1(0), 0, 0) = 0. \tag{4.4}$$

In view of the requirements (4.3), the following assumption is necessary.

HYPOTHESIS 2. *There is a solution* $y_0(\eta)$ *of* (4.4) *satisfying*

$$y_0(-\infty) = 0, \qquad y_0(\infty) = 1. \tag{4.5}$$

Note that Eq. (4.4) does not explicitly involve η. Hence, from the solution $y_0(\eta)$ we obtain many more solutions $y_0(\eta - C)$, C an arbitrary constant. Therefore, no generality is lost by supposing that $y_0'(0) \neq 0$; if this is not true, we replace $y_0(\eta)$ by $y_0(\eta - C)$, C being chosen so that it is true. We, therefore, have

$$y_0'(0) \neq 0. \tag{4.6}$$

One additional assumption will suffice for the existence of a transitional family.

HYPOTHESIS 3. $a(\eta) > 0$, *and*

$$\int_{-\infty}^{\infty} r(\eta) F_4(\Delta g(0) y_0''(\eta), \Delta g(0) y_0'(\eta), \Delta g(0) y_0(\eta) + g^1(0), 0, 0) y_0'(\eta)\, d\eta \neq 0,$$

where

$$r(\eta) \equiv \exp \int_0^\eta \frac{b(\bar\eta) - a'(\bar\eta)}{a(\bar\eta)}\, d\bar\eta, \tag{4.7}$$

$$a(\eta) \equiv G_1(y_0''(\eta), y_0'(\eta), y_0(\eta), 0, 0), \tag{4.8a}$$

and

$$b(\eta) \equiv G_2(\cdots). \tag{4.8b}$$

For future reference, we also define

$$c(\eta) = G_3(\cdots). \tag{4.8c}$$

In view of the nonuniqueness of solutions of (4.4), (4.5), it may be wondered whether Hypothesis 3 is satisfied for some choices of $y_0(\eta)$, but not others. In connection with the uniqueness proof in Section 5, it will be shown that in the usual case, this hypothesis is satisfied for all possible choices of y_0 if and only if it is satisfied for one of them.

THEOREM 4.1. *Under Hypotheses* 1–3, *there exists a family of solutions* $y(\eta, \epsilon)$, $\lambda(\epsilon)$ *of*

$$G(y_{\eta\eta}, y_\eta, y, \epsilon\eta + \lambda, \epsilon) = 0 \tag{4.9}$$

defined in some interval $|\epsilon| < \epsilon_1$, $-\infty < \eta < \infty$, *continuous in* ϵ *uniformly in* η, *satisfying* $\lambda(0) = 0$ *and* (4.3), *these latter limits being approached uniformly in* ϵ. *Hence, there exists a transitional family of solutions* (4.2) *of* (2.1) *satisfying* (4.1).

It will be convenient to work with the function $v = y - y_0$ rather than y. We thus seek solutions $v(\eta, \epsilon)$, $\lambda(\epsilon)$ of

$$G(y_0'' + v_{\eta\eta}, y_0' + v_\eta, y_0 + v, \epsilon n + \lambda, \epsilon) = 0 \tag{4.10}$$

satisfying

$$\lim_{|\eta|\to\infty} v(\eta, \epsilon) = 0, \tag{4.11a}$$

the approach being uniform in ϵ. We also impose the condition

$$v(0, \epsilon) = 0, \tag{4.11b}$$

so that $y(0, \epsilon) = \text{const}$. This has the effect of characterizing $\lambda(\epsilon)$ as that number such that $u(\lambda, \epsilon) = \theta w^1(\lambda, \epsilon) + (1 - \theta)\, w^2(\lambda, \epsilon)$, where $\theta = y_0(0)$.

The theorem will be proved with the aid of a series of lemmas developed below, which essentially demonstrate the applicability of Lemma 3.1 to the present problem. The first step is to interpret the left side of (4.10) as an operator from a suitable Banach space X into another space Y. For our purposes it is appropriate to use the following spaces, where r is the positive function defined by (4.7).

$$X \equiv H_2^{(r)} \cap C^2 \equiv \text{the space of functions } v(\eta)$$

with bounded continuous derivatives up to order two, for which the norm

$$\| v \|_X \equiv | v |_2 + \left(\sum_{k=0}^{2} \int_{-\infty}^{\infty} r(\eta) \mid v^{(k)}(\eta)|^2 \, d\eta \right)^{1/2}$$

is finite; $Y \equiv H_0^{(r)} \cap C^0 \equiv$ the space of bounded continuous functions with finite norm

$$\| v \|_Y \equiv | v |_0 + \left(\int_{-\infty}^{\infty} r(\eta)(v(\eta))^2 \, d\eta \right)^{1/2}.$$

LEMMA 4.2. *Under Hypotheses* 1–3, *the differential operator on the left of* (4.10) *defines an operator* $\mathbf{G}(v, \lambda, \epsilon)$ *from* $X \times R^1 \times R^1$ *into* Y.

Proof. If $v \in X$, quite clearly the function on the left of (4.10) (call it $f(\eta, \lambda, \epsilon)$) will be in C^0 for each λ, ϵ. The lemma will be proved if we can show that $\int_{-\infty}^{\infty} rf^2 \, d\eta < \infty$ for each λ and ϵ. Since (2.1) is satisfied by both $u = w_1(x, \epsilon)$ and $u = w_2(x \; \epsilon)$, we know that

$$G(0, 0, 1, \epsilon\eta + \lambda, \epsilon) \equiv G(0, 0, 0, \epsilon\eta + \lambda, \epsilon) \equiv 0. \tag{4.12}$$

Thus, by the mean value theorem we have

$$f(\eta, \lambda, \epsilon) = G_1(\theta y_{\eta\eta}, \theta y_\eta, \theta y, \epsilon\eta + \lambda, \epsilon) \, y'' + G_2(\cdots) \, y' + G_3(\cdots) \, y,$$

where $0 < \Theta(\eta) < 1$. Since F and G have derivatives bounded uniformly in $x = \epsilon\eta + \lambda$, each of these three coefficients G_i is bounded in terms of the magnitude of their first three arguments, uniformly in η. Therefore, we have a function $K_1(M_0)$ such that the following estimates hold for all $| y |_2 < M_0$ and $| \epsilon | < 1$:

$$| f(\eta, \lambda, \epsilon)| \leqslant K_1(M_0)(| y''(\eta)| + | y'(\eta)| + | y(\eta)|). \tag{4.13a}$$

Using the left equation in (4.12) similarly, we obtain

$$| f(\eta, \lambda, \epsilon)| \leqslant K_1(M_0)(| y''(\eta)| + | y'(\eta)| + | y(\eta) - 1 |). \tag{4.13b}$$

Setting $y = y_0 + v$, we have on the one hand, from (4.13a), that

$$\int_{-\infty}^{0} r(\eta)(f(\eta, \lambda, \epsilon))^2 \, d\eta$$
$$\leqslant K_2(M_0) \left\{ \int_{-\infty}^{0} r(\eta)[(y_0'')^2 + (y_0')^3 + (y_0)^2] \, d\eta + \| v \|_X^2 \right\};$$

and on the other hand, from (4.13b), that

$$\int_0^\infty r(\eta) f(\eta, \lambda, \epsilon))^2 \, d\eta$$
$$\leqslant K_2(M_0) \left\{ \int_0^\infty r(\eta)((y_0'')^2 + (y_0')^2 + (y_0 - 1)^2) \, d\eta + \| v \|_X^2 \right\}.$$

If the integrals on the right of these two inequalities are finite, we conclude that $\int_{-\infty}^{\infty} rf^2 \, d\eta < \infty$, hence $f \in Y$. This finiteness will be established by analyzing the asymptotic behavior of r and y_0.

Referring to (4.8) and (4.5), we define the constants $a^+ = a(\infty) = G_1(0, 0, 1, 0, 0)$; $a^- = a(-\infty) = G_1(0, 0, 0, 0, 0)$; with analogous definitions for $b^\pm$ and $c^\pm$. From (2.2) we obtain

$$a^\pm > 0; \qquad c^\pm < 0. \tag{4.14}$$

From (4.14) and Hypothesis 3 we see that $a(\eta)$ is positive and bounded away from zero. From this, (4.7), and the fact that $a'(\eta) \to 0$, we obtain that for any $\delta > 0$ there exists a constant C_δ with

$$r(\eta) \leqslant C_\delta \exp[(b^+/a^+) + \delta]\eta \qquad \text{for} \quad \eta \geqslant 0, \tag{4.15a}$$

$$r(\eta) \leqslant C_\delta \exp[(b^-/a^-) - \delta]\eta \qquad \text{for} \quad \eta \leqslant 0. \tag{4.15b}$$

Now consider the Eq. (4.4), satisfied by $y = y_0(\eta)$. Since $G(0, 0, 1, 0, 0) = 0$, one may use the mean value theorem to write it as

$$G_1(\theta y''_0, \theta y_0', 1 + \theta(y_0 - 1), 0, 0)\, y_0'' + G_2(\cdots)\, y_0' + G_3(\cdots)(y_0 - 1) = 0, \tag{4.16}$$

where $0 < \Theta(\eta) < 1$. Write this as $L^*(y_0 - 1) = 0$, where L^* is the linear operator on the left.

Let $\nu > 0$ be a constant such that $c^+ < c^+ + \nu < 0$ and let

$$\mu = (b^+ + ((b^+)^2 - 4a^+(c^+ + \nu))^{1/2}/2a^+) > 0,$$

so that $a^+\mu^2 - b^+\mu + c^+ = -\nu < 0$. Now notice that (4.5) implies that the coefficients of (4.16) approach a^+, b^+, c^+ respectively as $\eta \to \infty$. Therefore, for some large enough η_0,

$$G_1(\theta y_0'', \theta y_0', 1 + \theta(y_0 - 1), 0, 0)\, \mu^2 - G_2(\cdots)\mu + G_3(\cdots) < 0,$$

$G_3 < 0$, and $G_1 > 0$, for $\eta \geqslant \eta_0$.

Setting $V(\eta) = e^{-\mu(\eta - \eta_0)}$, we thus find that $L^* V \leqslant 0$ for $\eta \geqslant \eta_0$. If $k > 0$ is such that $| y_0(\eta_0) - 1 | \leqslant k$, we have from this and (4.16) that

$w \equiv kV(\eta) \pm (y_0(\eta) - 1)$ satisfies $L^*w \leqslant 0$, $w(\eta_0) \geqslant 0$, $w(\infty) = 0$. The maximum principle now tells us that $w \geqslant 0$, which means $|y_0 - 1| \leqslant ke^{-\mu(\eta-\eta_0)}$, for $\eta \geqslant \eta_0$. Equation (4.16) says $|y_0''| \leqslant C(|y_0 - 1| + |y_0'|)$ for some C. A standard interpolation inequality (see, e.g., [4, p. 114]), yields for each η_1, the following, where the suprema are taken over the interval $(\eta_1, \eta_1 + 1)$, and $\omega > 0$ is arbitrary:

$$\sup |y_0'(\eta)| \leqslant \omega \sup |y_0''(\eta)| + (1/2\omega) \sup |y_0(\eta) - 1|.$$

Choosing ω small enough and combining this with the above inequality, we obtain for $\eta_1 \geqslant \eta_0$,

$$\sup |y_0''| \leqslant C \sup |y_0 - 1| \leqslant Ce^{-\mu\eta_1},$$

with a similar exponential estimate for y_0'. Thus for some C,

$$|y_0 - 1|^2 + |y_0'|^2 + |y_0''|^2 \leqslant C \exp(-2\mu\eta) \tag{4.17}$$

for $\eta \geqslant \eta_0$.

In (4.15) we now choose $\delta > 0$ so small that $4(a^+)^2 \delta^2 \leqslant (b^+)^2 - 4a^+(c^+ + \nu)$. This inequality implies that $(b^+/a^+ + \delta) - 2\mu \leqslant -\delta < 0$, which, from (4.15a) and (4.17), in turn implies that $r(\eta)[(y_0'')^2 + (y_0')^2 + (y_0 - 1)^2]$ decays exponentially as $\eta \to \infty$. Thus, $\int_0^\infty rf^2\, d\eta < \infty$. A similar argument shows that $\int_{-\infty}^0 rf^2\, d\eta < \infty$. Thus $f \in Y$, and the lemma is proved.

LEMMA 4.3. *The operator* $\mathbf{G}$ *defined in Lemma* 4.2 *is continuously differentiable.*

Proof. The symbol C will always denote a constant independent of ϵ. Let v and $z \in X$, and $\| z \|_X < 1$. Expanding G in a Taylor series in its first three arguments with remainder Γ of order 2, we obtain

$$\begin{aligned}\mathbf{G}(v + z, \lambda, \epsilon) &= G(y_0'' + (v + z)'', y_0' + (v + z)', y_0 + (v + z), \epsilon\eta + \lambda, \epsilon)\\ &= \mathbf{G}(v, \lambda, \epsilon) + [G_1(y_0'' + v'', y_0' + v', y_0 + v, \epsilon\eta + \lambda, \epsilon)\, z''\\ &\quad + G_2(\cdots)\, z' + G_3(\cdots) z] + \Gamma(z'', z', z, \eta, \lambda, \epsilon),\end{aligned}$$

where for some C depending on v and its derivatives,

$$|\Gamma| \leqslant C(|z''| + |z'| + |z|)^2 \leqslant C|z|_2(|z''| + |z'| + |z|),$$

so that

$$\left(\int_{-\infty}^{\infty} r(\eta)\, \Gamma^2\, d\eta\right)^{1/2} \leqslant C\, |z|_2 \left(\int_{-\infty}^{\infty} r[(z'')^2 + (z')^2 + z^2]\, d\eta\right)^{1/2} \leqslant C \| z \|_X^2 .$$

Hence,

$$\| \Gamma \|_Y \leqslant C(\| z \|_X)^2.$$

Therefore, $\mathbf{G}$ is differentiable with respect to its first variable, and the derivative is given by $\mathbf{G}_1(v, \lambda, \epsilon; z) = G_1 z'' + G_2 z' + G_3 z$. To show that $\mathbf{G}_1$ depends continuously on v, λ, and ϵ, we note that

$$\| \mathbf{G}_1(v, \lambda, \epsilon; z) \|_Y$$
$$\leqslant C(| G_1(y_0'' + v'', y_0' + v', y_0 + v, \epsilon\eta + \lambda, \epsilon)|_0 + | G_2 |_0 + | G_3 |_0) \| z \|_X ,$$

so that the operator norm of $\mathbf{G}_1$ is estimated in terms of the suprema of the three coefficients. These in turn depend continuously on their five arguments, uniformly in η. It follows fairly easily that the mapping $(v, \lambda, \epsilon) \to \mathbf{G}_1(v, \lambda, \epsilon)$ is continuous with respect to the C^2 norm of v, hence, certainly with respect to the norm $\| v \|_X$. We omit the proof of continuous differentiability of $\mathbf{G}$ with respect to λ and ϵ, which follows a similar vein. This establishes the lemma.

LEMMA 4.4. *Let $v(\eta)$ be continuous in $[0, \infty)$, and satisfy*

$$Lv \equiv a^*(\eta)\, v'' + b^*(\eta) v' + c^*(\eta) v = y(\eta),$$

where the coefficients and y are bounded. Assume that for some $\eta_0 > 0$, $a^* > 0$ and $c^* \leqslant -\sigma < 0$ for $\eta > \eta_0$.

(a) *If v is bounded and $y \equiv 0$, then there exist positive constants C_1 and α, depending only on η_0, σ, and upper bounds for a^* and $| b^* |$, such that*

$$| v(\eta) | < C_1 | v |_0\, e^{-\alpha\eta}.$$

(b) *If $a^* \to a^+ > 0$, $b^* \to b^+$, $c^* \to c^+ < 0$ as $\eta \to \infty$, and*

$$\int_0^\infty [\exp(b^+/a^+ - \delta)\eta]\, v^2(\eta)\, d\eta < \infty \tag{4.18}$$

for some sufficiently small δ, then v is bounded.

Proof. First, consider case (b). Let

$$\mu = (-b^+ + ((b^+)^2 - 4a^+(c^+/2))^{1/2}/2a^+) > 0,$$

so $a^+\mu^2 + b^+\mu + c^+ = c^+/2 < 0$. Then $L(e^{\mu\eta}) = (a^*\mu^2 + b^*\mu + c^*) e^{\mu\eta} \leqslant (a^+\mu^2 + b^+\mu + c^+ + \gamma(\eta)) e^{\mu\eta}$, where γ is a positive function such that $\lim_{\eta\to\infty} \gamma(\eta) = 0$. Let η_1 be large enough so that $\eta_1 > \eta_0$, and for $\eta > \eta_1$, we have $c^+/2 + \gamma(\eta) < 0$ and

$$c^*(\eta) < c^+/2. \tag{4.19}$$

Then

$$L(e^{\mu\eta}) < 0, \qquad \eta > \eta_1 . \tag{4.20}$$

Furthermore,

$$\int_0^\infty [\exp(b^+/a^+ - \delta)\eta]\, e^{2\mu\eta}\, d\eta$$
$$= \int_0^\infty \exp\{-\delta + ((b^+)^2 - 2a^+c^+)^{1/2}/2a^+\}\eta\, d\eta = \infty,$$

provided $0 < \delta < (-c^+/2a^+)^{1/2}$. Therefore, for this choice of δ, the condition (4.18) implies the existence of a sequence $\{\eta_i\}$ with $\eta_i \to \infty$ and

$$| v(\eta_i)|\, e^{-\mu\eta_i} \equiv \beta_i \to 0.$$

For each $i > 1$, let

$$w_i(\eta) \equiv \beta_i e^{\mu\eta} - v(\eta) + 2\, | y |_0 (-c^+)^{-1} + | v(\eta_1)|\, e^{-\alpha(\eta-\eta_1)},$$

where α is a positive number. Then from (4.20) and (4.19),

$$Lw_i < -Lv + c^+ \,| y |_0 (-c^+)^{-1} + | v(\eta_1)|\, (a^*\alpha^2 - b^*\alpha + c^*)\, e^{-\alpha(\eta-\eta_1)} < 0,$$
$$\eta > \eta_1 ,$$

provided $\alpha > 0$ is chosen so small that $a^*\alpha^2 - b^*\alpha + c^* < 0$, $\eta > \eta_1$. Also

$$w_i(\eta_1) > 0$$

and

$$w_i(\eta_i) > | v(\eta_i)| - v(\eta_i) \geqslant 0.$$

By the maximum principle, we conclude that $w_i(\eta) \geqslant 0$ for $\eta \in (\eta_1 , \eta_i)$, so that

$$v(\eta) \leqslant \beta_i e^{\mu\eta} + 2\, | y |_0 (-c^+)^{-1} + | v(\eta_1)|\, e^{-\alpha(\eta-\eta_1)}, \qquad \eta \in (\eta_1 , \eta_i).$$

Letting $i \to \infty$, we have

$$v(\eta) \leqslant 2\, | y |_0 (-c^+)^{-1} + | v(\eta_1)|\, e^{-\alpha(\eta-\eta_1)}, \qquad \eta \in (\eta_1 , \infty). \tag{4.21}$$

A similar argument shows that $(-v)$, hence $| v |$, satisfies the same inequality. The continuity of v implies its boundedness on the remaining interval $[0, \eta_1]$.

Next consider case (a). Here we simply define μ as any positive number such that $(a^*\mu^2 + b^*\mu + c^*) < 0$ for $\eta > \eta_0$, so that $L(e^{\mu\eta}) < 0$. The numbers β_i and functions w_i are defined as before, except with $y \equiv 0$ and η_1 replaced by η_0. The conclusion (4.21) with the first term on the right missing holds as before. Clearly α depends only on the quantities indicated. This completes the proof.

We shall need some properties of the linear operator $\mathbf{P}: X \to Y$ given by

$$\mathbf{P}u \equiv \mathbf{G}_1(0, 0, 0; u) \equiv a(\eta)u'' + b(\eta)u' + c(\eta)u, \tag{4.22}$$

where the functions a, b, and c were defined by (4.8).

LEMMA 4.5. *There exists a function $\phi \in X$ such that the nullspace $\mathcal{N}(\mathbf{P})$ is spanned by ϕ, and the range*

$$\mathcal{R}(\mathbf{P}) = \left\{ y \in Y: \int_{-\infty}^{\infty} r(\eta)\, \phi(\eta)\, y(\eta)\, d\eta = 0 \right\}.$$

Proof. The function r (4.7) was chosen so that $\mathbf{P}u \equiv (1/r)[(rau')' + rcu]$, which is formally self-adjoint with respect to the scalar product $(u, v)_r \equiv \int_{-\infty}^{\infty} r(\eta)\, u(\eta)\, v(\eta)\, d\eta$. Let $\bar{\mathbf{P}}$ be a self-adjoint extension of $\mathbf{P}$ in the space $\mathcal{L}_2^r(-\infty, \infty)$ endowed with this weighted scalar product. A fundamental result by Weyl [7] yields that the spectrum of $-\bar{\mathbf{P}}$ is discrete below the constant $e_0 \equiv \liminf_{|\eta| \to \infty}(-c(\eta)) = \min[|c^-|, |c^+|] > 0$. Therefore [2, Theorems XIII.7.53–54] for each $\lambda^* < e_0$, $\mathcal{R}(-\bar{\mathbf{P}} - \lambda^* I)$ is closed, and λ^* is either a simple eigenvalue or in the resolvent set. In particular, this is true for $\lambda^* = 0$.

We differentiate Eq. (4.4) with respect to η. It is thereby seen that the function $\phi(\eta) = y_0'(\eta)$ satisfies $\mathbf{P}\phi = 0$, at least formally. It is in fact true strictly that $\bar{\mathbf{P}}\phi = 0$, since ϕ is in the domain of $\bar{\mathbf{P}}$, which is to say (see [3, p.274]) that ϕ, ϕ', and $\phi'' \in \mathcal{L}_2^r(-\infty, \infty)$. To see this, we first recall from the proof of Lemma 4.2 that $\phi = y_0'$ and $\phi' = y_0''$ behave well enough as $\eta \to \pm\infty$ so that they are in that space. Since $a\phi'' = b\phi' - c\phi$, b and c are bounded and a is bounded away from 0, we conclude that $\phi'' \in \mathcal{L}_2^r(-\infty, \infty)$. Thus ϕ is an eigenfunction with simple eigenvalue 0. Thus $\mathcal{N}(\bar{\mathbf{P}})$ is spanned by ϕ. But the same is, therefore, true of $\mathcal{N}(\mathbf{P})$ since $\mathbf{P} \subset \bar{\mathbf{P}}$. This proves the first part of the lemma.

For the second, we first use the fact that for self-adjoint operators such as $\bar{\mathbf{P}}$ with closed range $\mathcal{R}(\bar{\mathbf{P}}) = \mathcal{N}(\bar{\mathbf{P}})^\perp = \{y \in \mathcal{L}_2^r: (y, \phi)_r = 0\}$. Since $\mathcal{R}(\mathbf{P}) \subset \mathcal{R}(\bar{\mathbf{P}})$, we know that $\mathcal{R}(\mathbf{P})$ is contained in the set $\{y \in Y: \int r\phi y\, d\eta = 0\}$. Conversely, let y be a function in this set. To complete the proof we need to show that $y \in \mathcal{R}(\mathbf{P})$. Since y is in $\mathcal{R}(\bar{\mathbf{P}})$, there is a $u \in \mathcal{L}_2^{(r)}$ with $u'' \in \mathcal{L}_2^{(r)}$ such that $\bar{\mathbf{P}}u = y$. To prove that $y \in \mathcal{R}(\mathbf{P})$ it remains only to show that $u \in C^2$. We know that u and u' are continuous, simply because u'' is locally square integrable. Solving the differential equation $\bar{\mathbf{P}}u = y$ for u'', we, thus, determine that u'' is continuous as well. But we need to show that u, u', and u'' are bounded. The fact that u is bounded on $[0, \infty)$ follows from part (b) of Lemma 4.4 with $v = u$, $a^*(\eta) = a(\eta)$, etc. In fact, Hypothesis (4.18) with arbitrarily small δ follows from the fact that $u \in \mathscr{L}_2^{(r)}$ and (4.15b). An analogous result establishes the boundedness of u on $(-\infty, 0]$. The differential equation $\bar{\mathbf{P}}u = y$, together with the interpolation inequality used in Lemma 4.2, now yields that u' and u'' are bounded. Thus $u \in C^2$, and the proof is complete.

Proof of Theorem 4.1. We apply Lemma 3.1 with $M \equiv \mathbf{G}$, $m(v, \lambda, \epsilon) \equiv v(0)$, and the spaces X and Y as defined above. Lemmas 4.2 and 4.3 tell us that $M = \mathbf{G}$ is continuously differentiable in v, λ, and ϵ. It is immediate that m satisfies these same properties. We need to verify the hypotheses of Lemma 3.1.

(i) It is immediate from (4.4) that $\mathbf{G}(0, 0, 0) = 0$. Also clearly $m(0, 0, 0) = 0$.

(ii) This hypothesis is verified by Lemma 4.5. As indicated there, $\phi = y_0'$, and ϕ^* is the linear functional given by multiplication by $r\phi$ and integrating.

(iii) The derivative $\mathbf{G}_2(v, \lambda, \epsilon; 1)$ is, by the definition of G and $\mathbf{G}$, simply the ordinary derivative of

$$F(\Delta w(\epsilon\eta + \lambda, \epsilon)(y_0'' + v'') + \epsilon^2 w_{1xx}(\cdots) + 2\epsilon\Delta w_x(y_0' + v') \\ + \epsilon^2 \Delta w_{xx}(y_0 + v), \ldots)$$

with respect to λ. Taking this derivative and setting v, λ, and ϵ equal to 0, we obtain the following, after noting that (2.2b) implies $\Delta g(0) \neq 0$:

$$\begin{aligned}
&G_2(0, 0, 0; 1) \\
&\quad = F_1(\Delta g(0)\, y_0''(\eta), \Delta g(0)\, y_0'(\eta), \Delta g(0)\, y_0(\eta) + g'(0), 0, 0)\, y_0''(\eta)\, \Delta g'(0) \\
&\qquad + F_2(\cdots)\, y_0'(\eta)\, \Delta g'(0) + F_3(\cdots)(y_0(\eta)\, \Delta g'(0) + g^{1\prime}(0)) + F_4(\cdots) \\
&\quad = \frac{\Delta g'(0)}{\Delta g(0)} [a(\eta)\, y_0'' + b(\eta)\, y_0' + c(\eta)\, y_0] \\
&\qquad + \frac{g^{1\prime}(0)}{\Delta g(0)} G_3(y_0'', y_0', y_0, 0, 0) + F_4(\cdots) \\
&\quad = \frac{\Delta g'(0)}{\Delta g(0)} \mathbf{P} y_0 + \frac{g^{1\prime}(0)}{\Delta g(0)} c(\eta) + F_4(\cdots).
\end{aligned}$$

Condition (iii) of Lemma 3.1 can, therefore, be written

$$0 \neq (G_2(0, 0, 0; 1), \phi)_r = \frac{\Delta g'(0)}{\Delta g(0)} (\mathbf{P}y_0, \phi)_r + \frac{g^{1'}(0)}{\Delta g(0)} (c, \phi)_r + (F_4, \phi)_r .$$

But $(\mathbf{P}y_0, \phi)_r = (y_0, \mathbf{P}\phi)_r = 0$, and

$$(c, \phi)_r = \int_{-\infty}^{\infty} rc\phi \, d\eta = -\int_{-\infty}^{\infty} r[a\phi'' + b\phi'] \, d\eta = -\int_{-\infty}^{\infty} (ra\phi')' \, d\eta$$

$$= -\int_{-\infty}^{\infty} (ray_0'')' \, d\eta = -\lim_{\eta\to\infty} r(\eta) \, a(\eta) \, y_0''(\eta)$$

$$+ \lim_{\eta\to-\infty} r(\eta) \, a(\eta) \, y_0''(\eta),$$

if these limits exist. But estimates (4.15a) with small enough δ and (4.17) show the first limit is zero, and analogous estimates show the second is also zero. Thus, condition (iii) is simply $(\phi, F_4(\cdots))_r \neq 0$, which is guaranteed by Hypothesis 3.

(iv) In our setting, this condition is simply that $\phi(0) \neq 0$, which is guaranteed by (4.6).

The conclusion is that there exist solutions $v(x, \epsilon)$, $\lambda(\epsilon)$ of (4.10), (4.11b), defined and continuous with respect to ϵ in the norm of X for ϵ in some neighborhood of the origin, satisfying $v(x, 0) = \lambda(0) = 0$. Only the uniform limit relation (4.11a) remains to be verified.

Because of (4.12), we may use the mean value theorem to write (4.9) as

$$G_1(\theta y_{\eta\eta}, \theta y_\eta, 1 + \theta(y-1), \epsilon\eta + \lambda, \epsilon) \, y_{\eta\eta} + G_2(\cdots) \, y_\eta + G_3(\cdots)(y-1) = 0, \tag{4.23}$$

where $0 < \theta(\eta, \epsilon) < 1$. From the definition of G and Hypothesis 1, we know that

$$G_1(0, 0, 1, \epsilon\eta + \lambda, \epsilon)$$
$$= \Delta w(\epsilon\eta + \lambda, \epsilon) \, F_1(\epsilon^2 w_{xx}^2(\epsilon\eta + \lambda, \epsilon), \epsilon w_x{}^2(\cdots), w^2, \epsilon\eta + \lambda, \epsilon) > 0$$

and $G_3(0, 0, 1, \epsilon\eta + \lambda, \epsilon) < 0$ for ϵ sufficiently small. Moreover, the functions $\theta(y - 1) = \theta(y_0 - 1 + v)$, θy_η, and $\theta y_{\eta\eta}$ can be made as small as desired by taking η large enough and ϵ small enough, because $y_0 \to 1$ and v is uniformly small for ϵ small. Hence, there is a $\sigma > 0$, an $\eta_0 > 0$, and an $\epsilon_2 > 0$ such that the coefficients of (4.23) (call them a^*, b^*, c^*) satisfy the hypotheses of Lemma 4.4 for $|\epsilon| < \epsilon_2$. Also, a^*, b^*, and $|y - 1|_0$ will be bounded for ϵ in that interval. Conclusion (a) of that lemma thus yields that

$y(\eta, \epsilon) - 1 \to 0$ as $\eta \to \infty$, uniformly for $|\epsilon| < \epsilon_2$. A similar argument yields $y(\eta, \epsilon) \to 0$ as $\eta \to -\infty$. Since $y = y_0 + v$, this establishes (4.11a), and the theorem is proved.

5. The Question of Uniqueness

For purposes of this section, we make one further assumption.

Hypothesis 4. *Equation* (4.4) *may be solved for* y_0'', *yielding*

$$y_0'' = H(y_0', y_0), \tag{5.1}$$

where H has any required number of continuous derivatives.

Inequality (2.2b) now yields

$$H_2(0, 1) > 0, \qquad H_2(0, 0) > 0. \tag{5.2}$$

Having proved the existence of a transitional family of solutions in Section 4, we now inquire as to whether there is more than one. This is an important question, in view of the apparent arbitrariness in the construction of the family $u(x, \epsilon)$ in (4.2). Recall that the function $y_0(\eta) = y(\eta, 0)$ was merely required to satisfy (4.4)–(4.6). As noted, there are many functions which do so, since every function $y_0(\eta - C)$, for a constant C, satisfies (4.4) and (4.5). However, this arbitrariness in choice of y_0 does not in general lead to a multiplicity of families $u(x, \epsilon)$. The fact is that the function $\lambda(\epsilon)$ depends on the choice of y_0, and the final effect of this dependence is to cancel the freedom which was first apparent. One should properly consider two choices of solutions y_0 of (4.4) and (4.5) as equivalent if they differ by a shift in the independent variable. It turns out that each such equivalence class determines one and only one family $u(x, \epsilon)$ of type (4.2). The question remains as to how many equivalence classes exist. We shall show that under Hypotheses 1, 2, and 4, there is usually only one, although there may be two in exceptional cases. There is never more than two.

Theorem 5.1. *Under Hypotheses* 1–4, *there exist either one or two families of solutions* $u(x, \epsilon)$ *of the form* (4.2) *and* (4.3). *In the latter case, there is exactly one with the property that* $y(\eta, 0) \uparrow 1$ *as* $\eta \to \infty$ *and* $y(\eta, 0) \downarrow 0$ *as* $\eta \to -\infty$.

The proof relies upon the following lemmas.

LEMMA 5.2. *Let $f(t, z)$ be twice continuously differentiable, and satisfy $f(0, 0) = 0, f_1(0, 0) > 0$. Then the problem*

$$y' = f(t, y^{1/2}), \tag{5.3a}$$

$$y(0) = 0, \tag{5.3b}$$

has at most one nonnegative solution in the interval $0 \leqslant t \leqslant t_0$, for small enough t_0.

Proof. We consider three cases separately:

(1) $f_2(0, 0) > 0$. Then Theorem 2 of [1] applies. For that purpose we set $p(t, x) \equiv f(t, |x|^{1/2})$ and $g(t, x) \equiv 1$. The positivity of the two partial derivatives of f in a neighborhood of the origin, together with mean value theorem, insure that for some positive α, $p(t, x) \geqslant \alpha(t + |x|^{1/2})$ in that neighborhood when $t \geqslant 0$. Thus hypothesis (i) of the indicated theorem holds for $t > 0$. Also $0 \leqslant |\int_0^\xi (d\zeta/p(t, \zeta))| \leqslant |\int_0^\xi (d\zeta/\alpha(|\zeta|)^{1/2}) = (2/\alpha)|\xi|^{1/2}$, so (ii) is also satisfied. Since $p_1(0, 0) > 0$, we know that $p_1(t, x) > 0$ in a neighborhood of the origin. Hence (iii) follows. Finally, (iv) is true with $L = 0$. The conclusion of the theorem yields uniqueness for solutions of $y' = f(t, |y|^{1/2})$, $y(0) = 0$. Hence, uniqueness for nonnegative solutions of our problem follows.

(2) $f_2(0, 0) < 0$. Then for small t and $y \geqslant 0$, f is nonincreasing in y. It is well known that uniqueness holds when this is the case.

(3) $f_2(0, 0) = 0$. In this case we may write

$$f(t, y^{1/2}) = ta(t, y^{1/2}) + yb(t, y^{1/2}), \tag{5.4}$$

where a and b are differentiable functions with $a(0, 0) > 0$. Thus, for some positive constants ν_1, ν_2, we have $f(t, y^{1/2}) \geqslant \nu_1 t - \nu_2 y$, and any solution $y(t)$ is bounded below by the solution $z(t)$ of $z' = \nu_1 t - \nu_2 z$, $z(0) = 0$, i.e., $z(t) = (\nu_1/\nu_2^2)(e^{-\nu_2 t} - 1 + \nu_2 t) \geqslant (\nu_1/4)t^2$ for small t. Thus, any solution of (5.3) must satisfy $y(t) \geqslant \kappa t^2$ in $0 \leqslant t \leqslant \delta$, for some positive constants κ and δ. We shall prove uniqueness by showing that $f(t, y^{1/2})$ satisfies a Lipschitz condition in the region $t \geqslant 0, y \geqslant \kappa t^2$. From (5.4), we have, for $\eta > \xi \geqslant \kappa t^2$ and some τ_1, τ_2 with $\xi^{1/2} < \tau_i < \eta^{1/2}$, by the mean value theorem,

$$\begin{aligned} f(t, \eta^{1/2}) - f(t, \xi^{1/2}) &= ta_2(t, \tau_1)(\eta^{1/2} - \xi^{1/2}) + (\eta - \xi)\, b(t, \eta^{1/2}) \\ &\quad + \xi b_2(t, \tau_2)(\eta^{1/2} - \xi^{1/2}) \\ &= [a_2(t, \tau_1)(t/(\xi^{1/2} + \eta^{1/2})) \\ &\quad + b(t, \eta^{1/2}) + b_2(t, \tau_2)(\xi/(\xi^{1/2} + \eta^{1/2}))](\eta - \xi). \end{aligned}$$

The boundedness of a_2, b, and b_2, together with the lower bounds for ξ and η, imply that the quantity in brackets is bounded in absolute value, for ξ and η bounded. This establishes uniqueness in the third and final case.

LEMMA 5.3. *Under Hypotheses* 1, 2, *and* 4, *the problems* (4.4) *and* (4.5) *has either* (i) *a one parameter family of solutions of the form* $y_0(\eta - C)$ (C *an arbitrary constant*) *and no others, or* (ii) *two one-parameter families of solutions of this type, and no others. In the latter case, one of the families approaches the limit at* $-\infty$ *from below and that at* $+\infty$ *from above, and the other family has the opposite behavior.*

Remark. Hypothesis 3 is stated in terms of the function y_0, so its fulfillment conceivably could depend on the choice of y_0. However, if y_0 and $y_0{}^*$ are two "equivalent" choices; i.e., $y_0{}^*(\eta) = y_0(\eta - C)$, then Hypothesis 3 is satisfied with respect to y_0 if and only if it is satisfied with respect to $y_0{}^*$. To see this, we denote by a^*, b^*, r^*, and $F_4{}^*$ the functions entering into the hypothesis, computed with reference to $y_0{}^*$. Then $a^*(\eta) = a(\eta - C)$, $F_4{}^*(\eta) = F_4(\eta - C)$, and $r^*(\eta) = Kr(\eta - C)$, where

$$K = \exp \int_{-C}^{0} ((b - a')/a)\, d\eta.$$

The assertion follows by shifting the variable of integration. Thus, in case (i) of the above lemma, Hypothesis 3 is either fulfilled for all choices of y_0, or for none.

Proof. Let $Y(\eta)$ be any solution of (4.4) and (4.5). It must be monotone for large enough η. In fact, (4.16) holds with y_0 replaced by Y. Of course, in view of (5.1) we have $G_1 \equiv 1$, $G_2 \equiv -H_1$, and $G_3 \equiv -H_2$. From (5.2) we have $H_2(\theta Y', 1 + \theta(Y - 1)) > 0$ for large enough η, so in this range the maximum principle applies. It tells us that there can be no local maximum greater than 1, or local minimum less than 1. Likewise there can be no inflection point at which $Y' = 0$. Thus $Y' \neq 0$ for $\eta > \eta_0$, for some large enough η_0. Thus for $\eta > \eta_0$, the quantity $W = \frac{1}{2}(Y')^2$ is a well defined nonnegative function of Y, which vanishes only at $Y = 1$.

If the approach to the limit at ∞ is from below, we have $0 \leqslant Y' = (2W)^{1/2}$. Since $(dW/dY) = (Y'Y''/Y') = Y''$, W satisfies the equation

$$\frac{dW}{dY} = H((2W)^{1/2}, Y), \qquad Y \leqslant 1. \tag{5.5}$$

Let $t = 1 - Y$, and $f(t, W^{1/2}) \equiv -H((2W)^{1/2}, 1 - t)$. Then as a function of t, W satisfies $dW/dt = f(t, W^{1/2})$ in an interval $0 \leqslant t \leqslant \delta$, with $W(0) = 0$. From (5.2), we have $f_1(0, 0) > 0$, and from (4.12), $f(0, 0) = 0$. The hypotheses of Lemma 5.2 are, therefore, satisfied, and we conclude that these conditions determine $W(t)$ uniquely. If $\bar{Y}(\eta)$ is another solution of (4.4) and (4.5), approaching the limit at ∞ from below, we, therefore, conclude that $\frac{1}{2}(\bar{Y}')^2$ is the same function of $\bar{Y}$ as $\frac{1}{2}(Y')^2$ is of Y. In other words, $\bar{Y}' = Y'$ whenever $\bar{Y} = Y$, $\eta \geqslant \eta_0$. Let $\gamma < 1$ be sufficiently close to 1, and let $\eta_1, \bar{\eta}_1$ be such that $Y(\eta_1) = \gamma$, $\bar{Y}(\bar{\eta}_1) = \gamma$. Then the function $Y(\eta + (\bar{\eta}_1 - \eta_1))$ is a solution of (4.4) and (4.5) coinciding with $\bar{Y}$ at η_1, and whose derivative coincides with $\bar{Y}'$ at η_1. By uniqueness of solutions of the initial value problem for (5.1), we have $\bar{Y}(\eta) \equiv Y(\eta + (\bar{\eta}_1 - \eta_1))$ for all η. This shows that there can be no more than one family of solutions of the form indicated in the lemma, approaching the limit at ∞ from below.

On the other hand, suppose Y approaches its limit at ∞ from above. Then $Y' = -(2W)^{1/2}$, and W satisfies

$$(dW/dY) = H(-(2W)^{1/2}, Y), \qquad Y \geqslant 1.$$

An argument similar to the above shows that any other solution approaching from above must be of the form $Y(\eta - C)$.

In view of the above, we conclude that if all solutions of (4.4) and (4.5) approach the limit at ∞ from one side, we have case (i) in the lemma. But if (in rare cases) there exist solutions approaching from either side, then case (ii) holds. This exhausts all possibilities.

Now consider case (ii) in more detail. A similar argument applied to the limit at $-\infty$ implies that there are solutions approaching it from both sides; otherwise there could only be one family. Suppose there were two solutions $Y_1(\eta)$ and $Y_2(\eta)$ such that $Y_1 \downarrow 1$, $Y_2 \uparrow 1$ as $\eta \to \infty$, and $Y_1 \downarrow 0$, $Y_2 \uparrow 0$ as $\eta \to -\infty$. Then it is easy to see that for some k and some η_1, $Y_1(\eta_1 - k) = Y_2(\eta_1)$ and $Y_1'(\eta_1 - k) = Y_2'(\eta_1)$. Again by uniqueness of the initial value problem for (5.1), we have $Y_2(\eta) \equiv Y_1(\eta - k)$, which contradicts the assumed limiting behavior of Y_i. Thus, the two families must have the limiting behavior described in the lemma. This completes the proof.

Proof of Theorem 5.1. If case (i) in Lemma 5.3 holds, let $y_0(\eta)$ be any particular solution of (4.4)–(4.6), and let $u_0(x, \epsilon)$ be the family of solutions (4.2) constructed in Theorem 4.1, with $y(\eta, 0) = y_0(\eta)$. If case (ii) holds, let $y_0(\eta)$ and $y_1(\eta)$ be solutions of (4.4)–(4.6) representing the two different families indicated in Lemma 5.2. Then let $u^0(x, \epsilon)$ and $u^1(x, \epsilon)$ be the families of solutions (4.2) with $y(\eta, 0) = y_0(\eta)$ and $y_1(\eta)$, respectively.

Let $u^*(x, \epsilon)$ be any family of solutions of (2.1) of the form

$$u^*(x, \epsilon) \equiv w^1(x, \epsilon) + \tilde{y}(x/\epsilon, \epsilon)(w^2(x, \epsilon) - w^1(x, \epsilon)), \tag{5.6}$$

where $\tilde{y}(\infty, \epsilon) = 1$, $\tilde{y}(-\infty, \epsilon) = 0$. Clearly those of the form (4.2) and (4.3) are of this type; one need only set $\tilde{y}(\zeta, \epsilon) = y(\zeta - \mu(\epsilon), \epsilon)$, where $\mu(\epsilon) = \lambda(\epsilon)/\epsilon$;

μ is bounded, since $\lambda(0) = 0$. Conversely, (5.6) is of the form (4.2) with η replaced by $\zeta = x/\epsilon$, so $G(\tilde{y}_{\zeta\zeta}, \tilde{y}_\zeta, \tilde{y}, \epsilon\zeta, \epsilon) = 0$. Setting $\epsilon = 0$, we have that $\tilde{y}(\zeta, 0)$ satisfies (4.4) and (4.5). Thus, from Lemma 5.3, either $\tilde{y}(\zeta, 0) \equiv y_0(\zeta - k)$ for some k, or else $\tilde{y}(\zeta, 0) \equiv y_1(\gamma - k)$, the latter only being possible in the (exceptional) case (ii). Assuming that the former equation holds, we shall show that $u^*(x, \epsilon) \equiv u^0(x, \epsilon)$. On the other hand, by a similar argument, the other equation will imply that $u^*(x, \epsilon) \equiv u^1(x, \epsilon)$. This will prove the first assertion in the theorem. The second then follows easily from the contrasting behavior of $y_0(\eta)$ and $y_1(\eta)$ given in case (ii) of Lemma 5.2.

So assume that $\tilde{y}(\zeta, 0) = y_0(\zeta - k)$. Define $\mu^*(\epsilon)$ as the function of ϵ which satisfies $\tilde{y}(\mu^*(\epsilon), 0) = y_0(0)$, $\mu^*(0) = k$. Such a unique function is guaranteed to exist for sufficiently small $|\epsilon|$ by the implicit function theorem, since $\tilde{y}_\zeta(k, 0) = y_0'(0) \neq 0$, by (4.6). Next define

$$y^*(\eta, \epsilon) \equiv \tilde{y}(\eta + \mu^*(\epsilon), \epsilon),$$

so that

$$y^*(\eta, 0) \equiv y_0(\eta).$$

Setting $\eta = (x - \lambda^*(\epsilon))/\epsilon$, where $\lambda^* = \epsilon\mu^*$, we have

$$G(y^*_{\eta\eta}, y_\eta^*, y^*, \epsilon\eta + \lambda^*, \epsilon) \equiv 0. \tag{5.7}$$

Finally setting $v^*(\eta, \epsilon) \equiv y^*(\eta, \epsilon) - y^*(\eta, 0) \equiv y^*(\eta, \epsilon) - y_0(\eta)$, we have that (4.10) and (4.11) are satisfied with v replaced by v^* and λ by λ^*.

The argument used in the proof of Lemma 4.2 to establish that

$$\int_0^\infty r(\eta)[(y_0'')^2 + (y_0')^2 + (y_0 - 1)^2]\, d\eta < \infty$$

can also be used to prove, on the basis of (5.7) and $G(0, 0, 1, \epsilon\eta + \lambda^*, \epsilon) = 0$, that $\int_0^\infty r(\eta)[(y^*_{\eta\eta})^2 + (y_\eta^*)^2 + (y^* - 1)^2]\, d\eta < \infty$. Since $v^* = y^* - y_0$, we, thus, have $\int_0^\infty r(\eta)[(v^*_{\eta\eta})^2 + (v_\eta^*)^2 + (v^*)^2]\, d\eta < \infty$. This, with the analogous bound for the integral over $(-\infty, 0)$, implies that $v^* \in X$ for each ϵ. We, therefore, have

$$\mathbf{G}(v^*, \lambda^*, \epsilon) = 0, \tag{5.8}$$

$$v^*(0, \epsilon) = 0. \tag{5.9}$$

We have shown in Section 4 that the operators $M(v, \lambda, \epsilon) = \mathbf{G}(v, \lambda, \epsilon)$ and $m(v, \lambda, \epsilon) = v(0)$ satisfy the hypotheses of Lemma 3.1. The uniqueness statement of that lemma implies that there is only one solution (v^*, λ^*) of (5.8) and (5.9). Therefore our functions v^*, λ^* are precisely the functions

v and λ constructed in Theorem 4.1. According to the definitions of v^* and λ^*, this implies that $u^*(x, \epsilon) \equiv u^0(x, \epsilon)$. This completes the proof.

6. An Example

As an example, we consider the quasilinear equation

$$\epsilon^2(p(u, x)u')' - h(u, x) = 0, \tag{6.1}$$

where

$$p(u, x) > 0. \tag{6.2}$$

Hypothesis 1, in Section 2, is that the degenerate equation $h(u, x) = 0$ has two solutions $u = g^i(x)$, for which

$$h_1(g^i(x), x) \geqslant \kappa^2 > 0. \tag{6.3}$$

We suppose for simplicity that $g^1(0) = 0$ and $g^2(0) > 0$.

If we set $\sigma = \Delta g(0) > 0$, Hypothesis 2 takes the form of requiring that

$$\sigma(p(\sigma y_0, 0)\, y_0')' - h(\sigma y_0, 0) = 0 \tag{6.4a}$$

have a solution satisfying

$$y_0(-\infty) = 0, \qquad y_0(\infty) = 1. \tag{6.4b}$$

For any number η_1, the function $y_1(\eta) \equiv y_0(2\eta_1 - \eta)$ also satisfies (6.4a), and coincides with y_0 at $\eta = \eta_1$. If $y_0'(\eta_1) = 0$, then $y_1'(\eta_1)$ is also zero, and by uniqueness of the initial value problem for (6.4a), $y_0 \equiv y_1$, which would contradict (6.4b). Therefore we may safely assume that $y_0' \neq 0$, and so y_0 is monotone. (In particular, condition (4.6) is superfluous.) Setting $\sigma p(\sigma y_0, 0)\, y_0' \equiv Z$ and $\sigma y_0 \equiv V$, we know that Z must therefore be a well defined function of V, for $0 \leqslant V \leqslant \sigma$. In fact, (6.4) can be written in the form

$$\tfrac{1}{2} d(Z^2)/dV = p(V, 0)\, h(V, 0), \qquad 0 < V < \sigma, \tag{6.5a}$$

$$Z(0) = Z(\sigma) = 0. \tag{6.5b}$$

Integrating, we find that this problem has a real solution if and only if

$$\int_0^k p(V, 0)\, h(V, 0)\, dV \begin{cases} >0 & \text{for } k \in (0, \sigma), \\ =0 & \text{for } k = \sigma. \end{cases} \tag{6.6}$$

This easily verified condition thus replaces Hypothesis 2.

For Hypothesis 3, we calculate

$$a(\eta) = \sigma p(\sigma y_0(\eta), 0),$$
$$b(\eta) = 2\sigma^2 y_0'(\eta)\, p_1(\sigma y_0(\eta), 0) = 2a'(\eta).$$

Thus, $r(\eta) = \exp \int_0^\eta (a'/a)\, d\bar{\eta} = Ca(\eta) = C\sigma p(V, 0) > 0$, $(C > 0)$.

Since $F(\alpha, \beta, u, x, \epsilon) \equiv \alpha p(u, x) + \beta^2 p_1(u, x) + \epsilon p_2(u, x)\beta - h(u, x)$ and $ZZ_v/\sigma p = (p(\sigma y_0, 0)\, y_0')' = py_0'' + \sigma p_1(y_0')^2 = py_0'' + p_1 Z^2/\sigma p^2$, we have

$$\begin{aligned} F_4(\sigma y_0'', \sigma y_0, \sigma y_0, 0, 0) &= \sigma y_0'' p_2(V, 0) + p_{12}(V, 0)(\sigma y_0')^2 - h_2(V, 0) \\ &= p_2/p^2(ZZ_V - (p_1/p)\, Z^2) + p_{12}(Z/p)^2 - h_2(V, 0) \\ &= (1/p)\{(p_2/p) ZZ_V + ((pp_{12} - p_1 p_2)/p^2) Z^2\} - h_2(V, 0) \\ &= (1/p)\{q(V)\, W'(V) + 2q'(V)\, W(V)\} - h_2(V, 0) \\ &= (1/p)\{2(qW)' - qW'\} - h_2, \end{aligned}$$

where $q(V) \equiv p_2(V, 0)/p(V, 0)$, and $W(V) \equiv \frac{1}{2} Z^2(V)$.

Using $\sigma y_0'\, d\eta = dV$ and $r = C\sigma p$, we see that Hypothesis 3 now assumes the form

$$\int_0^\sigma [2(qW)' - qW' - ph_2]\, dV \neq 0.$$

The integral of the first term vanishes, since $W(0) = W(\sigma) = 0$. Also, from (6.5a), we have $qW'(V) = qp(V, 0)\, h(V, 0) = p_2(V, 0)\, h(V, 0)$. Hence, Hypothesis 3 reduces to

$$\int_0^\sigma (p_2(V, 0)\, h(V, 0) + p(V, 0)\, h_2(V, 0))\, dV \neq 0, \tag{6.7}$$

a condition which is again easily verified, since it does not require knowledge of y_0.

Hypothesis 4, of course, follows from (6.2).

Note that case (ii) in Lemma 5.3 is impossible, since $y_0(\eta)$ must be monotone. Therefore, Theorem 5.1 yields a single family of solutions. In all, we have the following theorem.

THEOREM 6.1. *Let $h(u, x)$ be such that $h(u, x) = 0$ has two distinct bounded solutions $u = g^i(x)$. Let $p(u, x)$, $h(u, x)$, $g^i(x)$ satisfy $g^1(0) = 0$,* (6.2), (6.3), *and* (6.6) (*where $\sigma = g^2(0) > 0$*), *and* (6.7). *Then there exist unique regular families of solutions $w^i(x, \epsilon)$ of* (6.1) *satisfying* (2.1), *and there exists a unique transitional family of solutions of* (6.1) *of the form* (4.2) *and* (4.3).

7. Asymptotic Approximations

We consider the question of constructing asymptotic expansions in powers of ϵ to approximate the solutions whose existence has been proved.

The proper procedure for such a construction in the case of the families $w^i(x, \epsilon)$ in Section 2 was in fact mentioned there. It takes the form $U = \sum \epsilon^n u_n(x)$ where the terms u_n are determined by (2.5), with $u_0(x) = g(x)$.

Once the expansion for w^i has been achieved, we may proceed to construct an expansion for the transitional family $u(x, \epsilon)$ as follows. First, a function $y_0(\eta)$ is determined as a solution of (4.4)–(4.6). An example of how this might be done was given in Section 6. Next, the equations (4.10) and (4.11) are considered for unknown functions $v(\eta, \epsilon)$ and $\lambda(\epsilon)$. We assume asymptotic expansions of the form

$$v = \sum_{n=1} \epsilon^n v_n(\eta); \qquad \lambda = \sum_{n=1} \epsilon^n \lambda_n$$

and attempt to determine the various terms by the conditions

$$\partial_\epsilon{}^k G(y_0''(\eta) + v_{\eta\eta}(\eta, \epsilon), y_0'(\eta) + v_\eta(\eta, \epsilon), y_0(\eta) + v(\eta, \epsilon), \epsilon\eta + \lambda(\epsilon), \epsilon)\,|_{\epsilon=0} = 0, \quad k = 1,...;$$

$$v_n(\pm\infty) = v_n(0) = 0.$$

Note that the exact form of G is not known, since the functions w^i are not known exactly. However, enough is known to determine these derivatives at $\epsilon = 0$. For $k = 1$, we obtain

$$G_1(y_0'', y_0', y_0, 0, 0)\, v_1'' + G_2(\cdots)\, v_1' + G_3(\cdots)\, v_1 + (\eta + \lambda_1)\, G_4(\cdots) + G_5(\cdots) = 0.$$

In view of (4.8) and (4.22), this can be written as

$$Pv_1 + \lambda_1 G_4(\cdots) + \eta G_4(\cdots) + G_5(\cdots) = 0. \tag{7.1}$$

We recall that P has a simple eigenvalue 0, with eigenfunction $\phi = y_0'$, and is self-adjoint with respect to a scalar product with weight function r (4.7). Thus, Eq. (7.1) can be solved for v_1 if and only if

$$\int_{-\infty}^{\infty} r(\eta)[\lambda_1 G_4 + \eta G_4 + G_5]\, y_0'(\eta)\, d\eta = 0,$$

or $\lambda_1 \int_{-\infty}^{\infty} r(\eta)\, G_4(\cdots)\, y_0'(\eta)\, d\eta =$ a known function. Hypothesis 3 says that the coefficient of λ_1 is different from zero, so λ_1 is determined uniquely. However, v_1 is not determined uniquely; rather, it is indeterminate to the extent of an arbitrary additive multiple of y_0'. Thus, if v_{10} is any particular solution, we must select the proper solution v_1 from the collection $v_{10}(\eta) + \alpha y_0'(\eta)$. But since $y_0'(0) \neq 0$, there is a unique way to select α so that $v_1(0) = 0$.

In this way, all the terms λ_n and v_n can be determined in succession. This yields an asymptotic expansion for $v(\eta, \epsilon)$. Setting $y(\eta, \epsilon) = y_0(\eta) + v(\eta, \epsilon)$ into (4.2) and using the previously obtained expansions for w^1 and w^2, we obtain the desired expansion for $u(x, \epsilon)$. The expansions for w^i are analogous to "outer" expansions of other singular perturbation problems, whereas that for y could be classed as an "inner" expansion. The expression for u contains both.

We shall dispense with the proof of validity of the above expansions. Such a proof could probably be devised using the following approach. Assuming (as we always have) enough regularity of F, it follows that $\mathbf{G}$ is also highly regular. Therefore, the solutions $v(\epsilon)$, $\lambda(\epsilon)$ of $\mathbf{G}(v, \lambda, \epsilon) = 0$ have a number of uniquely determinable derivatives at $\epsilon = 0$. These derivatives must correspond to the terms v_n and λ_n constructed above. Expanding $v(\epsilon)$ and $\lambda(\epsilon)$ in Taylor series in ϵ, we obtain the asymptotic expansions already obtained above, plus a remainder term which can be estimated. Estimating this term constitutes proof of the asymptotic nature of our series.

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