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Nuclear Physics B 886 (2014) 135–165
NUCLEAR PHYSICS B
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Quantization of a scalar field in two Poincaré patches of anti-de Sitter space and AdS/CFT

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Received 12 March 2014; accepted 23 June 2014

Available online 30 June 2014

Editor: Stephan Stieberger

Abstract

Two sets of modes of a massive free scalar field are quantized in a pair of Poincaré patches of Lorentzian anti-de Sitter (AdS) space, AdS_{d+1} ($d \geq 2$). It is shown that in Poincaré coordinates $(r, t, \vec{x})$, the two boundaries at $r = \pm\infty$ are connected. When the scalar mass m satisfies a condition $0 < \nu = \sqrt{(d^2/4) + (m\ell)^2} < 1$, there exist two sets of mode solutions to Klein–Gordon equation, with distinct fall-off behaviors at the boundary. By using the fact that the boundaries at $r = \pm\infty$ are connected, a conserved Klein–Gordon norm can be defined for these two sets of scalar modes, and these modes are canonically quantized. Energy is also conserved. A prescription within the approximation of semi-classical gravity is presented for computing two- and three-point functions of the operators in the boundary CFT, which correspond to the two fall-off behaviours of scalar field solutions.

1. Introduction

Quantization of scalar fields propagating in anti-de Sitter space was attempted in the past [1–3]. In [1] the problem of a time-like boundary at space-like infinity, through which data can propagate, is studied in a massless case by conformally mapping the spacetime in the global coordinates into upper hemisphere of Einstein static universe (ESU). By using the fact that AdS

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http://dx.doi.org/10.1016/j.nuclphysb.2014.06.022

space is mapped to a half of ESU, it was shown that there are two sets of mode functions, which are characterized by different boundary conditions and are orthonormal and form a complete set of basis by themselves, separately. It was concluded that only one of the two sets of mode functions can be quantized. In [2] this procedure is more elaborated and extended to massive scalars. In [3] mode functions for scalar fields in AdS space in both Poincaré coordinates and global coordinates are obtained. Group-theoretic analysis was performed in [4].

On the other hand, AdS/CFT duality was discovered in [5] and its precise definition has been developed [6–9]. To the string compactification on $\mathrm{AdS}_{d+1} \times \mathcal{M}_n$, there corresponds a conformal field theory (CFT) living on a space conformal to the d-dimensional boundary of the AdS. To each field Φ in the bulk there corresponds a local operator in the CFT. By fixing the boundary value of Φ and computing the effective action of the bulk theory, this effective action yields the generating functional of the operators in conformal field theory with the boundary value acting as the source function. In the semiclassical supergravity limit, one can compute the effective action by solving the classical equation of motion and just substituting the solution into the action.

In the case of a free scalar field ϕ of mass m, it falls off like $\phi \sim r^{-(d-\Delta_+)}\phi_0 + r^{-(d-\Delta_-)}\phi_1$ near the spacelike boundary $r \to \infty$. Here $\Delta_\pm = \frac{d}{2} \pm \nu$ and $\nu = \sqrt{(d^2/4) + m^2}$. When $\nu > 1$, only ϕ_0 acts as a source for an operator O_+ with a scaling dimension Δ_+ in CFT. When $0 < \nu < 1$, it is argued that either of the two operators O_+ or O_- with scaling dimensions Δ_+ and Δ_- can be considered in CFT. To compute two-point functions of O_+ one should take ϕ_0 as a source function and functionally differentiate the effective action with respect to ϕ_0 [7]. To compute two-point functions of O_-, however, one needs to Legendre transform the effective function with respect to ϕ_0 to obtain a generating functional [8]. This restriction of the holographic correspondence is argued to be related to the above peculiarity of the scalar field quantization in AdS space.

Meanwhile, in the context of AdS/CFT for 3d higher-spin gravity coupled to matter fields, it was found [10] that we can compute semi-classically two-point functions of two sets of single-trace operators in boundary CFT by introducing only one set of matter fields B and C. This motivates us to study whether we can quantize a scalar field in AdS space while keeping both two sets of scalar modes.

One of the purposes of this paper is to show that these two sets of scalar modes in AdS space can be quantized altogether by considering a coordinate system which is obtained by patching together a pair of Poincaré coordinates with radial coordinate $r > 0$ and $r < 0$, respectively, along the horizon ($r = 0$). The AdS space can be divided into two Poincaré patches. The boundary of AdS space is also divided into two. Usually, a scalar field is quantized only in one of the two Poincaré patches. In connection with AdS/CFT correspondence, however, conformal symmetry of boundary CFT has an origin in the isometry of AdS space. Although the metric in a pair of Poincaré coordinates is invariant under special conformal transformations, points in the two Poincaré patches are exchanged and a single Poincaré patch is not invariant. Hence it is not appropriate to restrict analysis of a field theory in AdS space to just within a single patch.[1,2]

In this paper, it is shown that the two patches can be joined together by matching the fluxes of a scalar field across the horizon and two boundaries, and that the united coordinate system admits two sets of scalar mode functions. The fluxes across the horizon vanishes, while those across

[1] In [12] a quotient space AdS_{d+1}/J, where J is an antipodal map $X_\mu \to -X_\mu$, is considered. This space is invariant under the isometry of AdS_{d+1}.

[2] EAdS space is one piece of the two disconnected hyperbolic spaces, and this single piece has the full conformal symmetry. This is in sharp contrast to the Lorentzian case.

the boundaries do not. These fluxes across the two boundaries, however, cancel out with each other. It is shown that in Poincaré coordinates for AdS_{d+1} ($d \geq 2$), the two boundaries $r = \pm\infty$ are connected. Hence the cancellation of the fluxes occurs on the connected boundaries of the hyperboloid. As a result, Klein–Gordon norm (3.9) is conserved. It is also shown that energy is conserved.

After canonical quantization of the scalar field, Wightman function for a scalar field in AdS space is computed by performing explicit integrations.[3] An allowed form of boundary conditions for a scalar field on the two boundaries is also identified. An interesting issue of AdS/CFT is the prescription for semi-classically computing two-point functions of O_+ and O_- for a scalar field theory with a mass in the range $-d^2/4 < m^2 < 1 - d^2/4$. To present this prescription is the second aim of this paper. It turns out that the (renormalized) action integral (in Euclidean AdS (EAdS) space) is given by a sum of bulk action and boundary terms:

$$I = \int_{-\infty}^{\infty} dr \int d^d y \sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2\right) + \lim_{r\to+\infty} \int_{r\text{ fixed}} d^d\vec{y}\sqrt{\gamma}\frac{1}{2}\Delta_-\phi^2$$
$$- \lim_{r\to-\infty} \int_{r\text{ fixed}} d^d\vec{y}\sqrt{\gamma}\frac{1}{2}\Delta_-\phi^2 - \lim_{r\to-\infty} \int_{r\text{ fixed}} d^d\vec{y}\sqrt{\gamma}\phi r\partial_r\phi. \tag{1.1}$$

Here r is the radial coordinate which takes the value in the range $-\infty < r < \infty$. $r = 0$ is the horizon and $r = \pm\infty$ are the two boundaries. Two Poincaré patches are also introduced in the EAdS space corresponding to the Lorentzian version. The metric is given by $ds^2 = dr^2/r^2 + r^2 d\vec{y}^2 = g_{\mu\nu}dy^\mu dy^\nu$ and γ_{ij} is an induced metric on the boundaries and $\sqrt{\gamma} = |r|^d$. The ϕ^2, $\phi\partial_r\phi$ terms on the boundaries are counterterms to cancel out the divergences which appear in calculation of the two-point functions. Two boundary values ϕ_+, ϕ_- of a scalar field will be used as source functions for the two-point functions in boundary CFT. Legendre transformation is not required. Calculation of three-point functions with our formalism is also outlined.

This paper is organized as follows. In Section 2 a global coordinates and Poincaré coordinates of AdS space are reviewed and peculiar properties of Lorentzian AdS space in Poincaré coordinates are discussed. A prescription for patching together two Poincaré charts is explained. In Section 3 Klein–Gordon (KG) equation will be solved in each Poincaré patch, and two kinds of mode functions in a pair of Poincaré patches are determined in such a way that KG norm is conserved. It is checked that the fluxes through the horizon vanish, and the fluxes at the boundaries cancel out. Conservation of energy is also shown. In Section 4 a scalar field operator is expanded into these modes, and canonical commutation relations are applied. In Section 5, Wightman function of a scalar field is computed explicitly. AdS/CFT correspondence for two-point functions will be studied in Section 6. Due to the properties of the mode functions obtained in Section 3, the solutions to the equation of motion on the pair of Poincaré patches have a peculiar parity property with respect to the radial coordinate r, which is modified by a parameter S. This fact allows us to write down a general solution ϕ in terms of two boundary values ϕ_+, ϕ_- of the scalar field. By assuming some form of boundary actions on the two boundaries, substituting the solution into the action, and adjusting the coefficients of the boundary terms to eliminate divergences as $|r| \to \infty$, we get a suitable generating functional of two-point functions. A prescription which makes both two point functions $\langle O_+O_+\rangle$ and $\langle O_-O_-\rangle$ positive is proposed. In Section 7

[3] Wightman function for a scalar field in AdS space was computed previously by solving a differential equation with respect to an AdS-invariant distance and matching its singularity with that of flat space [11].

a prescription for computing three-point functions in a bulk ϕ^3 theory is mentioned. Section 8 is devoted to a summary and discussions. In Appendix A, an explicit calculation of Wightman function is presented. In Appendix B a method for calculating integrals of products of the bulk–boundary propagators $K_{\Delta_\pm}$ is outlined.

2. AdS spacetime

2.1. Definition

A $(d+1)$-dimensional AdS spacetime AdS_{d+1} is defined by a constant negative curvature hyperboloid

$$X\cdot X \equiv -X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -\ell^2 \tag{2.1}$$

embedded in pseudo-Minkowski space $\mathbb{E}^{d,2}$. Here ℓ is an AdS radius. Line element in $\mathbb{E}^{d,2}$ induces a one on this hyperboloid.

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2. \tag{2.2}$$

There are several coordinate systems, and the global coordinates and the Poincaré ones are among them.[4]

Global coordinates are defined by

$$\begin{aligned} X_0 &= \ell \sec\rho \cos\tau, \\ X_i &= \ell \tan\rho\, \Omega_i \quad (i=1,\ldots,d) \\ X_{d+1} &= \ell \sec\rho \sin\tau, \end{aligned} \tag{2.3}$$

where radial coordinate ρ and time τ take values in ranges $0 \le \rho < \pi/2$ and $-\pi < \tau \le \pi$, and $\rho = \pi/2$ is a boundary. Spherical coordinates Ω_i satisfy $-1 \le \Omega_i \le 1$ and $\sum_i \Omega_i^2 = 1$. The line element is given by

$$ds^2 = \ell^2 \sec^2\rho \left(d\rho^2 - d\tau^2 + \sin^2\rho \sum_{i=1}^{d} d\Omega_i^2 \right). \tag{2.4}$$

To avoid time-like closed loops, one unwraps τ to have range $-\infty < \tau < \infty$, and works with a universal covering space, CAdS_{d+1}.

Poincaré coordinates are defined by

$$\begin{aligned} X_0 &= \frac{1}{2} z \left(1 + \frac{1}{z^2}\left(\ell^2 - t^2 + \vec{x}^2\right) \right), \\ X_d &= \frac{1}{2} z \left(1 + \frac{1}{z^2}\left(-\ell^2 - t^2 + \vec{x}^2\right) \right), \\ X_i &= \frac{\ell}{z} x^i, \\ X_{d+1} &= \frac{\ell}{z} t. \end{aligned} \tag{2.5}$$

[4] For review see, for example, [3,9].

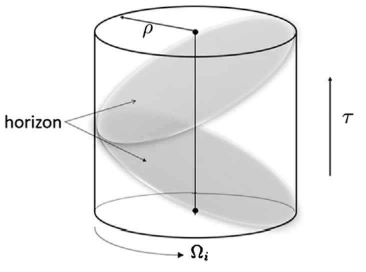

Fig. 1. The horizon in AdS is obtained by making two diagonal cuts through the cylinder.

Here t and x^i range between $-\infty$ and ∞, and radial coordinate z ranges over $0 \le z < \infty$. The line element is now given by

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right). \tag{2.6}$$

The boundary is at $z = 0$. There is also a Killing horizon at $z = \infty$. The time-like Killing vector becomes null at this horizon.

Poincaré coordinates cover only half of AdS_{d+1}, since $X_0 - X_d = 1/z > 0$. The remaining half is covered by coordinates (2.5) with $-\infty < z \le 0$. Usually, when AdS spacetime is studied in Poincaré patch, only a single patch is considered. However, as is explained below, it is necessary to consider a pair of Poincaré patches.

2.2. Two Poincaré patches

AdS space can be illustrated as an interior of a cylinder as in Fig. 1. The boundary of AdS is identified with the boundary of the cylinder. The horizons in AdS are obtained by making two diagonal cuts through the cylinder. The cuts divide AdS into two regions, each of which is covered by each of a pair of Poincaré coordinates. By using a pair of Poincaré coordinates, a single cover of AdS space is obtained. A simplified view (with only $\vec{x} = \vec{0}$ section) is given in Fig. 2. Here a new radial coordinate $r = \frac{1}{z}$ is introduced. This ranges over $-\infty < r < \infty$. The line element (2.6) is rewritten as

$$ds^2 = \ell^2\left(r^{-2}dr^2 + r^2\left(-dt^2 + d\vec{x}^2\right)\right) \equiv g_{\mu\nu}\,dx^\mu dx^\nu. \tag{2.7}$$

The boundaries are at $r = \pm\infty$ and the horizon is at $r = 0$. The metric (2.7) degenerates at the horizon $r = 0$, but there is no singularity in the curvature tensor $R_{\mu\nu\lambda\rho} = \ell^{-2}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\lambda\nu})$. The conformal boundary of AdS_{d+1} is a two-fold cover of conformally compactified Minkowski spacetime $\mathbb{E}^{d-1,1}$: $\partial(\mathrm{AdS}_{d+1}) = S^{d-1} \times S^1$ as in Fig. 1. And that of the universal cover is Einstein static universe: $\partial(\mathrm{CAdS}_{d+1}) = ESU_d$.

Furthermore, we need to take into account the flows of time t. Let us look at Fig. 3. The left Poincaré patch in Fig. 3 is also a single region due to periodicity in τ. The flows of time t are displayed. These flows are consistent with (2.5). The flows on the two Poincaré patches near the horizon are shifted with respect to each other by infinity, but we glue together the corresponding edges of the two Poincaré patches directly along the horizon. The resulting time coordinate is the one shown in Fig. 2.

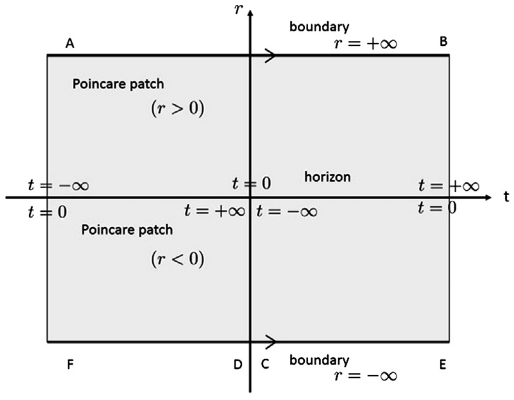

Fig. 2. AdS space is constructed by gluing two Poincaré patches at the horizon. The time flows on the two Poincaré patches near the horizon are shifted with respect to each other. Only the $\vec{x} = \vec{0}$ section is displayed.

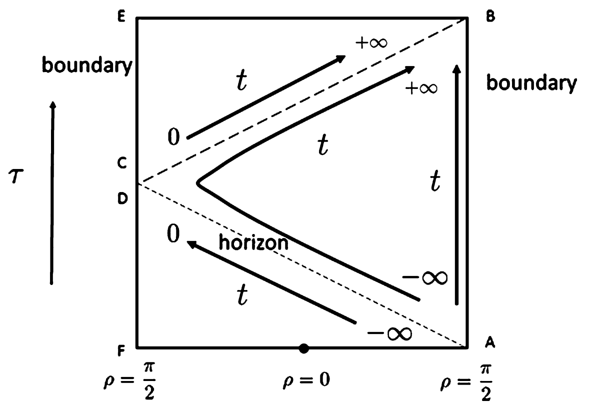

Fig. 3. Penrose diagram of AdS; The flows of time t are displayed. Points $A, \ldots, F$ correspond to those in Fig. 2.

In general, time variables in two different patches separated by a horizon do not need to coincide. In the next section, it will be shown that the fluxes of a scalar field across the horizon from each Poincaré patch vanish. Hence even if the time coordinates in the upper and lower patches are different, the fluxes are matched on both sides of the horizon.

2.3. Conformal symmetry of Poincaré patch

Importance of introducing a pair of Poincaré patches is understood by the following observation. A single set of Poincaré coordinates do not preserve the full isometry of AdS_{d+1} space, $SO(2, d)$, but only its subgroup $ISO(1, d-1) \times SO(1, 1)$ (Poincaré and dilatation symmetries). However, by introducing two Poincaré charts, a special conformal transformation,

$$t \to t' = \frac{t + (x^2 + r^{-2})a^0}{1 + a^2(x^2 + r^{-2}) + 2a \cdot x}, \tag{2.8}$$

$$\vec{x} \to \vec{x}' = \frac{\vec{x} + (x^2 + r^{-2})\vec{a}}{1 + a^2(x^2 + r^{-2}) + 2a \cdot x}, \tag{2.9}$$

$$r \to r' = r\big(1 + 2a \cdot x + a^2\big(x^2 + r^{-2}\big)\big), \tag{2.10}$$

also becomes a symmetry transformation of (2.7), and full conformal symmetry is realized. $a^i = (a^0, \vec{a})$ is a constant vector. ($x^2 \equiv -t^2 + \vec{x}^2$, $a \cdot x \equiv -a^0 t + \vec{a} \cdot \vec{x}$, etc.) The factor multiplying r on the right-hand side of (2.10) is not positive definite, and this transformation connects the two patches. The situation is completely different for EAdS. In this case a single Poincaré patch has a full conformal symmetry.

2.4. Boundaries at $r = +\infty$ and $r = -\infty$ are connected

Let us study the location of the conformal boundary in the Poincaré coordinates. By the definition of the hyperboloid (2.1) it is defined by $\sum_{i=1}^{d} X_i^2 \to \infty$, and given by $\rho = \pi/2$ in the global coordinates. In the Poincaré coordinates (2.5), it is given by

$$\sum_{i=1}^{d} X_i^2 = \frac{1}{4}h^2 r^2 + \ell^2 \vec{x}^2 r^2 - \frac{1}{2}h + \frac{1}{4r^2} \to +\infty. \tag{2.11}$$

Here h is a function $h(t, \vec{x}) \equiv t^2 - \vec{x}^2 + \ell^2$. Hence the boundary of the pair of Poincaré patches is composed of the following hypersurfaces:

1. $r \to \pm\infty$
2. $r = 0$
3. $|\vec{x}| \to \infty$ with $r \neq 0$
4. $|t| \to \infty$ with $r \neq 0$

The union of the above corresponds to the boundary of the global patch. Note that the horizon, and the spacial and even the temporal infinities are also part of the boundary. This last point is puzzling, because the conformal boundary in the global coordinates is time-like. This problem is not pursued in this paper. The structure of the boundary is illustrated in Fig. 4. Since all the parts of the boundary are connected, especially the boundaries at $r = +\infty$ and $r = -\infty$ at the same time t are connected.

In the case of AdS_2 space the coordinates $\vec{x}$ do not exist. The boundaries $r = \pm\infty$ are connected only through the lines $t = \pm\infty$. Hence in what follows we will consider AdS_{d+1} with $d \geq 2$.

3. Solutions to Klein–Gordon equation in a pair of Poincaré coordinates

In this section we consider a scalar field $\phi(r, t, \vec{x})$ of mass m in AdS spacetime in a pair of Poincaré coordinates $r > 0$ and $r < 0$. Action integral is defined by

$$S_{\mathrm{AdS}} = \int_{-\infty}^{\infty} dr \int dt\, d^{d-1}\vec{x} \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{3.1}$$

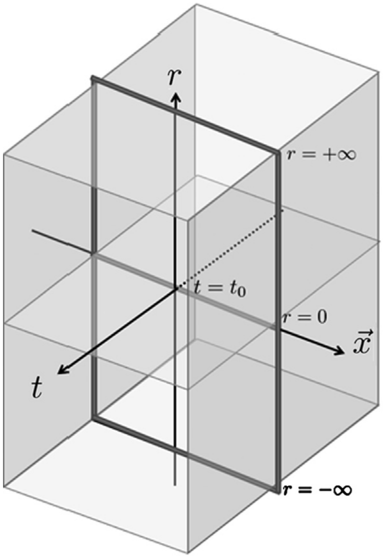

Fig. 4. Boundaries of a pair of Poincaré patches except for those at $t = \pm\infty$: two boundaries at $r = \pm\infty$ are connected. Thick lines are boundaries at $t = t_0$.

Solution will be constructed in such a way that the fluxes across the horizon vanish and those across the boundaries at $r = \pm\infty$ cancel out. The resulting solution will be shown to have the following structure in a pair of Poincaré patches. See (3.27)–(3.28).

$$\phi(r, t, \vec{x}) = \begin{cases} \varphi_+(r, t, \vec{x}) + \varphi_-(r, t, \vec{x}) & (r > 0), \\ S\varphi_+(-r, t, \vec{x}) - \frac{1}{S}\varphi_-(-r, t, \vec{x}) & (r < 0). \end{cases} \tag{3.2}$$

Here S is a real constant, and $\varphi_\pm(r, t, \vec{x})$ are functions defined for $r > 0$. As $|r| \to \infty$, $\varphi_\pm$ behaves as $|r|^{-\Delta_\mp}\phi_\pm(t, \vec{x})$ $(\Delta_\pm > 0)$. Although $\varphi_\pm$ generally oscillate rapidly near the horizon $r = 0$, if boundary values $\phi_\pm$ have compact supports, we have $\varphi_\pm \sim |r|^{\Delta_\pm}$ as $r \to \pm 0$. (Subsection 6.2.) Hence, ϕ vanishes at $r = \pm 0$ and $r = \pm\infty$, and in a coordinate $\tilde{\rho}$ $(r = \pm e^{\tilde{\rho}})$ in stead of r the solution is smooth on the entire hyperboloid.

In order to solve the equation of motion which is derived from the above action, we separate variables as

$$\phi(r, t, \vec{x}) = e^{-i\omega t + i\vec{k}\cdot\vec{x}}\chi(r). \tag{3.3}$$

Then $\chi(r)$ satisfies the equation

$$r^2\partial_r^2\chi + (d+1)r\partial_r\chi - m^2\ell^2\chi + \left(\omega^2 - \vec{k}^2\right)r^{-2}\chi = 0. \tag{3.4}$$

Two linearly independent solutions for non-integral ν is given by

$$\chi^\pm(r) = r^{-\frac{d}{2}} J_{\pm\nu}\left(\frac{\sqrt{\omega^2 - \vec{k}^2}}{r}\right), \tag{3.5}$$

where $J_\nu(z)$ is a Bessel function and

$$\nu=\sqrt{\frac{d^2}{4}+m^2\ell^2}. \tag{3.6}$$

We will restrict our attention to the case where ν is real and in the range $0<\nu<1$, because then mode functions with two different falloff behaviour can be obtained. For simplicity, we will set $\ell=1$ in what follows.

For $\omega^2-\vec{k}^2<0$, solutions (3.5) blow up exponentially at either side of the horizon $r=0$ and are non-normalizable. Thus from now on we will require $\omega^2-\vec{k}^2\geq 0$. In this case solutions oscillate near the horizon. The general solution to the Klein–Gordon equation can be written for $r>0$ and $r<0$ as

$$\phi_{\omega,\vec{k}}(r,t,\vec{x})=\begin{cases} e^{-i\omega t+i\vec{k}\cdot\vec{x}}(C_+(\omega,\vec{k})\psi_+(r,\omega,\vec{k})+C_-(\omega,\vec{k})\psi_-(r,\omega,\vec{k})) & (r>0),\\ e^{-i\omega t+i\vec{k}\cdot\vec{x}}(\tilde{C}_+(\omega,\vec{k})\psi_+(r,\omega,\vec{k})+\tilde{C}_-(\omega,\vec{k})\psi_-(r,\omega,\vec{k})) & (r<0).\end{cases} \tag{3.7}$$

Here the mode functions are defined by

$$\psi_\pm(r,\omega,\vec{k})=\begin{cases} 2^{\pm\nu}\Gamma(1\pm\nu)e^{\frac{i}{2}\pi(\frac{d}{2}\pm\nu)}r^{-\frac{d}{2}}J_{\pm\nu}\left(\frac{\sqrt{\omega^2-\vec{k}^2}}{r}\right) & (r>0),\\ 2^{\pm\nu}\Gamma(1\pm\nu)e^{-\frac{i}{2}\pi(\frac{d}{2}\pm\nu)}(-r)^{-\frac{d}{2}}J_{\pm\nu}\left(\frac{\sqrt{\omega^2-\vec{k}^2}}{-r}\right) & (r<0).\end{cases} \tag{3.8}$$

Because the metric (2.7) is degenerate at the horizon ($r=0$), the equation for ϕ is singular. So, the coefficients $\tilde{C}_\pm$ will be connected to $C_\pm$ in such a way that the fluxes are matched at the horizon and cancel out between the boundaries.

3.1. Klein–Gordon norm

The Klein–Gordon (KG) norm (ϕ_1,ϕ_2) for two modes $\phi_{1,2}$ is given by[5]

$$\begin{aligned}(\phi_1,\phi_2)&=\int_{-\infty}^{\infty}dr\int d^{d-1}\vec{x}\sqrt{-g}\frac{-i}{2}g^{tt}\left(\phi_1^*\partial_t\phi_2-\phi_2\partial_t\phi_1^*\right)\big|_{t=t_0\text{ fixed}}\\ &=\int_{-\infty}^{\infty}dr\int d^{d-1}\vec{x}\frac{i}{2}|r|^{d-3}\left(\phi_1^*\partial_t\phi_2-\phi_2\partial_t\phi_1^*\right)\big|_{t=t_0\text{ fixed}}.\end{aligned} \tag{3.9}$$

Although the KG current is divergenceless, for conservation of the norm (3.9), we need to impose some conditions on the solutions. We will show that this norm is conserved (*i.e.*, independent of t_0), if the coefficients satisfy the relations

$$\tilde{C}_+(\omega,\vec{k})=e^{i\pi\nu}C_+(\omega,\vec{k})Se^{i(\alpha+\frac{\pi}{2}d)}, \tag{3.10}$$

$$\tilde{C}_-(\omega,\vec{k})=-e^{-i\pi\nu}C_-(\omega,\vec{k})\frac{1}{S}e^{i(\alpha+\frac{\pi}{2}d)}. \tag{3.11}$$

Here α and S are real parameters.

When solution (3.7) is substituted into the norm (3.9) and $\vec{x}$ integral is performed, the norm is given by

[5] Here time $t=t_0$ is fixed. For the coordinate system in Fig. 2, constant-t hypersurfaces for $r>0$ and $r<0$ patches are not adjacent to each other at the horizon.

$$(\phi_1,\phi_2)=\frac{\omega_1+\omega_2}{2}(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}_1-\vec{k}_2)e^{i(\omega_1-\omega_2)t}\cdot\left(\int_0^\infty dr\, r^{d-3}\psi_1^*(r,\omega_1,\vec{k}_1)\psi_2(r,\omega_2,\vec{k}_2)+\int_{-\infty}^0 dr\,(-r)^{d-3}\psi_1^*(r,\omega_1,\vec{k}_1)\psi_2(r,\omega_2,\vec{k}_2)\right). \tag{3.12}$$

Now because ψ in (3.8) solves (3.4), ψ_1 and ψ_2 satisfy

$$r^{3-d}\partial_r\left(r^{d+1}\left(\psi_1^*\partial_r\psi_2-\psi_2\partial_r\psi_1^*\right)\right)=\left(\omega_1^2-\omega_2^2-\vec{k}_1^2+\vec{k}_2^2\right)\psi_1^*\psi_2, \tag{3.13}$$

and for $\omega_1^2-\omega_2^2\neq 0$, the norm (3.12) is expressed in terms of boundary values.[6]

$$(\phi_1,\phi_2)=(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}_1-\vec{k}_2)e^{i(\omega_1-\omega_2)t}\frac{1}{2(\omega_1-\omega_2)}\cdot\left(\left[r^{d+1}\left(\psi_1^*\partial_r\psi_2-\psi_2\partial_r\psi_1^*\right)\right]_0^\infty+\left[(-r)^{d+1}\left(\psi_1^*\partial_r\psi_2-\psi_2\partial_r\psi_1^*\right)\right]_{-\infty}^0\right). \tag{3.14}$$

The contributions from the boundaries $r=\pm\infty$ are computed by using $J_\nu(z)\sim(\Gamma(\nu+1))^{-1}(z/2)^\nu$ for $z\sim 0$. The result is

$$\begin{aligned}(\phi_1,\phi_2)|_{|r|=\infty}=\frac{1}{2(\omega_1-\omega_2)}&(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}_1-\vec{k}_2)e^{i(\omega_1-\omega_2)t}\\&\cdot\Bigg[2\nu C_+^*(\omega_1,\vec{k}_1)C_-(\omega_2,\vec{k}_2)e^{-\pi i\nu}\left(\frac{\omega_1^2-\vec{k}_1^2}{\omega_2^2-\vec{k}_2^2}\right)^{\frac{\nu}{2}}\\&-2\nu C_-^*(\omega_1,\vec{k}_1)C_+(\omega_2,\vec{k}_2)e^{\pi i\nu}\left(\frac{\omega_2^2-\vec{k}_2^2}{\omega_1^2-\vec{k}_1^2}\right)^{\frac{\nu}{2}}\\&+2\nu\tilde{C}_+^*(\omega_1,\vec{k}_1)\tilde{C}_-(\omega_2,\vec{k}_2)e^{\pi i\nu}\left(\frac{\omega_1^2-\vec{k}_1^2}{\omega_2^2-\vec{k}_2^2}\right)^{\frac{\nu}{2}}\\&-2\nu\tilde{C}_-^*(\omega_1,\vec{k}_1)\tilde{C}_+(\omega_2,\vec{k}_2)e^{-\pi i\nu}\left(\frac{\omega_2^2-\vec{k}_2^2}{\omega_1^2-\vec{k}_1^2}\right)^{\frac{\nu}{2}}\Bigg].\end{aligned} \tag{3.15}$$

This vanishes if $C_+=\tilde{C}_+=0$ or $C_-=\tilde{C}_-=0$, *i.e.*, if Dirichlet or Neumann boundary condition is imposed. There is, however, another solution. This norm also vanishes, if the following condition is satisfied.

$$C_+^*(\omega_1,\vec{k}_1)C_-(\omega_2,\vec{k}_2)=-e^{2\pi i\nu}\tilde{C}_+^*(\omega_1,\vec{k}_1)\tilde{C}_-(\omega_2,\vec{k}_2). \tag{3.16}$$

This new solution is possible, because a pair of Poincaré patches is introduced. As will be shown in the next subsection, a flux across one boundary matches that from another.

We now turn to the contributions to the norm from the horizon. These are obtained by using the asymptotic form $J_\nu(z)\sim\sqrt{2/\pi z}\cos(z-(2\nu+1)\pi/4)$ for $z\to\infty$. The contribution from the upper side of the horizon is given by

[6] We follow the techniques used in [13].

$$
\begin{aligned}
&(\phi_1, \phi_2)|_{r=+0} \\
&= \frac{1}{2(\omega_1-\omega_2)} e^{i(\omega_1-\omega_2)t} (2\pi)^{d-1} \delta^{(d-1)}(\vec{k}_1 - \vec{k}_2) \cdot \lim_{r\to 0} \Bigg[-\frac{2}{\pi} \big(4^{\nu}\Gamma(1+\nu)^2 N_+ \\
&\quad + 4^{-\nu}\Gamma(1-\nu)^2 N_-\big) \sin \frac{\sqrt{\omega_2^2 - \vec{k}_2^2} - \sqrt{\omega_1^2 - \vec{k}_1^2}}{r} \\
&\quad - \frac{2}{\pi}\left(\frac{\omega_2^2 - \vec{k}_2^2}{\omega_1^2 - \vec{k}_1^2}\right)^{\frac{1}{4}} M_1(\omega_1, \vec{k}_1; \omega_2, \vec{k}_2) + \frac{2}{\pi}\left(\frac{\omega_1^2 - \vec{k}_1^2}{\omega_2^2 - \vec{k}_2^2}\right)^{\frac{1}{4}} M_2(\omega_1, \vec{k}_1; \omega_2, \vec{k}_2) \\
&\quad + \frac{2}{\pi}\left(\frac{\omega_1^2 - \vec{k}_1^2}{\omega_2^2 - \vec{k}_2^2}\right)^{\frac{1}{4}} M_1^*(\omega_2, \vec{k}_2; \omega_1, \vec{k}_1) - \frac{2}{\pi}\left(\frac{\omega_2^2 - \vec{k}_2^2}{\omega_1^2 - \vec{k}_1^2}\right)^{\frac{1}{4}} M_2^*(\omega_2, \vec{k}_2; \omega_1, \vec{k}_1)\Bigg], \quad (3.17)
\end{aligned}
$$

where

$$
N_\pm = C_\pm^*(\omega_1, \vec{k}_1) C_\pm(\omega_2, \vec{k}_2), \quad (3.18)
$$

$$
\begin{aligned}
M_1(\omega_1, \vec{k}_1; \omega_2, \vec{k}_2) &= \Gamma(1+\nu)\Gamma(1-\nu) C_+^*(\omega_1, \vec{k}_1) C_-(\omega_2, \vec{k}_2) e^{-i\pi\nu} \\
&\quad \cdot \cos\left(\frac{\sqrt{\omega_1^2 - \vec{k}_1^2}}{r} - \frac{2\nu+1}{4}\pi\right) \sin\left(\frac{\sqrt{\omega_2^2 - \vec{k}_2^2}}{r} - \frac{-2\nu+1}{4}\pi\right),
\end{aligned} \quad (3.19)
$$

$$
\begin{aligned}
M_2(\omega_1, \vec{k}_1; \omega_2, \vec{k}_2) &= \Gamma(1+\nu)\Gamma(1-\nu) C_+^*(\omega_1, \vec{k}_1) C_-(\omega_2, \vec{k}_2) e^{-i\pi\nu} \\
&\quad \cdot \cos\left(\frac{\sqrt{\omega_2^2 - \vec{k}_2^2}}{r} - \frac{-2\nu+1}{4}\pi\right) \sin\left(\frac{\sqrt{\omega_1^2 - \vec{k}_1^2}}{r} - \frac{2\nu+1}{4}\pi\right).
\end{aligned} \quad (3.20)
$$

To simplify M_1 and M_2, we need to use some formulae for distributions: $\sin(\Lambda x)/(\pi x) \to \delta(x)$, $\cos(\Lambda x)/(\pi x) \to 0$ for $\Lambda \to +\infty$ [13]. In the limit $r \to +0$, functions $M_{1,2}$ can be simplified by using these formulae as

$$
\begin{aligned}
M_1(\omega_1, \vec{k}_1; \omega_2, \vec{k}_2) &= M_2(\omega_1, \vec{k}_1; \omega_2, \vec{k}_2) \\
&= \frac{1}{2}\Gamma(1+\nu)\Gamma(1-\nu) C_+^*(\omega_1, \vec{k}_1) C_-(\omega_2, \vec{k}_2) e^{-i\pi\nu} \\
&\quad \times \sin \frac{\sqrt{\omega_1^2 - \vec{k}_1^2} - \sqrt{\omega_2^2 - \vec{k}_2^2}}{r} \sin \frac{2\nu+1}{2}\pi.
\end{aligned} \quad (3.21)
$$

Since $(\omega_1 - \omega_2)^{-1} \sin \frac{\sqrt{\omega_1^2 - \vec{k}_1^2} - \sqrt{\omega_2^2 - \vec{k}_2^2}}{r} \to \pi \operatorname{sign}(\omega_1)\delta(\omega_1 - \omega_2)$, those terms which contain $M_{1,2}$ all vanish, and we get

$$
\begin{aligned}
(\phi_1, \phi_2)|_{r=+0} &= \operatorname{sign}(\omega_1)\delta(\omega_1 - \omega_2)(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}_1 - \vec{k}_2) \\
&\quad \cdot \left[4^{\nu}\Gamma(1+\nu)^2 \left|C_+(\omega_1, \vec{k}_1)\right|^2 + 4^{-\nu}\Gamma(1-\nu)^2 \left|C_-(\omega_1, \vec{k}_1)\right|^2\right].
\end{aligned} \quad (3.22)
$$

Contribution to the norm at the other side of the horizon, $(\phi_1, \phi_2)|_{r=-0}$, can be similarly computed. Finally, KG norm is independent of t and given by

$$(\phi_1, \phi_2) = \mathrm{sign}(\omega_1)\delta(\omega_1 - \omega_2)(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}_1 - \vec{k}_2) \cdot \left[4^{\nu}\Gamma(1+\nu)^2\left(1+S^2\right)\left|C_+(\omega_1, \vec{k}_1)\right|^2 + 4^{-\nu}\Gamma(1-\nu)^2\left(1+S^{-2}\right)\left|C_-(\omega_1, \vec{k}_1)\right|^2\right]. \tag{3.23}$$

3.2. Flux

Since the KG norm is conserved, the fluxes must cancel or vanish at the boundaries and the horizon. Let us check this. One can compute the flux across the horizon from the $r > 0$ patch.

$$J_{(+0)} = \int_{r=r_0\to+0,\ t=t_0} d^{d-1}\vec{x}\sqrt{-g}g^{rr}\frac{-i}{2}\left(\phi_1^*\partial_r\phi_2 - \phi_2\partial_r\phi_1^*\right). \tag{3.24}$$

One can show that this vanishes by using $\sqrt{-g}g^{rr} = r^{d+1}$ and $\phi \sim r^{-(d+1)/2}\cos(\cdots)$. A calculation similar to that used in deriving (3.23) must be done. Similarly, the flux at $r = -0$ also vanishes. The fluxes at $r = +\infty$, however, does not vanish. It is given by

$$J_{(+\infty)} = -i\nu(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}_1 - \vec{k}_2)e^{i(\omega_1-\omega_2)t} \cdot \left(\left(\frac{\omega_1^2 - \vec{k}_1^2}{\omega_2^2 - \vec{k}_2^2}\right)^{\frac{\nu}{2}} e^{-i\pi\nu}C_+^*(\omega_1, \vec{k}_1)C_-(\omega_2, \vec{k}_2) - \left(\frac{\omega_2^2 - \vec{k}_2^2}{\omega_1^2 - \vec{k}_1^2}\right)^{\frac{\nu}{2}} e^{i\pi\nu}C_-^*(\omega_1, \vec{k}_1)C_+(\omega_2, \vec{k}_2)\right). \tag{3.25}$$

This takes forms of interference terms between the two kinds of modes. By using (3.16) one can show that this is canceled by the out-going flux at $r = -\infty$.

$$J_{(-\infty)} = +i\nu(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}_1 - \vec{k}_2)e^{i(\omega_1-\omega_2)t} \cdot \left(-\left(\frac{\omega_1^2 - \vec{k}_1^2}{\omega_2^2 - \vec{k}_2^2}\right)^{\frac{\nu}{2}} e^{i\pi\nu}\tilde{C}_+^*(\omega_1, \vec{k}_1)\tilde{C}_-(\omega_2, \vec{k}_2) + \left(\frac{\omega_2^2 - \vec{k}_2^2}{\omega_1^2 - \vec{k}_1^2}\right)^{\frac{\nu}{2}} e^{-i\pi\nu}\tilde{C}_-^*(\omega_1, \vec{k}_1)\tilde{C}_+(\omega_2, \vec{k}_2)\right). \tag{3.26}$$

The above results might seem useless, because the two boundaries in Fig. 2 appear to be infinitely separated. As mentioned in Subsection 2.4, however, in AdS_{d+1} space with $d \geq 2$, the two boundaries at $r = +\infty$ and $r = -\infty$ are connected. In this way the total flux computed on the boundaries $r = \pm\infty$ cancels out at any time t.

To summarize, normalizable modes in the pair of Poincaré patches are given by

$$\phi_{\omega,\vec{k}}(r, t, \vec{x}) = e^{-i\omega t + i\vec{k}\cdot\vec{x}} \cdot \left[C_+(\omega, \vec{k})2^{\nu}\Gamma(1+\nu)e^{\frac{i}{2}\pi(\frac{d}{2}+\nu)}r^{-\frac{d}{2}}J_{\nu}\left(\sqrt{\omega^2 - \vec{k}^2}/r\right) + C_-(\omega, \vec{k})2^{-\nu}\Gamma(1-\nu)e^{\frac{i}{2}\pi(\frac{d}{2}-\nu)}r^{-\frac{d}{2}}J_{-\nu}\left(\sqrt{\omega^2 - \vec{k}^2}/r\right)\right] \tag{3.27}$$

for $r > 0$, and

$$\phi_{\omega,\vec{k}}(r,t,\vec{x}) = e^{-i\omega t+i\vec{k}\cdot\vec{x}}e^{i\alpha}\Big[SC_+(\omega,\vec{k})2^{\nu}\Gamma(1+\nu)e^{\frac{i}{2}\pi(\frac{d}{2}+\nu)}(-r)^{-\frac{d}{2}}J_{\nu}\Big(-\sqrt{\omega^2-\vec{k}^2}/r\Big) - \frac{1}{S}C_-(\omega,\vec{k})2^{-\nu}\Gamma(1-\nu)e^{\frac{i}{2}\pi(\frac{d}{2}-\nu)}(-r)^{-\frac{d}{2}}J_{-\nu}\Big(-\sqrt{\omega^2-\vec{k}^2}/r\Big)\Big] \tag{3.28}$$

for $r < 0$. Note that these mode functions are rapidly oscillating and blowing up near the horizon $r = 0$ like $\sim r^{\frac{1-d}{2}}\cos(\sqrt{\omega^2-\vec{k}^2}/r - (\pm 2\nu+1)\pi/4)$. However, this very rapid oscillation actually makes the mode functions cancel out and vanish at the horizon. We will show in Section 6 that the solution (6.12) to the boundary-value problem, constructed by smearing these mode functions by source functions which have compact supports, has a milder behaviour $\phi \sim |r|^{\Delta_\pm}$ for $r \to 0$, where $\Delta_\pm = d/2 \pm \nu$. By means of the coordinate[7] $\tilde{\rho} \equiv \log|r|$ this can be written as $\phi \sim e^{\Delta_\pm \tilde{\rho}}$, and ϕ asymptotes to zero exponentially near the horizon $\tilde{\rho} = -\infty$. In this sense, the mode functions are smoothly connected at the horizon.

3.3. *Conservation of energy*

In AdS_{d+1} there is a time-like Killing vector and by contracting this with a stress-energy tensor, a formally conserved energy can be defined. To obtain an exactly conserved energy, one needs to show that the energy-flux vanishes or cancels at the horizon and boundaries. In AdS space the Riemann scalar R is constant, $d(d+1)$ (in units $\ell = 1$), and a coupling $R\phi^2$ is equivalent to a mass term. Hence we may replace the mass squared m^2 by $m^2 - \xi d(d+1) + \xi R$ in the action. Here ξ is a constant and the conformal coupling corresponds to $\xi = \xi_c \equiv -(d-1)/(4d)$. We will leave ξ as a free parameter and fix its value below.[8]

The stress-energy tensor is, after substitution of the solution into the equation of motion, given by

$$T_{\mu\nu} = (1+2\xi)\partial_\mu\phi\partial_\nu\phi + 2\xi\phi\nabla_\mu\nabla_\nu\phi + \left(2\xi - \frac{1}{2}\right)m^2 g_{\mu\nu}\phi^2 - \frac{1}{2}(1+4\xi)g_{\mu\nu}\phi\nabla^\lambda\nabla_\lambda\phi - d^2\xi g_{\mu\nu}\phi^2. \tag{3.29}$$

Energy flux

$$\int_{t,r\text{ fixed}} d^{d-1}\vec{x}\sqrt{-g}g^{rr}T_{rt} \tag{3.30}$$

can be calculated as in the previous subsection for the particle number flux. T_{rt} is given by

$$T_{rt} = (1+2\xi)\partial_r\phi\partial_t\phi + 2\xi\phi\partial_r\partial_t\phi - \frac{2\xi}{r}\phi\partial_t\phi \tag{3.31}$$

[7] This variable $\tilde{\rho}$ is different from ρ of the global coordinates (2.3).

[8] In [2] it was shown that in the global coordinates of AdS space, energy of either Dirichlet or Neumann mode is conserved by choosing stress-tensor with a conformal coupling $\xi = \xi_c$.

and it is easily shown that the fluxes at $r = +0$ and $r = -0$ vanish. It turns out, however, that for general ξ, the energy-fluxes at $r = \pm\infty$ contain an infinity $|r|^{2\nu}$ associated with the modes ψ_-. This infinity can be removed by fine tuning ξ.

$$\xi = \frac{2\nu - d}{4(d - 2\nu + 1)} \tag{3.32}$$

Interestingly, at $\nu = \frac{1}{2}$, this agrees with the conformal value ξ_c presented above. There still remain finite ($\mathcal{O}(r^0)$) fluxes at the two boundaries. It can, however, be shown that the remaining fluxes at $r = +\infty$ and $r = -\infty$ cancel out completely by using (3.16), exactly as in the particle number flux. Hence the energy associated with both kinds of modes $\psi_\pm$ is conserved in the pair of Poincaré patches.

4. Mode expansion of ϕ and canonical commutation relations

In this section we will perform canonical quantization of a free scalar field in AdS_{d+1} in a pair of Poincaré coordinates. We use mode expansions (3.27) and (3.28). By replacing the coefficients $C_{+,-}$ by annihilation and creation operators, and integrating over ω and $\vec{k}$, we obtain the following operator:

$$\begin{aligned}\Phi(r,t,\vec{x}) = \int_{-\infty}^{\infty} d^{d-1}\vec{k} \int_{|\vec{k}|}^{\infty} d\omega \big[& e^{-i\omega t + i\vec{k}\cdot\vec{x}} \big(a_+(\omega,\vec{k})\hat{\psi}_+(r,\omega,\vec{k}) + a_-(\omega,\vec{k})\hat{\psi}_-(r,\omega,\vec{k}) \big) \\ & + e^{i\omega t - i\vec{k}\cdot\vec{x}} \big(a_+^\dagger(\omega,\vec{k})\hat{\psi}_+^*(r,\omega,\vec{k}) + a_-^\dagger(\omega,\vec{k})\hat{\psi}_-^*(r,\omega,\vec{k}) \big) \big].\end{aligned} \tag{4.1}$$

The integration region is restricted to $|\vec{k}| \le \omega$. This operator is defined for both $r > 0$ and $r < 0$. The functions $\hat{\psi}_\pm$ are obtained by slightly modifying $\psi_\pm$, and given by

$$\hat{\psi}_+(r,\omega,\vec{k}) = \begin{cases} 2^\nu \Gamma(1+\nu) r^{-\frac{d}{2}} J_\nu\big(\sqrt{\omega^2 - \vec{k}^2}/r\big) & (r > 0), \\ S e^{i\alpha} \cdot 2^\nu \Gamma(1+\nu)(-r)^{-\frac{d}{2}} J_\nu\big(-\sqrt{\omega^2 - \vec{k}^2}/r\big) & (r < 0), \end{cases} \tag{4.2}$$

$$\hat{\psi}_-(r,\omega,\vec{k}) = \begin{cases} 2^{-\nu} \Gamma(1-\nu) r^{-\frac{d}{2}} J_{-\nu}\big(\sqrt{\omega^2 - \vec{k}^2}/r\big) & (r > 0), \\ -\frac{1}{S} e^{i\alpha} \cdot 2^{-\nu} \Gamma(1-\nu)(-r)^{-\frac{d}{2}} J_{-\nu}\big(-\sqrt{\omega^2 - \vec{k}^2}/r\big) & (r < 0). \end{cases} \tag{4.3}$$

This operator and its canonical conjugate momentum $\Pi(r,t,\vec{x}) = |r|^{d-3}\partial_t \Phi$ must satisfy the canonical commutation relations: $[\Phi(r,t,\vec{x}), \Pi(r',t,\vec{x}')] = i\delta(r - r')\delta^{(d-1)}(\vec{x} - \vec{x}')$, $[\Phi(r,t,\vec{x}), \Phi(r',t,\vec{x}')] = 0$ and $[\Pi(r,t,\vec{x}), \Pi(r',t,\vec{x}')] = 0$. It can be shown that this is achieved by setting $\alpha = 0$ or π and imposing the following commutators. (Other commutators are vanishing.)

$$\begin{aligned} \big[a_+(\omega,\vec{k}), a_+^\dagger(\omega',\vec{k}')\big] &= \frac{1}{2^{1+2\nu}(2\pi)^{d-1}\Gamma(1+\nu)^2} \frac{1}{1+S^2} \delta(\omega - \omega')\delta^{(d-1)}(\vec{k} - \vec{k}'), \\ \big[a_-(\omega,\vec{k}), a_-^\dagger(\omega',\vec{k}')\big] &= \frac{1}{2^{1-2\nu}(2\pi)^{d-1}\Gamma(1-\nu)^2} \frac{S^2}{1+S^2} \delta(\omega - \omega')\delta^{(d-1)}(\vec{k} - \vec{k}'). \end{aligned} \tag{4.4}$$

Because $e^{i\alpha}$ is multiplied by S or $1/S$ in (4.2), (4.3), we can set $\alpha = 0$ by allowing S to take positive or negative values.

The role of parameter S is to specify the relative magnitude of the mode functions (4.2)–(4.3) in the two patches. One can replace ψ_- by $S^{-1}\tilde{\psi}_-$ and a_- by $S\tilde{a}_-$ without changing the form of (4.1). Then, S-dependences of $[a_+, a_+^\dagger]$ and $[\tilde{a}_-, \tilde{a}_-^\dagger]$ become the same: $1/(1+S^2)$. If one sets $S=0$, then the mode $\tilde{\psi}_-$ is quantized only in the patch with $r<0$, while ψ_+ is quantized only in the $r>0$ patch. At present we do not have an argument to determine S, and in this paper we will leave the value of S undetermined.

First let us consider $[\Phi(r,t,\vec{x}), \Phi(r',t,\vec{x}')]$. This is given as

$$\begin{aligned}&\left[\Phi(r,t,\vec{x}), \Phi\left(r',t,\vec{x}'\right)\right]\\&=\int d^{d-1}\vec{k}\int_{|\vec{k}|}^{\infty} d\omega(2\pi)^{1-d}\left(1+S^2\right)^{-1}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\\&\quad\cdot\left[2^{-1-2\nu}\frac{1}{\Gamma(1+\nu)^2}\left\{\hat{\psi}_+(r,\omega,\vec{k})\hat{\psi}_+^*\left(r',\omega,\vec{k}\right)-\hat{\psi}_+^*(r,\omega,-\vec{k})\hat{\psi}_+\left(r',\omega,-\vec{k}\right)\right\}\right.\\&\quad\left.+2^{-1+2\nu}\frac{S^2}{\Gamma(1-\nu)^2}\left\{\hat{\psi}_-(r,\omega,\vec{k})\hat{\psi}_-^*\left(r',\omega,\vec{k}\right)-\hat{\psi}_-^*(r,\omega,-\vec{k})\hat{\psi}_-\left(r',\omega,-\vec{k}\right)\right\}\right].\end{aligned} \tag{4.5}$$

For $rr'>0$, terms on the right-hand side cancel out completely. For $r>0$ and $r'<0$, we have

$$\begin{aligned}&\left[\Phi(r,t,\vec{x}), \Phi\left(r',t,\vec{x}'\right)\right]\\&=\int d^{d-1}\vec{k}\int_0^{\infty} d\mu\frac{\mu}{\sqrt{\vec{k}^2+\mu^2}}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{S}{2(1+S^2)}(2\pi)^{1-d}\\&\quad\cdot\left(e^{-i\alpha}-e^{i\alpha}\right)\left[J_\nu\left(\frac{\mu}{r}\right)J_\nu\left(\frac{\mu}{-r'}\right)+J_{-\nu}\left(\frac{\mu}{r}\right)J_{-\nu}\left(\frac{\mu}{-r'}\right)\right]\left(-rr'\right)^{-\frac{d}{2}}.\end{aligned} \tag{4.6}$$

Here we set $\omega=\sqrt{\vec{k}^2+\mu^2}$ and integration over ω is replaced by that over μ. This vanishes, if $e^{i\alpha}=\pm 1$. Similar result is obtained for $r<0$ and $r'>0$. By a similar analysis it can be shown that $[\Pi(r,t,\vec{x}), \Pi(r',t,\vec{x}')]=0$, if $e^{i\alpha}=\pm 1$.

Next we turn to $[\Phi(r,t,\vec{x}), \Pi(r',t,\vec{x}')]$. In this case we have

$$\begin{aligned}&\left[\Phi(r,t,\vec{x}), \Pi\left(r',t,\vec{x}'\right)\right]\\&=\int d^{d-1}\vec{k}\int_{|\vec{k}|}^{\infty} d\omega(2\pi)^{1-d}\left(1+S^2\right)^{-1}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}i\omega\left|r'\right|^{d-3}\\&\quad\cdot\left[2^{-1-2\nu}\frac{1}{\Gamma(1+\nu)^2}\left\{\hat{\psi}_+(r,\omega,\vec{k})\hat{\psi}_+^*\left(r',\omega,\vec{k}\right)+\hat{\psi}_+^*(r,\omega,-\vec{k})\hat{\psi}_+\left(r',\omega,-\vec{k}\right)\right\}\right.\\&\quad\left.+2^{-1+2\nu}\frac{S^2}{\Gamma(1-\nu)^2}\left\{\hat{\psi}_-(r,\omega,\vec{k})\hat{\psi}_-^*\left(r',\omega,\vec{k}\right)+\hat{\psi}_-^*(r,\omega,-\vec{k})\hat{\psi}_-\left(r',\omega,-\vec{k}\right)\right\}\right].\end{aligned} \tag{4.7}$$

For $rr' > 0$, this yields

$$\big[\Phi(r,t,\vec{x}),\Pi\big(r',t,\vec{x}'\big)\big] = \int d^{d-1}\vec{k}\int_0^\infty d\mu(2\pi)^{1-d}\big(1+S^2\big)^{-1}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}i\mu|r'|^{d-3} \cdot \big(rr'\big)^{-\frac{d}{2}}\left[J_\nu\left(\frac{\mu}{r}\right)J_\nu\left(\frac{\mu}{r'}\right)+S^2J_{-\nu}\left(\frac{\mu}{r}\right)J_{-\nu}\left(\frac{\mu}{r'}\right)\right]. \tag{4.8}$$

By using a formula

$$\int_0^\infty dx\,xJ_\nu(ax)J_\nu(bx) = \frac{1}{a^2-b^2}\big[x\big(J_\nu(ax)J'_\nu(bx)-J_\nu(bx)J'_\nu(ax)\big)\big]_0^\infty \tag{4.9}$$

and the identities for distributions used in the previous section, it can be shown that (4.8) agrees with $i\delta(r-r')\delta^{(d-1)}(\vec{x}-\vec{x}')$. For $r>0$ and $r'<0$ we have

$$\begin{aligned}&\big[\Phi(r,t,\vec{x}),\Pi\big(r',t,\vec{x}'\big)\big]\\ &= \int d^{d-1}\vec{k}\int_0^\infty d\mu(2\pi)^{1-d}\frac{S}{2(1+S^2)}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}i\mu|r'|^{d-3}\\ &\quad\cdot\big(e^{-i\alpha}+e^{i\alpha}\big)\big(-rr'\big)^{-\frac{d}{2}}\left[J_\nu\left(\frac{\mu}{r}\right)J_\nu\left(\frac{\mu}{-r'}\right)-J_{-\nu}\left(\frac{\mu}{r}\right)J_{-\nu}\left(\frac{\mu}{-r'}\right)\right].\end{aligned} \tag{4.10}$$

This vanishes due to (4.9). The commutators with $r<0$ and $r'>0$ also vanish.

5. Wightman function

In this section we will compute Wightman function for a scalar field in AdS_{d+1} space.

$$G\big(r,t,\vec{x};r',t',\vec{x}'\big) = \langle 0|\Phi(r,t,\vec{x})\Phi\big(r',t',\vec{x}'\big)|0\rangle. \tag{5.1}$$

Here $|0\rangle$ is a vacuum which is annihilated by a_+ and a_-.

By using the mode expansion (4.1) and the commutation relations (4.4), Wightman function is given by

$$\begin{aligned}&G\big(r,t,\vec{x};r',t',\vec{x}'\big) = G_+\big(r,t,\vec{x};r',t',\vec{x}'\big)+G_-\big(r,t,\vec{x};r',t',\vec{x}'\big),\\ &G_\pm = \int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\int_{|\vec{k}|}^\infty d\omega\frac{1}{2(1+S^2)}e^{-i\omega(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')}\\ &\quad\cdot\frac{1}{2^{\pm 2\nu}\Gamma(1\pm\nu)^2}\hat{\psi}_\pm(r,\omega,\vec{k})\hat{\psi}^*_\pm\big(r',\omega,\vec{k}\big),\end{aligned} \tag{5.2}$$

$G_\pm$ can be expressed as integrals (A.1) of a flat-space Wightman function integrated over a mass parameter μ. We will display the results for space-like separation of the plane ($\mathbb{E}^{d-1,1}$) coordinates.

$$\big(x-x'\big)^2 \equiv -\big(t-t'\big)^2+\big(\vec{x}-\vec{x}'\big)^2 > 0. \tag{5.3}$$

From the structure of the mode functions (4.2), (4.3), we have

$$G_+\big(r,t,\vec{x};r',t',\vec{x}'\big)=S^{\theta(-r)+\theta(-r')}G_+\big(|r|,t,\vec{x};\big|r'\big|,t',\vec{x}'\big), \tag{5.4}$$

$$G_-\big(r,t,\vec{x};r',t',\vec{x}'\big)=(-S)^{-\theta(-r)-\theta(-r')}G_-\big(|r|,t,\vec{x};\big|r'\big|,t',\vec{x}'\big). \tag{5.5}$$

By using some mathematical formulae in, for example [16], we can show that

$$G_+\big(r,t,\vec{x};r',t',\vec{x}'\big)=\frac{S^{\theta(-r)+\theta(-r')}}{1+S^2}\frac{\Gamma(\frac{d}{2}+\nu)}{2\pi^{\frac{d}{2}}\Gamma(1+\nu)}P^{\frac{d}{2}+\nu}{}_2F_1\left(\frac{d}{2}+\nu,\frac{1}{2}+\nu,1+2\nu;-4P\right), \tag{5.6}$$

$$G_-\big(r,t,\vec{x};r',t',\vec{x}'\big)=\frac{(-S)^{\theta(r)+\theta(r')}}{1+S^2}\frac{\Gamma(\frac{d}{2}-\nu)}{2\pi^{\frac{d}{2}}\Gamma(1-\nu)}P^{\frac{d}{2}-\nu}{}_2F_1\left(\frac{d}{2}-\nu,\frac{1}{2}-\nu,1-2\nu;-4P\right). \tag{5.7}$$

Here ${}_2F_1(a,b,c;z)$ is a hypergeometric function, and $\theta(x)$ is a step function ($\theta(x)=1$ for $x>0$ and 0 for $x<0$). P is defined by

$$P=\frac{1}{|rr'|}\frac{1}{(\frac{1}{|r|}-\frac{1}{|r'|})^2+(x-x')^2} \tag{5.8}$$

and related to the chordal distance $\sigma\equiv X\cdot X'+1$ by

$$P^{-1}=\begin{cases}-2\sigma & (rr'>0),\\ 2(\sigma-2) & (rr'<0).\end{cases} \tag{5.9}$$

Note that $\sigma=0$ for $X=X'$, and $\sigma=2$ for $X=-X'$. Hence P^{-1} vanishes either if the points coincide $X=X'$ ($rr'>0$), or if they are antipodal to each other, $X=-X'$ ($rr'<0$). The hypergeometric functions in (5.6) and (5.7) can be singular at $-4P=0,1,\infty$. As discussed above, condition $P=\infty$, which is equivalent to

$$\left(\frac{1}{|r|}-\frac{1}{|r'|}\right)^2+\big(x-x'\big)^2=0, \tag{5.10}$$

is satisfied for $X=\pm X'$ (coincident and antipodal points).[9] A singularity at $P=-1/4$ occurs for

$$\left(\frac{1}{|r|}+\frac{1}{|r'|}\right)^2+\big(x-x'\big)^2=0. \tag{5.11}$$

These singularities (5.10) and (5.11) are associated with a real charge and its image [12]. If $d\geq 2$, $P=0$ is not a singularity due to the pre-factors $P^{\frac{d}{2}\pm\nu}$ in (5.6) and (5.7). The result (5.6) is derived in Appendix A. When r' is sent to infinity, the above functions approach the bulk–boundary propagators: $G_\pm\sim\text{const}\cdot|r'|^{-\Delta_\pm}((1+r^2(x-x')^2)/|r|)^{-\Delta_\pm}$. For null and time-like separation ($P^{-1}\leq 0$), G is given by analytically continuing the above result by $i\epsilon$ prescription $t-t'\to t-t'-i\epsilon$. Feynman propagator iG_F is obtained from G by replacement $P^{-1}\to P^{-1}+i\epsilon$. Feynman propagator of a scalar field in a Poincaré patch of AdS space with a single type of modes was obtained in [11].

9 It can be shown by using (5.6)–(5.7) that singularities at $X=-X'$ cancel out between G_+ and G_-.

6. AdS/CFT correspondence

In the preceding sections we have learnt that a general solution ϕ to the K–G equation in a pair of Poincaré patches has the structure (3.2). As discussed at the end of Section 3, $\varphi_\pm$ in a coordinate $\tilde{\rho} \equiv \log|r|$ effectively asymptote to zero exponentially near the horizon $\tilde{\rho} \to -\infty$, and ϕ is smooth at the horizon, even if $\varphi_\pm$ are multiplied by S and $-1/S$ for $r < 0$. As we will see, this structure imposes some constraints on the boundary conditions for ϕ at $r = \pm\infty$. The above relations remind us of the connection between Fourier series expansion in an interval $(-\pi, \pi)$, and sinusoidal and cosinusoidal Fourier series expansions in a half interval $(0, \pi)$ [1–3]. In that case the sinusoidal one is odd under the reflection and the cosinusoidal one is even. Here this correspondence is modified by the extra factors S and $1/S$.

6.1. Wick rotation

In what follows we will switch to Euclidean Anti-de Sitter space (EAdS_{d+1}) by Wick rotation. In contrast to AdS_{d+1}, the quadric in $\mathbb{E}^{d+1,1}$ is composed of two hyperbolic spaces H^{d+1} (disconnected balls B_{d+1}). Each piece has $r > 0$ and $r < 0$, respectively. One of the two *is* EAdS_{d+1}. Hence one usually quantizes a scalar field in a single Poincaré patch. When the entire Lorentzian AdS space is considered, however, one cannot go from a Lorentzian signature to a Euclidean one, and then come back through analytical continuation. Our primary concern is to study a scalar field theory in Lorentzian AdS space, not in EAdS. We perform Wick rotations in order to make integrals which contain products of bulk–boundary propagators well-defined, when the UV divergence is regularized by cutoff $|r| =$ finite. Hence, in what follows, we will consider both pieces of the hyperbolic spaces, and glue together the two half spaces at the horizon, which is also part of the boundary. Then, we assume that the structure of the solution (3.2) is the same after Wick rotation, although the topology of the spacetime has changed by Wick rotation. The coordinates on the boundary will be denoted as $\vec{y}$ instead of $(\tau = it, \vec{x})$. The bulk action integral for the scalar field is given by

$$I_0 = \int_{-\infty}^{\infty} dr \int d^d y \sqrt{g} \left(-\frac{1}{2} \frac{1}{\sqrt{g}} \phi \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi \right) + \frac{1}{2} m^2 \phi^2 \right). \tag{6.1}$$

Here note that this action has an asymmetric form. This is different from the action in ordinary form by surface terms. This is arranged so that the on-shell value of I_0 vanishes [8]. Surface action integrals $I_\pm$ will also be introduced later, and the total action is $I = I_0 + I_+ + I_-$. The metric tensor is given by

$$ds_E^2 = g_{\mu\nu} dx^\mu dx^\nu = r^{-2} dr^2 + r^2 (d\vec{y})^2. \tag{6.2}$$

The equation of motion for ϕ in the bulk has solutions of a form $\phi \sim r^{-(d-\Delta)}$ as $|r| \to \infty$. There are two values of Δ

$$\Delta = \Delta_\pm \equiv \frac{d}{2} \pm \nu, \quad \nu = \sqrt{\left(\frac{d}{2}\right)^2 + m^2}. \tag{6.3}$$

BF bound [2] is given by $m^2 \geq -\frac{d^2}{4}$. When Δ_- satisfies the unitarity bound $\Delta_- \geq \frac{d-2}{2}$, ν will be in the range $0 < \nu < 1$.[10] If ν is in this range, there are two scalar operators O_+, O_- with scaling dimensions Δ_+, Δ_- in the boundary CFT. Discussion in this paper will be restricted to this case. Then $\Delta_- < \frac{d}{2} < \Delta_+$.

6.2. Green functions and solutions to boundary-value problem

Euclidean Green function $G_E(r, y; r', y')$ is obtained from Feynman propagator iG_F ($y = (\tau, \vec{x})$) by the relation

$$G_E\big(r, \tau, \vec{x}; r', \tau', \vec{x}'\big) = iG_F\big(r, -i\tau, \vec{x}; r', -i\tau', \vec{x}'\big) \tag{6.4}$$

The bulk–boundary Green functions are given by [7,15]

$$K_{\Delta_\pm}\big(\vec{y}, \vec{y}', r\big) = \frac{\Gamma(\Delta_\pm)}{\pi^{d/2}\Gamma(\Delta_\pm - \frac{d}{2})}\left(\frac{r}{1 + r^2(\vec{y} - \vec{y}')^2}\right)^{\Delta_\pm}. \tag{6.5}$$

Near the boundary ($r \to \infty$), these have the asymptotics:

$$K_{\Delta_\pm}\big(\vec{y}, \vec{y}', r\big) \to r^{-(d-\Delta_\pm)}\delta^{(d)}\big(\vec{y} - \vec{y}'\big) + r^{-\Delta_\pm}\frac{\Gamma(\Delta_\pm)}{\pi^{d/2}\Gamma(\Delta_\pm - \frac{d}{2})}\frac{1}{|\vec{y} - \vec{y}'|^{2\Delta_\pm}} + \cdots. \tag{6.6}$$

Due to (5.6)–(5.7) these are related to $G_\pm$ by

$$G_\pm\big(|r|, -i\tau, \vec{x}; \big|r'\big|, -i\tau', \vec{x}'\big) = \frac{1}{\pm 2(1 + S^{\pm 2})\nu}\frac{1}{|r'|^{\Delta_\pm}}K_{\Delta_\pm}\big(\vec{y}, \vec{y}', |r|\big), \quad r' \to \infty. \tag{6.7}$$

Then we can write down the general solution to the Klein–Gordon equation in the pair of Poincaré patches for EAdS_{d+1}:

$$\begin{aligned}\phi(r, \vec{y}) = {} & \frac{1}{1 + S^2}\left[\int d^d\vec{y}'\, K_{\Delta_+}\big(\vec{y}, \vec{y}', r\big)\phi_+\big(\vec{y}'\big) + S\int d^d\vec{y}'\, K_{\Delta_+}\big(\vec{y}, \vec{y}', r\big)\bar{\phi}_+\big(\vec{y}'\big)\right] \\ & + \frac{1}{1 + S^2}\left[S^2\int d^d\vec{y}'\, K_{\Delta_-}\big(\vec{y}, \vec{y}', r\big)\phi_-\big(\vec{y}'\big) \right.\\ & \left. - S\int d^d\vec{y}'\, K_{\Delta_-}\big(\vec{y}, \vec{y}', r\big)\bar{\phi}_-\big(\vec{y}'\big)\right] \quad (r > 0)\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}\phi(r, \vec{y}) = {} & \frac{1}{1 + S^2}\left[S\int d^d\vec{y}'\, K_{\Delta_+}\big(\vec{y}, \vec{y}', -r\big)\phi_+\big(\vec{y}'\big) + S^2\int d^d\vec{y}'\, K_{\Delta_+}\big(\vec{y}, \vec{y}', -r\big)\bar{\phi}_+\big(\vec{y}'\big)\right] \\ & + \frac{1}{1 + S^2}\left[-S\int d^d\vec{y}'\, K_{\Delta_-}\big(\vec{y}, \vec{y}', -r\big)\phi_-\big(\vec{y}'\big) \right.\\ & \left. + \int d^d\vec{y}'\, K_{\Delta_-}\big(\vec{y}, \vec{y}', -r\big)\bar{\phi}_-\big(\vec{y}'\big)\right] \quad (r < 0).\end{aligned} \tag{6.9}$$

[10] Values $\nu = 0, 1$ are not considered in this paper.

Here $\phi_\pm(\vec{y})$ and $\bar{\phi}_\pm(\vec{y})$ are boundary conditions at $r = +\infty$ and $r = -\infty$, respectively. According to (3.2), these functions must be related by

$$\bar{\phi}_+(\vec{y}) = S\phi_+(\vec{y}), \tag{6.10}$$

$$\bar{\phi}_-(\vec{y}) = -\frac{1}{S}\phi_-(\vec{y}). \tag{6.11}$$

After substituting the above into (6.8) we obtain

$$\phi(r,\vec{y}) = \begin{cases} \int d^d\vec{y}'\, K_{\Delta_+}(\vec{y},\vec{y}',r)\phi_+(\vec{y}') + \int d^d\vec{y}'\, K_{\Delta_-}(\vec{y},\vec{y}',r)\phi_-(\vec{y}') & (r>0), \\ S\int d^d\vec{y}'\, K_{\Delta_+}(\vec{y},\vec{y}',-r)\phi_+(\vec{y}') - \frac{1}{S}\int d^d\vec{y}'\, K_{\Delta_-}(\vec{y},\vec{y}',-r)\phi_-(\vec{y}') & (r<0). \end{cases} \tag{6.12}$$

Now the boundary conditions on ϕ are

$$\phi(r,\vec{y}) = \begin{cases} f_+(\vec{y})r^{-\Delta_-} + f_-(\vec{y})r^{-\Delta_+} + \mathcal{O}(r^{-2-\Delta_\pm}) & (r\to+\infty), \\ \bar{f}_+(\vec{y})(-r)^{-\Delta_-} + \bar{f}_-(\vec{y})(-r)^{-\Delta_+} + \mathcal{O}(r^{-2-\Delta_\pm}) & (r\to-\infty). \end{cases} \tag{6.13}$$

Here $f_\pm(\vec{y})$ and $\bar{f}_\pm(\vec{y})$ are functions which are determined in terms of $\phi_\pm(\vec{y})$:

$$f_+ = \phi_+(\vec{y}) + \frac{\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}\Gamma(-\nu)}\int \left|\vec{y}-\vec{y}'\right|^{-2\Delta_-}\phi_-(\vec{y}')\, d^d\vec{y}', \tag{6.14}$$

$$f_- = \phi_-(\vec{y}) + \frac{\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\nu)}\int \left|\vec{y}-\vec{y}'\right|^{-2\Delta_+}\phi_+(\vec{y}')\, d^d\vec{y}', \tag{6.15}$$

$$\bar{f}_+ = S\phi_+(\vec{y}) - \frac{1}{S}\frac{\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}\Gamma(-\nu)}\int \left|\vec{y}-\vec{y}'\right|^{-2\Delta_-}\phi_-(\vec{y}')\, d^d\vec{y}', \tag{6.16}$$

$$\bar{f}_- = -\frac{1}{S}\phi_-(\vec{y}) + S\frac{\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\nu)}\int \left|\vec{y}-\vec{y}'\right|^{-2\Delta_+}\phi_+(\vec{y}')\, d^d\vec{y}'. \tag{6.17}$$

The first terms are source functions and the second terms are 'responses' to the sources. In actual calculations of the asymptotics of a given solution, one cannot distinguish the two. For the integrals in f_- and $\bar{f}_-$, some regularization for the singularities at $\vec{y} = \vec{y}'$ will be necessary. Now the $\mathcal{O}(r^{-\Delta_-})$ and $\mathcal{O}(r^{-\Delta_+})$ terms in $\phi(r,\vec{y})$ are fixed on the boundaries, and in the derivation of the equation of motion, the variation of ϕ is at most $\delta\phi(r,\vec{y}) = \mathcal{O}(r^{-2-\Delta_-})$. Then the variations of the action on the boundaries vanish: $\int d^d\vec{y}\,\sqrt{g}\phi r\partial_r\delta\phi \to 0$, $\int d^d\vec{y}\,\sqrt{g}\delta\phi r\partial_r\phi \to 0$ $(r\to\pm\infty)$. Hence the variational problem is well-posed. To determine $f_\pm$ and $\bar{f}_\pm$ in terms of $\phi_\pm$, one needs to know $K_{\Delta_\pm}$.[11] In an asymptotically AdS space, such as the one in the presence of black holes, one would need to use a bulk–boundary propagator $K'_{\Delta_\pm}$ of a scalar field in such a background.

Let us now turn to the behaviour of the solution (6.12) near the horizon. We consider integrals $\int d^d\vec{y}'[r/(1+r^2(\vec{y}-\vec{y}')^2)]^{\Delta_\pm}\phi_\pm(\vec{y}')$. As far as the source functions $\phi_\pm(\vec{y})$ have compact supports, these integrals can be approximated as $r^{\Delta_\pm}\int d^d\vec{y}'\,\phi_\pm(\vec{y}')$ as $r\to 0$. Hence

$$\int d^d\vec{y}' K_{\Delta_+}(\vec{y},\vec{y}',r)\phi_+(\vec{y}') \sim \frac{\Gamma(\Delta_\pm)}{\pi^{\frac{d}{2}}\Gamma(\Delta_\pm-\frac{d}{2})}\left[\int d^d\vec{y}'\phi_\pm(\vec{y}')\right]r^{\Delta_\pm} \tag{6.18}$$

[11] The mode functions $\psi_\pm$ (3.8) have asymptotic behaviours $r^{-\frac{d}{2}\pm\nu}$, respectively. However, after integrating over the modes, each bulk–boundary propagator $K_{\Delta_\pm}$ acquires both power behaviours (6.6). In order to impose the boundary condition on the scalar field, one needs to use $K_{\Delta_\pm}$.

and the solution ϕ to the boundary-value problem behaves near the boundary as $\phi \sim r^{\Delta_\pm}$, although the mode functions (3.27) and (3.28) are blowing up and oscillating rapidly near the horizon. Then, ϕ and $(r\,\partial_r)^n\,\phi = \partial^n_{\tilde{\rho}}\,\phi$ vanish on the horizon, and the surface terms on the horizon are not required.

6.3. Two-point functions and boundary terms

According to AdS/CFT correspondence, in the semi-classical regime, the on-shell action of the scalar field in the AdS background is supposed to give generating functional of two-point functions of single-trace operators O_+ or O_- in boundary CFT. In this paper we will try to realize AdS/CFT correspondence for O_+ and O_- altogether at the same time. Since the bulk action I_0 (6.1) vanishes on shell, we need to introduce boundary terms (and counterterms). The choice of the boundary terms defines definite theories. Because there are two boundaries, we can introduce boundary action $I_\pm$ on each boundary. They must be local functionals of ϕ and its derivatives. We will consider the following form.

$$\begin{aligned} I_\pm &= \pm \lim_{r\to\pm\infty} \int\limits_{r=\text{fixed}} d^d\vec{y}|r|^d\big[\alpha_1\phi^2 + \alpha_2\phi r\partial_r\phi + \alpha_3(r\partial_r\phi)^2 + \alpha_4\phi(r\partial_r)^2\phi\big] \\ &= \pm \lim_{r\to\pm\infty} \int\limits_{r=\text{fixed}} d^d\vec{y}\big[\pm\alpha_2\sqrt{g}\phi g^{rr}\partial_r\phi + \sqrt{\gamma}\big\{\alpha_1\phi^2 + \alpha_3 g^{rr}(\partial_r\phi)^2 + \alpha_4 g^{rr}\phi\partial_r^2\phi\big\}\big]. \end{aligned} \tag{6.19}$$

Here α_i $(i = 1, 2, 3, 4)$ are constants, and γ_{ij} is an induced metric on the boundaries. It will turn out that the generating functional is universal up to a multiplicative constant, and we can set $\alpha_3 = \alpha_4 = 0$. These boundary terms are invariant under reparametrizations which keep the boundary unchanged.

We will substitute the general solution (6.8)–(6.9) into (6.19). It is necessary to evaluate integrals of a form $\lim_{r\to\infty}\int d^d\vec{y}\,r^d(r\partial_r)^n K_\Delta(\vec{y},\vec{y}_1,r)(r\partial_r)^{n'}K_{\Delta'}(\vec{y},\vec{y}_2,r)$ with $n, n' = 0, 1, 2$. The method will be explained in Appendix B. Then, the on-shell boundary action I_+ (6.19) is given by[12]

$$\begin{aligned} I_+ &= A_1\int d^d\vec{y}_1\int d^d\vec{y}_2\;\phi_+(\vec{y}_1)y_{12}^{-2\Delta_+}\phi_+(\vec{y}_2) \\ &\quad + A_2 r^{2\nu}\int d^d\vec{y}_1\int d^d\vec{y}_2\;\phi_+(\vec{y}_1)y_{12}^{-2\Delta_-}\phi_-(\vec{y}_2) \\ &\quad + \int d^d\vec{y}_1\int d^d\vec{y}_2\;\phi_-(\vec{y}_1)\big(A_3 r^{2\nu}y_{12}^{d-4\Delta_-} + A_4 y_{12}^{-2\Delta_-}\big)\phi_-(\vec{y}_2). \end{aligned} \tag{6.20}$$

The coefficients $A_1, \dots, A_4$ are given by

$$A_1 = \frac{2\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\nu)}\left[\alpha_1 - \frac{d}{2}\alpha_2 + \Delta_+\Delta_-\alpha_3 + \frac{1}{2}\big(\Delta_+^2 + \Delta_-^2\big)\alpha_4\right], \tag{6.21}$$

[12] We also evaluated these integrals using Fourier transforms of the bulk–boundary Green functions (6.5), $\tilde{K}_\Delta(\vec{k}) = 2^{\frac{d}{2}-\Delta+1}\Gamma(\Delta - \frac{d}{2})^{-1}r^{-\frac{d}{2}}|\vec{k}|^{\Delta-\frac{d}{2}}\mathrm{K}_{\frac{d}{2}-\Delta}(|\vec{k}|/r)$, with identical results. $\mathrm{K}_{d/2-\Delta}$ on the right-hand side is a McDonald function.

$$A_2 = \frac{2\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}\Gamma(-\nu)}\left[\alpha_1 - \Delta_-\alpha_2 + \Delta_-^2(\alpha_3+\alpha_4)\right], \tag{6.22}$$

$$A_3 = \frac{\Gamma(\frac{d}{2}-2\nu)\Gamma(\nu)^2}{\pi^{\frac{d}{2}}\Gamma(2\nu)\Gamma(-\nu)^2}\left[\alpha_1 - \Delta_-\alpha_2 + \Delta_-^2(\alpha_3+\alpha_4)\right], \tag{6.23}$$

$$A_4 = \frac{2\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}\Gamma(-\nu)}\left[\alpha_1 - \frac{d}{2}\alpha_2 + \Delta_+\Delta_-\alpha_3 + \frac{1}{2}\left(\Delta_+^2+\Delta_-^2\right)\alpha_4\right]. \tag{6.24}$$

This result can also be obtained more easily by using (6.6) and an integral formula

$$\int d^d\vec{y}\,|\vec{y}-\vec{y}_1|^{-2\Delta}|\vec{y}-\vec{y}_2|^{-2\Delta'} = y_{12}^{d-2\Delta-2\Delta'}\pi^{\frac{d}{2}}\frac{\Gamma(\Delta+\Delta'-\frac{d}{2})\Gamma(\frac{d}{2}-\Delta)\Gamma(\frac{d}{2}-\Delta)}{\Gamma(\Delta)\Gamma(\Delta')\Gamma(d-\Delta-\Delta')}, \tag{6.25}$$

which is a result of analytic continuation. Especially, the following identities hold.

$$\int d^d\vec{y}\,|\vec{y}-\vec{y}_1|^{-2\Delta_+}|\vec{y}-\vec{y}_2|^{-2\Delta_-} = 0, \tag{6.26}$$

$$\int d^d\vec{y}\,|\vec{y}-\vec{y}_1|^{-2\Delta_-}|\vec{y}-\vec{y}_2|^{-2\Delta_-} = \frac{\pi^{\frac{d}{2}}\Gamma(\nu)^2}{\Gamma(\Delta_-)^2\Gamma(2\nu)}y_{12}^{-d+4\nu}. \tag{6.27}$$

Those coefficients A_2 and A_3 in (6.24), which multiplies those terms divergent as $r\to\infty$, must vanish. These conditions put a constraint on the parameters α_i:

$$\alpha_1 - \Delta_-\alpha_2 + \Delta_-^2(\alpha_3+\alpha_4) = 0. \tag{6.28}$$

There are still free parameters in addition to an overall constant. Note that this finiteness prescription eliminates the coupling between ϕ_+ and ϕ_-.

Finally we get

$$-I_+ = \frac{4\nu}{\pi^{\frac{d}{2}}}(\alpha_2 - 2\Delta_-\alpha_3 - d\alpha_4)\int d^d\vec{y}_1\int d^d\vec{y}_2\left[\frac{\Gamma(\Delta_+)}{\Gamma(\nu)}y_{12}^{-2\Delta_+}\phi_+(\vec{y}_1)\phi_+(\vec{y}_2) + \frac{\Gamma(\Delta_-)}{\Gamma(-\nu)}y_{12}^{-2\Delta_-}\phi_-(\vec{y}_1)\phi_-(\vec{y}_2)\right]. \tag{6.29}$$

We believe that even if further boundary terms are introduced in (6.19), the result for $-I_+$ is unique up to an overall constant. From (6.29) we can read off the two-point functions $\langle O_\pm(y_1)O_\pm(y_2)\rangle$ in boundary CFT by means of functional differentiations of $-I_+$. This result shows that there is some kind of universality. Even if we add extra boundary terms to the action as in (6.19), after suitable renormalization, the result will be proportional to a universal generating function. There is, however, a serious problem in the present case. Since $\Gamma(-\nu)$ is negative for $0<\nu<1$, for any choice of α_2, α_3 and α_4, $\langle O_+(\vec{y}_1)O_+(\vec{y}_2)\rangle$ or $\langle O_-(\vec{y}_1)O_-(\vec{y}_2)\rangle$ necessarily turns out negative. This would imply that CFT would be non-unitary.

This problem is actually resolved, when we also use I_- as given in (6.19) with an overall negative sign with respect to I_+. Due to the relative coefficients in (6.12), we have from (6.29),

$$-I_- = -\frac{4\nu}{\pi^{\frac{d}{2}}}(\alpha_2 - 2\Delta_-\alpha_3 - d\alpha_4)\int d^d\vec{y}_1\int d^d\vec{y}_2\left[S^2\frac{\Gamma(\Delta_+)}{\Gamma(\nu)}y_{12}^{-2\Delta_+}\phi_+(\vec{y}_1)\phi_+(\vec{y}_2) + S^{-2}\frac{\Gamma(\Delta_-)}{\Gamma(-\nu)}y_{12}^{-2\Delta_-}\phi_-(\vec{y}_1)\phi_-(\vec{y}_2)\right]. \tag{6.30}$$

The sum $I = I_+ + I_-$ yields the following two point functions:

$$\langle O_+(\vec{y}_1)O_+(\vec{y}_2)\rangle = \frac{8\nu}{\pi^{\frac{d}{2}}}\frac{\Gamma(\Delta_+)}{\Gamma(\nu)}((\alpha_2 - 2\Delta_-\alpha_3 - d\alpha_4)(1 - S^2)y_{12}^{-2\Delta_+}, \tag{6.31}$$

$$\langle O_-(\vec{y}_1)O_-(\vec{y}_2)\rangle = \frac{8\nu}{\pi^{\frac{d}{2}}}\frac{\Gamma(\Delta_-)}{\Gamma(-\nu)}((\alpha_2 - 2\Delta_-\alpha_3 - d\alpha_4)\frac{(S^2-1)}{S^2}y_{12}^{-2\Delta_+}, \tag{6.32}$$

$$\langle O_+(\vec{y}_1)O_-(\vec{y}_2)\rangle = 0 \tag{6.33}$$

To reinstate unitarity, we need to adjust the parameters such that $(\alpha_2 - 2\Delta_-\alpha_3 - d\alpha_4)(1 - S^2) > 0$. If the bulk action (6.1) is rewritten into an ordinary symmetric form by partial integration, boundary terms $-\frac{1}{2}\int_{r\to+\infty} d^d\vec{y}\,|r|^d\phi r\partial_r\phi$ and $-\frac{1}{2}\int_{r\to-\infty} d^d\vec{y}\,|r|^d\phi r\partial_r\phi$ will appear, and to cancel the first term we must set $\alpha_2 = \frac{1}{2}$. In this case the second term is not canceled. Then, $\alpha_3 = \alpha_4 = 0$ and $\alpha_1 = \frac{1}{2}\Delta_-$ will be the simplest choice of parameters. Hence, $S^2 < 1$. The above prescription is different from the previous ones [5,7,8].

To summarize, after partial integration, the action integral is given by

$$\begin{aligned} I = &\int_{-\infty}^{\infty} dr \int d^d y\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2\right) \\ &+ \lim_{r\to+\infty}\int_{r\text{ fixed}} d^d\vec{y}\,\sqrt{\gamma}\frac{1}{2}\Delta_-\phi^2 - \lim_{r\to-\infty}\int_{r\text{ fixed}} d^d\vec{y}\,\sqrt{\gamma}\frac{1}{2}\Delta_-\phi^2 \\ &- \lim_{r\to-\infty}\int_{r\text{ fixed}} d^d\vec{y}\,\sqrt{\gamma}\phi r\partial_r\phi. \end{aligned} \tag{6.34}$$

7. Three-point functions

The Euclidean Green function satisfies

$$\left(\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\right) - m^2\right)G_E(x, x') = -\frac{1}{\sqrt{g}}\delta^{(d+1)}(x, x'). \tag{7.1}$$

Let us consider a $\lambda\phi^3$ interaction with λ being of order of $1/N$ [14]:

$$I_0 = \int_{-\infty}^{\infty} dr \int d^d\vec{y}\sqrt{g}\left(-\frac{1}{2}\phi\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\right)\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{3}\lambda\phi^3\right). \tag{7.2}$$

Equation of motion $\Delta\phi - m^2\phi = \lambda\phi^2$ can be solved by using the Green function G_E:

$$\begin{aligned} \phi(r, \vec{y}) = &\int d^d\vec{y}' K_{\Delta_+}(\vec{y}, \vec{y}', r)\phi_+(\vec{y}') + \int d^d\vec{y}' K_{\Delta_-}(\vec{y}, \vec{y}', r)\phi_-(\vec{y}') \\ &- \lambda\int d^d\vec{y}'\int_{-\infty}^{\infty} dr'\sqrt{g(r')}G_E(r, \vec{y}; r', \vec{y}')\phi(\vec{y}', r')^2 \quad (r > 0), \end{aligned} \tag{7.3}$$

$$\begin{aligned} \phi(r, \vec{y}) = S&\int d^d\vec{y}' K_{\Delta_+}(\vec{y}, \vec{y}', -r)\phi_+(\vec{y}') - \frac{1}{S}\int d^d\vec{y}' K_{\Delta_-}(\vec{y}, \vec{y}', -r)\phi_-(\vec{y}') \\ &- \lambda\int d^d\vec{y}'\int_{-\infty}^{\infty} dr'\sqrt{g(r')}G_E(r, \vec{y}; r', \vec{y}')\phi(\vec{y}', r')^2 \quad (r < 0). \end{aligned} \tag{7.4}$$

These equations can be solved by iterations. Up to the first order in λ, the solution is given as follows:

$$\begin{aligned}\phi(r,\vec{y}) = &\int d^d\vec{y}' K_{\Delta_+}(\vec{y},\vec{y}',r)\phi_+(\vec{y}') + \int d^d\vec{y}' K_{\Delta_-}(\vec{y},\vec{y}',r)\phi_-(\vec{y}') \\ &-\lambda\int_0^\infty dr' \int d^d\vec{y}' \sqrt{g(r')}G_E(r,\vec{y};r',\vec{y}')\Big[\int d^d\vec{y}'' K_{\Delta_+}(\vec{y}',\vec{y}'',r')\phi_+(\vec{y}'') \\ &+\int d^d\vec{y}'' K_{\Delta_-}(\vec{y}',\vec{y}'',r')\phi_-(\vec{y}'')\Big]^2 \\ &-\lambda\int_0^\infty dr' \int d^d\vec{y}' \sqrt{g(r')}G_E(r,\vec{y};-r',\vec{y}')\Big[S\int d^d\vec{y}'' K_{\Delta_+}(\vec{y}',\vec{y}'',r')\phi_+(\vec{y}'') \\ &-\frac{1}{S}\int d^d\vec{y}'' K_{\Delta_-}(\vec{y}',\vec{y}'',r')\phi_-(\vec{y}'')\Big]^2 + \mathcal{O}(\lambda^2) \quad (r>0),\end{aligned} \tag{7.5}$$

$$\begin{aligned}\phi(r,\vec{y}) = &\,S\int d^d\vec{y}' K_{\Delta_+}(\vec{y},\vec{y}',-r)\phi_+(\vec{y}') - \frac{1}{S}\int d^d\vec{y}' K_{\Delta_-}(\vec{y},\vec{y}',-r)\phi_-(\vec{y}') \\ &-\lambda\int_0^\infty dr' \int d^d\vec{y}' \sqrt{g(r')}G_E(r,\vec{y};r',\vec{y}')\Big[\int d^d\vec{y}'' K_{\Delta_+}(\vec{y}',\vec{y}'',r')\phi_+(\vec{y}'') \\ &+\int d^d\vec{y}'' K_{\Delta_-}(\vec{y}',\vec{y}'',r')\phi_-(\vec{y}'')\Big]^2 \\ &-\lambda\int_0^\infty dr' \int d^d\vec{y}' \sqrt{g(r')}G_E(r,\vec{y};-r',\vec{y}')\Big[S\int d^d\vec{y}'' K_{\Delta_+}(\vec{y}',\vec{y}'',r')\phi_+(\vec{y}'') \\ &-\frac{1}{S}\int d^d\vec{y}'' K_{\Delta_-}(\vec{y}',\vec{y}'',r')\phi_-(\vec{y}'')\Big]^2 + \mathcal{O}(\lambda^2) \quad (r<0).\end{aligned} \tag{7.6}$$

By substituting equation of motion into the bulk part (7.2) we get a total action

$$\begin{aligned}I = &\int_{-\infty}^{\infty} dr \int d^d\vec{y}\sqrt{g}\left(-\frac{1}{6}\lambda\phi^3\right) + \int_{r\to+\infty} d^d\vec{y}\, r^d\left(\frac{1}{2}\Delta_-\phi^2 + \frac{1}{2}r\phi\partial_r\phi\right) \\ &-\int_{r\to-\infty} d^d\vec{y}\,(-r)^d\left(\frac{1}{2}\Delta_-\phi^2 + \frac{1}{2}r\phi\partial_r\phi\right).\end{aligned} \tag{7.7}$$

The generating functional for three-point functions up to order $\mathcal{O}(\lambda^1)$ has two kinds of contributions: the bulk part and the boundary one.

The bulk part is obtained by substituting the solution for the free theory (6.12) into the bulk action in (7.7). The corresponding diagram [7] is presented in Fig. 5. The wavy lines are bulk–boundary propagators K_{Δ_σ}. There are actually a lot of terms and, a typical form of the terms is given by

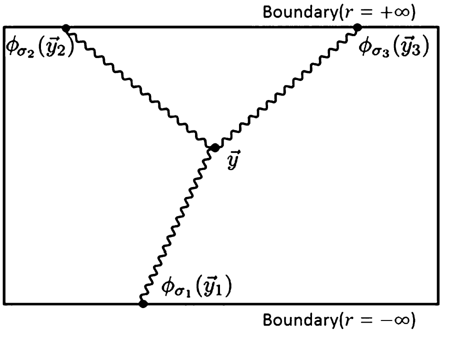

Fig. 5. Graph contributing to three-point functions.

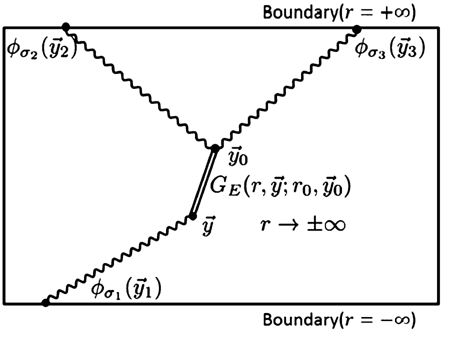

Fig. 6. Graph contributing to three-point functions via boundary terms.

$$I^{(3,\text{bulk})} = \lim_{r\to\infty} \lambda \int_0^\infty dr\, r^{d-1} \int d^d\vec{y} \int d^d\vec{y}_1 \int d^d\vec{y}_2 \int d^d\vec{y}_3\, K_{\Delta_1}(\vec{y},\vec{y}_1,|r|) \times K_{\Delta_2}(\vec{y},\vec{y}_2,|r|) K_{\Delta_3}(\vec{y},\vec{y}_3,|r|) \phi_{\sigma_1}(\vec{y}_1)\phi_{\sigma_2}(\vec{y}_2)\phi_{\sigma_3}(\vec{y}_3). \tag{7.8}$$

Here $\sigma_i = \pm$ and $\Delta_i \equiv \Delta_{\sigma_i}$. Integral of the form (7.8) is evaluated in [15] by using an inversion, as

$$I^{(3,\text{bulk})} = \lambda \int \frac{a}{|\vec{y}_1-\vec{y}_2|^{\Delta_1+\Delta_2-\Delta_3}|\vec{y}_1-\vec{y}_3|^{\Delta_1+\Delta_3-\Delta_2}|\vec{y}_2-\vec{y}_3|^{\Delta_2+\Delta_3-\Delta_1}} \times \phi_{\sigma_1}(\vec{y}_1)\phi_{\sigma_2}(\vec{y}_2)\phi_{\sigma_3}(\vec{y}_3) d^d\vec{y}_1\, d^d\vec{y}_2\, d^d\vec{y}_3, \tag{7.9}$$

where a is a constant given by

$$-\frac{\Gamma(\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3))\Gamma(\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1))\Gamma(\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2))}{2\pi^d\Gamma(\Delta_1-\frac{d}{2})\Gamma(\Delta_2-\frac{d}{2})\Gamma(\Delta_3-\frac{d}{2})} \times \Gamma\left(\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3-d)\right). \tag{7.10}$$

The boundary part of the generating function is obtained by substituting the solution (7.5)–(7.6) into the boundary terms in (7.7). The corresponding diagram is depicted in Fig. 6, and a typical form of the integrals is given by

$$\lambda \lim_{r\to+\infty} \int d^d\vec{y}\, r^d \int d^d\vec{y}_1\, \phi_{\sigma_1}(\vec{y}_1) \int_0^\infty dr_0\, r_0^{d-1} \int d^d\vec{y}_0\, F(\vec{y},\vec{y}_1,\vec{y}_0,r,r_0)$$
$$\times \int d^d\vec{y}_2\, K_{\sigma_2}(\vec{y}_0,\vec{y}_2,r_0)\phi_{\sigma_2}(\vec{y}_2) \int d^d\vec{y}_3\, K_{\sigma_3}(\vec{y}_0,\vec{y}_3,r_0)\phi_{\sigma_3}(\vec{y}_3). \tag{7.11}$$

Here F is defined by

$$\begin{aligned} F(\vec{y},\vec{y}_1,\vec{y}_0,r,r_0) \equiv{}& \Delta_- K_{\Delta_1}(\vec{y},\vec{y}_1,r) G_E(r,\vec{y};r_0,\vec{y}_0) \\ &+ \frac{1}{2} r\partial_r K_{\Delta_1}(\vec{y},\vec{y}_1,r) G_E(r,\vec{y};r_0,\vec{y}_0) \\ &+ \frac{1}{2} K_{\Delta_1}(\vec{y},\vec{y}_1,r) r\partial_r G_E(r,\vec{y};r_0,\vec{y}_0). \end{aligned} \tag{7.12}$$

The propagator $G_E = G_+ + G_-$ with (5.4)–(5.5) and (6.7) is to be substituted into (7.11). Upon substitution, each boundary term gives divergences. However, the linear combinations in the boundary terms work correctly, and the sum of all turns out finite. Moreover, the integral (7.11) can be explicitly carried out, and the result is proportional to the result of integral (7.9). The three-point functions are obtained by summing the bulk and boundary contributions. The details will be reported elsewhere. Here only the results of a three-point function of O_+ is presented:

$$\langle O_+(\vec{y}_1)O_+(\vec{y}_2)O_+(\vec{y}_3)\rangle = \lambda \frac{-1+5S^2-4S^3+2S^5}{1+S^2} \frac{\Gamma(\frac{\Delta_+}{2})^3 \Gamma(\frac{1}{2}(3\Delta_+ - d))}{\pi^{d/2}\Gamma(\nu)^3} \cdot \frac{1}{|\vec{y}_1-\vec{y}_2|^{\Delta_+}|\vec{y}_1-\vec{y}_3|^{\Delta_+}|\vec{y}_2-\vec{y}_3|^{\Delta_+}}. \tag{7.13}$$

8. Discussion

We showed a prescription for quantizing two sets of scalar modes in a pair of Poincaré patches of AdS space, and also presented a prescription for semi-classically obtaining two- and three-point functions in the boundary CFT. This is possible since the two boundaries at $r = \pm\infty$ are connected, as a result of which the KG norm is conserved. Needless to say, more analysis is necessary. This will be left to future study. There are a few comments.

If we want to quantize only a single set of scalar modes, or if $\nu > 1$ and only modes $\hat{\psi}_+$ in (4.2) are allowed, we can still do this in a pair of Poincaré patches. Mode expansion is (4.1) with only a_+ and $a_+^\dagger$ retained. Canonical commutation relation is $[a_+(\omega,\vec{k}), a_+^\dagger(\omega',\vec{k}')] = 2^{-1-2\nu}(2\pi)^{-d+1}\Gamma(1+\nu)^{-2}\delta(\omega-\omega')\delta^{(d-1)}(\vec{k}-\vec{k}')$. Wightman function is proportional to G_+ in (5.6), and the commutator $[\Phi(r,t,\vec{x}), \Pi(r',t,\vec{x}')]$ contains a term $iS\delta(r+r')\delta^{(d-1)}(\vec{x}-\vec{x}')$, which is harmless since a singularity at $r' = -r$ is beyond the horizon.

If the two operators O_+ and O_- are present in the boundary CFT, the sum of the scaling dimensions Δ_+ and Δ_- is d, and by using a composite operator $\int d^d\vec{y}\, O_+O_-$, a marginal deformation of the CFT may be considered. A prescription for realizing this deformation in our formalism in the form of an interpolating geometry in r direction is an interesting question.

In the study of this paper, a parameter S which parametrize quantization is introduced. The role of this parameter is to specify the relative magnitude of the mode functions in the two patches. We have not reached a concrete use of this degree of freedom yet. It is also discussed in Section 4 that by setting $S = 0$ one can quantize only one of the two sets of the scalar modes on each of the two Poincaré patches.

The procedure of this paper can be extended to the black hole geometry. Schwarzschild-AdS_{d+1} black hole solution in Poincaré coordinates is given by

$$ds^2 = \ell^2\left(-f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2\,d\vec{x}^2\right), \tag{8.1}$$

where

$$f(r) = r^2\left\{1 - \left|\frac{r_+}{r}\right|^d\right\}. \tag{8.2}$$

Event horizon is at $r = r_+ (> 0)$ and temperature is $T = \frac{d}{4\pi} r_+$. To quantize two sets of modes of a scalar field with mass m in the range $0 < (m\ell)^2 + \frac{d^2}{4} < 1$ in this background, we consider a pair of Poincaré patches with $r > 0$ and $r < 0$. Event horizons are at $r = \pm r_+$. The line element (8.1) is to be used in both patches. In Lorentzian space, the boundary conditions for scalar field at the event horizons must be in-going conditions. The fluxes across the horizons vanish due to $f(\pm r_+) = 0$. At the boundaries, the boundary conditions for the scalar field must be such that the fluxes at the boundaries cancel out. These will be (6.10) and (6.11). It is interesting to compute partition functions and entropies for the black hole geometries.

Appendix A. Calculation of Wightman function

By substituting $\omega = \sqrt{\vec{k}^2 + \mu^2} \equiv \omega(\vec{k}, \mu)$ into (5.2) with $r, r' > 0$, we obtain

$$G_+ = \frac{1}{(1+S^2)(rr')^{\frac{d}{2}}} \int_0^\infty d\mu\,\mu J_\nu\left(\frac{\mu}{r}\right) J_\nu\left(\frac{\mu}{r'}\right) G^{\text{Flat}}\left(t, \vec{x}, t', \vec{x}'; \mu\right). \tag{A.1}$$

Here

$$G^{\text{Flat}}\left(t, \vec{x}, t', \vec{x}'; \mu\right) = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1} 2\omega(\vec{k}, \mu)} e^{-i\omega(\vec{k},\mu)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')} \tag{A.2}$$

is a Wightman function for a free scalar field of mass μ in flat space. In the literature [11], it was argued that the coincident-point singularity of Wightman function (or, Feynman propagator) in AdS space should agree with that in flat space and this fixes its normalization. By using the fact that Wightman function is a function of AdS-invariant distance and satisfies a certain differential equation, Wightman function was determined. In this appendix, calculation of integral (A.1) is explicitly carried out.

For space-like separation of the plane coordinates, we can set $t - t' = 0$ by using the Lorentz symmetry of the integral. We also set $\vec{x}' = \vec{0}$. It suffices to consider the case $r, r' > 0$. To perform integration over $\vec{k}$, we note the following formulae [16]:

$$\int_0^\pi e^{iz\cos\theta} \sin^{2\nu}\theta\,d\theta = \sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)\left(\frac{2}{z}\right)^\nu J_\nu(z), \tag{A.3}$$

$$\int_0^\infty dx\,x^{\nu+1}\left(x^2 + y^2\right)^{-\mu-1} J_\nu(ax) = \frac{a^\mu y^{\nu-\mu}}{2^\mu \Gamma(\mu+1)} \mathrm{K}_{\nu-\mu}(ay) \quad \left[\text{for } 2\,\mathrm{Re}(\mu) + \frac{3}{2} > \mathrm{Re}(\nu) > -1\right]. \tag{A.4}$$

$\mathrm{K}_\nu(z)$ is McDonald function. By using these, we find that

$$\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{\sqrt{\vec{k}^2+\mu^2}} e^{i\vec{k}\cdot\vec{x}} = \frac{2}{(2\pi)^{\frac{d}{2}}} \left(\frac{|\vec{x}|}{\mu}\right)^{1-\frac{d}{2}} \mathrm{K}_{\frac{d}{2}-1}(\mu|\vec{x}|). \tag{A.5}$$

This leads to

$$G_+\big(r,\vec{x};r',\vec{0}\big) = \frac{1}{(2\pi)^{\frac{d}{2}}(1+S^2)(rr')^{\frac{d}{2}}} \int_0^\infty d\mu\, \mu \left(\frac{|\vec{x}|}{\mu}\right)^{1-\frac{d}{2}} \mathrm{K}_{\frac{d}{2}-1}(\mu|\vec{x}|) J_\nu\left(\frac{\mu}{r}\right) J_\nu\left(\frac{\mu}{r'}\right). \tag{A.6}$$

Now by using the formulae [16]

$$\begin{aligned} &\int_0^\infty dx\, x^{\mu+1} \mathrm{K}_\mu(ax) J_\nu(bx) J_\nu(cx) \\ &\quad = \frac{1}{\sqrt{2\pi}} a^\mu b^{-\mu-1} c^{-\mu-1} e^{-(\mu+\frac{1}{2})\pi i} \big(u^2-1\big)^{-\frac{1}{2}\mu-\frac{1}{4}} Q^{\mu+\frac{1}{2}}_{\nu-\frac{1}{2}}(u) \\ &\quad \big[2bcu = a^2+b^2+c^2,\ \mathrm{Re}(a) > |\mathrm{Im}(b)| + |\mathrm{Im}(c)|,\ \mathrm{Re}(\nu) > -1,\ \mathrm{Re}(\mu+\nu) > -1\big], \end{aligned} \tag{A.7}$$

$$\begin{aligned} Q^\mu_\nu(z) = {} & \frac{e^{\mu\pi i}\Gamma(\nu+\mu+1)\Gamma(\frac{1}{2})}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} \big(z^2-1\big)^{\frac{\mu}{2}} z^{-\nu-\mu-1} \\ & \times {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}, \nu+\frac{3}{2}; \frac{1}{z^2}\right), \end{aligned} \tag{A.8}$$

we obtain

$$\begin{aligned} G_+\big(r,\vec{x};r',\vec{0}\big) = {} & \frac{\Gamma(\nu+\frac{d}{2})}{2(S^2+1)\pi^{\frac{d}{2}}\Gamma(\nu+1)} \frac{1}{(rr')^{\nu+\frac{d}{2}}(\frac{1}{r^2}+\frac{1}{r'^2}+\vec{x}^2)^{\nu+\frac{d}{2}}} \\ & \times {}_2F_1\left(\frac{d}{4}+\frac{\nu+1}{2}, \frac{d}{4}+\frac{\nu}{2}, \nu+1; \frac{4}{(rr')^2} \frac{1}{(\frac{1}{r^2}+\frac{1}{r'^2}+\vec{x}^2)^2}\right). \end{aligned} \tag{A.9}$$

Finally by using a quadratic transform of a hypergeometric function [16],

$${}_2F_1(a,b,2b;2z) = (1-z)^{-a} {}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}, b+\frac{1}{2}; \left(\frac{z}{1-z}\right)^2\right), \tag{A.10}$$

we have

$$G_+\big(r,\vec{x};r',\vec{0}\big) = \frac{\Gamma(\nu+\frac{d}{2})}{2\pi^{\frac{d}{2}}(1+S^2)\Gamma(\nu+1)} P^{\nu+\frac{d}{2}} {}_2F_1\left(\frac{d}{2}+\nu, \nu+\frac{1}{2}, 2\nu+1; -4P\right). \tag{A.11}$$

P is defined in (5.8). Similar expression for G_- can be obtained by replacement $\nu \to -\nu$ in G_+ and multiplying the result by S^2. For example, for $d =$ even, $G = G_+ + G_-$ behaves near the singularity $P \to \infty$ as

$$G\left(r,\vec{x};r',\vec{0}\right)\to\frac{1}{4\pi^{\frac{d+1}{2}}}\Gamma\left(\frac{d-1}{2}\right)\left[rr'\left\{\left(\frac{1}{r}-\frac{1}{r'}\right)^2+\vec{x}^2\right\}\right]^{-\frac{d-1}{2}}. \tag{A.12}$$

This does not depend on S, ν or Δ. Hence normalization of the singularity of Wightman function cannot be used to fix the value of S. It can be checked that Eq. (A.11) with $S=0$ agrees with Eq. (7.4) for $iG_F(x,x')$ in [11] after replacements $\lambda_\pm=\Delta_\pm$, $n=d+1$, $a=\ell^{-1}=1$, $2/u=-4P$ and substitution $y^0-y'^0=0$.

Appendix B. Calculation of the integrals necessary for evaluating boundary actions $I_\pm$

Here the method for evaluating integrals of a form $\lim_{r\to\infty}\int d^d\vec{y}\,r^d(r\partial_r)^n K_\Delta(\vec{y},\vec{y}_1,r)(r\partial_r)^{n'}K_{\Delta'}(\vec{y},\vec{y}_2,r)$ will be explained for the cases $n=n'=0$. First, let us consider an integral,

$$L_1=\lim_{r\to\infty}\int d^d\vec{y}\,r^d K_{\Delta_+}(\vec{y},\vec{y}_1,r)K_{\Delta_+}(\vec{y},\vec{y}_2,r). \tag{B.1}$$

We use Feynman's parameter-integral formula

$$\frac{1}{X_1^{m_1}}\cdots\frac{1}{X_n^{m_n}}=\int_0^1 dt_1\cdots\int_0^1 dt_n\,\delta\left(\sum_i t_i-1\right)\frac{\prod_i t_i^{m_i-1}}{[\sum_i t_iX_i]^{\sum_i m_i}}\frac{\Gamma(\sum_i m_i)}{\prod_i\Gamma(m_i)}, \tag{B.2}$$

and perform y integration [15].

$$L_1=\frac{\Gamma(2\Delta_+-\frac{d}{2})}{\pi^{\frac{d}{2}}\Gamma(\Delta_+-\frac{d}{2})^2}r^{2\Delta_+}\int_0^1 dt\,\frac{t^{\Delta_+-1}(1-t)^{\Delta_+-1}}{[1+r^2t(1-t)y_{12}^2]^{2\Delta_+-\frac{d}{2}}}. \tag{B.3}$$

Here $\vec{y}_{12}=\vec{y}_1-\vec{y}_2$. In the $r\to\infty$ limit, regions near $t=0$ and $t=1$ have dominant contributions. These contributions from $0\le t\le\epsilon$ and $1-\epsilon\le t\le 1$ with $\epsilon=r^{-1}$ can be evaluated by setting $t=r^{-2}y_{12}^{-2}z$ or $t=1-r^{-2}y_{12}^{-2}z$ and replacing t integral by z integral. These two contributions have the same values and we have

$$\begin{aligned}&2\times\frac{\Gamma(2\Delta_+-\frac{d}{2})}{\pi^{\frac{d}{2}}\Gamma(\Delta_+-\frac{d}{2})^2}r^{2\Delta_+}\int_0^{ry_{12}^2}dz\,z^{\Delta_+-1}\left(r^{-2}y_{12}^{-2}\right)^{\Delta_+}\frac{1}{(1+z)^{2\Delta_+-\frac{d}{2}}}\\&\quad\to 2\frac{\Gamma(2\Delta_+-\frac{d}{2})}{\pi^{\frac{d}{2}}\Gamma(\Delta_+-\frac{d}{2})^2}B\left(\Delta_+,\Delta_+-\frac{d}{2}\right)y_{12}^{-2\Delta_+}.\end{aligned} \tag{B.4}$$

Here $B(a,b)$ is Euler's beta function. There is also a region of t which must be taken into account. For the region $\epsilon<t<1-\epsilon$, we can replace $1+r^2t(1-t)y_{12}^2$ in the denominator of (B.3) by $r^2t(1-t)y_{12}^2$. This yields a contribution proportional to

$$r^{-2\nu}y_{12}^{d-4\Delta_+}\int_\epsilon^{1-\epsilon}dt\left[t(1-t)\right]^{-1-\nu}\sim\frac{2}{\nu}r^{-\nu}y_{12}^{d-4\Delta_+}. \tag{B.5}$$

For $\nu>0$ this damps in the $r\to\infty$ limit. So finally, we obtain a finite result.

$$L_1 = \frac{2\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\Delta_+ - \frac{d}{2})} y_{12}^{-2\Delta_+}. \tag{B.6}$$

In the above calculation it is assumed that $\vec{y}_1 \neq \vec{y}_2$, and ultra local terms such as $\delta^{(d)}(\vec{y}_1 - \vec{y}_2)$ are neglected.

We then consider an integral

$$\begin{aligned} L_2 &= \lim_{r\to\infty} \int d^d\vec{y}\, r^d K_{\Delta_+}(\vec{y}, \vec{y}_1, r) K_{\Delta_-}(\vec{y}, \vec{y}_2, r) \\ &= \frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}}\Gamma(\Delta_+ - \frac{d}{2})\Gamma(\Delta_- - \frac{d}{2})} \int_0^1 dt\, \frac{r^d t^{\Delta_+ - 1}(1-t)^{\Delta_- - 1}}{[1 + r^2 t(1-t) y_{12}^2]^{\frac{d}{2}}}. \end{aligned} \tag{B.7}$$

Contribution from the region $0 \le t \le \epsilon$ to the integral is

$$r^{-2\nu} y_{12}^{-2\Delta_+} \int_0^{r y_{12}^2} dz\, z^{\Delta_+ - 1}(1+z)^{-\frac{d}{2}} \sim r^{-2\nu} y_{12}^{-2\Delta_+} \frac{1}{\nu}\big(r y_{12}^2\big)^{\nu} \to 0. \tag{B.8}$$

The one from the region $1 - \epsilon \le t \le 1$ is

$$r^{2\nu} y_{12}^{-2\Delta_-} \int_0^{r y_{12}^2} dz z^{\Delta_- - 1}(1+z)^{-\frac{d}{2}}. \tag{B.9}$$

Although the integral is finite in the limit $r \to \infty$, we need to take $O(r^{-\nu})$ correction into account because of the prefactor $r^{2\nu}$:

$$r^{2\nu} y_{12}^{-2\Delta_-} \left[B\left(\Delta_-, \frac{d}{2} - \Delta_-\right) - \frac{1}{\nu} r^{-\nu} y_{12}^{-2\nu} + O\big(r^{-\nu-1}\big) \right]. \tag{B.10}$$

From the region $\epsilon < t < 1 - \epsilon$, we obtain contribution

$$\int_\epsilon^{1-\epsilon} dt\, y_{12}^{-d} t^{\nu-1}(1-t)^{-\nu-1} \sim \frac{1}{\nu} y_{12}^{-d} r^{\nu}. \tag{B.11}$$

Hence we get

$$L_2 = \frac{\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}\Gamma(\Delta_- - \frac{d}{2})} y_{12}^{-2\Delta_-} r^{2\nu}. \tag{B.12}$$

A final example is

$$\begin{aligned} L_3 &= \lim_{r\to\infty} \int d^d\vec{y}\, r^d K_{\Delta_-}(\vec{y}, \vec{y}_1, r) K_{\Delta_-}(\vec{y}, \vec{y}_2, r) \\ &= \frac{\Gamma(2\Delta_- - \frac{d}{2})}{\pi^{\frac{d}{2}}\Gamma(\Delta_- - \frac{d}{2})^2} \int_0^1 dt\, \frac{r^d t^{\Delta_- - 1}(1-t)^{\Delta_- - 1}}{[1 + r^2 t(1-t) y_{12}^2]^{2\Delta_- - \frac{d}{2}}}. \end{aligned} \tag{B.13}$$

Actually, $\vec{y}$ integration converges only for $d - 4\Delta_- < 0$. If $d \ge 4$, this condition is satisfied, because $0 < \nu < 1$. Otherwise, ν must be in the range $0 < \nu < d/4$. At the end of the calculation, we will analytically continue the results in variable ν to its remaining region. Contribution to the integral from regions $0 \le t \le \epsilon$ and $1-\epsilon \le t \le 1$ is

$$2y_{12}^{-2\Delta_-} \int_0^{ry_{12}^2} dz\, z^{\Delta_- -1}(1+z)^{\frac{d}{2}-2\Delta_-}. \tag{B.14}$$

This is divergent as $r \to \infty$ and is expanded as

$$2y_{12}^{-2\Delta_-}\left[\frac{1}{\nu} r^\nu y_{12}^{2\nu} + B(\Delta_-, -\nu)\right]. \tag{B.15}$$

From integral in the region $\epsilon < t < 1-\epsilon$, we obtain

$$r^{2\nu} y_{12}^{d-4\Delta_-} \int_\epsilon^{1-\epsilon} dt\big[t(1-t)\big]^{\nu-1}. \tag{B.16}$$

This is expanded as

$$r^{2\nu} y_{12}^{d-4\Delta_-}\left\{B\left(\frac{d}{2}-\Delta_-, \frac{d}{2}-\Delta_-\right) - \frac{2}{\nu} r^{-\nu}\right\}. \tag{B.17}$$

Hence we get

$$L_3 = \frac{\Gamma(2\Delta_- - \frac{d}{2})\Gamma(\nu)^2}{\pi^{\frac{d}{2}}\Gamma(2\nu)\Gamma(-\nu)^2} r^{2\nu} y_{12}^{4\Delta_- -d} + 2\frac{\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}\Gamma(-\nu)} y_{12}^{-2\Delta_-}. \tag{B.18}$$

In a similar way cases $n, n' \neq 0$ can also be worked out.

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