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GALLAI THEOREMS FOR GRAPHS, HYPERGRAPHS, AND SET SYSTEMS

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1. Introduction

In 1959 Gallai [4] presented his now classical theorem, involving the vertex covering number α_0, the vertex independence number β_0, the edge covering number α_1, and the maximum matching (or edge independence) number β_1.

Theorem 1 (Gallai). *For any nontrivial, connected graph $G=(V, E)$ with p vertices,*

I. $\alpha_0+\beta_0=p$
II. $\alpha_1+\beta_1=p$.

Since then quite a large number of similar results and generalizations of this theorem have been obtained, which we will call 'Gallai Theorems'. A typical Gallai theorem has the form:

$$\alpha+\beta=p,$$

where α and β are numerical maximum or minimum functions of some type defined on the class of connected graphs and p denotes the number of vertices in a graph.

This paper is an attempt to collect and unify results of this type. In particular, we present two general theorems which encompass nearly all of the existing Gallai theorems. The first theorem is based on hereditary properties of set systems, while the second is based on partitions of vertices into subgraphs having

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treelike properties. We also present a variety of new Gallai theorems (one of which is not a corollary of either of the two above mentioned generalizations), as well as a number of other new results.

2. Gallai theorems for hereditary set systems

In this section we will prove a general theorem concerning hereditary properties of sets, from which one can obtain as corollaries a variety of Gallai theorems, including the original Gallai Theorem given above.

Let S be a finite set of n elements and let P be a *hereditary property* of the subsets of S, i.e. P is a function f from the power set of S to $\{0, 1\}$ such that $f(X) = 1$ and $X' \subseteq X$ implies $f(X') = 1$. If $f(X) = 1$ we say that subset X *has property* P and that X is a *P-set*; if $f(X) = 0$ then X is called a *P'-set.*

A transversal of a family $\mathcal{F}$ of sets is a set T such that $|T \cap F| \geq 1$ for each $F \in \mathcal{F}$.

Theorem 2. *Let $X \subseteq S$. Then X is a maximal P-set for some hereditary property P if and only if $S - X$ is a minimal transversal of the class of all P'-sets.*

Proof. Let X be a maximal P-set and Y any P'-set. Then $Y \cap (S - X) \neq \emptyset$, otherwise $Y \subseteq X$. But this would imply that Y is a P-set since P is a hereditary property. Hence, $S - X$ is a P'-set transversal. It remains to show that $S - X$ is a minimal transversal.

Suppose not. Then for some $u \in S - X$, $Z = S - X - \{u\}$ is a P'-set transversal. But $X \cup \{u\}$ does not intersect Z. Hence $X \cup \{u\}$ must be a P-set, contradicting the maximality of X.

Conversely, suppose that $S - X$ is a minimal transversal of the P'-sets. Then, since $S - X$ does not intersect X, X must be a P-set. We must show that X is a maximal P-set. Suppose for some $u \in S - X$, $X \cup \{u\}$ is a P-set. Then since there are no P'-sets contained in $X \cup \{u\}$, $S - X - \{u\}$ intersects all P'-sets, contradicting the minimality of $S - X$. Hence X is a maximal P-set, as required. □

Note that X is a minimal P'-set transversal if and only if X is a minimal transversal of the *minimal* P'-sets.

Suppose now that there is a positive weight $w(s)$ associated with each element $s \in S$. Define $\alpha^+(P)(\alpha^-(P))$ be the largest (smallest) sum of the weights of the elements in a minimal P'-set transversal; similarly, let $\beta^+(P)(\beta^-(P))$ be the largest (smallest) sum of the weights of the elements in a maximal P-set. The following weighted Gallai theorems follow immediately from Theorem 2.

Corollary 2a. *For any hereditary property P of a set system,*

(i) $\alpha^-(P) + \beta^+(P) = \sum_{s \in S} w(s)$

(ii) $\alpha^+(P) + \beta^-(P) = \sum_{s \in S} w(s)$

In stating the following corollaries of Theorem 2, we will assume that all elements in S have unit weight, so that the sum of the weights of the elements in a set X equals the cardinality of X. We note, however, that each of the following corollaries can be generalized to the arbitrary weighted case.

For example, Cockayne and Giles [2] noticed Corollary 2a(i) in the unweighted case.

Hedetniemi [7] also obtained special cases of Corollary 2a(i) when the basic set S is either the vertex set $V(G)$ or the edge set $E(G)$ of a graph $G = (V, E)$.

Part I of Gallai's theorem can be obtained from the unweighted version of Corollary 2a(i). We let $S = V(G)$ and we say that a set $X \subseteq S$ has property P if and only if $\langle X \rangle$ the subgraph induced by X is totally disconnected (i.e. X is an independent set of vertices). In this case, minimal P'-set transversals are minimal vertex sets which cover all minimal non-independent vertex sets, i.e. pairs of adjacent vertices. Thus, minimal P'-transversals are minimal vertex covers (sets of vertices which cover all edges). In this context, $\beta^-(P) = i(G)$, the independent domination number (equivalently, the smallest number of vertices in a maximal independent set), while $\alpha^+(P) = \alpha_0^+(G)$ is the maximum number of vertices in a minimal vertex cover.

Corollary 2b. *For any nontrivial connected graph G with p vertices,*
(i) (*Gallai* [4]) $\alpha_0 + \beta_0 = p$
(ii) (*McFall, Nowakowski* [14]) $\alpha_0^+ + \beta_0^- = p$.

Let set S be the edge set $E(G)$ of a graph G, where $|E(G)| = q$. Let a subset $X \subseteq S$ of edges have property P if and only if X is independent, i.e. X is a matching. A minimal P'-set in this case is a pair of adjacent edges. In this context, $\beta^+(P) = \beta_1(G)$, the matching number of G; while $\beta^-(P) = \beta_1^-(G)$ equals the smallest number of edges in a maximal matching (i.e. a minimaximal matching). The parameter $\alpha^-(P)$ can in this case be seen to equal $\alpha_0(L(G))$, the vertex covering number of the line graph $L(G)$ of G. We have:

Corollary 2c. *For any nontrivial connected graph G with p vertices and q edges,*
(i) (*Hedetniemi* [7] $\alpha_0(L(G)) + \beta_1(G) = q$
(ii) $\alpha_0^+(L(G)) + \beta_1^-(G) = q$.

Many Gallai theorems may be obtained by considering a class $\mathcal{G}$ of forbidden subgraphs, letting $S = V(G)$ (or $E(G)$) and saying that a set $X \subseteq S$ has property P if and only if the induced subgraph $\langle X \rangle$ contains no member of $\mathcal{G}$.

For example, let $\mathcal{G} = \{K_3\}$ and $S = V(G)$.

Corollary 2d. *Let α_3 denote the minimum number of vertices covering all the triangles of a graph G with p vertices, and let β_3 denote the maximum number of vertices in a set S such that $\langle S \rangle$ contains no triangles. Then*

$$\alpha_3 + \beta_3 = p.$$

Our next application for Theorem 2 provides another result on matching and introduces a new graph theory parameter called the *matchability number of a graph.* Let $S = V(G)$ and let a set $X = \{x_1, x_2, \ldots, x_t\} \subseteq V$ have property P if and only if X is *matchable,* i.e. there exists a set of t independent edges $(x_i, f(x_i))$, $i = 1, 2, \ldots, t$. Let $\alpha_m^+(\alpha_m^-)$ denote the maximum (minimum) number of vertices in a minimal transversal of non-matchable sets, and let $\beta^+(\beta^-)$ denote the maximum (minimum) number of vertices in a maximal matchable set. Notice that $\beta^+(P) = \beta_1(G)$ (the matching number).

Corollary 2e. *For any nontrivial, connected graph G with p vertices,*
(i) $\alpha_m^- + \beta_1 = p$
(ii) $\alpha_m^+ + \beta^- = p$.

From Corollary 2e(i) and Gallai's Theorem, Part II, we may immediately conclude

Corollary 2f. *For any nontrivial connected graph G.*

$$\alpha_m^-(G) = \alpha_1(G).$$

The *matchability number of a graph,* $\beta^-(G)$, i.e. the smallest cardinality of a maximal matchable set of vertices, appears to be a new parameter. It is in general neither equal to $\beta_1^-(G)$, the smallest number of edges in a maximal matching nor equal to $\beta_1(G)$, the matching number. This can be seen from the graphs G_1 and G_2 in Fig. 1. It is easy to see, in fact, that for any graph G,

$$\gamma'(G) = \beta_1^-(G) \leqslant \beta^-(G) \leqslant \beta^+(G) = \beta_1(G)$$

where $\gamma'(G)$ is the edge domination number of G. A graph G_1 with $\beta_1^- = 2 < \beta^- = 3$ and G_2 with $\beta^- = 2 < \beta_1 = 3$, where a $\beta_1^-(G_1)$-set is $\{(2,3), (4,6)\}$; and a $\beta^-(G)$-set is $\{2,3,4\}$; a $\beta^-(G_2)$-set is $\{3,4\}$ and a $\beta_1(G_2)$-set is $\{(1,2), (3,4), (5,6)\}$.

Next, we let $S = V(G)$ and let $X \subseteq S$ have property P if and only if X contains no closed neighborhood of G. We observe that a minimal transversal of the P'-sets is a minimal transversal of the set of closed neighborhoods, i.e. a minimal dominating set of G. The cardinalities of a largest and a smallest minimal dominating set are denoted $\Gamma(G)$ and $\gamma(G)$, respectively. Let $\beta_c^+(G)$ and $\beta_c^-(G)$ denote the cardinality of largest and smallest P-sets, respectively.

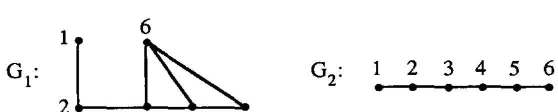

Fig. 1.

Corollary 2g. *For any nontrivial connected graph G with p vertices,*
(i) $\gamma + \beta_c^+ = p$
(ii) $\Gamma + \beta_c^- = p$.

Let $\varepsilon(G)$ denote the maximum number of pendant edges in a spanning forest of G. In [15] Nieminen proved

Theorem 3 (Nieminen). *For any non-trivial connected graph G with p vertices,*

$$\gamma + \varepsilon = p.$$

In [8] Hedetniemi observed a duality between Nieminen's result and Part II of Gallai's theorem as follows. A spanning star partition is a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of V such that for $1 \leq i \leq k$, $\langle V_i \rangle$ contains a *non-trivial* spanning star. Define $\beta_*^+(\beta_*^-)$ to equal the maximum (minimum) order of a spanning star partition of G. Also let $\alpha_1^+(\alpha_1^-)$ equal the maximum (minimum) order of minimal edge cover (a set of edges which covers all vertices of G).

Theorem 3′. *For any nontrivial connected graph G with p vertices,*
(i) (Hedetniemi) $\alpha_1^- + \beta_*^+ = p = \alpha_1 + \beta_1$ (*Gallai II*)
(ii) (Hedetniemi) $\alpha_1^+ + \beta_*^- = p = \varepsilon + \gamma$ (*Nieminen*).

Thus we conclude from Corollary 2g(i) and Theorems 2 and 3 that

Corollary 3a. *for any nontrivial connected graph G,*

$$\beta_c^+ = \varepsilon = \alpha_1^+.$$

Similar results can be obtained when closed neighborhoods are replaced by open neighborhoods. In this case minimial P'-set transversals are minimal total dominating sets (cf. Cockayne, et al. [3]).

As our last illustration of Theorem 2, we let $S = V(H)$, the vertex set of a hypergraph H with edge set $\mathscr{E}$. For a given positive integer k, we say that a set X of S has property P if and only if $|e \cap X| \leq k$ for each $e \in \mathscr{E}$. Minimal P'-transversals are minimal transversals of the sets of vertices which contain at least $k + 1$ vertices of some edge. The results in Corollary 2a(i) and (ii) in this context reduce to Corollaries 2b(i) and (ii) when H is a graph and $k = 1$.

Finally, we note that one can prove a Boolean dual of Theorem 2 which may have interesting special cases. Let Q be an *expanding* property on the subsets of a given set S, i.e. Q is a function f from the power set of S to $\{0, 1\}$ such that $f(X) = 1$ and $X' \supseteq X$ implies $f(X') = 1$.

Theorem 2′. *A set $X \subseteq S$ is a minimal Q-set if and only if $S - X$ is a maximal set whose union with any Q'-set is not S.*

3. Gallai theorems from spanning forests

An interesting variety of Gallai theorems can be obtained from a very elementary observation about spanning forests of a connected graph G. Let F be a spanning forest of a graph G with p vertices, let $e(F)$ denote the number of edges in F and $t(F)$ denote the number of connected components of F (i.e. the number of trees in the forest).

Proposition 4. *For any graph G with p vertices and any spanning forest F of G,*

$$t(F) + e(F) = p.$$

Next consider the class of all spanning forests satisfying some property P; let such a forest be called a P-forest. Let $t(P)$ denote the minimum number of trees in a (spanning) P-forest, and let $e(P)$ denote the maximum number of edges in a P-forest.

Corollary 4a. *For any graph G with p vertices and any property P of a spanning forest*

$$t(P) + e(P) = p.$$

A number of properties P of spanning forests lead to interesting Gallai theorems.

Let P_1 denote the property that every tree in a forest has at most two vertices (equivalently, has diameter $\leqslant 1$). Then it can be seen that $t(P_1) = \alpha_1(G)$, the vertex covering number, and $e(P_1) = \beta_1(G)$, the matching number.

Corollary 4b. *For any connected graph G with p vertices,*

$$t(P_1) + e(P_1) = \alpha_1 + \beta_1 = p \text{ (Gallai, II)}$$

Next, let P_2 denote the property that every tree in a forest F has diameter $\leqslant 2$. In this case every tree in F is a star ($K_{1,n}$) and it can be seen that

$t(P_2) = \gamma$, the domination number, and
$e(P_2) = \varepsilon$, the pendant edge number

(the maximum number of pendant edges in a spanning forest).

Corollary 4c. *For any connected graph G with p vertices,*

$$t(P_2) + e(P_2) = \gamma + \varepsilon = p \text{ (Nieminen).}$$

One can observe from Corollary 4c, Corollary 2g, and Theorem 3 that there are four equivalent definitions of the domination number of a graph $G = (V, E)$:

(i) the minimum number of vertices in a set D such that every vertex in $V - D$ is adjacent to at least one vertex in D;

(ii) the minimum number of vertices in a transversal of the closed neighborhoods of G;
(iii) the minimum order of a spanning star partition of the vertices of G (note; this does not allow isolated vertices);
(iv) the minimum number of trees in a spanning forest in which every tree has diameter ≤ 2 (note: this allows isolated vertices).

Finally, let P_* denote the property that every tree in a forest is a path. In this case $t(P_*)$ equals the minimum number of paths whose union is $V(G)$, and $e(P_*)$ is the maximum number of edges in a path decomposition of $V(G)$.

Corollary 4d. *For any graph G with p vertices,* $t(P_*) + e(P_*) = p$.

Notice that $t(P_*) = 1$ if and only if G has a Hamiltonian path.

4. Other Gallai theorems

Additional Gallai theorems can be found in the literature which involve a wide variety of graphical parameters. In this section we mention a few of these. We also include a new theorem, which apparently is not a corollary of any of the previous results.

Let $\gamma_c(G)$, the *connected domination number* of G, equal the minimum number of vertices in a dominating set D such that $\langle D \rangle$ is a connected subgraph. Let $\varepsilon_T(G)$ equal the maximum number of pendant edges in a spanning tree of G.

Theorem 5 (Hedetniemi and Laskar [9]). *For a connected graph G with p vertices,*

$$\gamma_c(G) + \varepsilon_T(G) = p.$$

Let $\lambda(G)$, the *edge connectivity* of G, be the minimum number of edges in a set S, such that $G - S$ is disconnected or K_1. Definite $\beta_\lambda(G)$ to be the maximum number of edges in a set S that $\langle S \rangle$ is disconnected.

Theorem 6 (Hedetniemi [6]). *For a connected graph G with q edges,*

$$\lambda(G) + \beta_\lambda(G) = q.$$

This result follows from Corollary 2a.

Let $\lambda^+(G)$ denote the maximum cardinality of a minimal set of edges S, such that $G - S$ is disconnected or K_1, and let $\beta_\lambda^-(G)$ denoted the minimum number of edges in a maximal set of edges S such that $\langle S \rangle$ is disconnected. Then the following theorem also results from Corollary 2a.

Theorem 7 (Peters et al. [16]). *For any non-trivial connected graph G with q edges,*

$$\lambda^+(G) + \beta_\lambda^-(G) = q$$

Similar results hold for vertex-connectivity. Let $\kappa(G)$ denote the *vertex connectivity* of G, i.e. minimum number of vertices in a set S, such that $G - S$ is either disconnected or K_1. Let $\beta_\kappa(G)$ denote the maximum number of vertices in a set S such that $\langle S \rangle$ is disconnected or K_1.

Theorem 8 (Hedetniemi [6]). *For a connected graph G with p vertices,*

$$\kappa(G) + \beta_\kappa(G) = p.$$

If we replace $\kappa(G)$ by $\kappa^+(G)$ and $\beta_\kappa(G)$ by $\beta_\kappa^-(G)$ in the above theorem, we obtain another Gallai theorem, where $\kappa^+(G)$ is the maximum number of vertices in a minimal set S, such that $V - S$ is disconnected or K_1 and $\beta_\kappa^-(G)$ is the minimum number of vertices in a maximal set S, such that $\langle S \rangle$ is disconnected.

Theorem 9 (Hare, Laskar, Peters [5]). *For a connected graph G with p vertices,*

$$\kappa^+(G) + \beta_\kappa^-(G) = p.$$

We now introduce two new parameters, $\alpha_{1k}(G)$ *and* $\beta_{1k}(G)$ of a graph G and prove a Gallai theorem involving these parameters, which apparently is not a corollary of any of the previous results. As a matter of fact, if $k = 1$, then Gallai theorem II results.

Let $k \leq \delta$, the minimum degree of G. Definite $\alpha_{1k}(G)$ to be the minimum number of edges in an edge set $X \subseteq E$ such that, for every $v \in V(G)$, $\deg v$ in $\langle X \rangle$ is at least k. Define $\beta_{1k}(G)$ to be the maximum number of edges in an edge set $X \subseteq E$ such that, for every $v \in V(G)$, $\deg v$ in $\langle X \rangle$ is at most k. Note that for $k = 1$, $\alpha_{11}(G) = \alpha_1(G)$ is the edge-covering number and $\beta_{11}(G) = \beta_1(G)$ is the matching number of G.

Theorem 10. *For a connected graph G with p vertices and* $k \leq \delta$,

$$\alpha_{1k}(G) + \beta_{1k}(G) = kp.$$

Proof. Let X be an α_{1k}-set, i.e. X is a set of $\alpha_{1k}(G)$ edges such that for each $v \in V(G)$, $\deg v$ in $\langle X \rangle$ is at least k.

Let $A = \{v \mid \deg v \text{ in } \langle X \rangle = k\}$
$B = \{v \mid \deg v \text{ in } \langle X \rangle > k\}$.

Let $A = \{v_1, v_2, \ldots, v_s\}$ and $B = \{u_1, u_2, \ldots, u_t\}$. Suppose the degree of

$u_i \in B$ in $\langle X \rangle$ is $k+\lambda_i$, $i=1,2,\ldots,t$. Note that each $\lambda_i > 0$. Due to the minimality of $\langle X \rangle$, it follows that B is independent. Let Y denote the set of edges obtained from X by deleting λ_i edges from each $u_i \in B$, $i=1,2,\ldots,t$. Then for each $v_i \in A$, $\deg v_i$ in $\langle Y \rangle$ is at most k, and for each $u_i \in B$ $\deg u_i$ in $\langle Y \rangle$ is exactly k. In other words, for each $v \in V(G) \deg v$ in $\langle Y \rangle$ is at most k, and hence, $\beta_{1k}(G) \geq |Y|$. Counting the degrees of each $v \in V(G)$ in $\langle X \rangle$ and $\langle Y \rangle$ we get

$$\begin{aligned} sk + \sum_{i=1}^{t} (k+\lambda_i) + tk + ks - \sum_{i=1}^{t} \lambda_i \\ = 2k(t+s) \\ = 2kp. \end{aligned}$$

Hence, the sum of the numbers of edges in X and Y is kp and we have

$$\alpha_{1k}(G) + \beta_{1k}(G) \geq |X| + |Y| = kp.$$

To show that $\alpha_{1k}(G) + \beta_{1k}(G) \leq kp$, we let X be a β_{1k}-set, i.e. a set of $\beta_{1k}(G)$ edges such that for each $v \in V(G)$, $\deg v$ in $\langle X \rangle$ is at most k. Let

$$A = \{v \mid \deg v \text{ in } \langle X \rangle = k\}$$
$$B = \{v \mid \deg v \text{ in } \langle X \rangle < k\}.$$

Let $A = \{v_1, v_2, \ldots, v_s\}$ and $B = \{u_1, u_2, \ldots, u_t\}$. Suppose $\deg u_i$ in $\langle X \rangle$ is $k-\lambda_i$, $i=1,2,\ldots,t$. Note that $\lambda_i > 0$. Due to the maximality of X, it follows that if $u_iu_j \in X$, then $u_iu_j \in E$. Now, as before we construct a set Y of edges from X so that for each $v \in V(G)$, $\deg v$ in $\langle Y \rangle$ is at least k. This can be done by adding λ_i edges to each vertex $u_i \in B$, $i=1,2,\ldots,t$, and these edges join vertices of B with A. Note that $|Y| \geq \alpha_{1k}(G)$. Counting the degrees of vertices in $\langle X \rangle$ and $\langle Y \rangle$ we get

$$\begin{aligned} ks + \sum_{i=1}^{t} (k-\lambda_i) + kt + \sum_{i=1}^{t} \lambda_i + sk \\ = 2k(s+t) \\ = 2kp. \end{aligned}$$

Thus the number of edges in X and $Y = kp$. But, $\alpha_{1k}(G) + \beta_{1k}(G) \leq |X| + |Y| = kp$. □

Corollary 10a. *For a connected graph G with p vertices,*

$\alpha_{11}(G) + \beta_{11}(G) = \alpha_1(G) + \beta_1(G) = p$. *(Gallai)*

$\alpha_{11}(G) = \alpha_1(G)$, $\beta_{11}(G) = \beta_1(G)$ *and*

$\alpha_{11} + \beta_{11} = \alpha_1 + \beta_1 = 1 \cdot p = p$.

The following theorem is a natural extension of the Gallai Theorem II. The proof is essentially the same as that of Theorem 10.

Theorem 11. *Let f be an integer-valued function defined on $V(G)$ and suppose $f(v) \leqslant deg(v)$ for each $v \in V(G)$. Define $\alpha_{1f}(G)$ to be the minimum number of edges in an edge set $X \subseteq E(G)$ such that, for every $v \in V(G)$, $deg(v)$ in $\langle X \rangle$ is at least $f(v)$. Similarly $\beta_{1f}(G)$ is defined. Then*

$$\alpha_{1f}(G) + \beta_{1f}(G) = \sum_{v \in V(G)} f(v)$$

In [13], Laskar and Sherk showed that Gallai theorems can be found in the study of projective planes, as follows.
Let Π denote a projective plane of order n,
Let P denote the set of points of Π, and
Let L denote the set of lines of Π.
For any integer $m \geqslant 1$, let P_m denote a set of points of Π, such that for every $1 \in L$,

$$|1 \cap P_m| \geqslant m.$$

Similarly, let M_m denote a set of points of Π, such that for every $1 \in L$,

$$|1 \cap M_m| \leqslant m.$$

Finally, let α_m equal the minimum cardinality of all P_m-sets and let β_m equal the maximum cardinality of all M_m-sets.

Theorem 12 (Laskar, Sherk).

$$\alpha_m + \beta_{n+1-m} = n^2 + n + 1, \quad 1 \leqslant m \leqslant n.$$

We also note that a dual version of Theorem 11 also holds.

In [10] and [11], Hedetniemi and Laskar propose a bipartite theory of graphs, a theory in which many of the standard results in graph theory have counterparts for bipartite graphs. Among the results obtained in this bipartite theory are several 'bipartite' Gallai theorems, as follows.

Let $G = (X, Y, E)$ denote a bipartite graph where edges only join vertices in X with vertices in Y. Given an arbitrary graph $G = (V, E)$ we can construct three bipartite graphs of some interest, as illustrated in Fig. 2.

Let $G = (X, Y, E)$ be a bipartite graph.

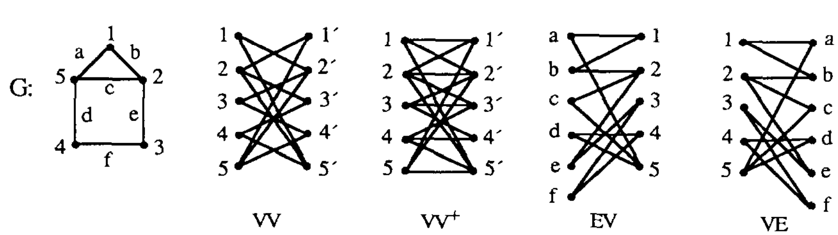

Fig. 2.

A set $S \subseteq X$ is a *Y-dominating set* if for every $y \in Y$ there is an $x \in S$ such that $(x, y) \in E$.
A set $S \subseteq X$ is *hyper-independent* if there does not exist a $y \in Y$ whose open neighborhood $N(y) \subseteq S$.
A set $S \subseteq X$ is *hyper-dominating* if for every $x \in X - S$ there does not exist a $y \in N(x)$ such that

$$N(y) \subseteq S \cup \{x\}.$$

Finally, define $\beta_h(G)$ to equal the maximum order of a hyper-independent set, and $\gamma_Y(G)$ to equal the minimum order of a Y-dominating set of G.

Theorem 13 (Hedetniemi, Laskar). *For any graph $G = (V, E)$, where $|V| = p$ and $|E| = q$,*
(i) $\gamma_t(G) + \beta_h(VV) = p$
(ii) $\gamma(G) + \beta_h(VV^+) = p$
(iii) $\alpha_1(G) + \beta_h(EV) = q$
(iv) $\alpha_0(G) + \beta_h(VE) = p$

A variety of new Gallai theorems can be generated from an extension of Theorem 2 which involves a generalization of the concepts of minimality and maximality. These ideas were introduced in [1]. We omit proofs here. We use the notation of Section 2.

A subset X of S is called a *k-minimal P-set* if X is a P-set but for all ℓ satisfying $1 \leq \ell \leq k$, all ℓ-subsets U of X and all $(\ell - 1)$-subsets R of S, $(X - U) \cup R$ is not a P-set. We note that 1-minimality is the usual concept of minimality.

Similarly X is a *k-maximal P-set* if X is a P-set but the addition of any ℓ elements to X where $\ell \leq k$ followed by the removal of any $\ell - 1$ elements, forms a set which is not a P-set. Let P be a hereditary property on S and let the new hereditary property Q on S be defined by:

X is a Q-set if and only if X is a transversal of the class of P'-sets.

Theorem 14 (Bollobás, Cockayne and Mynhardt). *X is a k-maximal P-set if and only if $S - X$ is a k-minimal Q-set.*

Let $\alpha_k(S, P)$ equal the minimum cardinality of a k-minimal P-set, and let $\beta_k(S, P)$ equal the maximal cardinality of k-maximal P-set.

Corollary 14a (Bollobás, Cockayne, Mynhardt). *Let P be a hereditary property on the subsets of a set S. Then*

$$\alpha_k(S, P) + \beta_k(S, P) = |S|.$$

Note that for $k = 1$, P-sets are independent sets of vertices and, again, we get

Gallai's theorem

$$\alpha_0 + \beta_0 = p.$$

As a final note, we mention that a matroid generalization of Gallai's theorem has been obtained by Kajitani and Ueno [12]. In particular, they establish a one to one correspondence between independent parity sets in a partition matroid M, induced by incidences of edges in the subdivision graph $S(G)$ of a graph G, and matchings in G.

Given this correspondence, it is easy to see that the complement of an independent parity set in the dual matroid M^* of M corresponds to a covering of G and vice versa. From this observation the following interesting result can be proved, a corollary of which is Gallai's Theorem II.

Theorem 15 (Kajitani and Ueno). *Let n be the number of vertices of a connected graph $G=(V, E)$.*

(i) *Suppose that X is a maximal matching of G. If Y is a covering of G which contains X and is minimal in the coverings containing X, then*

$$|X| + |Y| = n.$$

(ii) *Suppose that Y is a minimal covering of G. If X is a matching of G which is contained in Y and maximal in the matchings contained in Y, then*

$$|X| + |Y| = n.$$

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