Matrices that preserve the value of the generalized matrix function of the upper triangular matrices

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Abstract

Let H be a subgroup of the symmetric group $S_n$ and $\chi$ an irreducible character of $H$. In this paper we give conditions that characterize matrices that leave invariant the value of a given generalized matrix function on the set of upper triangular matrices. In some cases we describe completely these matrices.

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1. Introduction

Let $S_n$ be the symmetric group of degree $n$. Let $F$ be an arbitrary field of characteristic zero and $c$ a function, not identically zero, from $S_n$ into $F$. If $X = [x_{ij}]$ is an $n \times n$ matrix over $F$, the generalized Schur function $d_c(X)$ is defined by [5,6]

$$d_c(X) = \sum_{\sigma \in S_n} c(\sigma) \prod_{i=1}^n x_{i\sigma(i)}.$$

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Let $c$ coincide with a character $\chi$ of a subgroup $H$ of $S_n$ and be zero in $S_n \setminus H$. We then denote $d_c(X)$ by $d^H_c(X)$ and say that it is the generalized matrix function associated with $H$ and $\chi$.

Let $M_n(F)$ be the linear space of $n$-square matrices with elements in $F$. In [3] Marcus and Chollet defined the set, $\mathcal{S}(H, \chi)$, of the matrices $A \in M_n(F)$ satisfying

$$d^H_c(AX) = d^H_c(X)$$

for all $X \in M_n(F)$.

They showed that this set is a subgroup of the multiplicative group of nonsingular matrices, $GL(n, F)$.

In [7] Oliveira and Dias da Silva characterize completely this subgroup: If $\theta \in S_n$, we denote by $P(\theta)$ the $n \times n$ permutation matrix whose $(i, j)$ entry is

$$P(\theta)_{ij} = \delta_{i\theta(j)}, \quad i, j \in \{1, \ldots, n\}.$$ 

Let $H^{id}_T$ be the subgroup of $H$ generated by those transpositions, $\tau$, of $H$ such that $\chi(\tau) = -\chi(id)$ and denote by $Z(H, \chi)$ the set of elements $\pi$ of $H$ satisfying

$$\chi(id)\chi(\pi\sigma) = \chi(\pi)\chi(\sigma)$$

for all $\sigma$ in $H$.

**Theorem 1.1.** The matrix $A$ belongs to $\mathcal{S}(H, \chi)$ if and only if

$$A = MP(\gamma),$$

where

1. $M = [m_{ij}], m_{ij} = 0$ whenever $i$ and $j$ belong to different orbits of $H^{id}_T$,
2. $\gamma \in Z(H, \chi)$,
3. $\det(M) = \frac{\chi(\gamma)}{\chi(id)}$.

Here we study the set, $\mathcal{S}(H, \chi)$, of the matrices $A \in M_n(F)$ satisfying

$$d^H_c(AX) = d^H_c(X)$$

for all $X \in T^U_n(F)$, where $T^U_n(F)$ is the set of $n$-square upper triangular matrices.

Since $I_n \in \mathcal{S}(H, \chi)$, $\mathcal{S}(H, \chi)$ is a nonempty set. It is also easy to see that $\mathcal{S}(H, \chi) \subseteq \mathcal{S}(H, \chi)$.

Let $\sigma \in H$ such that $\chi(\sigma^{-1}) \neq 0$. We denote by $V_{\sigma}(H, \chi)$ the set of matrices $L \in T^U_n(F)$ (the set of $n$-square lower triangular matrices) with diagonal elements equal to $1$, satisfying

$$d^H_c(P(\sigma)LX) = d^H_c(P(\sigma)X)$$

for all $X \in T^U_n(F)$. 


In Section 2 we give a description of the matrices in the set $\mathcal{T}(H, \chi)$ and establish a relationship between the set $\mathcal{T}(H, \chi)$ and the sets $V_\sigma(H, \chi)$. In Section 3, using the Murnaghan–Nakayama rule [1] we present, in some cases, a complete description of the matrices in $V_\sigma(S_n, \chi)$.

2. The matrices in $\mathcal{T}(H, \chi)$

The purpose of this section is to characterize matrices in the set $\mathcal{T}(H, \chi)$, where $H$ is a subgroup of $S_n$ and $\chi$ is a character of $H$. The main results are the following:

Theorem 2.1

$$\mathcal{T}(H, \chi) = \bigcup_{\sigma \in H, \chi(\sigma^{-1}) \neq 0} \{ P(\sigma)L R : L \in V_\sigma(H, \chi), R \in T_n^U(F), \det(R) = \frac{\chi(id)}{\chi(\sigma^{-1})} \}.$$  

Let $\sigma \in H$ such that $\chi(\sigma) \neq 0$. Denote by $H_\sigma^T$ the subgroup of $H$ generated by those transpositions, $\tau$, of $H$ satisfying $\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1})$, i.e., $\chi(\sigma^{-1}\tau) = \chi(\sigma^{-1})\epsilon(\tau)$.

Theorem 2.2. Let $L = [l_{ij}] \in T_n^L(F)$ with diagonal elements equal to 1 and let $\sigma \in H$ such that $\chi(\sigma) \neq 0$. If $L \in V_\sigma(H, \chi)$ then $l_{ij} = 0$ whenever $i$ and $j$ belong to different orbits of $H_\sigma^T$.

There are some cases where the converse of Theorem 2.2 also holds.

Theorem 2.3. Let $L = [l_{ij}] \in T_n^L(F)$ with diagonal elements equal to 1 and let $\sigma \in Z(H, \chi)$. Then $L \in V_\sigma(H, \chi)$ if and only if $l_{ij} = 0$ whenever $i$ and $j$ belong to different orbits of $H_\sigma^T$.

Let $x$ be an indeterminate over the field $F$ and $E^{(i)}+x^{(j)}$ the matrix obtained from the identity matrix by adding $x$ times column $j$ to column $i$.

Proposition 2.4. Let $L = E^{(k)}+x^{(k+1)} \in T_n^L(F)$ with $x \neq 0$ and $\sigma \in H$ such that $\chi(\sigma) \neq 0$. Then $L \in V_\sigma(H, \chi)$ if and only if the transposition $(k, k+1) \in H$ and $\chi((k, k+1)\sigma^{-1}) = -\chi(\sigma^{-1})$.

We start by proving some easy results.

Proposition 2.5. If $A \in \mathcal{T}(H, \chi)$, then $\det(A) \neq 0$.

The proof of this proposition is analogous to Theorem 2.1 in [7].
Proof. Assume $A$ is singular. There is a nonzero column matrix

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

such that $Ax = 0$. Let $p \in \{1, \ldots, n\}$ be the largest integer such that $x_p \neq 0$. Let $X$ be the matrix those $p$th column is $x$, $x_{ii} = 1$, for $i \neq p$, the remaining elements being zero. Then $X \in T_n^U(F)$ and $AX$ has a zero column. Thus $d_H^H(AX) = 0$ but $d_H^H(X) = \chi(id)x_p \neq 0$, violating the definition of $\mathcal{F}(H,\chi)$. □

Remarks

(1) In [3], it was proved that the set $\mathcal{S}(H,\chi)$ is a subgroup of $GL(n, F)$. In general, the set $\mathcal{T}(H,\chi)$ is not a subgroup of $GL(n, F)$.

Example 2.6. Let $\chi$ be the irreducible character of $S_4$ such that $\chi(\sigma) = F(\sigma) - 1$, i.e.,

$$\chi = \begin{bmatrix} 1^4 & 2 \, 1^2 & 3 \, 1 & 2^2 & 4 \\
3 & 1 & 0 & -1 & -1 \end{bmatrix}$$

It is easy to prove that the matrices

$$A = \begin{bmatrix} 0 & 3 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 3 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \end{bmatrix}$$

belong to $\mathcal{F}(S_4,\chi)$. But the product

$$AA' = \begin{bmatrix} 3 & 3 & 0 & 0 \\
3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \end{bmatrix}$$

is such that

$$-9 = d_{S_4}^S(AA' I_4) \neq d_{S_4}^S(I_4) = 3.$$ 

So, $AA' \notin \mathcal{F}(S_4,\chi)$.

(2) If $A \in \mathcal{F}(H,\chi)$ and $R \in T_n^U(F)$ with $\det(R) = 1$, then $AR \in \mathcal{F}(H,\chi)$.

Proof. Let $X \in T_n^U(F)$. Since $R \in T_n^U(F)$ then $RX \in T_n^U(F)$. Because $A \in \mathcal{F}(H,\chi)$,

$$d_H^H(ARX) = d_H^H(RX) = \chi(id)\det(RX) = \chi(id)\det(X) = d_H^H(X).$$

So, $AR \in \mathcal{F}(H,\chi)$. □

(3) If $A \in \mathcal{F}(H,\chi)$ then $d_H^H(A) = \chi(id)$.

Proof. Since $I_n \in T_n^U(F)$ and $A \in \mathcal{F}(H,\chi)$,

$$d_H^H(A I_n) = d_H^H(I_n) = \chi(id).$$ □
(4) If $H = S_n$ and $\chi = \epsilon$ is the alternating character then
$$\mathcal{T}(S_n, \epsilon) = \{ A \in M_n(F) : \det(A) = 1 \}.$$ 

**Proposition 2.7.** Let $H$ be a subgroup of $S_n$ and $\chi$ be a character of $H$. Then a matrix $A$ is in $\mathcal{T}(H, \chi)$ if and only if there exist $\sigma \in H$ such that $\chi(\sigma) \neq 0$ and $L \in T_n^U(F)$ with diagonal elements equal to 1 satisfying

(i) $L^{-1} P(\sigma^{-1}) A \in T_n^U(F),$
(ii) $\det(A) = \frac{\chi(id)\epsilon(\sigma)}{\chi(\sigma^{-1})},$
(iii) $d_H^\chi(P(\sigma)LZ) = \chi(\sigma^{-1})\det(Z)$, for all $Z \in T_n^U(F)$.

**Proof.** Let $A \in \mathcal{T}(H, \chi)$. It is well known that there is a permutation $\sigma$ in $S_n$ such that $A = P(\sigma)LR$, where $L \in T_n^L(F)$ with diagonal elements equal to 1 and $R \in T_n^U(F)$. Then, $L^{-1} P(\sigma^{-1}) A = R \in T_n^U(F)$. So, we have (i). For $X \in T_n^U(F)$, since $A \in \mathcal{T}(H, \chi)$,
$$d_H^\chi(AX) = d_H^\chi(X).$$

Then,
$$d_H^\chi(P(\sigma)LRX) = d_H^\chi(X).$$
Setting $Z = RX$, since $L$, $A$ are nonsingular, $R$ is nonsingular and $Z$ is arbitrary in $T_n^U(F)$. So we get
$$d_H^\chi(P(\sigma)LZ) = d_H^\chi(R^{-1}Z).$$
If $Z = I_n$, then
$$d_H^\chi(P(\sigma)L) = d_H^\chi(R^{-1}) = \chi(id) \prod_{i=1}^n r_i^{-1} \neq 0.$$
But,
$$d_H^\chi(P(\sigma)L) = \begin{cases} \chi(\sigma^{-1}) & \text{if } \sigma \in H, \\ 0 & \text{if } \sigma \notin H. \end{cases}$$
Then, $\sigma \in H$ and $\chi(\sigma^{-1}) \neq 0$. Therefore
$$\chi(\sigma^{-1}) = \chi(id) \prod_{i=1}^n r_i^{-1}. $$
Consequently, $\det(R) = \frac{\chi(id)\epsilon(\sigma)}{\chi(\sigma^{-1})}$ and
$$\det(A) = \det(P(\sigma)LR) = \epsilon(\sigma)\det(R) = \frac{\chi(id)\epsilon(\sigma)}{\chi(\sigma^{-1})}$$
and we have (ii).
Now, because
\[ d_H^\chi (R^{-1}Z) = \chi(id)\det(R^{-1})\det(Z) = \chi(\sigma^{-1})\det(Z) \]
we get
\[ d_H^\chi (P(\sigma)LZ) = d_H^\chi (P(\sigma)LRR^{-1}Z) = d_H^\chi (AR^{-1}Z) = d_H^\chi (R^{-1}Z) = \chi(\sigma^{-1})\det(Z). \]
Consequently, we have (iii).

Conversely, let \( A, L, P(\sigma) \) be matrices satisfying the conditions (i)–(iii). Let \( X \in T_n^U(F) \). Using (i) and (iii) we get
\[ d_H^\chi (AX) = d_H^\chi (P(\sigma)LX) = \chi(\sigma^{-1})\det(L^{-1}P(\sigma^{-1})AX) = \chi(\sigma^{-1})\epsilon(\sigma)\det(A)\det(X). \]
Using (ii), we can conclude that
\[ d_H^\chi (AX) = \chi(id)\det(X) = d_H^\chi (X). \]
Consequently, \( A \in \mathcal{F}(H, \chi). \)

It is easy to prove Theorem 2.1 using the last proposition and the fact that, if \( \sigma \in H \) then
\[ d_H^\chi (P(\sigma)Z) = \chi(\sigma^{-1})\det(Z) \]
for all \( Z \in T_n^U(F) \).

Now, for each \( \sigma \in H \) such that \( \chi(\sigma^{-1}) \neq 0 \), we are going to study matrices in \( V_\sigma(H, \chi) \).

**Proof of Theorem 2.2.** Let \( k \) be an integer such that \( k \in \{1, \ldots, n\} \) and for all \( n \geq s > k, (k, s) \notin H^n_F \).

We are going to prove that \( l_{k+1} = \cdots = l_n = 0 \).

Using \( X = E^{(k+1)+x(k)} \), since \( L \in V_\sigma(H, \chi) \) and \( X \in T_n^U(F) \),
\[ d_H^\chi (P(\sigma)LX) = \chi(\sigma^{-1})\det(X) = \chi(\sigma^{-1}). \]
But,
\[ d_H^\chi (P(\sigma)LX) = \begin{cases} \chi(\sigma^{-1})(1 + xl_{k+1}) & \text{if } (k, k+1) \notin H, \\ \chi(\sigma^{-1}) + (\chi(\sigma^{-1}) + xl_{k+1}) & \text{if } (k, k+1) \in H. \end{cases} \]
Because \( (k, k+1) \notin H^n_F \) and \( x \) is arbitrary then \( l_{k+1} = 0 \).
Next, using $X = E^{(k+2)+x(k)}$ then

$$\chi(\sigma^{-1}) = d_H^X(P(\sigma)LX) = \begin{cases} 
\chi(\sigma^{-1})(1 + xl_{k+2k}) & \text{if } (k, k + 2) \notin H, \\
\chi(\sigma^{-1}) + (\chi(\sigma^{-1}) + (\chi((k, k + 2)\sigma^{-1}))(xl_{k+2k}) & \text{if } (k, k + 2) \in H.
\end{cases}$$

Because $(k, k + 2) \notin H$ and $x$ is arbitrary then $l_{k+2k} = 0$.

Then with $X = E^{(k+3)+x(k)}$ we prove that $l_{k+3k} = 0$, etc. In this way we can show that $l_{k+1k}, l_{k+2k}, \ldots, l_{nk}$ are equal to zero.

Now let $r$ be the largest integer such that $1 \leq r < k$ and $(k, r) \in H$. Taking $X = E^{(r+1)+x(r)}$ we prove $l_{r+1r} = 0$. Then with $X = E^{(r+2)+x(r)}$ we prove that $l_{r+2r} = 0$, etc. Consequently we have $l_{r+1r} = \cdots = l_{k-1r} = l_{k+1r} = \cdots = l_{nr} = 0$.

Therefore we can conclude that $l_{iu} = 0$ if $u, k$ belong to the same orbit of $H$ and $i, k$ belong to different orbits of $H$. Since $k$ is arbitrary, we get the theorem. □

In general, the converse of Theorem 2.2 does not hold.

**Example 2.8.** Let $\chi$ be the irreducible character of $S_5$ such that

$$\begin{array}{cccccc}
1 & 2 & 1 & 3 & 1 & 2 \\
5 & 4 & 1 & 4 & 2 & 3
\end{array}$$

Using $\sigma = (1, 2)$, it is easy to see that $(1, 2) \in (S_5)^{(1,2)}$. Taking $L = E^{(1)+1(2)}$, $L$ is such that $l_{ij} = 0$ whenever $i, j$ belong to different orbits of $(S_5)^{(1,2)}$. But, using $X = E^{(2)+1(1)}$,

$$d_S^X(P(1, 2)LX) = \chi(id) + 2\chi(1, 2) = 7 \neq 1 = \chi(1, 2) = d_S^X(P(1, 2)X).$$

Consequently, $L \notin V_{(1,2)}(S_5, \chi)$.

**Proposition 2.9.** If $\sigma \in Z(H, \chi)$ then

$$V_{\sigma}(H, \chi) = V_{id}(H, \chi).$$

**Proof.** Let $Z \in T_n^U(F)$. Using the definition of $d_H^X$, we have

$$d_H^X(P(\sigma)LZ) = \sum_{\pi \in H} \chi(\pi) \prod_{i=1}^n (P(\sigma)LZ)_{\pi(i)} = \sum_{\pi \in H} \chi(\pi) \prod_{j=1}^n (LZ)_{j\pi(j)}. $$

Since $\chi(\rho^{-1}) = \chi(\sigma^{-1})$ and $\rho \in Z(H, \chi)$, we can conclude that

$$d_H^X(P(\sigma)LZ) = \frac{\chi(\sigma^{-1})}{\chi(id)} \sum_{\rho \in H} \chi(\rho) \prod_{j=1}^n (LZ)_{j\rho(j)} = \frac{\chi(\sigma^{-1})}{\chi(id)} d_H^X(LZ).$$
Consequently, $L \in V_\sigma(H, \chi)$ if and only if \( \frac{z^{(\sigma^{-1})}}{\chi(id)} d_X^{(\sigma^{-1})} d_X^T(Z) = \frac{z^{(\sigma^{-1})}}{\chi(id)} d_X^{(\sigma^{-1})} d_X^T(Z) \) if and only if $L \in V_{id}(H, \chi)$. □

Proof of Theorem 2.3. Using Theorem 2.2, if $L = [l_{ij}] \in V_\sigma(H, \chi)$, then $l_{ij} = 0$ whenever $i, j$ belong to different orbits of $H^T_\sigma$. Conversely, let $L = [l_{ij}] \in T_n(F)$ with diagonal elements equal to 1 such that $l_{ij} = 0$ whenever $i, j$ belong to different orbits of $H^T_\sigma$. Since $\sigma \in Z(H, \chi)$, using the last proposition, we are prove that $L \in V_{id}(H, \chi)$. Because $H^T_\sigma = H_{id}^T$, by Theorem 1.1, $L \in S(H, \chi)$. But, $S(H, \chi) \subseteq T(H, \chi)$, consequently, $L \in V_{id}(H, \chi) = V_{\sigma}(H, \chi)$. □

Proof of Proposition 2.4. Let $L = E^{(k+1)}(k)$ with $x \neq 0$. Let $Z \in T_n(F)$. Using the definition of $d_X^T$, if $(k, k+1) \in H$ we have

\[
d_X^T(P(\sigma)LZ) = \sum_{\rho \in H} \chi(\rho^{-1}) \prod_{j=1}^{n} (LZ)_{j\rho(j)}
\]

\[
= \chi^{(\sigma^{-1})} \prod_{i=1, i \neq k+1}^{n} z_{ii}(z_{k+1}k+1 + xz_{kk+1})
+ \chi((k, k+1)\sigma^{-1}) \prod_{i=1, i \neq k+1}^{n} z_{ii}(xz_{kk+1}).
\]

If $\sigma \in H$ is such that $\chi((k, k+1)\sigma^{-1}) = -\chi(\sigma^{-1})$ then $d_X^T(P(\sigma)LZ) = \chi(\sigma^{-1})\det(Z)$. Consequently, $L \in V_{\sigma}(H, \chi)$. Conversely, if $L \in V_{\sigma}(H, \chi)$, taking $Z = E^{(k+1)}(k)$ we have,

\[
d_X^T(P(\sigma)LZ) = \chi(\sigma^{-1})
\]

and

\[
d_X^T(P(\sigma)LZ) = \begin{cases} 2\chi(\sigma^{-1}) & \text{if } (k, k+1) \notin H, \\ 2\chi(\sigma^{-1}) + \chi((k, k+1)\sigma^{-1}) & \text{if } (k, k+1) \in H. \end{cases}
\]

So, $(k, k+1) \in H$ and $\chi((k, k+1)\sigma^{-1}) = -\chi(\sigma^{-1})$. □

3. Murnaghan–Nakayama rule and $V_{\sigma}(S_n, \chi)$

In this section we characterize some sets $V_{\sigma}(S_n, \chi)$ where $\chi$ is an irreducible character of $S_n$ and $\sigma \in S_n$ is such that $\chi(\sigma) \neq 0$.

We define a partition $\alpha$ of $n$ as $\alpha = (\alpha_1, \ldots, \alpha_r)$ where the $\alpha_i$’s are integers, $\alpha_1 \geq \cdots \geq \alpha_r \geq 0$, and $\alpha_1 + \cdots + \alpha_r = n$. We do not distinguish between two
partitions that differ by a sequence of zeros. If \( \alpha = (\alpha_1, \ldots, \alpha_r) \) is a partition of \( n \) and \( \alpha_r > 0 \), we say that \( r \) is the length of \( \alpha \). Each partition \( \alpha = (\alpha_1, \ldots, \alpha_r) \) of length \( r \) is related to a Young diagram, denoted by \([\alpha]\), which consists of \( r \) left justified rows of boxes. The number of boxes in the \( i \)th row is \( \alpha_i \).

If \( \alpha = (\alpha_1, \ldots, \alpha_r) \) is a partition of \( n \), the \( \alpha_1 \)-tuple \( \alpha' = (\alpha'_1, \ldots, \alpha'_\alpha_1) \) \([2]\), defined by

\[
\alpha'_i = \lfloor \{ j : \alpha_j \geq i \} \rfloor
\]
is also a partition of \( n \) called the conjugate partition of \( \alpha \).

We say that a Young diagram is symmetric if it is associated with a partition \( \alpha \) such that \( \alpha = \alpha' \).

It is well known that the irreducible characters of \( S_n \) are in a bijective correspondence with the ordered partitions of \( n \). We identify the irreducible character \( \lambda \) with the partition that corresponds to \( \lambda \). If \( \lambda \) is an irreducible character of \( S_n \), the character \( \lambda' \) such that

\[
\lambda'(\sigma) = \epsilon(\sigma)\lambda(\sigma)
\]

for all \( \sigma \in S_n \) is an irreducible character called the character associated with \( \lambda \). If \( \lambda = \lambda' \) we say that \( \lambda \) is self-associated.

The main results of this section are:

\textbf{Theorem 3.1.} Let \( \chi \) be an irreducible character of \( S_n \). Then

\[
\bigcup_{\sigma \in S_n, \chi(\sigma) \neq 0} V_\sigma(S_n, \chi) = \{I_n\}
\]

if and only if

\[ \chi = 1 \text{ or } \chi \text{ is self-associated.} \]

\textbf{Theorem 3.2.} Let \( \chi = (n-1, 1) \) be the irreducible character of \( S_n \) with \( n > 3 \). Let \( \sigma \in S_n \) be a cycle with length \( n - 2 \) and \( L = [l_{ij}] \in T_n^2(F) \) with diagonal elements equal to 1. Then

\[ L \in V_\sigma(S_n, \chi) \]

if and only if \( L \) satisfies the condition:

"For \( r > p \), if there exists an integer \( k \) such that \( p \leq k \leq r \) and \( \sigma(k) \neq k \) then \( l_{rp} = 0 \)."

\textbf{Theorem 3.3.} Let \( \chi = (s, 1^{n-s}) \) be the irreducible character of \( S_n \) satisfying

(i) \( s - 1 > n - s \geq 1 \),
(ii) if \( s = 6 \) then \( n \notin \{9, 10\} \),
(iii) if \( s \) is odd and \( s \geq 5 \) then \( 2(n - s) \neq s - 1 \).
Let $\sigma \in S_n$ be a cycle with length $s - 1$ such that
\[ \{ j : \sigma(j) \neq j \} = \{ u, u + 1, \ldots, u + s - 2 \} \]
for some integer $u < n - s + 2$. Let $L = [l_{ij}] \in T_n^I(F)$ with diagonal elements equal to 1. Then
\[ L \in V_\sigma(S_n, \chi) \]
if and only if $L$ satisfies the condition:
"For $r > p$, if there exists an integer $k$ such that $p \leq k \leq r$ and $\sigma(k) \neq k$ then $l_{rp} = 0$".

We start by proving some auxiliary lemmas:

**Lemma 3.4.** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ be a partition of $n$ with $\alpha_2 > 1$ and length $p$. Let $t$ be the largest integer such that $\alpha_t > 1$ and $\alpha' = (\alpha'_1, \ldots, \alpha'_p)$ be the conjugate partition of $\alpha$. If
\[ (\alpha_2 - 1, \ldots, \alpha_t - 1) = (\alpha_2 - 1, \ldots, \alpha_t - 1)' \]
then $\alpha_l = \alpha'_l$ for $l \in \{2, \ldots, t\}$.

**Proof.** Let $\beta = (\beta_1, \ldots, \beta_{t-1})$ be the partition $(\alpha_2 - 1, \ldots, \alpha_t - 1)$ and $\beta' = (\beta'_1, \ldots, \beta'_{t-1})$ be the conjugate partition of $\beta$.

By definition
\[ \beta'_u = |\{ i : \beta_i \geq u \}| = |\{ i : \alpha_{i+1} \geq u + 1 \}| \]
for $u \in \{1, \ldots, \beta_{t-1}\}$. □

**Claim.** $\beta'_u = \alpha'_u + 1$, for $u \in \{1, \ldots, \beta_{t-1}\}$.

**Proof.** Suppose that $a = \alpha'_u + 1$. Then $a$ is the largest integer such that $\alpha_a \geq u + 1$. Consequently, $\alpha_{a+1} < u + 1$. Therefore, $\beta'_u = a - 1 = \alpha'_u - 1$. □

Since $\beta = \beta'$, we have $t - 1 = \beta_{t-1}$ and
\[ \alpha_{t+1} - 1 = \beta_t = \beta'_t = \alpha'_{t+1} - 1 \]
for $l \in \{1, \ldots, t - 1\}$. Consequently, $\alpha_l = \alpha'_l$ for $l \in \{2, \ldots, t\}$. □

**Lemma 3.5.** Let $D$ be the Young diagram associated with the partition $\alpha = (\alpha_1, \ldots, \alpha_p)$ with length $p$ such that $\alpha_2 > 1$. Let $t$ be the largest integer such that $\alpha_t > 1$. If $D$ is not symmetric but the diagram obtained by omitting all the boundary boxes of $D$ is, then there is a unique way of omitting the regular boundary of length
\[ \max(\alpha_2 + p - 2, \alpha_1 + t - 2) \]
and the diagram so obtained is not symmetric.
Proof. Let $D_1$ be the Young diagram obtained by omitting all the boundary boxes of $D$. Since $D_1 = [(\alpha_2 - 1, \ldots, \alpha_t - 1)]$ is symmetric then, using Lemma 3.4,
\[\alpha_l = \alpha'_l\]
for $l \in \{2, \ldots, t\}$, where $\alpha' = (\alpha'_1, \ldots, \alpha'_p)$ is the conjugate partition of $\alpha$. In particular,
\[\alpha_2 = t.\]
Since $D$ is not symmetric, then $\alpha_1 > p$ or $\alpha_1 < p$. Consequently, there is a unique way of omitting the regular boundary of length $\max\{\alpha_2 + p - 2, \alpha_1 + t - 2\}$.
Let $D_2$ be the Young diagram so obtained.
If $\alpha_1 > p$ then $\max\{\alpha_2 + p - 2, \alpha_1 + t - 2\} = \alpha_1 + t - 2$. So, $[(\alpha_2 - 1, \ldots, \alpha_t - 1, 1, \alpha_{t+1}, \ldots, \alpha_p)] = D_2$. Since $\alpha_2 = t < p$ then $\alpha_2 - 1 < p$. Consequently, $D_2$ is not symmetric.
If $\alpha_1 < p$ then $\max\{\alpha_2 + p - 2, \alpha_1 + t - 2\} = \alpha_2 + p - 2$. So, $[(\alpha_1, \alpha_1 - 1, \ldots, \alpha_1)] = D_2$. Since $\alpha_1 \geq \alpha_2 = t$ then $\alpha_1 > t - 1$. Consequently, $D_2$ is not symmetric. □

Lemma 3.6. Let $\chi = (1^p)$ be the irreducible character of $S_p$ where $p \geq 2$. Let $\sigma, (i, i+1) \in S_p$, then
\[\chi(\sigma(i, i+1)) = -\chi(\sigma) \neq 0.\]
Proof. Since $\chi = \epsilon$ is the alternating character of $S_p$, the result follows. □

Lemma 3.7. Let $\chi = (p + a, p + 1, 1^l)$ be the irreducible character of $S_m$ such that $a > l + 1$ and $p = l + 2$. Then there exist $\sigma \in S_m$ and a transposition $(i, i+1) \in S_m$ such that $\chi(\sigma) \neq 0$ and
\[\chi(\sigma(i, i+1)) = -\chi(\sigma).\]
Proof. The proof follows from an exhaustive consideration of cases, which we list:

1. $l = 0$. In this case, $a > 1$ and $p = 2$.
   (1.1) $a = 2$,
   (1.2) $a = 3$,
   (1.3) $a = 4$,
   (1.4) $a \geq 5$.
2. $l \geq 1$. In this case, $a > l + 1$ and $p = l + 2$.
   (2.1) $a \neq l + 3$,
   (2.2) $a = l + 3, l \geq 2$,
   (2.3) $a = l + 3, l = 1$.
Details are omitted. □
Lemma 3.8. Let \( \chi = (p - s, 1^s) \) be the irreducible character of \( S_p \) such that \( p - s \geq 2, s \geq 1 \) and \( p - s \neq s + 1 \). Then there exist \( \sigma \in S_p \) and a transposition \( (i, i + 1) \in S_p \) such that \( \chi(\sigma) \neq 0 \) and
\[
\chi(\sigma(i, i + 1)) = -\chi(\sigma).
\]

Proof. Let \( D \) be the Young diagram associated with the partition \( (p - s, 1^s) \). We divide the proof into two cases:

1. \( p - s > s + 1 \).
   Consider the permutation of \( S_p \), \((1, \ldots, p - s - 1)\). Since this cycle is a cycle with length \( p - s - 1 \) and \( p - s > s + 1 \) then \( p - s - 1 > s \). So, there is a unique way of omitting \( p - s - 1 \) boundary boxes of \( D \). If \( D_1 \) is the Young diagram obtained, then \( D_1 = [(1^{s+1})] \). Therefore, by Murnaghan–Nakayama rule we have
   \[
   \chi((1, \ldots, p - s - 1)(p - s, p - s + 1)) = -1 = -\chi((1, \ldots, p - s - 1)).
   \]

2. \( p - s < s + 1 \).
   Since \( p - s - 1 < s \), there is a unique way of omitting \( s \) boundary boxes of \( D \). Let \( D_2 \) be the Young diagram obtained, then \( D_2 = [(p - s)] \). Therefore, by the Murnaghan–Nakayama rule we have
   \[
   \chi((1, \ldots, s)(s + 1, \ldots, p)(s, s + 1)) = (-1)^s = -\chi((1, \ldots, s)(s + 1, \ldots, p)).
   \]

Lemma 3.9. Let \( \chi = (p, 2, 1^l) \) be the irreducible character of \( S_m \) such that \( p \geq 2 \) and \( p \neq l + 2 \). Then there exist \( \sigma \in S_m \) and a transposition \( (i, i + 1) \in S_m \) such that \( \chi(\sigma) \neq 0 \) and
\[
\chi(\sigma(i, i + 1)) = -\chi(\sigma).
\]

Proof. Let \( D \) be the Young diagram associated with the partition \( (p, 2, 1^l) \). We divide the proof into two cases:

1. \( p > l + 2 \).
   Consider the permutation of \( S_m \), \((1, \ldots, p)\). Since this cycle is a cycle with length \( p \) and \( p > l + 2 \) then there is a unique way of omitting \( p \) boundary boxes of \( D \). If \( D_1 \) is the Young diagram so obtained, then \( D_1 = [(1^{l+2})] \). Therefore, by the Murnaghan–Nakayama rule we have
   \[
   \chi((1, \ldots, p)(p + 1, p + 2)) = -1 = -\chi((1, \ldots, p)).
   \]

2. \( p < l + 2 \).
   Consider the permutation \( \rho = (1, \ldots, l + 2) \) with length \( l + 2 \). So, there is a unique way of omitting \( l + 2 \) boundary boxes of \( D \). Let \( D_2 \) be the Young diagram so obtained, then \( D_2 = [(p)] \). Because \( l + p + 1 \) is the number of the boundary
boxes of $D$ and is the length of the permutation $\rho(l + 3, \ldots, l + p + 1)(l + 2, l + 3)$, by the Murnaghan–Nakayama rule we have
\[
\chi(\rho(l + 3, \ldots, l + p + 1)(l + 2, l + 3)) = -l^{l+1} = -\chi(\rho(l + 3, \ldots, l + p + 1)).
\]
□

**Lemma 3.10.** Let $\chi = (p + a, p + 1, 1^l)$ be the irreducible character of $S_m$ such that $p \geq 2$ and $a \geq 1$. Then there exist $\sigma \in S_m$ and a transposition $(i, i + 1) \in S_m$ such that $\chi(\sigma) \neq 0$ and $\chi(\sigma(i, i + 1)) = -\chi(\sigma)$.

**Proof.** Again, the proof involves an exhaustive consideration of cases which we list:

1. $l + 1 = a$.
   1.1) $l \geq 1$, $p \neq l + 2$,
   1.2) $l \geq 1$, $p = l + 2$, $l$ even,
   1.3) $l \geq 1$, $p = l + 2$, $l$ odd,
   1.4) $l = 0$, $p = 2$,
   1.5) $l = 0$, $p = 3$,
   1.6) $l = 0$, $p \geq 4$.
2. $l + 1 < a$.
   2.1) $p \neq l + 2$,
   2.2) $p = l + 2$.
3. $l + 1 > a$.

Details will be omitted. □

**Proof of Theorem 3.1.** **Sufficiency.** If $\chi$ is self-associated then $\chi(\pi) = 0$ whenever $\epsilon(\pi) = -1$. Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$. Suppose there exists a transposition $\rho \in S_n$ such that $\chi(\sigma \rho) = -\chi(\sigma)$.

Since $\chi(\sigma) \neq 0$ then $\epsilon(\sigma) = 1$. Consequently, $\epsilon(\sigma \rho) = -1$ and $\chi(\sigma \rho) = 0$. Contradiction. Therefore, $(S_n)^T = \{id\}$ and using Theorem 2.2 we have, $\bigcup_{\sigma \in S_n, \chi(\sigma) \neq 0} V_\sigma(S_n, \chi) = \{I_n\}$.

Since if $\chi = (m)$ then $\chi(\rho) = 1$ for all $\rho \in S_n$ and so $(S_n)^T = \{id\}$. Using Theorem 2.2 we have, $\bigcup_{\sigma \in S_n, \chi(\sigma) \neq 0} V_\sigma(S_n, \chi) = \{I_n\}$.

**Necessity.** Suppose that $\chi \neq (m)$ and $\chi$ is not self-associated. Let $D$ be the Young diagram associated with the partition $\alpha = \chi$. Because $\chi$ is not self-associated, $D$ is not symmetric. If not all boxes of $D$ are boundary boxes, we do the following procedure:
Procedure

Let \( U \) be the Young diagram associated with the partition \((a_1, \ldots, a_p)\) with length \( p \) such that \( U \) is not symmetric and not all boxes of \( U \) are boundary boxes. Let \( t \) be the largest integer such that \( a_t > 1 \).

We omit all the boundary boxes of \( U \), if the diagram so obtained is not symmetric.

We omit \( \max\{a_2 + p - 2, a_1 + t - 2\} \) boxes of \( U \), otherwise.

Using Lemma 3.5 we see that the diagram obtained in each step of the procedure is not symmetric.

Repeat this procedure until the Young diagram so obtained, \( P \), has all boxes in the boundary. This diagram is associated with one of the following partitions:

(1) \((1^p)\) with \( p \geq 2 \),
(2) \((p)\) with \( p \geq 2 \),
(3) \((p - s, 1^s)\) with \( p - s \geq 2 \), \( s \geq 1 \) and \( p - s \neq s + 1 \).

The result now follows from an exhaustive consideration of cases. \( \square \)

Lemma 3.11. Let \( \chi = (r, 1^{n-r}) \) be the irreducible character of \( S_n \) with \( r > 1 \), \( n \geq 3 \) and \( n - r \neq 0 \). Let \( \chi_1 = (r - 1, 1^{n-r}) \) and \( \chi_2 = (r, 1^{n-r-1}) \) be irreducible characters of \( S_{n-1} \). Then

\[ \chi(id) = \chi_1(id) + \chi_2(id). \]

Proof. Immediate from the Murnaghan–Nakayama rule. \( \square \)

Given two partitions \( \rho = (\rho_1, \ldots, \rho_r) \), \( \pi = (\pi_1, \ldots, \pi_t) \) of \( n \), the symbol \( \rho > \pi \) denotes strict majorization that is, \( r \leq t \)

\[ \sum_{i=1}^{k} \rho_i \geq \sum_{i=1}^{k} \pi_i, \quad 1 \leq k \leq r, \]

and there exists \( j \leq r \) such that

\[ \sum_{i=1}^{j} \rho_i > \sum_{i=1}^{j} \pi_i. \]

Lemma 3.12. Let \( \chi = (s, 1^{n-s}) \), \( \lambda = (t, 1^{n-t}) \) be distinct irreducible characters of \( S_n \) with \( n \geq 6 \). If \( \chi' > \lambda > \chi \)

then

\[ \lambda(id) > \chi(id) + 2. \]
Proof. We proof the result by induction on $n$. When $n = 6$, we have 6 irreducible characters of $S_6$ satisfying the conditions of the lemma, $\lambda_1 = (6)$, $\lambda_2 = (5, 1)$, $\lambda_3 = (4, 1^2)$, $\lambda_4 = (3, 1^3)$, $\lambda_5 = (2, 1^4)$, $\lambda_6 = (1^6)$.

In this case,

\[
\begin{align*}
\lambda'_6(id) &= \lambda_1(id) = 1 = \lambda'_1(id) = \lambda_6(id), \\
\lambda'_5(id) &= \lambda_2(id) = 5 = \lambda'_2(id) = \lambda_5(id), \\
\lambda'_4(id) &= \lambda_3(id) = 10 = \lambda'_3(id) = \lambda_4(id)
\end{align*}
\]

and

\[
\lambda'_6 > \lambda'_5 > \lambda'_4 > \lambda_4 > \lambda_5 > \lambda_6.
\]

Therefore, if $\lambda'_i > \lambda'_j > \lambda_i$ for $i, j \in \{4, 5, 6\}$, then $\lambda_j(id) > \lambda_i(id) + 2$.

So, suppose that the result is true for $n = k$. Then, if $\chi$ and $\lambda$ are two distinct irreducible characters of $S_k$, with $k \geq 6$, satisfying the conditions of the lemma, such that $\chi > \lambda > \chi$ then $\lambda(id) > \chi(id) + 2$.

Let $\alpha = (a, 1^{k+1-a})$, $\beta = (b, 1^{k+1-b})$ be two distinct irreducible characters of $S_{k+1}$ such that

\[
\alpha' > \beta > \alpha
\]

then, $k + 2 - a > b > a$. Since $a \geq 1$, then $b > a \geq 1$. Therefore, $b > 1$.

But $k + 2 - a \leq k + 2 - 1 = k + 1$. Since $b < k + 2 - a \leq k + 1$ then $b < k + 1$. This is $k + 1 - b \neq 0$. Using Lemma 3.11 we have

\[
\beta(id) = \beta_1(id) + \beta_2(id),
\]

where $\beta_1 = (b, 1^{k-b})$ and $\beta_2 = (b - 1, 1^{k+1-b})$.

We have to consider several cases:

1. $a = 1$. In this case, $\alpha = (1^{k+1})$. Let $\alpha_1 = (1^k)$.

   1.1 If $b < k$, then $\alpha'_1 > \beta_1 > \alpha_1$. Using the induction hypothesis we have $\beta_1(id) > \alpha_1(id) + 2 = 1 + 2$. Therefore, $\beta(id) > \beta_1(id) > 1 + 2 = \alpha(id) + 2$.

   1.2 If $b = k$, since $k \geq 6$, then $\alpha'_1 > \beta_2 > \alpha_1$. Using the induction hypothesis we have $\beta_2(id) > \alpha_1(id) + 2 = 1 + 2$. Therefore, $\beta(id) > \beta_2(id) > 1 + 2 = \alpha(id) + 2$.

2. $a > 1$. Let $\alpha_2 = (a, 1^{k-a})$, $\alpha_3 = (a - 1, 1^{k+1-a})$ be irreducible characters. Since

   \[
   k + 2 - a > b > a \text{ and then } k + 2 - a > b - 1 > a - 1 \text{ and } \alpha'_1 > \beta_2 > \alpha_3, \text{ then } k + 1 - a \geq b > a.
   \]

   2.1 If $b = k + 1 - a$, then $\alpha_2(id) = \beta'_1(id) = \beta_1(id)$. By induction hypothesis,

   \[
   \beta(id) = \beta_1(id) + \beta_2(id) > \alpha_2(id) + \alpha_3(id) + 2.
   \]

   Because $a < b < k + 1$, using Lemma 3.11, we get $\alpha_2(id) + \alpha_3(id) = \alpha(id)$.

   Therefore, $\beta(id) > \alpha(id) + 2$.
(2.2) If \( b < k + 1 - a \) then \( \alpha' > \beta > \alpha_2 \). By induction hypothesis,
\[
\beta(id) = \beta_1(id) + \beta_2(id) > \alpha_2(id) + 2 + \alpha_3(id) + 2 > \alpha(id) + 2.
\]
So we get the result. □

Remark. For \( n = 5 \) the irreducible characters of \( S_5 \), satisfying the conditions of Lemma 3.12, are \( \lambda_1 = (5) \), \( \lambda_2 = (4, 1) \), \( \lambda_3 = (3, 1^2) \), \( \lambda_4 = (2, 1^3) \), \( \lambda_5 = (1^5) \). In this case we have
\[
\begin{align*}
\lambda_1(id) &= 1 = \lambda_5(id), \\
\lambda_2(id) &= 4 = \lambda_4(id), \\
\lambda_3(id) &= 6.
\end{align*}
\]
For \( n = 4 \) the irreducible characters of \( S_4 \), satisfying the conditions of Lemma 3.12, are \( \lambda_1 = (4) \), \( \lambda_2 = (3, 1) \), \( \lambda_3 = (2, 1^2) \), \( \lambda_4 = (1^4) \). In this case,
\[
\begin{align*}
\lambda_1(id) &= 1 = \lambda_4(id), \\
\lambda_2(id) &= 3 = \lambda_3(id).
\end{align*}
\]
For \( n = 3 \) the irreducible characters of \( S_3 \), satisfying the conditions of Lemma 3.12, are \( \lambda_1 = (3) \), \( \lambda_2 = (2, 1) \), \( \lambda_3 = (1^3) \). In this case,
\[
\begin{align*}
\lambda_1(id) &= 1 = \lambda_3(id), \\
\lambda_2(id) &= 2.
\end{align*}
\]
For \( n = 2 \) the irreducible characters of \( S_2 \), satisfying the conditions of Lemma 3.12, are \( \lambda_1 = (2) \), \( \lambda_2 = (1^2) \). In this case,
\[
\lambda_1(id) = 1 = \lambda_2(id).
\]
For \( n = 1 \) the irreducible character of \( S_1 \), satisfying the conditions of Lemma 3.12, is \( \lambda_1 = (1) \) and \( \lambda_1(id) = 1 \).

Let \( \chi \) be an irreducible character of \( S_n \) and \( \sigma \in S_n \) such that \( \chi(\sigma) \neq 0 \). Denote by \( R_\sigma \) the set of transpositions \( (a, b) \in S_n \) such that
\[
\chi(\sigma(a, b)) = -\chi(\sigma).
\]

**Proposition 3.13.** Let \( \chi = (s, 1^{n-s}) \) be the irreducible character of \( S_n \) where
\[
1. s - 1 > n - s \geq 1, \\
2. 2s \geq 5 \text{ and } s \text{ is odd then } 2(n - s) \neq s - 1, \\
3. \text{if } s = 6 \text{ then } n \notin [9, 10].
\]
Let \( (a, b) \) be a transposition of \( S_n \) and \( \sigma \in S_n \) be a cycle of length \( s - 1 \) such that \( \chi(\sigma) \neq 0 \). Then,
\[
(a, b) \in R_\sigma \text{ if and only if } \sigma(a) = a, \sigma(b) = b.
\]
Proof. Sufficiency. If \( \sigma(a) = a \) and \( \sigma(b) = b \) then \( \sigma \) and \( (a, b) \) are disjoint permutations. Since the length of \( \sigma \) is \( s-1 \) and \( s-1 > n-s \), using the Murnaghan–Nakayama rule, \( \chi(\sigma) = 1 \) and \( \chi(\sigma(a, b)) = -1 \). So, \( (a, b) \in R_\sigma \).

Necessity is proved by contradiction through the consideration of several cases and subcases which we list:

1. \( \sigma(a) \neq a \) and \( \sigma(b) = b \),
2. \( \sigma(a) = a \) and \( \sigma(b) \neq b \),
3. \( \sigma(a) \neq a \) and \( \sigma(b) \neq b \). In this case, \( \sigma(a, b) = \sigma_1\sigma_2 \) where \( \sigma_1, \sigma_2 \) are two disjoint cycles, \( \sigma_i \) with length \( r \), \( \sigma_2 \) with length \( t \) and \( r + t = s - 1 \), \( r \geq 1 \), \( t \geq 1 \), \( r \equiv t \).

(3.1) \( t = 1 \). Since \( s-1 > n-s \) then \( s-2 \geq n-s \).

(3.1.1) \( s-2 > n-s \),

(3.1.2) \( s-2 = n-s \),

(3.2) \( t > 1 \).

(3.2.1) \( r \geq t > n-s \),

(3.2.2) \( r > n-s \geq t \),

(3.2.3) \( n-s \geq r = t \),

(3.2.4) \( n-s \geq r > t \).

Details are omitted. \( \square \)

Proof of Theorem 3.2. Since \( n > 3 \) then, if \( n-1 \geq 5 \) and \( n-1 \) is odd we have \( 2(n-(n-1)) = 2 \neq n-2 \). If \( n-1 = 6 \) then \( n \notin [9, 10] \). Using Proposition 3.13, if \( (a, b) \) is a transposition of \( S_n \), then \( (a, b) \in R_\sigma \) if and only if \( \sigma(a) = a \) and \( \sigma(b) = b \).

Since \( \sigma \) is a cycle of length \( n-2 \), there are only two integers \( u, v \in \{1, \ldots, n\} \), \( u > v \) such that \( \sigma(u) = u \) and \( \sigma(v) = v \). Consequently, \( R_\sigma = \{(u, v)\} \) and \( (S_n)^T = (\langle u, v \rangle) \).

Necessity. Suppose that \( L = \{l_{ij}\} \in V_\sigma(S_n, \chi) \). By Theorem 2.2, if \( a > b, a, b \in \{1, \ldots, n\} \) and \( \sigma(a) \neq a \) or \( \sigma(b) \neq b \) then \( l_{ab} = 0 \).

Suppose there exists an integer \( k \) such that \( u > k > v \) and \( \sigma(k) \neq k \). Let \( Z \) be the matrix whose \((v+1)\)th column is the \( v \)th column of \( I_n \) and the \( u \)th column of \( Z \) is the \((v+1)\)th column of \( I_n \), the remaining columns of \( Z \) are the columns of \( I_n \). Then

\[
\chi^S(P(\sigma)LZ) = (\chi(\sigma^{-1}(v+1, u))) + (\chi(\sigma^{-1}(v+1, u, v)))l_{uv}.
\]

Since \( \sigma^{-1}(v+1) \neq v+1 \) and \( \sigma^{-1}(u) = u \) then \( \sigma^{-1}(v+1, u) \) is a cycle of length \( n-1 \). Using the Murnaghan–Nakayama rule,

\[
\chi(\sigma^{-1}(v+1, u)) = 0.
\]

But \( \chi(\sigma^{-1}(v+1, u, v)) \) is a cycle of length \( n \), then \( \chi(\sigma^{-1}(v+1, u, v)) = -1 \). Therefore,

\[
\chi^S(P(\sigma)LZ) = -l_{uv}.
\]

Since \( L \in V_\sigma(S_n, \chi) \),

\[
-l_{uv} = d^S(\sigma)p(\sigma)Z = d^S(\sigma)Z = 0.
\]

Consequently, \( l_{uv} = 0 \) and we have the condition.
Sufficiency. Let $L = [l_{ij}]$ be a matrix satisfying the condition of the theorem. Then
\[ L = \begin{cases} 
I_n & \text{if } u \neq v + 1, \\
I_n + E^v + l_{a,s}(u) & \text{if } u = v + 1.
\end{cases} \]

Let $X \in T^n$. If $u \neq v + 1$, $a^n (P(\sigma)LX) = d^n_s (P(\sigma)LX) = d^n_s (P(\sigma)X)$. If $u = v + 1$, $a^n (P(\sigma)LX) = \chi(\sigma^{-1}) \prod_{s=1}^n x_{ss} = d^n_s (P(\sigma)X)$. Consequently, $L \in V_\sigma(S_n, \chi)$. $\square$

**Proof of Theorem 3.3.** Using Proposition 3.13, we see that $(S_n)^\chi_\pi$ is generated by those transpositions, $(a \ b)$ such that $\sigma (a) = a$ and $\sigma (b) = b$. Consequently, if $\pi \in (S_n)^\chi_\pi$, $\pi$, $\sigma^{-1}$ are disjoint permutations and by Murnaghan–Nakayama rule we have, $\chi (\sigma^{-1}) = \epsilon (\pi) \chi (\sigma^{-1}) = \epsilon (\pi)$.

**Necessity.** Using Theorem 2.2, if $a > b$, $a$, $b \in \{1, \ldots, n\}$ and $\sigma (a) \neq a$ or $\sigma (b) \neq b$ then $l_{ab} = 0$.

Suppose that $i > j$, $i, j \in \{1, \ldots, n\}$ and there exists $i > k > j$ such that $\sigma (k) \neq k$. We are going to prove that $l_{ij} = 0$.

Using the hypothesis of the theorem, then $j < u$ and $i > u + s - 2$. Let $t$, $f$ two integers, $t, f \in \{u, \ldots, u + s - 2\}$ such that $t < f$ and $\sigma (t) = f$. We are seeing that $l_{u + s - 1 \ u - 1} = 0$.

Let $Z$ be the matrix those $r$th column is the $(u - 1)$th column of $I_n$, the $f$th column of $Z$ is the $r$th column of $I_n$ and the $(u + s - 1)$th column of $Z$ is the $f$th column of $I_n$, the remaining columns of $Z$ are the columns of $I_n$. Then

\[ d^n_s (P(\sigma)LZ) = \chi (\sigma^{-1}(t, \ f, \ u + s - 1)) \]
\[ + \chi (\sigma^{-1}(u - 1, \ t, \ f, \ u + s - 1)))l_{u + s - 1 \ u - 1}. \]

Since $\sigma (t) = f$ and $\sigma^{-1}(t, \ f, \ u + s - 1)$ is a cycle with length $s - 1$, using the Murnaghan–Nakayama rule,

\[ \chi (\sigma^{-1}(t, \ f, \ u + s - 1)) = 1. \]

But $\chi (\sigma^{-1}(u - 1, \ t, \ f, \ u + s - 1))$ is a cycle with length $s$ and $n - s \geq 1$, then $\chi (\sigma^{-1}(u - 1, \ t, \ f, \ u + s - 1)) = 0$. Since $L \in V_\sigma(S_n, \chi)$,

\[ -l_{u + s - 1 \ u - 1} = d^n_s (P(\sigma)LZ) = d^n_s (P(\sigma)Z) = 0. \]

Consequently, $l_{u + s - 1 \ u - 1} = 0$.

Now, let $Z$ be the matrix those $r$th column is the $(u - 2)$th column of $I_n$, the $f$th column of $Z$ is the $r$th column of $I_n$ and the $(u + s - 1)$th column of $Z$ is the $f$th column of $I_n$, the remaining columns of $Z$ are the columns of $I_n$. Then we can conclude that $l_{u + s - 1 \ u - 2} = 0$. In this way we can show that $l_{u + s - 1 \ u} = \cdots = l_{u + s - 1 \ u - 1} = 0$.

Next, in the same way we prove that $l_{u + s - 1 \ u + 1} = \cdots = l_{u + s - 1 \ u + s - 1} = 0$.

Therefore we can conclude that $l_{ij} = 0$. 

Sufficiency. Let $L$ be a matrix satisfying the condition of the Theorem. Then
\[
L = \begin{bmatrix}
L_1 & 0 & 0 \\
0 & L_2 & 0 \\
0 & 0 & L_3
\end{bmatrix},
\]
where $L_1 \in T^L_{u-1}(F)$ with diagonal elements equal to 1, $L_2 = I_{n-2}$ and $L_3 \in T^L_{n-u-s+3}(F)$ with diagonal elements equal to 1. Let $Z \in T^U_u(F)$,
\[
Z = \begin{bmatrix}
Z_1 & 0 & 0 \\
0 & Z_2 & 0 \\
0 & 0 & Z_3
\end{bmatrix},
\]
where $Z_1 \in T^U_{u-1}(F)$, $Z_2 \in T^U_{n-2}(F)$ and $Z_3 \in T^U_{n-u-s+3}(F)$. Since $\chi(\sigma^{-1}) = 1$ and $\chi(\sigma^{-1} \rho) = \epsilon(\rho)$ if $\rho \in (S_n)_T$, then
\[
d^S_{\chi}(P(\sigma)LZ) = \chi(\sigma^{-1}) (\det(L_1 Z_1) \det(Z_2) \det(L_3 Z_3)) = \chi(\sigma^{-1}) \det(Z).
\]
Then $L \in V_\sigma(S_n, \chi)$. □

Remark. With the same arguments we have used before, we can also prove the lower triangular matrices version of this paper.

References