Available online at www.sciencedirect.com

ScienceDirect

Theoretical Computer Science 391 (2008) 75–89

Theoretical Computer Science

www.elsevier.com/locate/tcs

Shuffle operations on discrete paths☆

S. Brlek, G. Labelle, A. Lacasse*

LaCIM, Université du Québec à Montréal, C.P. 8888 Succursale "Centre-Ville", Montréal (QC), Canada H3C 3P8

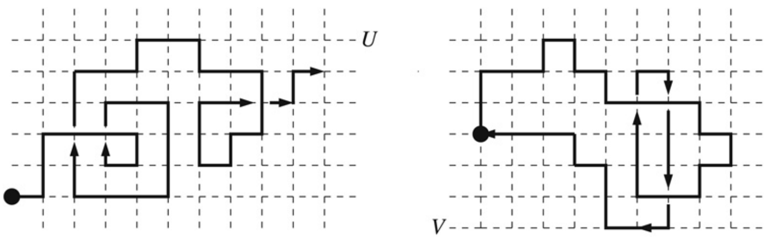
Abstract

We consider the shuffle operation on paths and study some parameters. In the case of square lattices, shuffling with a particular periodic word (of period 2) corresponding to paperfoldings reveals some characteristic properties: closed paths remain closed; the area and perimeter double; the center of gravity moves under a 45° rotation and a $\sqrt{2}$ zoom factor. We also observe invariance properties for the associated Dragon curves. Moreover, replacing square lattice paths by paths involving $2k\pi/N$-turns, we find analogous results using more general shuffles.

Keywords: Lattice paths; Polygonal paths; Discrete regions; Shuffle; Dragon curve

1. Introduction

We consider first the square lattices, identified with the discrete plane $\mathbb{Z} \times \mathbb{Z}$. A path in the square lattice is a polygonal path made of the elementary unit translations $a = (1, 0), \overline{a} = (-1, 0), b = (0, 1), \overline{b} = (0, -1)$. A finite path w is therefore a word on the alphabet $\Sigma = \{a, \overline{a}, b, \overline{b}\}$, also known as the Freeman chain code [6,7]. For instance the open path (a) in the figure below is coded by the word



$$U = abbaaa\overline{b}\overline{a}bbaa\overline{b}\overline{b}\overline{b}\overline{a}\overline{a}\overline{a}bbbbaabaa\overline{b}aa\overline{b}\overline{b}\overline{a}\overline{b}\overline{a}bbaaaba,$$

while the closed path (b) in the above figure is coded by the word

$$V = bbaaba\overline{b}a\overline{b}aaa\overline{b}a\overline{b}\overline{a}\overline{b}\overline{a}\overline{a}bbbba\overline{b}\overline{b}\overline{b}\overline{b}\overline{a}\overline{a}bb\overline{a}b\overline{a}\overline{a}\overline{a}.$$

☆ With the support of NSERC (Canada).

* Corresponding author.
E-mail addresses: brlek@lacim.uqam.ca (S. Brlek), labelle.gilbert@lacim.uqam.ca (G. Labelle), annie.lacasse@gmail.com (A. Lacasse).

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2007.10.032

Observe that crossings are quite ambiguous in the geometric representation. Fortunately, the codings provided by the words representation are not.

The paper is organized as follows. We introduce in Section 2.1 a notion of perfect shuffle on square lattice paths encoded by their sequence of turns. This encoding has been used by the authors in [2] for providing an elementary proof of a geometrical result obtained by Daurat and Nivat [4]. Section 2.2 deals with dragon-like curves associated with this kind of shuffles. Section 2.3 is devoted to the study of some parameters related to iterative steps towards dragon curves. Finally, in Section 3, we generalize our results to other kinds of shuffle operations on paths involving $2k\pi/N$-turns, $k = 0, \ldots, N-1$. Special emphasis is put on the behavior of area and center of gravity under shuffle operations.

2. Shuffles on square lattice paths

From now on, a path w of length n, $|w| = n$, is a function $w : [1..n] \longrightarrow \Sigma$ and is written $w = w_1 w_2 \cdots w_n$ where w_i is the ith letter, $1 \le i \le n$. Also, the number of occurrences of a given letter α in the word w is denoted $|w|_\alpha$. The set of n-length paths over Σ is denoted Σ^n, the set of all paths is Σ^*, and for later use $\Sigma^{\ge 2}$ is the set of paths of length at least 2. Note that the *reversal* $\widetilde{w}$ of $w \in \Sigma^n$ is the unique word satisfying $\widetilde{w}_i = w_{n-i+1}$, $1 \le i \le n$. Moreover, we recall the usual *shuffle product* (see [8]). Let u and v be two elements of Σ^* written as $u = u_1 u'$ and $v = v_1 v'$ where u_1 and v_1 belong to Σ. Then one recursively defines the shuffle product $u \sqcup\!\sqcup v$ of u and v by

$$u \sqcup\!\sqcup v = u_1(u' \sqcup\!\sqcup v) + v_1(u \sqcup\!\sqcup v'),$$

with $\epsilon \sqcup\!\sqcup w = w \sqcup\!\sqcup \epsilon = w$, for every $w \in \Sigma^*$. For example, taking $u = ab$ and $v = aab$, one can check that $(ab) \sqcup\!\sqcup (aab) = abaab + 3aabab + 6aaabb$.

2.1. Perfect shuffles and word of turns

Definition 1. The *perfect shuffle* $\sqcup\!\sqcup^* : \Sigma^{n+1} \times \Sigma^n \longrightarrow \Sigma^*$ is defined by $u \sqcup\!\sqcup^* v = u_1 v_1 u_2 v_2 \cdots u_n v_n u_{n+1}$.

On a square grid, a path can also be described by a sequence of left and right turns along with forward and backward steps. Using the alphabet Σ of unit steps, we define the corresponding set of movements by

$$\begin{aligned}
V_L &= \{ab, b\overline{a}, \overline{a}\overline{b}, \overline{b}a\} && \text{is the set of left turns;}\\
V_R &= \{ba, a\overline{b}, \overline{b}\overline{a}, \overline{a}b\} && \text{is the set of right turns;}\\
V_F &= \{aa, \overline{aa}, bb, \overline{bb}\} && \text{is the set of forward steps;}\\
V_B &= \{a\overline{a}, \overline{a}a, b\overline{b}, \overline{b}b\} && \text{is the set of backward steps.}
\end{aligned}$$

These sets correspond respectively to the basic movements in the left (L), right (R), forward (F) and back (B) directions.

Note that each path $w = w_1 w_2 \cdots w_n$ is completely determined up to translation by its initial step and a word called the *word of turns* on the alphabet $\Sigma_d = \{L, F, R, B\}$. Indeed let $g : \Sigma^2 \longrightarrow \Sigma_d$, be defined by

$$g(u) = \begin{cases} L & \text{if } u \in V_L,\\ F & \text{if } u \in V_F,\\ R & \text{if } u \in V_R,\\ B & \text{if } u \in V_B. \end{cases} \tag{1}$$

Then g is extended to a function on arbitrary paths $\varphi : \Sigma^{\ge 2} \longrightarrow \Sigma \times \Sigma_d^*$ defined by the sequence

$$\varphi(w) = \left(w_1, \prod_{k=1}^{n-1} g(w_k w_{k+1}) \right), \tag{2}$$

where n is the length of the word w and the product is the concatenation. Since we are only interested in the geometric properties of paths, we drop the starting step (which amounts to work with the equivalence class determined by the cyclic permutations). The following lemma is straightforward.

Lemma 2. *Let $w \in \Sigma^*$ be a closed path, then*
(i) *w is of even length: $w = w_1 w_2 \cdots w_{2n}$ for some integer $n \geq 1$;*
(ii) *$\varphi(w)$ is of odd length.*

2.2. Dragon curve

We first define a word associated with a general closed polygonal path in the complex plane.

Definition 3. Let $u = (u_0, u_1, \ldots, u_{m-1})$, $u_k = x_k + \mathrm{i}y_k$, be the vertex sequence of a closed polygonal path in the complex plane, starting at u_0. The word of turns of u is defined by the word of complex numbers $d_1 d_2 \cdots d_{m-1}$ given by

$$d_1 = \frac{\Delta u_1}{\Delta u_0}, \qquad d_2 = \frac{\Delta u_2}{\Delta u_1}, \qquad d_3 = \frac{\Delta u_3}{\Delta u_2}, \ldots, \qquad d_{m-1} = \frac{\Delta u_{m-1}}{\Delta u_m}, \tag{3}$$

where $\Delta u_k = u_{k+1} - u_k$ and $u_m = u_0$ by convention.

In other words, considering the normalization $u_0 = 0$ and $u_1 = 1$ we can write

$$\begin{array}{ll} u_0 = 0, & \Delta u_0 = 1, \\ u_1 = 1, & \Delta u_1 = d_1, \\ u_2 = 1 + d_1, & \Delta u_2 = d_1 d_2, \\ u_3 = 1 + d_1 + d_1 d_2, & \Delta u_3 = d_1 d_2 d_3, \\ \cdots & \cdots \\ u_{m-1} = 1 + d_1 + \cdots + d_1 d_2 \cdots d_{m-2} & \Delta u_{m-1} = d_1 d_2 \cdots d_{m-1}. \end{array} \tag{4}$$

In the present context, let $u_k = x_k + \mathrm{i}y_k$, for $k = 0, 1, \ldots, 2n-1$, be the vertices of a closed lattice path made of horizontal or vertical unit steps. Let $w = w_1 w_2 \cdots w_{2n}$ be a word over the alphabet Σ of unit steps and let $\varphi(w) = d_1 d_2 \cdots d_{2n-1}$ be its coding on $\Sigma_d = \{L, R, F, B\}$ according to the bijection (2) describing the succession of turns in the boundary of the closed path.

Consider L as the complex number i (turn left 90°), R as the complex number $\bar{\mathrm{i}}$ (turn right 90°), F as the number 1 (forward, no turn) and B as the number -1 (backward, turn 180°). Then the above notion of perfect shuffle to the word of turns $d_1 d_2 \cdots d_{2n-1}$ gives a new word

$$(LR)^n \sqcup\!\sqcup^* d_1 d_2 \cdots d_{2n-1} = L d_1 R d_2 L d_3 \cdots L d_{2n-1} R = \mathrm{i} d_1 \bar{\mathrm{i}} d_2 \mathrm{i} d_3 \bar{\mathrm{i}} d_4 \cdots \mathrm{i} d_{2n-1} \bar{\mathrm{i}}$$

which is the word of turns of another polygonal path $(v_0, v_1, \ldots, v_{4n-1})$, denoted $S(u_0, u_1, \ldots, u_{2n-1}) = S(u)$, given explicitly by the following families of equations (using complex multiplication, induction and the fact that $\mathrm{i}\bar{\mathrm{i}} = 1$)

$$S: \begin{cases} v_{4k} = (1+\mathrm{i})u_{2k}, & \Delta v_{4k} = \Delta u_{2k}, \\ v_{4k+1} = (1+\mathrm{i})u_{2k} + \Delta u_{2k}, & \Delta v_{4k+1} = \mathrm{i}\Delta u_{2k}, \\ v_{4k+2} = (1+\mathrm{i})u_{2k+1}, & \Delta v_{4k+2} = \mathrm{i}\Delta u_{2k+1}, \\ v_{4k+3} = (1+\mathrm{i})u_{2k+1} + \mathrm{i}\Delta u_{2k+1}, & \Delta v_{4k+3} = \Delta u_{2k+1}, \end{cases} \tag{5}$$

for $k = 0, 1, \ldots, n-1$.

Define now an associated operator on closed paths whose iterates converge to the classical Dragon curves (see Fig. 1).

The shuffle may also be iterated and all iterations produce non-intersecting closed paths. We have the following shuffle iterations 4, 7 and 10 for the cross $RLLRLLRLLRL$ and LLL.

Definition 4. The Dragon operator is defined by

$$\mathfrak{D} = \frac{1}{1+\mathrm{i}} S.$$

More precisely,

$$\mathfrak{D}(u_0, u_1, \ldots, u_{2n-1}) = (w_o, w_1, \ldots, w_{4n-1}),$$

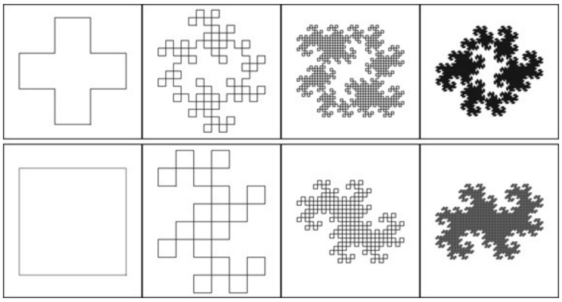


Fig. 1. Shuffle iterations on the cross and the square (up to a scaling factor).

where,

$$\mathfrak{D} : \begin{cases} w_{4k} = u_{2k}, & \Delta w_{4k} = \frac{1}{2}(1-\mathrm{i})\Delta u_{2k}, \\ w_{4k+1} = u_{2k} + \frac{1}{2}(1-\mathrm{i})\Delta u_{2k}, & \Delta w_{4k+1} = \frac{1}{2}(1+\mathrm{i})\Delta u_{2k}, \\ w_{4k+2} = u_{2k+1}, & \Delta w_{4k+2} = \frac{1}{2}(1+\mathrm{i})\Delta u_{2k+1}, \\ w_{4k+3} = u_{2k+1} + \frac{1}{2}(1+\mathrm{i})\Delta u_{2k+1}, & \Delta w_{4k+3} = \frac{1}{2}(1-\mathrm{i})\Delta u_{2k+1}, \end{cases} \tag{6}$$

for $k = 0, 1, \ldots, n-1$.

2.3. Geometric properties

The following version of Green's Theorem [3,9] is sufficient to start our analysis of some parameters associated with the operators S and $\mathfrak{D}$.

Theorem 5. *Let $P(x, y)$, $Q(x, y)$ be two continuously differentiable functions on an open set containing a simply connected region Ω bounded by simple piecewise continuously differentiable positively oriented curve Γ. Then*

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \int_{\Gamma} P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y.$$

We deduce the corresponding complex version of Green's Theorem.

Theorem 6. *For $u = x + \mathrm{i}y$, let $f(u) = A + \mathrm{i}B = A(x, y) + \mathrm{i}B(x, y)$. Then,*

$$\int_{\Gamma} f(u)\mathrm{d}u = -\iint_{\Omega} \left(\frac{\partial B}{\partial x} + \frac{\partial A}{\partial y} \right) \mathrm{d}x\mathrm{d}y + \mathrm{i} \iint_{\Omega} \left(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} \right) \mathrm{d}x\mathrm{d}y.$$

Proof. Left to the reader. □

Examples. For $f(u) = \bar{u} = x - \mathrm{i}y$, we have

$$\int_{\Gamma} \bar{u}\mathrm{d}u = \iint_{\Omega} (0+0)\mathrm{d}x\mathrm{d}y + \mathrm{i} \iint_{\Omega} (1-(-1))\mathrm{d}x\mathrm{d}y = 2\mathrm{i} \iint_{\Omega} \mathrm{d}x\mathrm{d}y = 2\mathrm{i}a,$$

where a denotes the (signed) area of Ω. It follows that

$$a = \frac{1}{2} \Im \int_{\Gamma} \bar{u}\mathrm{d}u, \tag{7}$$

$$c = \frac{\mathrm{i}}{4a} \overline{\int_{\Gamma} \bar{u}^2 \mathrm{d}u}, \tag{8}$$

$$m = \frac{1}{4} \Im \int_{\Gamma} |u|^2 \bar{u}\mathrm{d}u - |c|^2 a, \tag{9}$$

where c is the center of gravity, m is the moment of inertia of Ω and $\Im z$ is the imaginary part of z. More precisely, we have

$$a = \frac{1}{2}\int_{\Gamma}(x\mathrm{d}y - y\mathrm{d}x),$$

$$c = x^* + \mathrm{i}y^* = \frac{\iint_{\Omega} x\mathrm{d}x\mathrm{d}y}{\iint_{\Omega}\mathrm{d}x\mathrm{d}y} + \mathrm{i}\frac{\iint_{\Omega} y\mathrm{d}x\mathrm{d}y}{\iint_{\Omega}\mathrm{d}x\mathrm{d}y},$$

$$m = \iint_{\Omega}\left((x-x^*)^2 + (y-y^*)^2\right)\mathrm{d}x\mathrm{d}y = \iint_{\Omega}(x^2+y^2)\mathrm{d}x\mathrm{d}y - (x^{*2}+y^{*2})a.$$

Note that the center of gravity and the moment of inertia are defined only if $a \neq 0$.

To implement the complex version of Green Theorem in the context of a polygonal closed path having vertices $u_0, u_1, \ldots, u_{m-1}$ the integral $\int_{\Gamma} f(u)\mathrm{d}u$ can be evaluated as follows:

Lemma 7. *Define* $f^*(u,s) = \int_0^1 f(u+ts)\mathrm{d}t$. *Then*

$$\int_{\Gamma} f(u)\mathrm{d}u = \sum_{\nu=0}^{m-1} f^*(u_\nu, \Delta u_\nu)\Delta u_\nu.$$

Proof. Let $u = u_\nu + t\Delta u_\nu$ and $\mathrm{d}u = \Delta u_\nu \mathrm{d}t$. Then we have,

$$\int_{\Gamma} f(u)\mathrm{d}u = \sum_{\nu=0}^{m-1}\int_{[u_\nu,u_{\nu+1}]} f(u)\mathrm{d}u = \sum_{\nu=0}^{m-1}\int_0^1 (f(u_\nu + t\Delta u_\nu)\mathrm{d}t)\Delta u_\nu = \sum_{\nu=0}^{m-1} f^*(u_\nu, \Delta u_\nu)\Delta u_\nu. \quad \square$$

Using Lemma 7, the above formulas (7) and (8) for a and c take the following forms

$$a = \frac{1}{2}\Im\sum_{\nu=0}^{m-1}\overline{u_\nu}\Delta u_\nu, \tag{10}$$

$$c = \frac{\mathrm{i}}{4a}\sum_{\nu=0}^{m-1}\left({u_\nu}^2 + u_\nu\Delta u_\nu + \frac{1}{3}(\Delta u_\nu)^2\right)\overline{\Delta u_\nu}. \tag{11}$$

Theorem 8. *Let* $u = (u_0, u_1, \ldots, u_{2n-1})$ *be a closed path made of unit steps in the complex plane starting at* u_0. *Then the paths* $S(u)$ *and* $\mathfrak{D}(u)$ *are also closed. Moreover,*

(a) area $(S(u)) = 2\,\text{area}\,(u)$, (b) $c(S(u)) = (1+\mathrm{i})\cdot c(u)$,

(c) area $(\mathfrak{D}(u)) = \text{area}\,(u)$, (d) $c(\mathfrak{D}(u)) = c(u)$,

where S *and* $\mathfrak{D}$ *denote respectively the Shuffle and Dragon operators.*

Proof. The path u is closed if and only if $\sum_{\nu=0}^{2n-1}\Delta u_\nu = 0$. Let $v = S(u)$. It follows from (5) that

$$\sum_{\nu=0}^{4n-1}\Delta v_\nu = \sum_{k=0}^{n-1}(\Delta u_{2k} + \mathrm{i}\Delta u_{2k} + \Delta u_{2k+1} + \mathrm{i}\Delta u_{2k+1}) = (1+\mathrm{i})\sum_{\nu=0}^{2n-1}\Delta u_\nu = 0$$

which shows that $S(u)$ is closed. Similarly $\mathfrak{D}(u)$ is also closed.

We now prove (c). Let $a = \text{area}\,(u)$ and $A = \text{area}\,(\mathfrak{D}(u))$. Then by (10), applied to $w = \mathfrak{D}(u)$, we get using (6), that

$$A = \frac{1}{2}\Im\sum_{\nu=0}^{4n-1}\overline{w_\nu}\Delta w_\nu$$

$$= \frac{1}{2}\Im \sum_{k=0}^{n-1}\left(\overline{u_{2k}}\Delta u_{2k} + \overline{u_{2k+1}}\Delta u_{2k+1} + \frac{\mathrm{i}}{2}(|\Delta u_{2k}|^2 - |\Delta u_{2k+1}|^2)\right)$$

$$= \frac{1}{2}\Im \sum_{\nu=0}^{2n-1}\overline{u_{\nu}}\Delta u_{\nu} + \frac{1}{4}\sum_{k=0}^{n-1}\left(|\Delta u_{2k}|^2 - |\Delta u_{2k+1}|^2\right) = a$$

since $|\Delta u_{2k}| = |\Delta u_{2k+1}| = 1$ for $k = 0, 1, \ldots, n-1$.

Similarly, for (d), let c and C be the center of gravity of respectively u and $\mathfrak{D}(u)$. Let Γ be the perimeter of u and $\mathfrak{D}(\Gamma)$ the perimeter of $\mathfrak{D}(u)$. Then we obtain,

$$\begin{aligned}\int_{\mathfrak{D}(\Gamma)} \overline{w}^2 \mathrm{d}w - \int_{\Gamma} \overline{u}^2 \mathrm{d}u &= \sum_{k=0}^{n-1} \mathrm{i}\left(\overline{u_{2k}}|\Delta u_{2k}|^2 - \overline{u_{2k+1}}|\Delta u_{2k+1}|^2 + \frac{1}{2}|\Delta u_{2k}|^2\overline{\Delta u_{2k}} - \frac{1}{2}|\Delta u_{2k+1}|^2\overline{\Delta u_{2k+1}}\right)\\ &\quad - \frac{1}{6}\sum_{k=0}^{n-1}\left(|\Delta u_{2k}|^2\overline{\Delta u_{2k}} + |\Delta u_{2k+1}|^2\overline{\Delta u_{2k+1}}\right)\\ &= \sum_{k=0}^{n-1} \mathrm{i}\left(\overline{u_{2k}} - \overline{u_{2k+1}} + \frac{1}{2}\overline{\Delta u_{2k}} - \frac{1}{2}\overline{\Delta u_{2k+1}}\right) - \frac{1}{6}\sum_{k=0}^{n-1}\left(\overline{\Delta u_{2k}} + \overline{\Delta u_{2k+1}}\right)\end{aligned}$$

since $|\Delta u_{2k}| = |\Delta u_{2k+1}| = 1$. The last sum is 0 since the path is closed,

$$\sum_{k=0}^{n-1}\overline{\Delta u_{2k}} + \overline{\Delta u_{2k+1}} = \sum_{\nu=0}^{2n-1}\overline{\Delta u_{\nu}} = \overline{0} = 0.$$

The other sum is also 0 since it can be rewritten as

$$\frac{\mathrm{i}}{2}\sum_{k=0}^{n-1}(\overline{u_{2k}} - \overline{u_{2k+2}}) = \frac{\mathrm{i}}{2}((u_0 + u_2 + \cdots + u_{2n-2}) - (u_2 + \cdots + u_{2n-2} + u_{2n})) = 0$$

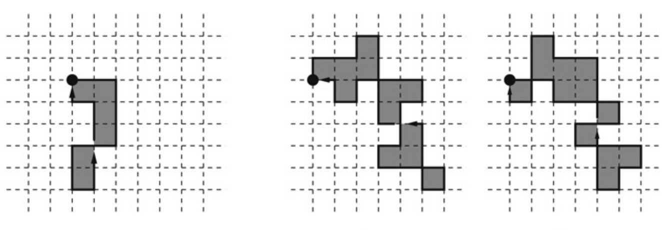
since $u_0 = u_{2n}$.

Hence, using the fact that $A = a$, we have

$$C = \frac{\mathrm{i}}{4A}\overline{\int_{\mathfrak{D}(\Gamma)} \overline{w}^2 \mathrm{d}w} = \frac{\mathrm{i}}{4A}\overline{\int_{\Gamma} \overline{u}^2 \mathrm{d}u} = \frac{\mathrm{i}}{4a}\overline{\int_{\Gamma} \overline{u}^2 \mathrm{d}u} = c.$$

Finally, (a) and (b) follow from the fact that $S(u) = (1+\mathrm{i})\mathfrak{D}(u)$ which shows that $S(u)$ is obtained from $\mathfrak{D}(u)$ using a 45° rotation and a zoom by a factor $\sqrt{2} = |1+\mathrm{i}|$. □

Example. Let $\varphi(w) = FRFFRFLFLLFFFLR$ with $a(\varphi(w)) = 6$. Then as shown below, we have $a((RL)^8 \sqcup\!\sqcup^* \varphi(w)) = a((LR)^8 \sqcup\!\sqcup^* \varphi(w)) = 12$.



$\varphi(w) = FRFFRFLFLLFFFLR$ $(RL)^8 \sqcup\!\sqcup^* \varphi(w)$ $(LR)^8 \sqcup\!\sqcup^* \varphi(w)$

3. Generalization to arbitrary shuffles and rational turns

In the previous section we dealt with 0°, 90°, 180°, 270° turns in square lattice paths and very special shuffles leading to the classical Dragon curves. We now work with more general words of turns and shuffles having angles $2k\pi/N$, $k = 0, \ldots, N-1$. The emphasis is now put on conditions preserving area and center of gravity of their associated Dragon operators.

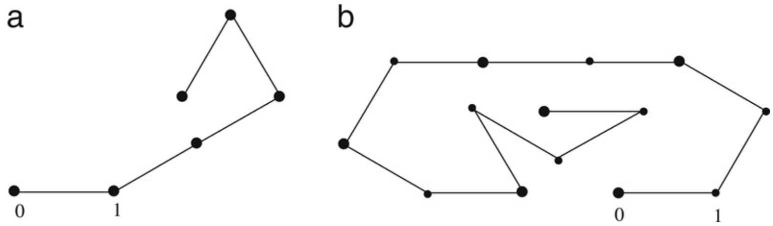


Fig. 2. (a) $d = \omega 1 \omega^3 \omega^4$ (b) $\delta \sqcup\!\sqcup^* d = \omega^2\omega^3 \cdot 1 \cdot \omega^2 \cdot \omega \cdot \omega^7\omega^2\omega^5 \sqcup\!\sqcup^* d$.

3.1. Shuffling words and paths on roots of unity alphabets

Let $\omega = e^{2\pi i/N}$ be a primitive N-th root of 1 and replace the above alphabets Σ and Σ_d by

$$\Sigma = \Sigma_d = \{1, \omega, \omega^2, \ldots, \omega^{N-1}\}.$$

The corresponding paths and words of turns are simply words on these new alphabets where formulas (3) and (4) hold and for which $u = (u_0, u_1, \ldots, u_{n-1})$ denotes the vertex sequence of the polygonal path. Note that these new paths are *equilateral* in the sense that all their sides have same length.

For instance, let $N = 12$ and $\Sigma = \Sigma_d = \{1, \omega, \omega^2, \ldots, \omega^{11}\}$ where $\omega = e^{2\pi i/12}$. Consider the word of turns $d = \omega 1 \omega^3 \omega^4 \in \Sigma_d^*$. Then the corresponding polygonal path is given by Fig. 2a.

Let $d_1 d_2 \cdots d_{n-1} \in \Sigma_d^*$ be the word of turns of a closed equilateral polygonal path with vertex sequence $u = (u_0, u_1, \ldots, u_{n-1})$. Consider a *fixed* factorization of $\delta \in \Sigma_d^*$ into subwords (possibly empty)

$$\delta = \delta_0 \cdot \delta_1 \cdot \delta_2 \cdots \delta_{n-1}, \quad \delta_k \in \Sigma_d^*, \tag{12}$$

where $\cdot$ denotes concatenation and each subword δ_k is given by

$$\delta_k = \delta_{k,1} \cdots \delta_{k,m_k}, \quad \delta_{k,j} \in \Sigma_d. \tag{13}$$

The factorization (12) gives rise to the new *perfect shuffle* $\sqcup\!\sqcup^*$ (a special term of the shuffle product $\sqcup\!\sqcup$) defined by

$$\delta \sqcup\!\sqcup^* d = \delta_0 \cdot d_1 \cdot \delta_1 \cdot d_2 \cdot \delta_2 \cdot d_3 \cdots \delta_{n-2} \cdot d_{n-1} \cdot \delta_{n-1}.$$

Assuming the normalization $u_0 = 0$, $u_1 = 1$ this new word is the word of turns of another polygonal path $v = S(u)$ with $v_0 = 0$ and $v_1 = 1$.

For example, let $\Sigma_d = \{1, \omega, \omega^2, \ldots, \omega^{11}\}$ and consider the word $d = \omega 1 \omega^3 \omega^4 \in \Sigma_d^*$. Under the factorization $\delta = \omega^2\omega^3 \cdot 1 \cdot \omega^2 \cdot \omega \cdot \omega^7\omega^2\omega^5$, we have

$$\delta \sqcup\!\sqcup^* d = (\omega^2\omega^3) \cdot \omega \cdot (1) \cdot 1 \cdot (\omega^2) \cdot \omega^3 \cdot (\omega) \cdot \omega^4 \cdot (\omega^7\omega^2\omega^5)$$

which corresponds to the path of Fig. 2b. We observe that in Fig. 2b big dots (except the first one, which is always 0) correspond to the reading of the d_k's in the word $\delta \sqcup\!\sqcup^* d$. As we can see, the polygon associated with these big dots has not the same shape as the original polygonal path of Fig. 2a.

3.2. General shuffle and dragon operators

In the present section we develop conditions on the factorization (12) of δ for which $S(u)$ is also closed, admits an associated Dragon operator $\mathfrak{D}(u)$ and

$$\text{area}\,(S(u)) = s \cdot \text{area}(u),$$
$$c(S(u)) = t \cdot c(u),$$

where $s \in \mathbb{R}$ and $t \in \mathbb{C}$ are constants (independent of u).

Definition 9. A Dragon operator $\mathfrak{D}$ associated with a factorization δ given by (12) and (13) is a transformation of the form

$$w = \mathfrak{D}(u) = \kappa \cdot S(u),$$

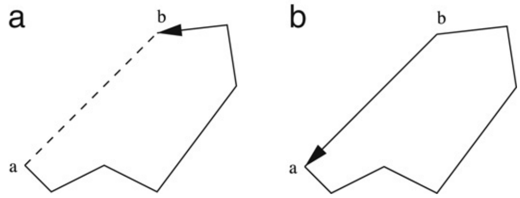


Fig. 3. (a) An $[a, b]$-ear ε (b) the corresponding closure ε°.

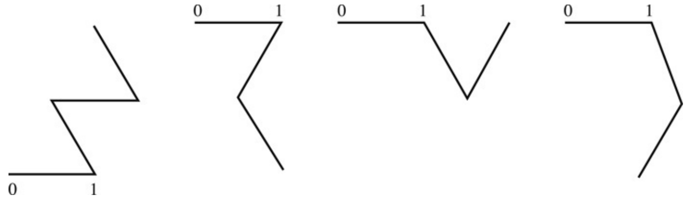


Fig. 4. The ears $\varepsilon(\delta_0)$, $\varepsilon(\delta_1)$, $\varepsilon(\delta_2)$, $\varepsilon(\delta_3)$ associated with the subwords $\delta_0, \delta_1, \delta_2, \delta_3$.

where S is the associated Shuffle operator and $\kappa \in \mathbb{C}$ (depending on the factorization) is such that the polygon u appears as a subpolygon of w. More precisely,

$$w_0 = u_0 = 0, \qquad w_{m_0+1} = u_1 = 1, \qquad w_{m_0+m_1+2} = u_2, \ldots, \qquad w_{m_0+m_1+\cdots+m_{n-1}+n} = u_{n-1}.$$

Note that these conditions are very restrictive on the given factorization of δ.

Definition 10. Consider an oriented segment $[a, b]$ in the complex plane. An $[a, b]$-ear is a path ε with vertex sequence

$$(\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}), \quad m \geq 0$$

satisfying $\alpha_0 = a$ and $\alpha_{m+1} = b$ (see Fig. 3a).

The complex number $b - a$ is called the opening of the ear ε and is denoted $\check{\varepsilon}$. Moreover, the closure of an ear ε is the closed path ε° having the vertex sequence $(\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \alpha_0)$ (see Fig. 3b).

Note that a closed polygonal path is an ear with 0 opening and that every $[a, b]$-ear ε (resp ε°), with $a \neq b$, can be normalized to an [0, 1]-ear $T_{a,b}\ \varepsilon$ (resp $T_{a,b}\ \varepsilon^\circ$) by a transformation of the form

$$z \to \frac{z - a}{b - a}$$

and more generally to a $[a', b']$-ear, where a', b' are arbitrary distinct complex numbers. In particular, each word $\gamma = \gamma_1\gamma_2 \cdots \gamma_m \in \Sigma_d^*$ gives rise to a corresponding ear $\varepsilon = \varepsilon(\gamma)$ having the vertex sequence

$$(0, 1, 1 + \gamma_1, 1 + \gamma_1 + \gamma_1\gamma_2, \ldots, 1 + \gamma_1 + \gamma_1\gamma_2 + \cdots + \gamma_1\gamma_2 \cdots \gamma_m)$$

with opening $\check{\varepsilon} = \check{\varepsilon}(\gamma) = 1 + \gamma_1 + \gamma_1\gamma_2 + \cdots + \gamma_1\gamma_2 \cdots \gamma_m$.

As an illustration of the dragon operator and the associated ear, let $N = 12$. Then the factorization $\delta = \delta_0 \cdot \delta_1 \cdot \delta_2 \cdot \delta_3 \in \Sigma_d^*$, where

$$\delta_0 = \omega^4\omega^8\omega^4, \qquad \delta_1 = \omega^8\omega^2, \qquad \delta_2 = \omega^{10}\omega^4, \qquad \delta_3 = \omega^{10}\omega^{10}$$

admits a dragon operator. This can be seen from the following example. Let $u = (u_0, u_1, u_2, u_3)$ be the closed equilateral polygonal path corresponding to the word $d = \omega^3\omega^3\omega^3 \in \Sigma_d^*$ (see Fig. 5a) and let $\varepsilon(\delta_0)$, $\varepsilon(\delta_1)$, $\varepsilon(\delta_2)$, $\varepsilon(\delta_3)$ be the ears associated respectively to the subwords $\delta_0, \delta_1, \delta_2, \delta_3$ (see Fig. 4).

The shuffle $v = S(u)$ is illustrated in Fig. 5b. For the general situation, the analysis is as follows: glueing (via affine transformations) the successive ears

$$\varepsilon(\delta_0), \varepsilon(\delta_1), \ldots, \varepsilon(\delta_k), \ldots, \varepsilon(\delta_{n-1})$$

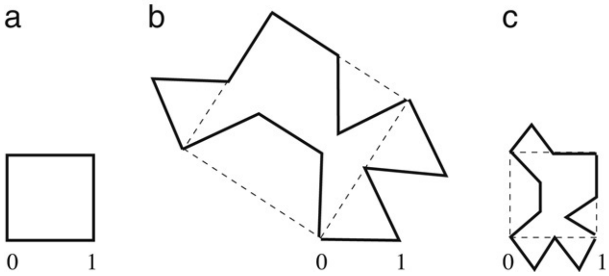


Fig. 5. (a) Original closed path u (b) path $S(u)$ (c) path $\mathfrak{D}(u)$.

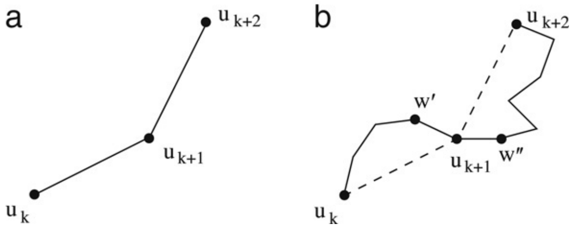


Fig. 6. Three consecutive vertices: (a) of u, (b) of $w = \mathfrak{D}(u)$.

to the corresponding equal sides

$$[u_0, u_1], [u_1, u_2], \ldots, [u_k, u_{k+1}], \ldots, [u_{n-1}, u_0],$$

we obtain ears $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1}$ satisfying

$$\begin{aligned} &\text{(i) } \varepsilon_k = u_k + (\Delta u_k)\varepsilon(\delta_k)/\check{\varepsilon}(\delta_k), \\ &\text{(ii) } \check{\varepsilon}_k = \Delta u_k, \\ &\text{(iii) } |\check{\varepsilon}_k| = |\Delta u_k| = 1, \end{aligned} \tag{14}$$

for $k = 0, 1, \ldots, n-1$ (see Fig. 5c).

We deduce that the necessary conditions for the existence of the dragon operator $\mathfrak{D}$ are

$$\kappa = \frac{1}{\check{\varepsilon}(\delta_0)}, \qquad \mathfrak{D}(u) = \frac{1}{\check{\varepsilon}(\delta_0)} S(u)$$

and

$$|\check{\varepsilon}(\delta_0)| = |\check{\varepsilon}(\delta_1)| = \cdots = |\check{\varepsilon}(\delta_{n-1})|, \tag{15}$$

where $|z|$ denotes the modulus of the complex number z. Note that the polygon $S(u)$ is similar to the polygon $\mathfrak{D}(u)$.

Now let u_k, u_{k+1}, u_{k+2} be three consecutive vertices of u (see Fig. 6a) where

$$d_k = \frac{u_{k+2} - u_{k+1}}{u_{k+1} - u_k},$$

and consider the corresponding three consecutive vertices w', u_{k+1}, w'' of $w = \mathfrak{D}(u)$ (see Fig. 6b).

When we reach the letter d_k, reading the word $\delta \sqcup^* d$, we must also have

$$d_k = \frac{w'' - u_{k+1}}{u_{k+1} - w'}.$$

In other words, we must have

$$\frac{u_{k+1} - w'}{u_{k+1} - u_k} = \frac{w'' - u_{k+1}}{u_{k+2} - u_{k+1}},$$

which is equivalent in terms of [0, 1]-ears (see Fig. 7) to the equality $\overrightarrow{0\text{A}} = \overrightarrow{\text{B}1}$.

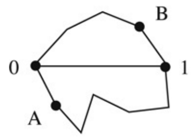


Fig. 7. Corresponding [0, 1]-ears.

Some computations give

$$\mathrm{A} = \frac{1}{\breve{\varepsilon}(\delta_{k+1})},$$
$$\mathrm{B} = \frac{1 + \delta_{k,1} + \delta_{k,1}\delta_{k,2} + \cdots + \delta_{k,1}\delta_{k,2}\cdots\delta_{k,m_k-1}}{\breve{\varepsilon}(\delta_k)} = 1 - \frac{\delta_{k,1}\delta_{k,2}\cdots\delta_{k,m_k}}{\breve{\varepsilon}(\delta_k)}.$$

Hence, the following conditions must hold

$$\breve{\varepsilon}(\delta_k) = \delta_{k,1}\delta_{k,2}\cdots\delta_{k,m_k}\breve{\varepsilon}(\delta_{k+1}) \in \mathbb{C}, \quad k = 0, 1, \ldots, n-1, \quad \delta_n = \delta_0. \tag{16}$$

Now let $\alpha = \alpha_1 \cdots \alpha_p, \beta \in \Sigma_d^*$. Then we define a binary relation $\prec$ by

$$\alpha \prec \beta \iff \breve{\varepsilon}(\alpha) = \alpha_1\alpha_2\cdots\alpha_p\breve{\varepsilon}(\beta) \in \mathbb{C}.$$

Condition (16) can be rewritten as

$$\delta_0 \prec \delta_1 \prec \cdots \prec \delta_k \prec \cdots \prec \delta_{n-1} \prec \delta_0, \tag{C1}$$

and note that condition (C1) implies condition (15).

Lemma 11. *If the factorization $\delta = \delta_0 \cdot \delta_1 \cdot \delta_2 \cdots \delta_{n-1}$ satisfies the condition (C1) then S transforms closed paths into closed paths and*

$$\mathfrak{D}(u) = \frac{1}{\breve{\varepsilon}(\delta_0)} S(u)$$

is a well-defined Dragon operator.

Proof. It remains only to show that closed paths are transformed into closed paths. Consider first the case $n = 3$. Let $d = d_1 d_2 \in \Sigma_d^*$ and $\delta = \delta_0\delta_1\delta_2 \in \Sigma_d^*$, where

$$\delta_0 = \delta_{0,1}\delta_{0,2},$$
$$\delta_1 = \delta_{1,1},$$
$$\delta_2 = \delta_{2,1}\delta_{2,2}\delta_{2,3}.$$

Note that the closure of $u = (u_0, u_1, u_2)$ is equivalent to $\breve{\varepsilon}(d) = 1 + d_1 + d_1 d_2 = 0$.

We have,

$$\delta \sqcup\!\sqcup^* d = (\delta_{0,1}\delta_{0,2}) \cdot d_1 \cdot (\delta_{1,1}) \cdot d_2 \cdot (\delta_{2,1}\delta_{2,2}\delta_{2,3})$$

and the vertices of $S(u)$ are the complex numbers $(v_0, v_1, \ldots, v_9)$. It is easily shown that

$$v_0 = 0,$$
$$v_1 = 1,$$
$$v_2 = 1 + \delta_{0,1},$$
$$v_3 = \breve{\varepsilon}(\delta_0),$$
$$v_4 = \breve{\varepsilon}(\delta_0) + \delta_{0,1}\delta_{0,2}d_1,$$
$$v_5 = \breve{\varepsilon}(\delta_0)(1 + d_1),$$
$$v_6 = \breve{\varepsilon}(\delta_0) + \breve{\varepsilon}(\delta_0)d_1 + \delta_{0,1}\delta_{0,2}d_1\delta_{1,1}d_2,$$
$$v_7 = \breve{\varepsilon}(\delta_0) + \breve{\varepsilon}(\delta_0)d_1 + \delta_{0,1}\delta_{0,2}d_1\delta_{1,1}d_2 + \delta_{0,1}\delta_{0,2}d_1\delta_{1,1}d_2\delta_{2,1},$$

$$v_8 = \check{\varepsilon}(\delta_0) + \check{\varepsilon}(\delta_0)d_1 + \delta_{0,1}\delta_{0,2}d_1\delta_{1,1}d_2 + \delta_{0,1}\delta_{0,2}d_1\delta_{1,1}d_2\delta_{2,1} + \delta_{0,1}\delta_{0,2}d_1\delta_{1,1}d_2\delta_{2,1}\delta_{2,2},$$
$$v_9 = \check{\varepsilon}(\delta_0)(1 + d_1 + d_1d_2) = \check{\varepsilon}(\delta_0)\cdot 0 = 0.$$

In the general case, the same kind of pattern occurs and we have,

$$\check{\varepsilon}(\delta_0)(1 + d_1 + d_1d_2 + \cdots + d_1d_2\cdots d_{n-1}) = 0. \quad \square$$

3.3. Technical lemmas and main theorem

Before stating the main result (Theorem 12) of this section we need the following five lemmas concerning ears, additivity of area and center of gravity, and existence of Dragon curves.

Lemma 12. *Let u be a closed polygonal path and $w = \mathfrak{D}(u)$. Then*

$$\int_u f(z)\mathrm{d}z = \int_w f(z)\mathrm{d}z$$

if and only if

$$\Sigma_{k=0}^{n-1} \int_{\varepsilon_k^o} f(z)\mathrm{d}z = 0,$$

where ε_k denotes the $[u_k, u_{k+1}]$-ear associated with $\mathfrak{D}$, $k = 0, 1, \ldots, n-1$ (see Fig. 5c).

Proof. We have the following path decompositions,

$$w = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{n-1}$$
$$u = [u_0, u_1] + [u_1, u_2] + \cdots + [u_{n-2}, u_{n-1}] + [u_{n-1}, u_0].$$

Hence,

$$\begin{aligned}\int_w f - \int_u f &= \left(\int_{\varepsilon_0} f - \int_{[u_0,u_1]} f\right) + \left(\int_{\varepsilon_1} f - \int_{[u_1,u_2]} f\right) + \cdots + \left(\int_{\varepsilon_{n-1}} f - \int_{[u_{n-1},u_0]} f\right)\\ &= \int_{\varepsilon_0^\circ} f + \int_{\varepsilon_1^\circ} f + \cdots + \int_{\varepsilon_{n-1}^\circ} f. \quad \square\end{aligned}$$

The next lemma is a complex version of the classical result about the center of gravity of compound objects [5].

Lemma 13. *Consider two closed polygonal paths p_1, p_2 having a common side in opposite directions. Let $p = p_1 + p_2$ be the resulting polygonal path obtained by deleting the common side. Then,*

$$a(p) = a(p_1) + a(p_2), \qquad \mathrm{cg}(p) = \frac{a(p_1)\mathrm{cg}(p_1) + a(p_2)\mathrm{cg}(p_2)}{a(p_1) + a(p_2)}$$

provided that $a(p_1) \neq 0$, $a(p_2) \neq 0$ and $a(p_1) + a(p_2) \neq 0$. Note that $a(p)$ and $\mathrm{cg}(p)$ denote the signed area and signed center of gravity of p.

Proof. We have,

$$a(p) = \frac{1}{2}\Im \int_p \bar{z}\mathrm{d}z = \frac{1}{2}\Im\left(\int_{p_1} \bar{z}\mathrm{d}z + \int_{p_2} \bar{z}\mathrm{d}z\right) = a(p_1) + a(p_2)$$

and

$$\begin{aligned}\mathrm{cg}(p) &= \frac{\mathrm{i}}{4a}\overline{\int_p \bar{z}^2\mathrm{d}z} = \frac{\mathrm{i}}{4a}\left(\overline{\int_{p_1} \bar{z}^2\mathrm{d}z} + \overline{\int_{p_2} \bar{z}^2\mathrm{d}z}\right)\\ &= \frac{\mathrm{i}}{4a}\left(\frac{4a_1}{\mathrm{i}}\mathrm{cg}(p_1) + \frac{4a_2}{\mathrm{i}}\mathrm{cg}(p_2)\right)\end{aligned}$$

$$= \frac{a_1\mathrm{cg}(p_1) + a_2\mathrm{cg}(p_2)}{a}$$
$$= \frac{a_1\mathrm{cg}(p_1) + a_2\mathrm{cg}(p_2)}{a_1 + a_2}. \quad \square$$

Lemma 14. *If the Dragon operator is well-defined then*

(a) $a(\mathfrak{D}(u)) = a(u) + \Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ)$,

(b) $\mathrm{cg}(\mathfrak{D}(u)) = \dfrac{a(u)\mathrm{cg}(u) + \Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ)\mathrm{cg}(\varepsilon_k^\circ)}{a(u) + \Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ)}$.

Lemma 15. *Let $\gamma = \gamma_1 \cdots \gamma_m \in \Sigma_d^*$ and $\varepsilon^\circ(\gamma)$ be its associated closed ear having the vertex sequence $(z_0, z_1, \ldots, z_{m+1})$ where $z_0 = 0$, $z_1 = 1$ and $z_k = 1 + \gamma_1 + \gamma_1\gamma_2 + \cdots + \gamma_1\gamma_2\cdots\gamma_{k-1}$. Then the signed area of $\varepsilon^\circ(\gamma)$ is given by*

(i) $a(\varepsilon^\circ(\gamma)) = \dfrac{1}{2}\Im\Sigma_{1\le i\le j\le m}\gamma_{i\cdots j} \in \mathbb{R}$

where $\gamma_{i\cdots j} = \gamma_i\gamma_{i+1}\cdots\gamma_j \in \mathbb{C}$.

Moreover, the signed center of gravity of $\varepsilon^\circ(\gamma)$ is given by

(ii) $\mathrm{cg}(\varepsilon^\circ(\gamma)) = \dfrac{1}{3}\dfrac{\Sigma_{\nu=1}^{m}(z_\nu + z_{\nu+1})\Im\Sigma_{j=1}^{\nu}\gamma_{j\cdots\nu}}{\Im\Sigma_{1\le i\le j\le m}\gamma_{i\cdots j}} \in \mathbb{C}$.

Proof. (i) From (10), we have,

$$\begin{aligned} a &= \frac{1}{2}\Im\Sigma_{j=0}^{m+1}\overline{z_j}\Delta z_j \\ &= \frac{1}{2}\Im(\Sigma_{j=1}^{m}\overline{(1 + \gamma_1 + \gamma_1\gamma_2 + \cdots + \gamma_1\gamma_2\cdots\gamma_{i-1} + \cdots + \gamma_1\gamma_2\cdots\gamma_{j-1})}\gamma_1\cdots\gamma_j \\ &\quad - (1 + \overline{\gamma_1} + \overline{\gamma_1\gamma_2} + \cdots + \overline{\gamma_1\cdots\gamma_m})(1 + \gamma_1 + \gamma_1\gamma_2 + \cdots + \gamma_1\cdots\gamma_m)) \\ &= \frac{1}{2}\Im\left(\Sigma_{j=1}^{m}\Sigma_{i=1}^{j}\gamma_i\gamma_{i+1}\cdots\gamma_j\right) \\ &= \frac{1}{2}\Im\Sigma_{1\le i\le j\le m}\gamma_{i\cdots j}. \end{aligned}$$

(ii) For $k = 1, 2, \ldots, m$, let p_k be the triangle with vertices z_0, z_k, z_{k+1}, then $\varepsilon^\circ(\gamma) = p_0 + p_1 + \cdots + p_m$ (sum of paths). By Lemma 13, we have

$$\mathrm{cg}(\varepsilon^\circ(\gamma)) = \frac{a(p_1)\mathrm{cg}(p_1) + a(p_2)\mathrm{cg}(p_2) + \cdots + a(p_m)\mathrm{cg}(p_m)}{a(p_1) + a(p_2) + \cdots a(p_m)}$$

and we conclude using (i) and the fact that

$$\mathrm{cg}(p_k) = \frac{1}{3}(z_0 + z_k + z_{k+1}) = \frac{z_k + z_{k+1}}{3}. \quad \square$$

Lemma 16. *Consider two sequences of complex constants $q_0, q_1, \ldots, q_{n-1}$ and $r_0, r_1, \ldots, r_{n-1}$ satisfying*

$$q_k - r_k + r_{k-1} = 0, \quad \text{for } k = 0, 1, \ldots, n-1,$$

where $r_{-1} = r_{n-1}$ by convention. Then, for every closed path u, with $u_0 = u_n$:

$$\Sigma_{k=0}^{n-1}(q_k u_k + r_k\Delta u_k) = 0.$$

Proof. Since $u_0 = u_n$, we have

$$\begin{aligned} \Sigma_{k=0}^{n-1} r_k\Delta u_k &= \Sigma_{k=0}^{n-1} r_k(u_{k+1} - u_k) \\ &= \Sigma_{k=0}^{n-1} r_k u_{k+1} - \Sigma_{k=0}^{n-1} r_k u_k \end{aligned}$$

$$= \Sigma_{k=0}^{n-1} r_{k-1} u_k - \Sigma_{k=0}^{n-1} r_k u_k$$
$$= \Sigma_{k=0}^{n-1} (r_{k-1} - r_k) u_k.$$

Hence,

$$\Sigma_{k=0}^{n-1}(q_k u_k + r_k \Delta u_k) = \Sigma_{k=0}^{n-1}(q_k + r_{k-1} - r_k) u_k = 0. \quad \square$$

We now state and prove our main result which gives conditions for the preservation of area and center of gravity of the Dragon operator.

Theorem 17. *Consider a factorization* $\delta = \delta_0 \cdot \delta_1 \cdot \delta_2 \cdots \delta_{n-1} \in \Sigma_d^*$ *satisfying condition* (C1) *that is*

$$\delta_0 \prec \delta_1 \prec \cdots \prec \delta_{n-1} \prec \delta_0.$$

Then, for every closed path $u = (u_0, u_1, \ldots, u_{n-1})$*, the associated Dragon operator* $\mathfrak{D}$ *preserves area, that is* $\text{area}(\mathfrak{D}(u)) = \text{area}(u)$*, if the factorization satisfies*

$$\Sigma_{k=0}^{n-1} a(\varepsilon^\circ(\delta_k)) = 0, \tag{C2}$$

where

$$a(\varepsilon^\circ(\delta_k)) = \frac{1}{2} \Im \Sigma_{1 \le i \le j \le m} \gamma_{i \cdots j} \in \mathbb{R}.$$

Moreover, $\mathfrak{D}$ *preserves center of gravity, that is* $\text{cg}(\mathfrak{D}(u)) = \text{cg}(u)$*, if the following conditions are also satisfied*

$$a\left(\varepsilon^\circ(\delta_k)\right)\left(1 - \frac{1}{\check{\varepsilon}(\delta_k)} \text{cg}\left(\varepsilon^\circ(\delta_k)\right)\right) + a\left(\varepsilon^\circ(\delta_{k-1})\right) \frac{\text{cg}\left(\varepsilon^\circ(\delta_{k-1})\right)}{\check{\varepsilon}(\delta_{k-1})} = 0, \quad k = 0, \ldots, n-1, \tag{C3}$$

where

$$\text{cg}(\varepsilon^\circ(\delta_k)) = \frac{1}{3} \frac{\Sigma_{\nu=1}^m (z_\nu + z_{\nu+1}) \Im \Sigma_{j=1}^\nu \gamma_{j \cdots \nu}}{\Im \Sigma_{1 \le i \le j \le m} \gamma_{i \cdots j}},$$

and $\check{\varepsilon}(\delta_k)$ *is the opening of* δ_k*.*

Proof. By Lemma 14(a),

$$a(\mathfrak{D}(u)) = a(u) \iff \Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ) = 0$$
$$\iff \Sigma_{k=0}^{n-1} a(\varepsilon^\circ(\delta_k)) = 0$$

since

$$a(\varepsilon_k^\circ) = \frac{1}{|\check{\varepsilon}(\delta_k)|^2} a(\varepsilon^\circ(\delta_k)), \quad k = 0, \ldots n-1$$

and $|\check{\varepsilon}(\delta_k)|$ are independent of k.

For the center of gravity $\text{cg}(\mathfrak{D}(u))$, Lemma 14(b) gives

$$\text{cg}(\mathfrak{D}(u)) = \text{cg}(u) \iff \frac{a(u)\text{cg}(u) + \Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ)\text{cg}(\varepsilon_k^\circ)}{a(u) + \Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ)} = \text{cg}(u)$$
$$\iff \frac{a(u)\text{cg}(u) + \Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ)\text{cg}(\varepsilon_k^\circ)}{a(u)} = \text{cg}(u) \quad \text{(by (C2))}$$
$$\iff \text{cg}(u) + \Sigma_{k=0}^{n-1} \frac{a(\varepsilon_k^\circ)}{a(u)} \text{cg}(\varepsilon_k^\circ) = \text{cg}(u)$$
$$\iff \Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ)\text{cg}(\varepsilon_k^\circ) = 0.$$

But by (14)(i),

$$\varepsilon_k^\circ = u_k + \Delta u_k \varepsilon(\delta_k)/\check{\varepsilon}(\delta_k).$$

Hence,

$$\mathrm{cg}(\varepsilon_k^\circ) = u_k + \frac{\Delta u_k}{\check{\varepsilon}(\delta_k)}\mathrm{cg}(\varepsilon^\circ(\delta_k)).$$

This implies that

$$\Sigma_{k=0}^{n-1} a(\varepsilon_k^\circ)\mathrm{cg}(\varepsilon_k^\circ) = 0$$

is equivalent to

$$\Sigma_{k=0}^{n-1} a(\varepsilon^\circ(\delta_k))u_k + \frac{a(\varepsilon^\circ(\delta_k))\mathrm{cg}(\varepsilon^\circ(\delta_k))}{\check{\varepsilon}(\delta_k)}\Delta u_k = 0$$

and we conclude, using Lemma 16 with

$$q_k = a(\varepsilon^\circ(\delta_k)) \quad \text{and} \quad r_k = \frac{a(\varepsilon^\circ(\delta_k))\mathrm{cg}(\varepsilon^\circ(\delta_k))}{\check{\varepsilon}(\delta_k)}. \quad \square$$

Corollary 18. *Under conditions* (C1)–(C3)*, the Shuffle operator satisfies*

$$a(S(u)) = |\check{\varepsilon}(\delta_0)|^2\, a(u),$$
$$\mathrm{cg}(S(u)) = \check{\varepsilon}(\delta_0)\, \mathrm{cg}(u).$$

for any closed polygonal path $u = (u_0, u_1, \ldots, u_{n-1})$.

3.4. Associated Dragon curves

In order to define limiting Dragon curves we must be able to iterate Dragon operators similarly as in Section 2.2. Consider a fixed factorization

$$\alpha = \alpha_0 \cdot \alpha_1 \cdots \alpha_{p-1},$$

where $\alpha_k = \alpha_{k,1} \cdots \alpha_{k,m_k} \in \Sigma_d^*$ and such that p divides $m_0 + m_1 + \cdots + m_{p-1}$.

Then for every word

$$d = d_1 \cdots d_{pn-1} \tag{17}$$

we take the word $\delta = \alpha^n$ with factorization

$$\delta = \alpha_0 \cdot \alpha_1 \cdots \alpha_{p-1} \cdot \alpha_0 \cdot \alpha_1 \cdots \alpha_{p-1} \cdots \alpha_0 \cdot \alpha_1 \cdots \alpha_{p-1}$$

and consider the perfect shuffle $\delta \sqcup\!\sqcup^* d = \alpha^n \sqcup\!\sqcup^* d$.

The perfect shuffle defines a Shuffle operator S on closed polygonal paths $u = (u_0, u_1, \ldots, u_{pn-1})$ associated with words of the form (17). It is easy to see that $v = S(u)$ contains $n[(m_0 + 1) + \cdots + (m_{p-1} + 1)]$ sides. Then since p divides $m_0 + m_1 + \cdots + m_{p-1}$, we have

$$v = (v_0, v_1, \ldots, v_{pN-1}),$$

where $N = \frac{(m_0+1)+\cdots+(m_{p-1}+1)}{p}$ is an integer.

Hence we can iterate S on polygonal paths containing a multiple of p sides. Given a closed polygonal path $u = (u_0, u_1, \ldots, u_{pn-1})$ we can also iterate

$$\mathfrak{D} = \frac{1}{\check{\varepsilon}(\alpha_0)} S$$

by the formula

$$\mathfrak{D}^k(u) = \frac{1}{(\check{\varepsilon}(\alpha_0))^k} S^k(u).$$

The Dragon curve associated with α and u is the limiting curve of the sequence $\mathfrak{D}^k(u)$ as k goes to infinity.

4. Concluding remarks

There exists infinite families of Dragon curves arising from factorizations

$$\alpha = \alpha_0 \cdot \alpha_1 \cdots \alpha_{p-1}$$

satisfying conditions (C1)–(C3) and such that p divides $m_0 + \cdots + m_{p-1}$. Maple programs can be implemented to find such factorizations.

As the reader can check, we have the special such family for $p = 2$: take $\alpha = \alpha_0 \cdot \alpha_1$ where

$$\alpha_0 = \lambda = \lambda_1 \lambda_2 \cdots \lambda_m \in \Sigma_d^*,$$
$$\alpha_1 = \overline{\widetilde{\lambda}} = \bar{\lambda}_m \bar{\lambda}_{m-1} \cdots \bar{\lambda}_1 \in \Sigma_d^*$$

is the conjugate (in $\mathbb{C}$) reversal of λ.

Acknowledgement

This paper is an extended and enhanced version of a paper presented in Liege (Belgium) during the conference CANT'2006 [1].

References

[1] S. Brlek, G. Labelle, A. Lacasse, Shuffle operations on lattice paths, in: Proc. International School and Conference on Combinatorics, Automata and Number Theory, CANT'2006, Liège, Belgium, 8–19 May 2006, Prépublication no 06.002.
[2] S. Brlek, G. Labelle, A. Lacasse, Properties of the contour path of discrete sets, in: 9th International Conference on Developments in Language Theory, DLT'2005, Internat. J. Found. Comput. Sci. 17 (3) (2006) 543–556 (special issue). World Scientific Publishing Company.
[3] S. Brlek, G. Labelle, A. Lacasse, Incremental algorithms based on discrete Green theorem, in: Proc. International Conference on Discrete Geometry and Computer Imagery, DGCI'2003, Naples, Italy, 18–21 November 2003, in: Springer Lecture Notes in Computer Science, vol. 2886, 2003, pp. 277–287.
[4] A. Daurat, M. Nivat, Salient and reentrant points of discrete sets, in: Proc. International Workshop on Combinatorial Image Analysis, IWCIA'03, Palermo, Italy, 14–16 May 2003.
[5] R.P. Feynman, R.B. Leighton, M. Sands, The Feynman Lectures on Physics, vol. 1, Addison-Wesley, 1977.
[6] H. Freeman, On the encoding of arbitrary geometric configurations, IRE Trans. Electron. Comput. 10 (1961) 260–268.
[7] H. Freeman, Boundary encoding and processing, in: B.S. Lipkin, A. Rosenfeld (Eds.), Picture Processing and Psychopictorics, Academic Press, New York, 1970, pp. 241–266.
[8] M. Lothaire, Combinatorics on Words, Addison, Wesley, 1983.
[9] W.G. McCallum, D. Hughes-Hallett, A.M. Gleason, et al., Multivariable Calculus, John Wiley and Sons, New York, 1997.