Available online at www.sciencedirect.com

SCIENCE DIRECT

J. Math. Anal. Appl. 314 (2006) 689–700

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Jordan zero-product preserving additive maps on operator algebras ☆

L. Zhao [1], J. Hou *

Department of Mathematics, Shanxi University, Taiyuan 030000, PR China

Received 30 April 2004

Available online 28 April 2005

Submitted by Z.-J. Ruan

Abstract

Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an additive surjective map between some operator algebras such that $AB + BA = 0$ implies $\Phi(A)\Phi(B) + \Phi(B)\Phi(A) = 0$. We show that, under some mild conditions, Φ is a Jordan homomorphism multiplied by a central element. Such operator algebras include von Neumann algebras, C*-algebras and standard operator algebras, etc. Particularly, if H and K are infinite-dimensional (real or complex) Hilbert spaces and $\mathcal{A} = \mathcal{B}(H)$ and $\mathcal{B} = \mathcal{B}(K)$, then there exists a nonzero scalar c and an invertible linear or conjugate-linear operator $U : H \to K$ such that either $\Phi(A) = cUAU^{-1}$ for all $A \in \mathcal{B}(H)$, or $\Phi(A) = cUA^*U^{-1}$ for all $A \in \mathcal{B}(H)$.

Keywords: Operator algebras; Jordan zero-products; Jordan homomorphisms

1. Introduction

Let Φ be a map between two rings. We say that Φ is zero-product preserving if $\Phi(A)\Phi(B) = 0$ whenever $AB = 0$; we say that Φ is Jordan zero-product preserving if $\Phi(T)\Phi(S) + \Phi(S)\Phi(T) = 0$ whenever $TS + ST = 0$. The study of zero-product pre-

☆ This work is supported partially by NNSFC and PNSFS.

* Corresponding author.

E-mail address: jhou@dns.sxtu.edu.cn (J. Hou).

[1] Present address: Institute of Mathematics, Fudan University, Shanghai 200433, PR China.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.04.015

serving additive or linear maps between operator algebras is a topic which attracts much attention of many authors, and it turns out, in many cases, a map preserves zero-products if and only if it is a central element multiple of a ring homomorphism (see, for example, [2,8, 10] and the references therein). We know that many operator spaces bear a Jordan algebra structure. It is interesting to ask whether or not we can characterize the Jordan zero-product preservers.

Let $\mathcal{A}$, $\mathcal{B}$ be Jordan rings. Recall that an additive map $\mathcal{J}:\mathcal{A}\to\mathcal{B}$ is said to be a Jordan ring homomorphism if $\mathcal{J}(TS+ST)=\mathcal{J}(T)\mathcal{J}(S)+\mathcal{J}(S)\mathcal{J}(T)$ for all elements T,S in $\mathcal{A}$. In case that $\mathcal{A}$ and $\mathcal{B}$ are Jordan algebras over a field and the underlying field has characteristic not 2, $\mathcal{J}$ is a Jordan ring homomorphism if and only if $\mathcal{J}(T^2)=\mathcal{J}(T)^2$ for all T in the domain. It is trivial to see that a Jordan ring homomorphism multiplied by a central element does preserves Jordan zero-products. In this paper, we consider the converse problem and characterize additive (or linear) Jordan zero-product preserving maps between some operator algebras and show that such maps arise in the standard way.

The same question was firstly considered in [8], there the present authors characterized the additive surjections which preserves Jordan zero-products in both directions on $\mathcal{B}(H)$, the von Neumann algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H, and on $\mathcal{S}(H)$, the real Jordan algebra of all self-adjoint operators in $\mathcal{B}(H)$, respectively. The results got there are closely related to the square-zero preservers. Recall that Φ is said to preserve Jordan zero-products in both directions if $\Phi(A)\Phi(B)+\Phi(B)\Phi(A)=0\Leftrightarrow AB+BA=0$. In this paper, we omit the assumption of "in both directions" and use a different approach to show that, for real or complex Hilbert space H, every Jordan zero-product preserving additive surjection on $\mathcal{B}(H)$ has either the form $\Phi(A)=cUAU^{-1}$ for all $A\in\mathcal{B}(H)$, or the form $\Phi(A)=cUA^*U^{-1}$ for all $A\in\mathcal{B}(H)$, where c is a nonzero scalar, U is a bounded invertible linear or conjugate-linear operator and $A^*\in\mathcal{B}(H)$ is the adjoint of A. We also prove that the bounded linear surjections preserving Jordan zero-products between von Neumann algebras, or between C^*-algebras, or between the real subspace of self-adjoint elements of C*-algebras, have the form of $T\mathcal{J}$, where $\mathcal{J}$ is a Jordan homomorphism and T is an invertible central element. Finally we give a similar characterization of unital additive surjections between standard operator algebras on (real or complex) Banach spaces which preserve Jordan zero-products in both directions. It turns out, such additive maps take one of the following nice forms: isomorphisms, anti-isomorphisms, conjugate isomorphisms and conjugate anti-isomorphisms.

2. The cases of $\mathcal{B}(H)$ and von Neumann algebras

Let $\mathcal{B}(H)$ and $\mathcal{B}(K)$ be the algebras of all bounded linear operators on the infinite-dimensional (real or complex) Hilbert spaces H and K, respectively. The following main result shows that every Jordan zero-product preserving additive surjective map between $\mathcal{B}(H)$ and $\mathcal{B}(K)$ is in fact a scalar multiple of an isomorphism, or an anti-isomorphism, or a conjugate isomorphism, or a conjugate anti-isomorphism.

Theorem 2.1. *Let H and K be* (*real or complex*) *infinite-dimensional Hilbert spaces. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a Jordan zero-product preserving additive surjection. Then there exists a nonzero scalar c and an invertible bounded linear or conjugate-linear operator $U : H \to K$ such that either $\Phi(A) = cUAU^{-1}$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = cUA^*U^{-1}$ for all $A \in \mathcal{B}(H)$* (*in the real case, U is linear*)*.*

Proof. Let $P \in \mathcal{B}(H)$ with $P^2 = P$. Since $P(I - P) + (I - P)P = 0$, we have $\Phi(P)\Phi(I - P) + \Phi(I - P)\Phi(P) = 0$, and consequently,

$$\Phi(I)\Phi(P) + \Phi(P)\Phi(I) = 2\Phi(P)^2.$$

Thus we have

$$\Phi(P)^2\Phi(I) + \Phi(P)\Phi(I)\Phi(P) = 2\Phi(P)^3$$

and

$$\Phi(I)\Phi(P)^2 + \Phi(P)\Phi(I)\Phi(P) = 2\Phi(P)^3.$$

These together imply that

$$\Phi(I)\Phi(P)^2 = \Phi(P)^2\Phi(I).$$

Similarly, it follows from

$$\Phi(I)^2\Phi(P) + \Phi(I)\Phi(P)\Phi(I) = 2\Phi(I)\Phi(P)^2$$

and

$$\Phi(P)\Phi(I)^2 + \Phi(I)\Phi(P)\Phi(I) = 2\Phi(P)^2\Phi(I)$$

that

$$\Phi(P)\Phi(I)^2 = \Phi(I)^2\Phi(P).$$

Since every infinite-dimensional Hilbert space has infinite multiplicity, by [5], every bounded linear operator on an infinite-dimensional Hilbert space is an algebraic sum of finite many idempotents (a sum of at most five idempotents if the space is complex [14, Theorem 5]). Hence we have $\Phi(A)\Phi(I)^2 = \Phi(I)^2\Phi(A)$ holds for every $A \in \mathcal{B}(H)$. Therefore, by the surjectivity of Φ,

$$\Phi(I)^2 = \lambda I$$

for some scalar λ.

Let $T, S \in \mathcal{B}(H)$ with $ST = 0$. For any idempotent P, it follows from $TP(I - P)S + (I - P)STP = 0$ that $\Phi(TP)\Phi((I - P)S) + \Phi((I - P)S)\Phi(TP) = 0$. Thus

$$\Phi(TP)\Phi(S) + \Phi(S)\Phi(TP) = \Phi(TP)\Phi(PS) + \Phi(PS)\Phi(TP) \tag{2.1}$$

holds for every idempotent P. On the other hand, $T(I - P)PS + PST(I - P) = 0$ implies that $\Phi(T(I - P))\Phi(PS) + \Phi(PS)\Phi(T(I - P)) = 0$, and hence,

$$\Phi(T)\Phi(PS) + \Phi(PS)\Phi(T) = \Phi(TP)\Phi(PS) + \Phi(PS)\Phi(TP) \tag{2.2}$$

for every idempotent P. Combining (2.1) and (2.2), we get

$$\Phi(TP)\Phi(S) + \Phi(S)\Phi(TP) = \Phi(T)\Phi(PS) + \Phi(PS)\Phi(T)$$

for every idempotent P. Hence for every $A \in \mathcal{B}(H)$,

$$\Phi(TA)\Phi(S) + \Phi(S)\Phi(TA) = \Phi(T)\Phi(AS) + \Phi(AS)\Phi(T). \tag{2.3}$$

Take $T = Q$ and $S = I - Q$ for some $Q \in \mathcal{B}(H)$ with $Q^2 = Q$. Then $ST = 0$ and from (2.3), one gets $\Phi(QA)\Phi(I - Q) + \Phi(I - Q)\Phi(QA) = \Phi(Q)\Phi(A(I - Q)) + \Phi(A(I - Q))\Phi(Q)$. Thus we see that

$$\begin{aligned}&\Phi(QA)\Phi(I) + \Phi(I)\Phi(QA) - \Phi(Q)\Phi(A) - \Phi(A)\Phi(Q)\\&\quad = \Phi(QA)\Phi(Q) + \Phi(Q)\Phi(QA) - \Phi(Q)\Phi(AQ) - \Phi(AQ)\Phi(Q).\end{aligned}$$

On the other hand, taking $T = I - Q$ and $S = Q$, we obtain from (2.3) another equation

$$\begin{aligned}&\Phi(I)\Phi(AQ) + \Phi(AQ)\Phi(I) - \Phi(A)\Phi(Q) - \Phi(Q)\Phi(A)\\&\quad = \Phi(Q)\Phi(AQ) + \Phi(AQ)\Phi(Q) - \Phi(QA)\Phi(Q) - \Phi(Q)\Phi(QA).\end{aligned}$$

Hence

$$\Phi(QA + AQ)\Phi(I) + \Phi(I)\Phi(QA + AQ) = 2\big(\Phi(Q)\Phi(A) + \Phi(A)\Phi(Q)\big)$$

holds for every idempotent Q. This further implies that

$$\Phi(AB + BA)\Phi(I) + \Phi(I)\Phi(AB + BA) = 2\big(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)\big) \tag{2.4}$$

holds for every $B \in \mathcal{B}(H)$. Multiplying (2.4) from left and right by $\Phi(I)$ respectively, we see that

$$\begin{aligned}&\Phi(I)^2\Phi(AB + BA) + \Phi(I)\Phi(AB + BA)\Phi(I)\\&\quad = 2\Phi(I)\big(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)\big)\end{aligned}$$

and

$$\begin{aligned}&\Phi(I)\Phi(AB + BA)\Phi(I) + \Phi(AB + BA)\Phi(I)^2\\&\quad = 2\big(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)\big)\Phi(I).\end{aligned}$$

These two equations, together with the fact that $\Phi(I)^2 = \lambda I$, entail that

$$\Phi(I)\big(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)\big) = \big(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)\big)\Phi(I). \tag{2.5}$$

Let $A = B$ in (2.4) and (2.5); then

$$\Phi(I)\Phi(A^2) + \Phi(A^2)\Phi(I) = 2\Phi(A)^2, \tag{2.6}$$

$$\Phi(I)\Phi(A)^2 = \Phi(A)^2\Phi(I). \tag{2.7}$$

By the surjectivity of Φ, Eq. (2.7) implies that $\Phi(I)$ commutes with all idempotent operators and hence there must exist a scalar μ such that $\Phi(I) = \mu I$. While Eq. (2.6) tells that $\mu \neq 0$. Let $c = \frac{1}{\mu}$ and $\Psi(\cdot) = c\Phi(\cdot)$, then $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is an additive surjection preserving Jordan zero-products and $\Psi(I) = I$. Moreover, for every $A \in \mathcal{B}(H)$, $\Psi(A^2) = \Psi(A)^2$, which implies that Ψ is a Jordan ring homomorphism. Since $\mathcal{B}(K)$ is prime, one sees that Ψ is either a ring homomorphism or a ring anti-homomorphism.

Therefore, Φ is a scalar multiple of a surjective ring homomorphism or a surjective ring anti-homomorphism.

We will show that Φ is injective. Without loss of generality we assume that Φ is a surjective ring homomorphism. We first claim that the null space of Φ is closed. For every $0 \neq y \in K$, define $\ker_y(\Phi) = \{T \in \mathcal{B}(H) \mid \Phi(T)y = 0\}$, which is obviously a left ideal of $\mathcal{B}(H)$ and $\ker(\Phi) = \bigcap_{y \in K} \ker_y(\Phi)$. If $\mathcal{L}$ is a left ideal such that $\ker_y(\Phi)$ is a proper subset of $\mathcal{L}$, then $\Phi(\mathcal{L})y$ is a nonzero invariant linear manifold of K. It follows that $\Phi(\mathcal{L})y = K$. So, there exists $T \in \mathcal{L}$ such that $\Phi(T)y = y$. For any $S \in \mathcal{B}(H)$, we have $S - ST \in \ker_y(\Phi) \subset \mathcal{L}$. This implies that $S \in \mathcal{L}$ since $ST \in \mathcal{L}$. Therefore, we have $\mathcal{L} = \mathcal{B}(H)$, and consequently, $\Phi_y(\Phi)$ is a maximal left ideal of $\mathcal{B}(H)$. It follows that $\ker_y(\Phi)$ is closed and hence $\ker(\Phi)$ is closed, as desired. The rest of arguments is similar to that in [10, Lemma 2]. For the completeness, we give the details here. Note that the set of ring two-sided ideals coincides with the set of algebraic two-sided ideals in $\mathcal{B}(H)$. Thus, if Φ is not injective, then the kernel of Φ is a closed two-sided ideal which contains the ideal consisting of all compact operators. Suppose the (Hilbert space) dimension of H is $\aleph_H$, which is an infinite cardinal number. For each infinite cardinal number $\aleph \leqslant \aleph_H$, let

$$I_\aleph = \bigl\{T \in \mathcal{B}(H) \mid \dim M < \aleph \text{ holds for all closed linear subspaces } M \subseteq \operatorname{range}(T)\bigr\}.$$

Then $I_\aleph$ is a closed two-sided ideal of $\mathcal{B}(H)$ and every closed two-sided ideal of $\mathcal{B}(H)$ arises in this way [3, Section 17]. In particular, $I_{\aleph_H}$ is the largest one. Therefore, Φ induces a ring isomorphism from the quotient algebra $\mathcal{B}(H)/\ker\Phi$ onto $\mathcal{B}(K)$. This implies that there is an element $A \in \mathcal{B}(H)$ such that $A + \ker\Phi$ is a single element of $\mathcal{B}(H)/\ker\Phi$ (an element T in an algebra $\mathcal{A}$ is single if, for any $S, R \in \mathcal{A}$, $STR = 0$ will imply $ST = 0$ or $TR = 0$). It is a well-known result due to Erdos (see [4] or [6]) that, for a C*-algebra $\mathcal{A}$, there exists a faithful representation (π, H_1) of $\mathcal{A}$ such that an element $T \in \mathcal{A}$ is a single element if and only if $\pi(T)$ is of rank one on H_1, and consequently, $\dim T\mathcal{A}T = 1$. Hence $(A + \ker\Phi)\mathcal{B}(H)(A + \ker\Phi) = A\mathcal{B}(H)A + \ker\Phi$ is of dimension one modulo $\ker\Phi$. Let $\aleph \leqslant \aleph_H$ be the infinite cardinal number such that $\ker\Phi = I_\aleph$. Then the range of A contains a close subspace of dimension $\aleph$. By halving this subspace into two, each of dimension $\aleph$, we see that $A\mathcal{B}(H)A$ contains two elements linearly independent modulo $I_\aleph$, a contradiction. So, Φ is injective.

Hence we have shown that Φ is a scalar multiple of a ring isomorphism or a ring anti-isomorphism from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$. Thus, as a well-known fact, Φ has the desired form stated in the theorem, completing the proof. □

When the maps are linear, we have more neat conclusion.

Corollary 2.2. *Let H and K be* (*real or complex*) *infinite-dimensional Hilbert spaces. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a Jordan zero-product preserving linear surjection. Then there exists a nonzero scalar c and an invertible bounded linear operator $U : H \to K$ such that either $\Phi(A) = cUAU^{-1}$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = cUA^{\mathrm{tr}}U^{-1}$ for all $A \in \mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to an arbitrarily fixed orthonormal basis of H.*

Note that every surjective algebraic homomorphism from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$ is automatically continuous. This fact is used in the proof of [10, Lemma 2] to show that such homomorphism is also injective. It is clear from ring theory that the automatical continuity is not true for ring homomorphisms if H is complex and finite-dimensional. In fact, every ring automorphism ϕ of the complex algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices has the form $\phi(T) = AT_\tau A^{-1}$ $\forall T = (t_{ij}) \in M_n(\mathbb{C})$, here $A \in M_n(\mathbb{C})$ is nonsingular, τ is a field automorphism of $\mathbb{C}$ and $T_\tau = (\tau(t_{ij}))$. ϕ is continuous if and only if τ is the identity or the conjugation. However, Theorem 2.1 implies that every surjective ring homomorphism from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$ is automatically continuous if both H and K are infinite-dimensional.

The method used in the proof of Theorem 2.1 is almost valid for general von Neumann algebra case. But we have to restrict our attention on bounded linear maps.

Theorem 2.3. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a Jordan zero-product preserving bounded linear surjection between von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. Then $\Phi(I)$ is an invertible central element and there exist a central idempotent E of $\mathcal{N}$ and a homomorphism $\Phi_1 : \mathcal{M} \to E\mathcal{N}$ as well as an anti-homomorphism $\Phi_2 : \mathcal{M} \to (I - E)\mathcal{N}$ such that*

$$\Phi(A) = \Phi(I)\big(\Phi_1(A) \dotplus \Phi_2(A)\big)$$

for all $A \in \mathcal{M}$.

Proof. Note that the linear span of projections is norm dense in a von Neumann algebra. Checking the proof of Theorem 2.1, and using the continuity of Φ, one can get

$$\Phi(I)\Phi(A^2) + \Phi(A^2)\Phi(I) = 2\Phi(A)^2$$

and

$$\Phi(I)\Phi(A)^2 = \Phi(A)^2\Phi(I)$$

for every $A \in \mathcal{M}$. Thus $\Phi(I) \neq 0$ is in the center of $\mathcal{N}$ since Φ is surjective and every element in a von Neumann algebra is a sum of at most four square elements. It follows that $\Phi(I)\Phi(A^2) = \Phi(A)^2$ for all $A \in \mathcal{M}$. It is also clear that $\Phi(I)$ is invertible. Let $\Psi(\cdot) = \Phi(I)^{-1}\Phi(\cdot)$. Then Ψ is a Jordan algebraic homomorphism. Since von Neumann algebras are local matrix rings, by [9, Theorem 7], there exists a central idempotent E of $\mathcal{N}$ such that $E\Psi$ is a homomorphism and $(I - E)\Psi$ is an anti-homomorphism. □

Recall that a von Neumann algebra is called properly infinite if it contains no nonzero finite central projection. Since every element in a properly infinite von Neumann algebra is a sum of at most five idempotents [14], a similar argument as that in the proof of Theorem 2.1 yields the following

Theorem 2.4. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra and Φ a Jordan zero-product preserving additive surjection from $\mathcal{M}$ onto a von Neumann algebra $\mathcal{N}$. Then $\Phi(I)$ is an invertible central element and there exist a central idempotent E of $\mathcal{N}$ and a ring homomorphism $\Phi_1 : \mathcal{M} \to E\mathcal{N}$ as well as an ring anti-homomorphism $\Phi_2 : \mathcal{M} \to (I - E)\mathcal{N}$ such that*

$$\Phi(A) = \Phi(I)\big(\Phi_1(A) \dotplus \Phi_2(A)\big)$$

for all $A \in \mathcal{M}$.

3. The case of C*-algebras

Now we turn to the C*-algebra case. Since the linear sums of projections are dense in a unital C*-algebra of real rank zero [1], a similar argument as that in the proof of Theorem 2.3 shows that every bounded linear surjection from a C*-algebra of real rank zero onto a C*-algebra is a Jordan homomorphism multiplied by an invertible central element. However, to work with general C*-algebras requires more efforts.

If $\mathcal{A}$ is a unital C*-algebras, we denote by I the unit of $\mathcal{A}$ and $\mathcal{A}_{\mathrm{sa}}$ the real linear space of all self-adjoint elements in $\mathcal{A}$. It is obvious that $\mathcal{A}_{\mathrm{sa}}$ is a real Jordan algebra. Note that every surjective Jordan ring homomorphism from a unital ring onto a ring is unital.

The following are main results in this section.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be* C**-algebras with $\mathcal{A}$ unital. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a surjective bounded linear map preserving the Jordan zero-products. Then $\mathcal{B}$ is unital, $\Phi(I)$ is an invertible central element of $\mathcal{B}$, and there is a bounded surjective Jordan homomorphism $\mathcal{J}$ from $\mathcal{A}$ onto $\mathcal{B}$ such that*

$$\Phi(A) = \Phi(I)\mathcal{J}(A)$$

for all $A \in \mathcal{A}$.

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be* C**-algebras with $\mathcal{A}$ unital. Let $\Phi : \mathcal{A}_{\mathrm{sa}} \to \mathcal{B}_{\mathrm{sa}}$ be a surjective bounded real linear map preserving the Jordan zero-products. Then $\mathcal{B}$ is unital, $\Phi(I)$ is an invertible central element of $\mathcal{B}$, and there is a bounded surjective unital Jordan homomorphism $\mathcal{J}$ from $\mathcal{A}_{\mathrm{sa}}$ onto $\mathcal{B}_{\mathrm{sa}}$ such that*

$$\Phi(S) = \Phi(I)\mathcal{J}(S)$$

for all $S \in \mathcal{A}$.

Our proofs of these two theorems based on the following lemma, which models itself on [2, Lemma 4.4].

Lemma 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be* C**-algebras with $\mathcal{A}$ unital. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bounded linear map such that $\Phi(S)\Phi(T) + \Phi(T)\Phi(S) = 0$ for $S, T \in \mathcal{A}_{\mathrm{sa}}$ with $ST + TS = 0$. Then for any $S \in \mathcal{A}_{\mathrm{sa}}$, we have*

(1) $\Phi(I)\Phi(S)^2 = \Phi(S)^2\Phi(I)$,
(2) $\Phi(I)\Phi(S^2) + \Phi(S^2)\Phi(I) = 2\Phi(S)^2$.

Proof. Identify the C*-subalgebra of $\mathcal{A}$ generated by I and S with $C(\Lambda)$, where $\Lambda \subseteq [-\|S\|, \|S\|]$ is the spectrum of S, and $C(\Lambda)$ is the algebra of all continuous complex functions defined on Λ. Denote again by Φ the bidual map of Φ from $C(\Lambda)^{**}$ into $\mathcal{B}^{**}$. For each positive integer n and each integer k, let

$$\Lambda_{n,k} = \bigl(k/n, (k+1)/n\bigr] \cap \Lambda.$$

Pick an arbitrary point $x_{n,k}$ from each nonempty $\Lambda_{n,k}$. Set $x_{n,k}=\infty$ to be the isolated point at infinity of $\Lambda_\infty=\Lambda\cup\{\infty\}$ if $\Lambda_{n,k}=\emptyset$. For any $f\in C(\Lambda)$, using the convention $f(\infty)=0$, we have

$$f=\lim_{n\to\infty}\sum_{k\in\mathbb{Z}} f(x_{n,k})1_{\Lambda_{n,k}},$$

where $1_{\Lambda_{n,k}}$ is the characteristic function of the Borel set $\Lambda_{n,k}$, and the limit of the finite sums converges uniformly on Λ. In particular, for every fixed positive integer n we have

$$1=\sum_{k\in\mathbb{Z}}1_{\Lambda_{n,k}}.$$

For two disjoint nonempty sets $\Lambda_{n,j}$ and $\Lambda_{n,k}$, we can find two sequences $\{f_m\}_m$ and $\{g_m\}_m$ in $C(X)$ such that $f_{m+p}g_m=0$ for $m,p=0,1,\ldots,$ $f_m\to 1_{\Lambda_{n,j}}$ and $g_m\to 1_{\Lambda_{n,k}}$ pointwise on Λ. By the weak* continuity of Φ, we see that

$$\begin{aligned}&\Phi(1_{\Lambda_{n,j}})\Phi(g_m)+\Phi(g_m)\Phi(1_{\Lambda_{n,j}})\\&\quad=\lim_{p\to\infty}\bigl(\Phi(f_{m+p})\Phi(g_m)+\Phi(g_m)\Phi(f_{m+p})\bigr)=0\end{aligned}\tag{3.1}$$

for all $m=1,2,\ldots.$ Thus

$$\begin{aligned}&\Phi(1_{\Lambda_{n,j}})\Phi(1_{\Lambda_{n,k}})+\Phi(1_{\Lambda_{n,k}})\Phi(1_{\Lambda_{n,j}})\\&\quad=\lim_{m\to\infty}\bigl(\Phi(1_{\Lambda_{n,j}})\Phi(g_m)+\Phi(g_m)\Phi(1_{\Lambda_{n,j}})\bigr)=0.\end{aligned}$$

Consequently, for each positive integer n and each integer j we have

$$\begin{aligned}&\Phi(1)\Phi(1_{\Lambda_{n,j}})+\Phi(1_{\Lambda_{n,j}})\Phi(1)\\&\quad=\sum_{k\in\mathbb{Z}}\bigl(\Phi(1_{\Lambda_{n,k}})\Phi(1_{\Lambda_{n,j}})+\Phi(1_{\Lambda_{n,j}})\Phi(1_{\Lambda_{n,k}})\bigr)=2\Phi(1_{\Lambda_{n,j}})^2.\end{aligned}\tag{3.2}$$

From (3.2), we have $\Phi(1_{\Lambda_{n,j}})^2\Phi(1)=\Phi(1)\Phi(1_{\Lambda_{n,j}})^2$, it follows that

$$\begin{aligned}\Phi(f)^2\Phi(1)&=\lim_{n\to\infty}\Bigl(\sum_{k\in\mathbb{Z}}f(x_{n,k})\Phi(1_{\Lambda_{n,k}})\Bigr)^2\Phi(1)\\&=\lim_{n\to\infty}\sum_{k\in\mathbb{Z}}f(x_{n,k})^2\Phi(1_{\Lambda_{n,k}})^2\Phi(1)\\&=\Phi(1)\lim_{n\to\infty}\sum_{k\in\mathbb{Z}}f(x_{n,k})^2\Phi(1_{X_{n,k}})^2\\&=\Phi(1)\Phi(f)^2.\end{aligned}$$

On the other hand, it follows from (3.1) and (3.2) that

$$\begin{aligned}2\Phi(f)^2&=\lim_{n\to\infty}2\Bigl(\sum_{k\in\mathbb{Z}}f(x_{n,k})\Phi(1_{\Lambda_{n,k}})^2\Bigr)\\&=\lim_{n\to\infty}2\sum_{k\in\mathbb{Z}}f(x_{n,k})^2\Phi(1_{\Lambda_{n,k}})^2\end{aligned}$$

$$= \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} f(x_{n,k})^2 \big(\Phi(1)\Phi(1_{\Lambda_{n,k}}) + \Phi(1_{\Lambda_{n,k}})\Phi(1)\big)$$
$$= \Phi(1)\Phi(f^2) + \phi(f^2)\Phi(1).$$

Hence the conclusion holds. □

Proof of Theorem 3.1. Replacing S by $S+T$ with $S, T \in \mathcal{A}_{\mathrm{sa}}$ in (1) and (2) of Lemma 3.3, we have

$$\Phi(I)\big(\Phi(S)\Phi(T) + \Phi(T)\Phi(S)\big) = \big(\Phi(S)\Phi(T) + \Phi(T)\Phi(S)\big)\Phi(I),$$
$$\Phi(I)\Phi(ST + TS) + \Phi(ST + TS)\Phi(I) = 2\big(\Phi(S)\phi(T) + \Phi(T)\Phi(S)\big).$$

For each $A \in \mathcal{A}$, write $A = S + iT$ with $S, T \in \mathcal{A}_{\mathrm{sa}}$. Applying above equations and the linearity of Φ, we get

$$\Phi(I)\Phi(A)^2 = \Phi(A)^2\Phi(I) \tag{3.3}$$

and

$$\Phi(I)\Phi(A^2) + \Phi(A^2)\Phi(I) = 2\Phi(A)^2 \tag{3.4}$$

hold for all $A \in \mathcal{A}$. Since every element in a C*-algebra is an algebraic sum of square elements and Φ is surjective, from (3.3), we know that $\Phi(I)$ is in the center of $\mathcal{B}$. Hence it follows from (3.4) that $\Phi(I)\mathcal{B} = \mathcal{B}$. In particular, $\Phi(I)E = \Phi(I)$ for some $E \in \mathcal{B}$. So,

$$\Phi(A)^2E = \Phi(A^2)\Phi(I)E = \Phi(A^2)\Phi(I) = \Phi(A)^2, \quad \forall A \in \mathcal{A}.$$

Thus $BE = B$ for all $B \in \mathcal{B}$. Similarly, $EB = B$ for all $B \in \mathcal{B}$. This implies that $\mathcal{B}$ is unital with unit E and it follows from $\Phi(I)\mathcal{B} = \mathcal{B}$ that $\Phi(I)$ is invertible.

Let $\mathcal{J}(A) = \Phi(I)^{-1}\Phi(A)$ for all $A \in \mathcal{A}$; then it is easy to verify that $\mathcal{J}$ is a surjective bounded Jordan homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. □

Proof of Theorem 3.2. Define $\Psi : \mathcal{A} \to \mathcal{B}$ such that $\Psi(A) = \Phi(S) + i\Phi(T)$ for all $A \in \mathcal{A}$ with the decomposition $A = S + iT$, $S, T \in \mathcal{A}_{\mathrm{sa}}$, then Ψ is surjective. By Lemma 3.3, it is easily checked that

$$\Psi(I)\Psi(A)^2 = \Psi(A)^2\Psi(I) \quad \text{and} \quad \Psi(I)\Psi(A^2) + \Psi(A^2)\Psi(I) = 2\Psi(A)^2.$$

The rest of the proof is the same as the proof of Theorem 3.1 and we omit it. □

4. The case of standard operator algebras on Banach spaces

All cases considered in Sections 2 and 3 are operator $*$-subalgebras or Jordan $*$-subalgebras on Hilbert spaces. Now let us turn to the case of standard operator algebras on real or complex Banach spaces.

Let X, Y be infinite dimensional Banach spaces over the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Denote by $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. Recall that a standard operator algebra on X is a norm closed subalgebra of $\mathcal{B}(X)$ which contains the identity and all finite-rank operators. In this subsection, we describe additive surjections between standard operator algebras on X and Y respectively which preserve Jordan zero-products in both directions. Let X' denote the dual of X and A' the adjoint of A for $A \in \mathcal{B}(X)$.

Theorem 4.1. *Let X, Y be real or complex infinite-dimensional Banach spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a unital additive surjection. If Φ preserves Jordan zero-products in both directions, then either*

(1) *there exist a bijective bounded linear or conjugate linear operator $U : X \to Y$ such that*

$$\Phi(A) = UAU^{-1}$$

for all $A \in \mathcal{A}$, or

(2) *there exist a bijective bounded linear or conjugate linear operator $U : X' \to Y$ such that*

$$\Phi(A) = UA'U^{-1}$$

for all $A \in \mathcal{A}$. In this case, X and Y are reflexive.

Proof. It is trivial to verify that Φ is injective. We proceed in steps.

Step 1. Φ preserves idempotents and rank-one idempotents in both directions.

If $P \in \mathcal{A}$ is an idempotent, then $P(I - P) + (I - P)P = 0$. This implies $\Phi(P)(I - \Phi(P)) + (I - \Phi(P))\Phi(P) = 0$, that is, $\Phi(P) = \Phi(P)^2$. Consequently, $\Phi(P)$ is an idempotent. Suppose that P is rank-one while $\Phi(P)$ is not rank-one. Then $\Phi(P)$ can be written as a sum of an idempotent and a rank-one idempotent in $\mathcal{B}$. Since Φ^{-1} satisfies the same hypotheses as Φ, what we have just proved shows that the rank-one idempotent P can also be written as a sum of two nonzero idempotents. This is a contradiction.

Step 2. Φ preserves rank-one operators in both directions. In particular, Φ preserves rank-one nilpotent in both directions.

Let P be an idempotent of rank-one, then for every nonzero $\lambda \in \mathbb{C}$, we have $(\lambda P)(I - P) + (I - P)(\lambda P) = 0$, which implies that $2\Phi(\lambda P) = \Phi(\lambda P)\Phi(P) + \Phi(P)\Phi(\lambda P)$. Since $\Phi(P)$ is a rank-one idempotent, one gets

$$\Phi(\lambda P)\Phi(P) = \Phi(P)\Phi(\lambda P)\Phi(P) = \Phi(P)\Phi(\lambda P).$$

It follows that $\Phi(\lambda P) = \Phi(P)\Phi(\lambda P)\Phi(P)$, which implies that $\Phi(\lambda P)$ is of rank-one. Especially, there exists $f_P(\lambda) \in \mathbb{C}$ such that $\Phi(\lambda P) = f_P(\lambda)\Phi(P)$.

If $A = x \otimes f$ is a nilpotent of rank-one, then there exist $f_1 \in X'$ such that $f_1(x) = 1$. Let $f_2 = f_1 - f$. Obviously $P_i = x \otimes f_i$ $(i = 1, 2)$ are rank-one idempotents and $A = P_1 - P_2 = x \otimes f_1 - x \otimes f_2$. Suppose that $\Phi(P_i) = y_i \otimes g_i$, by Step 1, $g_i(y_i) = 1$. Notice that $P = \frac{1}{2}(P_1 + P_2)$ is a rank-one idempotent. So $\Phi(P) = \frac{1}{2}(y_1 \otimes g_1) + (y_2 \otimes g_2)$ is a rank-one idempotent. It is clear that either y_1, y_2 are linear dependent or g_1, g_2 are linear dependent. Without loss of generality, assume $y_1 = y_2 = y$; then $\Phi(A) = y \otimes g_1 - y \otimes g_2$, which is a nilpotent of rank-one.

Step 3. Either

(a) there exists a bijective bounded linear or conjugate-linear operator $U : X \to Y$ such that

$$\Phi(A) = UAU^{-1}$$

for every finite rank operator $A \in \mathcal{B}(X)$, or

(b) there exists a bijective bounded linear or conjugate-linear operator $U : X' \to Y$ such that

$$\Phi(A) = UA'U^{-1}$$

for every finite rank operator $A \in \mathcal{B}(X)$. In this case, X and Y are reflexive.

Since Φ is additive and preserves rank-one operators, rank-one idempotent and rank-one nilpotent in both directions, the assertion follows easily from [11,13] (also see [7]).

Step 4. For every operator $A \in \mathcal{A}$ and rank-one idempotent $R \in \mathcal{B}(X)$, $\Phi(RAR) = \Phi(R)\Phi(A)\Phi(R)$.

By Step 3, for every finite rank operator $A_0 \in \mathcal{B}(X)$, we have

$$\Phi(RA_0R) = \Phi(R)\Phi(A_0)\Phi(R).$$

We have to prove that above equation holds for every $A \in \mathcal{A}$.

Let $R = z \otimes h$ and $P \in \mathcal{B}(X)$ with $P = x \otimes f$ a rank-one idempotent, where $x, z \in X$ and $f, h \in X'$. By [12, Lemma 3.5], there exist nilpotents $S = x \otimes g$ and $T = y \otimes f$ of rank-one with $y \in X$, $g \in X'$ such that $P = ST$. Furthermore, $Q = TS = y \otimes g$ is a idempotent of rank-one disjoint with P, and R is a linear combination of P, Q, S and T. For every $A \in \mathcal{A}$, let $B = (I - P - Q)A(I - P - Q)$; then we have $PB = QB = SB = TB = 0$ and $BP = BQ = BS = BT = 0$. Consequently, $RB = BR = 0$. By the property of Φ, one gets $\Phi(R)\Phi(B) + \Phi(B)\Phi(R) = 0$. Since $\Phi(R)$ is an idempotent, a simple computation shows that $\Phi(R)\Phi(B)\Phi(R) = 0$. Use the fact that $A - B$ is of finite rank, we get

$$\Phi(RAR) = \Phi(R(A - B)R) = \Phi(R)\Phi(A - B)\Phi(R) = \Phi(R)\Phi(A)\Phi(R).$$

Step 5. Either $\Phi(A) = UAU^{-1}$ for every $A \in \mathcal{A}$ or $\Phi(A) = UA'U^{-1}$ for every $A \in \mathcal{A}$.

Suppose that for the operators of finite rank the case (a) of Step 3 holds. Let $A \in \mathcal{A}$. For any $z \in X$ and $h \in X'$ with $h(z) = 1$, $R = z \otimes h \in \mathcal{B}(X)$ is an idempotent of rank-one, and by Step 4, we have

$$\tau\big(h(Az)\big)URU^{-1} = \tau\big(h\big(U^{-1}\Phi(A)Uz\big)\big)URU^{-1},$$

where τ is the identity or the conjugation of $\mathbb{C}$. This yields

$$h(Az) = h\big(U^{-1}\Phi(A)Uz\big). \tag{4.1}$$

Fix z for a moment. Then (4.1) holds for every $h \in X'$ with $h(z) = 1$ and so, for every $h \in X'$ by linearity. Thus, $Az = U^{-1}\Phi(A)Uz$ is valid for every $z \in X$ and the case (1) of the theorem is proved.

Now, assume that the case (b) of Step 3 holds true for every operators of finite rank. Then for every $z \in X$ and $h \in X'$ with $h(z) = 1$, by Step 4, we get

$$\tau\big(h(Az)\big)U(x \otimes h)'U^{-1} = \tau\big(h\big(\big(U^{-1}\Phi(A)U\big)'z\big)\big)$$

and therefore

$$h(Az) = h\big(\big(U^{-1}\Phi(A)U\big)'z\big).$$

Using similar arguments as above, we obtain $A = (U^{-1}\Phi(A)U)'$. Consequently, the case (2) of the theorem holds true. □

References

[1] L.G. Brown, G.K. Pedersen, C*-algebras of real rank zero, J. Funct. Anal. 99 (1991) 131–149.

[2] M.A. Chebotar, W.-F. Ke, P.-H. Lee, N.-C. Wong, Mappings preserving zero products, Studia Math. 155 (2003) 77–94.

[3] J.B. Conway, A Course in Operator Theory, Grad. Stud. Math. vol. 21, Amer. Math. Soc., Providence, RI, 2000.

[4] J.A. Erdos, On certain elements in C*-algebras, Illinois J. Math. 15 (1971) 628–693.

[5] Hadwin Lunch Bunch, Local multiplications on algebras spanned by idempotents, Linear Multilinear Algebra 37 (1994) 259–263.

[6] J. Hou, M. Gao, Elements of rank one and multiplications on von Neumann algebras, Chinese J. Contemp. Math. 15 (1994) 367–374.

[7] J. Hou, J. Cui, Introduction to the Linear Maps on Operator Algebra, Science Press, Beijing, 2002 (in Chinese).

[8] J. Hou, L. Zhao, Zero-product preserving additive maps on symmetric operator spaces and self-adjoint operator spaces, Linear Algebra Appl. 399 (2005) 235–244.

[9] N. Jacobson, C. Rickart, Jordan homomorphisms of rings, Trans. Amer. Math. Soc. 69 (1950) 479–502.

[10] W.-F. Ke, B.-R. Li, N.-C. Wong, Zero product preserving maps of operator-valued functions, Proc. Amer. Math. Soc. 132 (2004) 1979–1985.

[11] B. Kuzma, Additive mappings decreasing rank one, Linear Algebra Appl. 384 (2002) 175–187.

[12] M. Omladič, On operator preserving commutativity, J. Funct. Anal. 66 (1986) 105–122.

[13] M. Omladič, P. Šemrl, Linear mappings preserving operators of rank one, Linear. Algebra Appl. 182 (1993) 239–256.

[14] C. Pearcy, D. Topping, Sums of small numbers of idempotents, Michigan Math. J. 14 (1967) 453–465.