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On the transitive matrices over distributive lattices ☆

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Abstract

A matrix is called a lattice matrix if its elements belong to a distributive lattice. For a lattice matrix A of order n, if there exists an $n \times n$ permutation matrix P such that $F = PAP^{\mathrm{T}} = (f_{ij})$ satisfies $f_{ij} \not< f_{ji}$ for $i > j$, then F is called a canonical form of A. In this paper, the transitivity of powers and the transitive closure of a lattice matrix are studied, and the convergence of powers of transitive lattice matrices is considered. Also, the problem of the canonical form of a transitive lattice matrix is further discussed.

AMS classification: 15

Keywords: Lattice matrix; Transitive matrix; Transitive closure; Power; Convergence; Canonical form

1. Introduction

Transitive lattice matrices are an important type of lattice matrices which represent transitive L-relations [5] (or transitive V-relations [15]). Since the beginning of the 1980s, several authors have studied this type of matrices for some special cases of distributive lattices. In 1982, Kim [13] introduced the concept of transitive binary

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Boolean matrices and in 1983, Hashimoto [7] introduced the concept of transitive fuzzy matrices and considered the convergence of powers of transitive fuzzy matrices. A transitive fuzzy matrix represents a fuzzy transitive relation [3,10,21] and fuzzy transitive relations play an important role in clustering, information retrieval, preference, and so on [15,17,18]. In [8], Hashimoto gave the canonical form of a transitive fuzzy matrix. In [19], Tan considered the convergence of powers of transitive lattice matrices.

In this paper, we continue to study transitive lattice matrices. In Section 3, we shall discuss the transitivity of powers and the transitive closure of a lattice matrix. In Section 4, we shall consider the convergence of powers of transitive lattice matrices. In Section 5, we shall further discuss the problem of the canonical form of a transitive lattice matrix.

2. Definitions and preliminary lemmas

Let $(P, \leqslant)$ be a poset and $a, b \in P$. If $a \leqslant b$ or $b \leqslant a$ then a and b are called *comparable*. Otherwise, a and b are called *incomparable*, in notation, $a \| b$. If for any $a, b \in P$, a and b are comparable, then P is called a *chain*. An *unordered poset* is a poset in which $a \| b$ for all $a \neq b$. A chain C in a poset P is a nonempty subset of P, which, as a subposet, is a chain. An *antichain* C in a poset P is nonempty subset which, as a subposet, is unordered. The *width* of a poset P, denoted by $\omega(P)$, is n, where n is a natural number, iff there is an antichain in P of n elements and all antichains in P have $\leqslant n$ elements. A poset $(L, \leqslant)$ is called a *lattice* if for all a, b in L, the greatest lower bound and the least upper bound of a and b exist. It is clear that any chain is a lattice, which is called a *linear lattice*.

Let $(L, \leqslant)$ be a lattice. The least upper bound (or join) and the greatest lower bound (or meet) of a and b in L will be denoted by $a \vee b$ and $a \wedge b$, respectively. It is clear that if $(L, \leqslant)$ is a linear lattice (especially, the fuzzy algebra [0,1] or the binary Boolean algebra $B_1 = \{0, 1\}$) then $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for all a and b in L.

Let $(L, \leqslant, \vee, \wedge)$ be a lattice and $\phi \neq X \subseteq L$. X is call a *sublattice* of L if for any $a, b \in X$, $a \vee b$ and $a \wedge b \in X$. It is clear that if $\{X_\lambda \mid \lambda \in \Gamma\}$ is a set of sublattices of L, then $Y = \bigcap_{\lambda \in \Gamma} X_\lambda$ is sublattice of L whenever $Y \neq \phi$. Let X be a nonempty subset of L. Define the sublattice generated by X to be the intersection of all sublattices of L which contain X and denote it by $L(X)$. Let $(L', \leqslant, \vee, \wedge)$ be a lattice. A map ϕ from L to L' is call a *homomorphism* if $\phi(x \vee y) = \phi(x) \vee \phi(y)$ and $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ for all x, y in L. An injective homomorphism is called a *monomorphism* or an *embedding* of L into L'. In this case, we say L may be embedded into L'.

Let $(L, \leqslant, \vee, \wedge)$ be a lattice and $a, b \in L$. The least element x in L satisfying $b \vee x \geqslant a$ is called the *relative lower pseudocomplement* of b in a, and is denoted by $a - b$. If for any pair of elements a, b in L, $a - b$ exists, then L is said to be a *dually Brouwerian lattice*.

Remark 2.1. If L is a linear lattice with least element 0 and $a, b \in L$, then $a - b = \begin{cases} a & \text{if } a > b \\ 0 & \text{if } a \leqslant b \end{cases}$. In particular, if L is the fuzzy algebra [0,1] then the operation "$-$" coincides with the operation "$\ominus$" defined in [7]. If L is a Boolean lattice, then $a - b = a \wedge b'$, where b' is the complement element of b in L.

In this paper, the lattice $(L, \leqslant, \vee, \wedge)$ is always supposed to be a distributive lattice with the least and greatest elements 0 and 1, respectively.

The following lemmas will be used in this paper.

Lemma 2.1. *The sublattice generated by a finite set of elements of the lattice L is finite.*

Lemma 2.2. *Each finite distributive lattice can be embedded into a finite Boolean lattice.*

The proofs of Lemmas 2.1 and 2.2 can be found in [1].

Lemma 2.3 [20, Lemma 2.2]. *Let L be a dually Brouwerian lattice. Then for any a, b, c in L, we have*

(1) $a - b \leqslant a$;
(2) $a \leqslant b \Rightarrow a - b = 0$;
(3) $b \leqslant c \Rightarrow a - b \geqslant a - c$ *and* $b - a \leqslant c - a$;
(4) $a - (b \wedge c) = (a - b) \vee (a - c)$;
(5) $a - (b \vee c) \leqslant (a - b) \wedge (a - c)$;
(6) $(a \wedge b) - c \leqslant (a - c) \wedge (b - c)$;
(7) $(a - b) \vee (b - c) = (a \vee b) - (b \wedge c)$.

Lemma 2.4. *Let L be a dually Brouwerian lattice such that for any a, b, c in L, $(a \wedge b) - c = (a - c) \wedge (b - c)$. Then for any a, b, c in L, we have*

(1) $(a - b) \wedge (b - a) = 0$;
(2) $(a - b) \wedge (b - c) \leqslant (a \wedge b) - c$.

Proof. (1) Since

$$\begin{aligned} 0 &\leqslant (a - b) \wedge (b - a) \\ &\leqslant (a - (a \wedge b)) \wedge (b - (a \wedge b)) \quad \text{(by Lemma 2.3(3))} \\ &= (a \wedge b) - (a \wedge b) = 0 \quad \text{(by Lemma 2.3(2))}, \end{aligned}$$

we have

$$(a - b) \wedge (b - a) = 0.$$

This proves (1).

(2) Since

$$
\begin{aligned}
&((b-a)\wedge(b-c))\vee((a\wedge b)-c)\\
&\quad=((b-a)\wedge(b-c))\vee((a-c)\wedge(b-c))\\
&\quad=(b-c)\wedge((b-a)\vee(a-c))\\
&\quad=(b-c)\wedge((a\vee b)-(a\wedge c))\quad\text{(by Lemma 2.3(7))}\\
&\quad=b-c\quad\text{(because }(a\vee b)-(a\wedge c)\geqslant b-c\text{)},
\end{aligned}
$$

we have

$$
\begin{aligned}
(a-b)\wedge(b-c)&=(a-b)\wedge(((b-a)\wedge(b-c))\vee((a\wedge b)-c))\\
&=((a-b)\wedge(b-a)\wedge(b-c))\vee((a-b)\wedge((a\wedge b)-c))\\
&=(a-b)\wedge((a\wedge b)-c)\quad\text{(because }(a-b)\wedge(b-a)=0\text{)}\\
&\leqslant(a\wedge b)-c.
\end{aligned}
$$

This proves (2). □

Now let $(L, \leqslant, \vee, \wedge)$ be a distributive lattice and $M_n(L)$ the set of all $n \times n$ matrices over L (*lattice matrices*). For any A in $M_n(L)$, we shall denote by a_{ij} or A_{ij} the element of L which stands in the (i, j)th entry of A. We denote by E_{ij} the matrix all of whose entries are zero excepts its (i, j)th entry, which is 1. A matrix P in $M_n(L)$ is called a *permutation matrix* if exactly one of the elements of its every row and every column is 1 and the others are 0.

For any A, B, C in $M_n(L)$ and a in L, we define:

$A \vee B = C$ iff $c_{ij} = a_{ij} \vee b_{ij}$ for i, j in $N = \{1, 2, \ldots, n\}$;
$A \wedge B = C$ iff $c_{ij} = a_{ij} \wedge b_{ij}$ for i, j in N;
$A^{\mathrm{T}} = C$ iff $c_{ij} = a_{ji}$ for i, j in N;
$\triangledown A = A \wedge A^{\mathrm{T}}$;
$AB = C$ iff $c_{ij} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj})$ for i, j in N;
$aA = C$ iff $c_{ij} = a \wedge a_{ij}$ for i, j in N;
$A \leqslant B$ iff $a_{ij} \leqslant b_{ij}$ for i, j in N and $A \geqslant B$ iff $B \leqslant A$;
$I_n = (\delta_{ij})$, where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ for i, j in N.

The following properties will be used in this paper.

(a) $M_n(L)$ is a semigroup with the identity element I_n with respect to the multiplication;
(b) $(M_n(L), \vee, \cdot)$ is a semiring and for A, B, C, D in $M_n(L)$ if $A \leqslant B$ and $C \leqslant D$ then $AC \leqslant BD$.

(c) For any A, B, C in $M_n(L)$, $A(B \wedge C) \leqslant (AB) \wedge (AC)$ and $(B \wedge C)A \leqslant (BA) \wedge (CA)$.

Properties (a) and (b) can be found in [4]. Since $B \wedge C \leqslant B$ and $B \wedge C \leqslant C$ for any B and C in $M_n(L)$, by (b), we have that $A(B \wedge C) \leqslant AB$, $A(B \wedge C) \leqslant AC$, $(B \wedge C)A \leqslant BA$ and $(B \wedge C)A \leqslant CA$, and so $A(B \wedge C) \leqslant (AB) \wedge (AC)$ and $(B \wedge C)A \leqslant (BA) \wedge (CA)$. Thus property (c) follows.

For any A in $M_n(L)$, the *powers* of A are defined as follows:

$$A^\circ = I_n, \quad A^l = A^{l-1}A, \quad l \in Z_+,$$

where Z_+ denotes the set of all positive integers.

The (i, j)th entry of A^l is denoted by $a_{ij}^{(l)}$.

If L is a dually Brouwerian lattice and $A, B, C \in M_n(L)$, then we can define:

$A - B = C$ iff $c_{ij} = a_{ij} - b_{ij}$ for i, j in N;

$\Delta A = A - A^{\mathrm{T}}$.

Lemma 2.5 [4, Corollary 1.1]. *For any A in $M_n(L)$, the sequence*

$$A, A^2, \ldots, A^l, \ldots \tag{2.1}$$

is ultimately periodic.

For the sequence (2.1), let $k = k(A)$ and $d = d(A)$ be the least integers $k \geqslant 0$ and $d \geqslant 1$ such that $A^k = A^{k+d}$. The integers $k(A)$ and $d(A)$ are called the *index* and the *period* of A. Clearly, the sequence (2.1) is of the form

$$A, A^2, \ldots, A^{k(A)-1} | A^{k(A)}, \ldots, A^{k(A)+d(A)-1} | A^{k(A)}, \ldots, A^{k(A)+d(A)-1} | \ldots \tag{2.2}$$

It is well known from the theory of semigroups (see e.g. [9]) that the set $G(A) = \{A^{k(A)}, A^{k(A)+1}, \ldots, A^{k(A)+d(A)-1}\}$ is a cyclic group with respect to the multiplication. The identity element of $G(A)$ is A^r for some r with $k(A) \leqslant r \leqslant k(A) + d(A) - 1$. More precisely, let $\beta \geqslant 1$ be the uniquely determined integer such that $k(A) \leqslant \beta d(A) \leqslant k(A) + d(A) - 1$. Then $r = \beta d(A)$.

Let $A \in M_n(L)$. A is called *convergent* if $d(A) = 1$ and in this case $k(A)$ is called the *convergent index* of A; A is called *nilpotent* if there exists some integer $k \geqslant 1$ such that $A^k = O$(the zero matrix). It is clear that if A is nilpotent then A converges to the zero matrix and in this case $k(A)$ is called the *nilpotent index* of A.

Lemma 2.6 [4, Corollary 5.2]. *Let $A \in M_n(L)$. Then A is nilpotent iff $A^n = O$, i.e., $k(A) \leqslant n$.*

Let $A \in M_n(L)$, A is called *transitive* if $A^2 \leqslant A$; A is called *idempotent* if $A^2 = A$. Denote by $r = r(A)$ the least integer $r \geqslant 1$ such that A^r is idempotent and $t = t(A)$ the least integer $t \geqslant 1$ such that A^t is transitive. Clearly, $t(A) \leqslant r(A)$.

Let $B \in M_n(L)$. The matrix B is called the *transitive closure* of A if B is transitive and $A \leqslant B$, and for any transitive matrix C in $M_n(L)$ with $A \leqslant C$ we have $B \leqslant C$. The transitive closure of A is denoted by A^+.

Lemma 2.7 [2, Lemma 2]. *For any A in $M_n(L)$, we always have*

$$\bigvee_{s>n} A^s \leqslant \bigvee_{l=1}^{n} A^l.$$

Lemma 2.8. *For any A in $M_n(L)$, we have $A^+ = \bigvee_{l=1}^{n} A^l$.*

Proof. Let $B = \bigvee_{l=1}^{n} A^l$. Then $A \leqslant B$ and $B^2 = (\bigvee_{l=1}^{n} A^l)^2 = (\bigvee_{l=1}^{n} A^l) \vee (\bigvee_{s=n+1}^{2n} A^s)$. But $\bigvee_{s=n+1}^{2n} A^s \leqslant \bigvee_{s>n} A^s \leqslant \bigvee_{l=1}^{n} A^l$ (by Lemma 2.7), we have $B^2 \leqslant (\bigvee_{l=1}^{n} A^l) \vee (\bigvee_{l=1}^{n} A^l) = B$, i.e., B is transitive.

Let now C be any transitive matrix in $M_n(L)$ with $A \leqslant C$. Then $C^l \leqslant C$ (by the transitivity of C) and $A^l \leqslant C^l$ for any positive integer l, and so $B = \bigvee_{l=1}^{n} A^l \leqslant \bigvee_{l=1}^{n} C^l \leqslant C$. By the definition of the transitive closure, we have $B = A^+$. This completes the proof. □

Lemma 2.9. *Let L be a dually Brouwerian lattice and $A \in M_n(L)$. Then*

$$A = \Delta A \vee \triangledown A.$$

Proof. Let $S = \Delta A \vee \triangledown A$. Then for all i, j in N,

$$\begin{aligned} s_{ij} &= (\Delta A)_{ij} \vee (\triangledown A)_{ij} = (a_{ij} - a_{ji}) \vee (a_{ij} \wedge a_{ji}) \\ &\leqslant a_{ij} \text{ (because } a_{ij} - a_{ji} \leqslant a_{ij} \text{ and } a_{ij} \wedge a_{ji} \leqslant a_{ij}). \end{aligned}$$

On the other hand,

$$\begin{aligned} s_{ij} &= (a_{ij} - a_{ji}) \vee (a_{ij} \wedge a_{ji}) \\ &= ((a_{ij} - a_{ji}) \vee a_{ij}) \wedge ((a_{ij} - a_{ji}) \vee a_{ji}) \geqslant a_{ij}. \end{aligned}$$

Therefore $s_{ij} = a_{ij}$ for all i, j in N. i.e., $S = A$. This completes the proof. □

3. Transitivity of powers of a lattice matrix

In this section, we shall discuss the transitivity of the powers of a lattice matrix A in $M_n(L)$.

The following propositions can be found in [19].

Proposition 3.1 [19, Propositions 3.2 and 3.4]. *If A^s, $s \geqslant 1$, is transitive, then*

(1) $A^{r(A)} \leqslant A^s$. *More generally,* $A^{r(A)} \leqslant A^{s+ld(A)}$ *for any integer* $l \geqslant 0$;
(2) $d(A)|s$. *In particular,* $d(A)|t(A)$.

Proposition 3.2 [19, Proposition 3.3]. *The group* $G(A) = \{A^{k(A)}, A^{k(A)+1}, \ldots, A^{k(A)+d(A)-1}\}$ *contains exactly one transitive matrix* (*namely* $A^{r(A)}$).

Proposition 3.3. *Let* $A \in M_n(L)$. *If* A^s *is transitive then*

(1) $A^+ \wedge I_n = A^s \wedge I_n$;
(2) *If* $d(A) > 1$, *then none of the matrices*

$$A^{s+1}, A^{s+2}, \ldots, A^{s+d(A)-1}$$

is transitive. *In particular*, *none of the matrices*

$$A^{t(A)+1}, A^{t(A)+2}, \ldots, A^{t(A)+d(A)-1}$$

is transitive.

Proof. (1) For any integer $l > 0$, we have $A^{ls} \leqslant A^s$ since A^s is transitive, and so $a_{ii}^{(ls)} \leqslant a_{ii}^{(s)}$ for all i in N. Since $a_{ii}^{(l)} \leqslant a_{ii}^{(ls)}$, we have $a_{ii}^{(l)} \leqslant a_{ii}^{(s)}$, and so $\bigvee_{l=1}^{n} a_{ii}^{(l)} \leqslant a_{ii}^{(s)}$ for all i in N. i.e., $A^+ \wedge I_n \leqslant A^s \wedge I_n$. On the other hand, we have $A^s \wedge I_n \leqslant A^+ \wedge I_n$ since $A^s \leqslant A^+$. Therefore $A^+ \wedge I_n = A^s \wedge I_n$. This completes the proof of (1).

(2) If $A^{s+\lambda}$ $(1 \leqslant \lambda \leqslant d(A) - 1)$ were transitive, then Proposition 3.1(2) could imply $d(A)|s$ and $d(A)|(s + \lambda)$, which is impossible. This proves (2). $\square$

Proposition 3.4. *If* $I_n \leqslant A^+$, *then the sequence* (2.1) *contains a unique transitive matrix*, *namely* $A^{r(A)}$.

Proof. Let A^s be transitive $(s \geqslant 1)$. By Proposition 3.3(1) and the hypothesis $I_n \leqslant A^+$, we have $A^s \geqslant I_n$, and so $A^s = A^s I_n \leqslant A^s A^s = A^{2s}$. On the other hand, by the transitivity of A^s, we have $A^{2s} \leqslant A^s$. Hence $A^s = A^{2s}$, and since there is a unique idempotent in the sequence (2.1), we have $A^s = A^{r(A)}$. This completes the proof. $\square$

By Proposition 3.1(2), we have that $d(A)|t(A)$ and $d(A)|r(A)$ and by Proposition 3.3(2), we know that all transitive matrices in the sequence (2.1) are contained in the set $\{A^{t(A)}, A^{t(A)+d(A)}, \ldots, A^{r(t)}\}$, but, in general, we cannot state that the all matrices in this set are transitive.

Proposition 3.5. *For any* A *in* $M_n(L)$, *the integer* $d(A) = |G(A)|$ *is the greatest common divisor of all integers* $s > 0$ *such that* A^s *is transitive*.

Proof. Consider the (formally infinite) sequence

$$A^{t(A)}, A^{t(A)+d(A)}, \ldots A^{r(A)} = A^{t(A)+(l-1)d(A)}, A^{t(A)+ld(A)}, \ldots$$

where $r(A) = t(A) + (l-1)d(A)$. Since the greatest common divisor (g.c.d) of the integers $t(A), t(A) + d(A), t(A) + 2d(A), \ldots$, is exactly the number $d(A)$, we have the proposition. $\square$

In the end of this section, we mention two transitive matrices which are intimately connected with any matrix A in $M_n(L)$.

Proposition 3.6. *Let* $A \in M_n(L)$, $\sigma(A) = \bigvee_{l=0}^{d(A)-1} A^{k(A)+l}$ *and* $\tau(A) = \bigwedge_{l=0}^{d(A)-1} A^{k(A)+l}$. *Then*

(1) $\sigma(A) = A^{n-1}A^+ = (A^+)^n$;
(2) $\sigma(A)$ *and* $\tau(A)$ *are transitive*.

Proof. (1) Since

$$\begin{aligned}\sigma(A)A &= \bigvee_{l=1}^{d(A)} A^{k(A)+l}\\ &= \bigvee_{l=0}^{d(A)-1} A^{k(A)+l} \quad (\text{Note that } A^{k(A)+d(A)} = A^{k(A)})\\ &= \sigma(A),\end{aligned}$$

we have that $\sigma(A)A^l = \sigma(A)$ for any integer $l \geqslant 1$.

Therefore

$$\sigma(A)A^+ = \sigma(A)\cdot(A \vee A^2 \vee \cdots \vee A^n) = \bigvee_{l=1}^{n} \sigma(A)A^l = \sigma(A). \tag{3.1}$$

By Lemma 2.8, we have $AA^+ \leqslant (A^+)^2 \leqslant A^+$. This implies

$$A^+ \geqslant AA^+ \geqslant A^2A^+ \geqslant \cdots \tag{3.2}$$

In the following we will prove that $A^{n-1}A^+ = A^lA^+$ for every integer $l \geqslant n-1$.

For any i, j in N, let T be any term of the (i, j)th entry $a_{ij}^{(n)}$ of A^n. Then T is of the form $a_{ii_1} \wedge a_{i_1i_2} \wedge \cdots \wedge a_{i_{n-1}j}$, where $1 \leqslant i_1, i_2, \ldots, i_{n-1} \leqslant n$. Since the number of the indices $i, i_1, i_2, \ldots, i_{n-1}, j$ is $n+1$, there must be two indices i_u and i_v such that $i_u = i_v$ for some u and $v (u < v)$ (taking $i_0 = i$ and $i_n = j$). Therefore

$$\begin{aligned}T &= a_{ii_1} \wedge a_{i_1i_2} \wedge \cdots \wedge a_{i_{n-1}j}\\ &= a_{ii_1} \wedge \cdots \wedge a_{i_{u-1}i_u} \wedge (a_{i_ui_{u+1}} \wedge \cdots \wedge a_{i_{v-1}i_u}) \wedge a_{i_ui_{v+1}} \wedge \cdots \wedge a_{i_{n-1}j}\\ &= a_{ii_1} \wedge \cdots \wedge a_{i_{u-1}i_u} \wedge (a_{i_ui_{u+1}} \wedge \cdots \wedge a_{i_{v-1}i_u})\\ &\quad \wedge (a_{i_ui_{u+1}} \cdots a_{i_{v-1}i_u}) \wedge a_{i_ui_{v+1}} \wedge \cdots \wedge a_{i_{n-1}j}\\ &\leqslant a_{ij}^{(n+(v-u))} \leqslant \bigvee_{l=n+1}^{2n} a_{ij}^{(l)},\end{aligned}$$

and so $a_{ij}^{(n)} \leqslant \bigvee_{l=n+1}^{2n} a_{ij}^{(l)}$ for all i, j in N. i.e., $A^n \leqslant A^{n+1} \vee \cdots \vee A^{2n}$. Thus

$$A^{n-1}A^+ = A^n \vee A^{n+1} \vee \cdots \vee A^{2n-1} \leqslant A^{n+1} \vee A^{n+2} \vee \cdots \vee A^{2n} = A^n A^+.$$

Since $A^n A^+ \leqslant A^{n-1}A^+$ (by (3.2)), we have $A^{n-1}A^+ = A^n A^+$, and so

$$A^{n-1}A^+ = A^l A^+ \tag{3.3}$$

for every integer $l \geqslant n-1$.

Now

$$\begin{aligned}\sigma(A) &= A^{k(A)} \vee \cdots \vee A^{k(A)+d(A)-1}\\ &= A^{k(A)+\alpha d(A)} \vee \cdots \vee A^{k(A)+\alpha d(A)+d(A)-1}\end{aligned}$$

for any integer $\alpha \geqslant 0$.

Choose α such that $l = k(A) + \alpha d(A) \geqslant n-1$, we then have that

$$\begin{aligned}\sigma(A) &= \sigma(A)A^+ \text{ (by (3.1))}\\ &= (A^l \vee A^{l+1} \vee \cdots \vee A^{l+d(A)-1})A^+\\ &= (A^l A^+) \vee (A^{l+1}A^+) \vee \cdots \vee (A^{l+d(A)-1}A^+)\\ &= (A^{n-1}A^+) \vee (A^{n-1}A^+) \vee \cdots \vee (A^{n-1}A^+) \quad \text{(by (3.3))}\\ &= A^{n-1}A^+.\end{aligned}$$

Since

$$\begin{aligned}(A^+)^n &= (A \vee A^2 \vee \cdots \vee A^n)^{n-1}A^+\\ &= (A^{n-1} \vee A^n \vee \cdots \vee A^{n(n-1)})A^+\\ &= (A^{n-1}A^+) \vee (A^n A^+) \vee \cdots \vee (A^{n(n-1)}A^+)\\ &= A^{n-1}A^+ \quad \text{(by (3.3))},\end{aligned}$$

we have $\sigma(A) = (A^+)^n$. This proves (1).

(2) Since

$$(\sigma(A))^2 = (A^+)^{2n} = ((A^+)^2)^n \leqslant (A^+)^n = \sigma(A),$$

we have that $\sigma(A)$ is transitive.

Since

$$(\tau(A))^2 = \tau(A)\left(\bigwedge_{l=0}^{d(A)-1} A^{k(A)+l}\right) \leqslant \bigwedge_{l=0}^{d(A)-1} (\tau(A)A^{k(A)+l})$$

and for any l in $\{0, 1, \ldots, d(A) - 1\}$

$$\tau(A)A^{k(A)+l} = \left(\bigwedge_{s=0}^{d(A)-1} A^{s+k(A)}\right) A^{k(A)+l} \leqslant \bigwedge_{s=0}^{d(A)-1} A^{s+l+2k(A)}$$

$$= \bigwedge_{t=0}^{d(A)-1} A^{k(A)+t} = \tau(A),$$

we have $(\tau(A))^2 \leqslant \tau(A)$, i.e., $\tau(A)$ is transitive. This proves (2). □

Corollary 3.1. *If $I_n \leqslant A^+$, then $\sigma(A) = A^+$.*

Proof. Since $I_n \leqslant A^+$, we have $A^+ \leqslant (A^+)^2$, and so $A^+ = (A^+)^2$. Therefore $(A^+)^n = A^+$, and so $\sigma(A) = (A^+)^n$ (by Proposition 3.6(1))$= A^+$. This completes the proof. □

4. Convergence of powers of transitive lattice matrices

In this section, we shall discuss the convergence of powers of transitive matrices in $M_n(L)$.

Theorem 4.1. *Let $A, C \in M_n(L)$. If A is transitive and $A \wedge I_n \leqslant C \leqslant A$, then*

(1) *C converges to $C^{k(C)}$ with $k(C) \leqslant n$.*
(2) *If A satisfies $\bigvee_{i=1}^{n}(a_{ij} \vee a_{ji}) \leqslant a_{jj}$ for some j in N, then C converges to $C^{k(C)}$ with $k(C) \leqslant n - 1$.*
(3) *If C satisfies $\bigvee_{i=1}^{n}(c_{ij} \vee c_{ji}) \leqslant c_{jj}$ for some j in N, then C converges to $C^{k(C)}$ with $k(C) \leqslant n - 1$.*

Proof. First, we have that $a_{ii} \leqslant c_{ii} \leqslant a_{ii}$ for all i in N since $A \wedge I_n \leqslant C \leqslant A$, and so for all i in N

$$a_{ii} = c_{ii} \tag{4.1}$$

(1) We know that any term T of the (i, j)th entry $c_{ij}^{(n)}$ of C^n is of the form $c_{ii_1} \wedge c_{i_1i_2} \wedge \cdots \wedge c_{i_{n-1}j}$. Since the number of indices $i, i_1, i_2, \ldots, i_{n-1}, j$ is $n + 1$, there must be two indices i_u and i_v such that $i_u = i_v$ for some u and v $(u < v)$ (taking $i_0 = i$ and $i_n = j$). Then $T \leqslant c_{i_ui_{u+1}} \wedge \cdots \wedge c_{i_{v-1}i_u}$ and $T \leqslant c_{ii_1} \wedge c_{i_1i_2} \wedge \cdots \wedge c_{i_{u-1}i_u} \wedge c_{i_ui_{v+1}} \wedge \cdots \wedge c_{i_{n-1}j}$. Since A is transitive, we have $A \geqslant A^k$ for all $k \geqslant 1$, and so $a_{ij} \geqslant a_{ij}^{(k)}$ for all i, j in N and all $k \geqslant 1$. Thus

$$\begin{aligned}c_{i_u i_u}^{(v-u-1)} &\geqslant c_{i_u i_u} = a_{i_u i_u} \quad \text{(by (4.1))}\\ &\geqslant a_{i_u i_u}^{(v-u)} \geqslant c_{i_u i_u}^{(v-u)} \quad (\text{because } C \leqslant A)\\ &\geqslant c_{i_u i_{u+1}} \wedge \cdots \wedge c_{i_{v-1} i_u} \geqslant T.\end{aligned}$$

Let T_1 be any term of $c_{i_u i_u}^{(v-u-1)}$. Then T_1 is of the form $c_{i_u t_1} \wedge c_{t_1 t_2} \wedge \cdots \wedge c_{t_{v-u-2} i_u}$ for some $t_1, t_2, \ldots, t_{v-u-2}$ in N, and so that $c_{ii_1} \wedge c_{i_1 i_2} \wedge \cdots \wedge c_{i_{u-1} i_u} \wedge T_1 \wedge c_{i_u i_{v+1}} \wedge \cdots \wedge c_{i_{n-1} j}$ is a term of $c_{ij}^{(n-1)}$. Therefore $c_{ij}^{(n-1)} \geqslant c_{ii_1} \wedge c_{i_1 i_2} \wedge \cdots \wedge c_{i_{u-1} i_u} \wedge T_1 \wedge c_{i_u i_{v+1}} \wedge \cdots \wedge c_{i_{n-1} j}$ for any term T_1 of $c_{i_u i_u}^{(v-u-1)}$, and so

$$c_{ij}^{(n-1)} \geqslant c_{ii_1} \wedge c_{i_1 i_2} \wedge \cdots \wedge c_{i_{u-1} i_u} \wedge c_{i_u i_u}^{(v-u-1)} \wedge c_{i_u i_{v+1}} \wedge \cdots \wedge c_{i_{n-1} j} \geqslant T.$$

Since $c_{ij}^{(n-1)} \geqslant T$ for every term T of $c_{ij}^{(n)}$, we have $c_{ij}^{(n)} \leqslant c_{ij}^{(n-1)}$, i.e., $C^n \leqslant C^{n-1}$. Certainly $C^{n+1} \leqslant C^n$.

On the other hand, since

$$\begin{aligned}c_{ij}^{(n+1)} &\geqslant c_{ii_1} \wedge c_{i_1 i_2} \wedge \cdots \wedge c_{i_{u-1} i_u} \wedge c_{i_u i_u} \wedge c_{i_u i_{u+1}} \wedge \cdots \wedge c_{i_{n-1} j}\\ &= T \wedge c_{i_u i_u} = T \quad (\text{because } T \leqslant c_{i_u i_u}),\end{aligned}$$

we have $c_{ij}^{(n+1)} \geqslant c_{ij}^{(n)}$, i.e., $C^{n+1} \geqslant C^n$. Since $C^{n+1} \leqslant C^n$, we have $C^n = C^{n+1}$ and $k(C) \leqslant n$. This proves (1).

(2) By the proof of (1), we have $C^n \leqslant C^{n-1}$. In the following we shall show that $C^{n-1} \leqslant C^n$. It is clear that any term T of the (i, j)th entry $c_{ij}^{(n-1)}$ of C^{n-1} is of the form $c_{ii_1} \wedge c_{i_1 i_2} \wedge \cdots \wedge c_{i_{n-2} j}$. Let $i_0 = i$ and $i_{n-1} = j$.

(a) If $i_u = i_v$ for some u and v $(u < v)$, then

$$\begin{aligned}c_{i_u i_u}^{(v-u)} &\geqslant c_{i_u i_{u+1}} \wedge \cdots \wedge c_{i_{v-1} i_u}\\ &\geqslant c_{ii_1} \wedge \cdots \wedge c_{i_{u-1} i_u} \wedge c_{i_u i_{u+1}} \wedge \cdots \wedge c_{i_{v-1} i_u} \wedge c_{i_u i_{v+1}} \wedge \cdots \wedge c_{i_{n-2} j}\\ &= T,\end{aligned}$$

and so

$$c_{i_u i_u} = a_{i_u i_u} \geqslant a_{i_u i_u}^{(v-u)} \geqslant c_{i_u i_u}^{(v-u)} \quad (\text{because } C \leqslant A) \geqslant T.$$

Then

$$\begin{aligned}c_{ij}^{(n)} &\geqslant c_{ii_1} \wedge c_{i_1 i_2} \wedge \cdots \wedge c_{i_{u-1} i_u} \wedge c_{i_u i_u} \wedge c_{i_u i_{u+1}} \wedge \cdots \wedge c_{i_{n-2} j}\\ &= T \wedge c_{i_u i_u} = T.\end{aligned}$$

(b) Suppose that $i_u \neq i_v$ for all $u \neq v$. By the hypothesis, $\bigvee_{l=1}^{n}(a_{li_m} \vee a_{i_m l}) \leqslant a_{i_m i_m}$ for some m. Then, by (4.1), we have $c_{i_m i_m} = a_{i_m i_m} \geqslant T$, and so

$$c_{ij}^{(n)} \geqslant c_{ii_1} \wedge c_{i_1 i_2} \wedge \cdots \wedge c_{i_{m-1} i_m} \wedge c_{i_m i_m} \wedge c_{i_m i_{m+1}} \wedge \cdots \wedge c_{i_{n-2} j} \geqslant T.$$

Therefore, $c_{ij}^{(n)} \geqslant T$ for every term T of $c_{ij}^{(n-1)}$, and so $c_{ij}^{(n-1)} \leqslant c_{ij}^{(n)}$, i.e., $C^{n-1} \leqslant C^n$. Thus $C^{n-1} = C^n$, i.e., $k(C) \leqslant n-1$. This proves (2).

(3) The proof of (3) is similar to that of (2). $\square$

As a special case of Theorem 4.1, we obtain the following corollary.

Corollary 4.1. *If $A \in M_n(L)$ is transitive, then*

(1) *A converges to $A^{k(A)}$ with $k(A) \leqslant n$;*
(2) *If A satisfies $\bigvee_{i=1}^{n}(a_{ij} \vee a_{ji}) \leqslant a_{jj}$ for some j in N, then A converges to $A^{k(A)}$ with $k(A) \leqslant n-1$.*

Remark 4.1. Corollary 4.1(1) is Theorem 5.1(1) in [19].

Remark 4.2. If L is the fuzzy algebra [0,1], then Theorem 4.1(2) and (3) become Theorems 2 and 3 in [7], respectively.

In the following, the lattice L will be supposed to be a dually Brouwerian lattice and satisfy the following conditions:

For any a, b, c in L,

$$a - (b \vee c) = (a - b) \wedge (a - c) \tag{CD$_1$}$$

and

$$(a \wedge b) - c = (a - c) \wedge (b - c) \tag{CD$_2$}$$

Such lattices are abundant: for example, every Boolean lattice, a complete linear lattice, the direct product of a finite number of complete linear lattices, and especially the fuzzy algebra [0,1] and $[0, 1]^n$ are all such kind of lattices.

Lemma 4.1. *If $A \in M_n(L)$ is transitive, then*

(1) *ΔA is transitive and nilpotent;*
(2) *∇A is idempotent.*

Proof. Since A is transitive, we have that for any i, j in N. $a_{ij}^{(2)} \leqslant a_{ij}$, i.e., $\bigvee_{k=1}^{n}(a_{ik} \wedge b_{kj}) \leqslant a_{ij}$, and so

$$a_{ik} \wedge a_{kj} \leqslant a_{ij} \tag{4.2}$$

for all i, j, k in N.

(1) For any i, j, k in N,

$$\begin{aligned}&(\Delta A)_{ik} \wedge (\Delta A)_{kj}\\&\quad = (a_{ik} - a_{ki}) \wedge (a_{kj} - a_{jk})\end{aligned}$$

$$
\begin{aligned}
&\leqslant (a_{ik} - (a_{kj} \wedge a_{ji})) \wedge (a_{kj} - (a_{ji} \wedge a_{ik})) \quad \text{(by (4.2) and Lemma 2.3(3))} \\
&= ((a_{ik} - a_{kj}) \vee (a_{ik} - a_{ji})) \wedge ((a_{kj} - a_{ji}) \vee (a_{kj} - a_{ik})) \\
&\quad \text{(by Lemma 2.3(4))} \\
&= ((a_{ik} - a_{kj}) \wedge (a_{kj} - a_{ji})) \vee ((a_{ik} - a_{kj}) \wedge (a_{kj} - a_{ik})) \\
&\quad \vee ((a_{ik} - a_{ji}) \wedge (a_{kj} - a_{ji})) \vee ((a_{kj} - a_{ik}) \wedge (a_{ik} - a_{ji})) \\
&\leqslant ((a_{ik} \wedge a_{kj}) - a_{ji}) \vee ((a_{ik} \wedge a_{kj}) - a_{ji}) \vee ((a_{ik} \wedge a_{kj}) - a_{ji}) \\
&\quad \text{(by Lemma 2.4 and (CD}_2\text{))} \\
&\leqslant a_{ij} - a_{ji} = (\Delta A)_{ij}.
\end{aligned}
$$

Therefore, $(\Delta A)^{(2)}_{ij} = \bigvee_{k=1}^{n}((\Delta A)_{ik} \wedge (\Delta A)_{kj}) \leqslant (\Delta A)_{ij}$. i.e., $(\Delta A)^2 \leqslant \Delta A$. In the following we shall show that ΔA is nilpotent.

Clearly, any term T of the (i, j)th entry $(\Delta A)^{(n)}_{ij}$ of $(\Delta A)^n$ is of the form $(\Delta A)_{ii_1} \wedge (\Delta A)_{i_1 i_2} \wedge \cdots \wedge (\Delta A)_{i_{n-1} j}$, where $1 \leqslant i_1, i_2, \ldots, i_{n-1} \leqslant n$. Then there must be two indices i_u and i_v such that $i_u = i_v$ for some u and v $(u < v)$ (taking $i_0 = i$ and $i_n = j$), and so

$$
\begin{aligned}
T &\leqslant (\Delta A)_{i_u i_{u+1}} \wedge \cdots \wedge (\Delta A)_{i_{v-1} i_u} \\
&\leqslant (\Delta A)^{(v-u)}_{i_u i_u} \leqslant (\Delta A)_{i_u i_u} \quad \text{(by the transitivity of } \Delta A) \\
&= a_{i_u i_u} - a_{i_u i_u} = 0.
\end{aligned}
$$

Therefore $(\Delta A)^{(n)}_{ij} = 0$ for all i, j in N. i.e., $(\Delta A)^n = O$. This proves (1).

(2) For any i, j in N,

$$
\begin{aligned}
(\nabla A)^{(2)}_{ij} &= \bigvee_{k=1}^{n}((\nabla A)_{ik} \wedge (\nabla A)_{kj}) = \bigvee_{k=1}^{n}(a_{ik} \wedge a_{ki} \wedge a_{kj} \wedge a_{jk}) \\
&= \bigvee_{k=1}^{n}((a_{ik} \wedge a_{kj}) \wedge (a_{jk} \wedge a_{ki})) \leqslant \bigvee_{k=1}^{n}(a_{ij} \wedge a_{ji}) = (\nabla A)_{ij}.
\end{aligned}
$$

On the other hand, since

$$
\begin{aligned}
(\nabla A)_{ij} &= a_{ij} \wedge a_{ji} \leqslant a_{ii} \quad \text{(by (4.2))} \\
&= a_{ii} \wedge a_{ii} = (\nabla A)_{ii},
\end{aligned}
$$

we have

$$
(\nabla A)^{(2)}_{ij} = \bigvee_{k=1}^{n}((\nabla A)_{ik} \wedge (\nabla A)_{kj}) \geqslant (\nabla A)_{ii} \wedge (\nabla A)_{ij} = (\nabla A)_{ij}.
$$

Therefore $(\nabla A)_{ij} = (\nabla A)^{(2)}_{ij}$, i.e., $\nabla A = (\nabla A)^2$. This proves (2). □

Remark 4.3. By Lemmas 2.9 and 4.1, we have that if $A \in M_n(L)$ is triansitive then A can be expressed as a join of a nilpotent matrix and an idempotent matrix in $M_n(L)$.

Theorem 4.2. *If $A \in M_n(L)$ is transitive, then A converges to $A^{k(A)}$ with $k(A) \leqslant k(\Delta A)$.*

Proof. By Lemma 2.9, $A = \Delta A \vee \triangledown A$. Let $M = \Delta A$ and $S = \triangledown A$. Since A is transitive, by Lemma 4.1, we have that M is nilpotent and S is idempotent. Therefore $M^l = O$ and $S^2 = S$, where $l = k(M) \leqslant n$. Now we consider the matrix $A^l = (M \vee S)^l$. Let T be any term of the expansion for $(M \vee S)^l$. Then T is of the form $T = M^{\alpha_1} S^{\beta_1} \cdots M^{\alpha_r} S^{\beta_r}$ for some nonnegative integers $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$ with $(\alpha_1 + \cdots + \alpha_r) + (\beta_1 + \cdots + \beta_r) = l$. If $\beta_1 = \cdots = \beta_r = 0$, then $T = M^{\alpha_1} \cdots M^{\alpha_r} = M^l = O$. If there exist some $\beta_t (1 \leqslant t \leqslant r)$ such that $\beta_t > 0$, then $S^{\beta_t + 1} = S^{\beta_t}$ since $S^2 = S$. In this case, $T = M^{\alpha_1} S^{\beta_1} \cdots M^{\alpha_t} S^{\beta_t + 1} \cdots M^{\alpha_r} S^{\beta_r}$. But $M^{\alpha_1} S^{\beta_1} \cdots M^{\alpha_t} S^{\beta_t + 1} \cdots M^{\alpha_r} S^{\beta_r}$ is also a term of the expansion for $(M \vee S)^{l+1} = A^{l+1}$, we have that $T \leqslant A^{l+1}$ for any term T of the expansion for $(M \vee S)^l = A^l$, and so $A^l \leqslant A^{l+1}$. On the other hand, we have $A^{l+1} \leqslant A^l$ since A is transitive. Thus $A^l = A^{l+1}$. This completes the proof. $\square$.

In [7] Hashimoto obtained the following Theorem.

Theorem 4.3 [7, Theorem 1]. *If A is an $n \times n$ transitive fuzzy matrix, then*

$$(A - AQ)^n = (A - AQ)^{n+1}$$

for any $n \times n$ fuzzy matrix Q.

Remark 4.4. Theorem 4.3 means that if A is an $n \times n$ transitive fuzzy matrix then the matrix $A - AQ$ converges to $(A - AQ)^{k(A-AQ)}$ with $k(A - AQ) \leqslant n$ for any $n \times n$ fuzzy matrix Q.

Theorem 4.4. *Let L be a complete linear lattice and $A \in M_n(L)$ be transitive. Then*

$$(A - AQ)^n = (A - AQ)^{n+1} \tag{4.3}$$

for any $n \times n$ matrix Q over L.

Proof. Similar to that of Theorem 1 in [7]. $\square$

In the following we shall prove that the equality (4.3) hold true for $n \times n$ transitive matrices over any distributive and dually Brouwerian lattice with the condition (CD_1).

To do this, we need some notations and lemmas.

Let B_k be a finite Boolean lattice and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote its atoms. It is clear that $|B_k| = 2^k$. For any a in B_k, the lth *constituent* of a, $a_{(l)}$, is in $B_1 = \{0, 1\}$, such that $a_{(l)} = 1$ if and only if $a \geqslant \sigma_l$. Evidently, $a = \bigvee_{l=1}^{k}(\sigma_l \wedge a_{(l)})$.

It is easy to verify that for a, b in B_k and $l \in \{1, 2, \ldots, k\}$, $(a \vee b)_{(l)} = a_{(l)} \vee b_{(l)}$, $(a \wedge b)_{(l)} = a_{(l)} \wedge b_{(l)}$ and $(a - b)_{(l)} = a_{(l)} - b_{(l)}$.

For any $m \times n$ matrix $A = (a_{ij})$ over B_k, the lth *constituent* of A, $A_{(l)}$, is an $m \times n$ matrix over B_1 whose (i, j)th entry is $a_{ij(l)}$. Evidently

$$A = \bigvee_{l=1}^{k} \sigma_l A_{(l)}.$$

Lemma 4.2 [14, Proposition 2.1]. *If $A = \bigvee_{l=1}^{k}\sigma_l C_{(l)}$ and $C_{(l)}$ are all $(0, 1)$ matrices, then $C_{(l)} = A_{(l)}$ for all $1 \leqslant l \leqslant k$.*

Lemma 4.3. *For all $m \times n$ matrices A and B over B_k, we have*

$$\begin{aligned}(A \vee B)_{(l)} &= A_{(l)} \vee B_{(l)} \quad \text{and}\\ (A - B)_{(l)} &= A_{(l)} - B_{(l)} \quad \text{for all } 1 \leqslant l \leqslant k.\end{aligned}$$

The proof is trivial. □

Lemma 4.4 [14, Proposition 2.2]. *For all $m \times n$ matrices A and all $n \times s$ matrices B over B_k, we have $(AB)_{(l)} = A_{(l)}B_{(l)}$ for all $1 \leqslant l \leqslant k$.*

Lemma 4.5. *Let $A \in M_n(B_k)$. Then A is transitive if and only if $A_{(l)}$ is transitive for all $1 \leqslant l \leqslant k$.*

The proof is trivial. □

Theorem 4.5. *Let L be a dually Brouwerian lattice with the condition (CD_1), and $A \in M_n(L)$ be transitive.*

Then

$$(A - AQ)^n = (A - AQ)^{n+1} \tag{4.4}$$

for any Q in $M_n(L)$.

Proof. Let $A = (a_{ij})$, $Q = (d_{ij}) \in M_n(L)$ and A be transitive.

Let $S(A, Q) = \{a_{ij}, d_{ij}, a_{ij} - a_{st}, a_{ij} - d_{st}, 1 \leqslant i, j, s, t, \leqslant n\}$ and $L(A, Q)$ denote the sublattice of L generated by $S(A, Q)$. By Lemma 2.1, $L(A, Q)$ is a finite distributive lattice, and so $L(A, Q)$ may be embedded in some finite Boolean lattice B_k (by Lemma 2.2). Therefore, A and Q may be regarded as matrices over B_k. Furthermore, since

$$
\begin{aligned}
(A - AQ)_{ij} &= a_{ij} - (AQ)_{ij} = a_{ij} - \left(\bigvee_{s=1}^{n}(a_{is} \wedge d_{sj})\right) \\
&= \bigwedge_{s=1}^{n}(a_{ij} - (a_{is} \wedge d_{sj})) \quad \text{(by the condition (CD}_1\text{))} \\
&= \bigwedge_{s=1}^{n}((a_{ij} - a_{is}) \vee (a_{ij} - d_{sj})) \quad \text{(by Lemma 2.3(4))},
\end{aligned}
$$

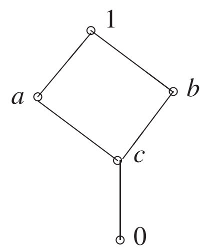
we have $(A - AQ)_{ij} \in L(A, Q) \subseteq B_k$, and so $A - AQ \in M_n(B_k)$. Since A is transitive, $A_{(l)}$ is transitive for all $1 \leqslant l \leqslant k$ (by Lemma 4.5). Then

$$
\begin{aligned}
(A - AQ)^n &= \sum_{l=1}^{k} \sigma_l((A - AQ)^n)_{(l)} \\
&= \sum_{l=1}^{k} \sigma_l(A_{(l)} - A_{(l)}Q_{(l)})^n \quad \text{(by Lemmas 4.3 and 4.4)} \\
&= \sum_{l=1}^{k} \sigma_l(A_{(l)} - A_{(l)}Q_{(l)})^{n+1} \\
&\quad \text{(by Theorem 4.4 and the transitivity of } A_{(l)}\text{)} \\
&= (A - AQ)^{n+1}.
\end{aligned}
$$

This completes the proof. □

Remark 4.5. The condition (CD_1) for the lattice L in Theorem 4.5 is necessary.

Example 4.1. Consider the lattice $L = \{0, a, b, c, 1\}$ whose diagram is as follows:



It is easy to verify that L is a distributive and dually Brouwerian lattice in which the condition (CD_1) is not true.

Let now

$$
A = \begin{bmatrix} c & a \\ b & c \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} c & c \\ c & c \end{bmatrix} \in M_2(L).
$$

Then

$$A^2 = \begin{bmatrix} c & a \\ b & c \end{bmatrix} \cdot \begin{bmatrix} c & a \\ b & c \end{bmatrix} = \begin{bmatrix} c & c \\ c & c \end{bmatrix} \leqslant A,$$

which means that A is transitive.

Next, we compute $R = A - AQ$, we have

$$A - AQ = \begin{pmatrix} c & a \\ b & c \end{pmatrix} - \begin{pmatrix} c & a \\ b & c \end{pmatrix} \cdot \begin{pmatrix} c & c \\ c & c \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

$$(A - AQ)^2 = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad \text{and} \quad (A - AQ)^3 = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}.$$

It is clear that $(A - AQ)^2 \neq (A - AQ)^3$.

5. On canonical form of a transitive matrix

The problem for the canonical form of a lattice matrix was first appeared in the work [11]. Let A be an $n \times n$ lattice matrix. If there exists an $n \times n$ permutation matrix P such that $F = PAP^{\mathrm{T}} = (f_{ij})$ satisfies $f_{ij} \nless f_{ij}$ for $i > j$ then F is called a *canonical form* of A. In [11], Kim and Roush posed the canonical problem for an idempotent fuzzy matrix.

Problem A. For an idempotent fuzzy matrix E does there exists a permutation matrix P such that $F = PEP^{\mathrm{T}}$ satisfies $f_{ij} \geqslant f_{ji}$ for $i > j$?

Problem A was solved by Kim and Roush in the work [12]. Furthermore, Hashimoto [8] presented the canonical form of a transitive fuzzy matrix and obtained the following result.

Theorem 5.1 [8, Theorem 2]. *For a transitive fuzzy matrix A there exists a permutation matrix P such that $T = (t_{ij}) = PAP^{\mathrm{T}}$ satisfies $t_{ij} \geqslant t_{ji}$ for $i > j$.*

In 1986, Peng [16] introduced the concept of transitive matrices over a lattice and posed the following problem.

Problem B. For a transitive matrix A over a lattice does there exists a permutation matrix P such that $F = PAP^{\mathrm{T}}$ satisfies $f_{ij} \nless f_{ji}$ for $i > j$?

Problem B was solved by Hao in the work [6] by giving an example in the negative.

In this section, we will give further discussion for Problem B in the case of distributive lattices.

Theorem 5.2. *Let L be a distributive lattice and n be an integer with $n \geqslant 4$. Then for any $n \times n$ transitive matrix A over L there exists an $n \times n$ permutation matrix P such that $F = PAP^{\mathrm{T}}$ satisfies $f_{ij} \nless f_{ji}$ for $i > j$ if and only if L is a linear lattice.*

Proof. *Necessity*: Suppose that L is not a linear lattice. Then $\omega(L) \geqslant 2$, and so that there must be two elements a and b in L such that $a \| b$. Therefore $a \wedge b < a < a \vee b$ and $a \wedge b < b < a \vee b$. Now let

$$A = aE_{12} \vee bE_{23} \vee aE_{34} \vee bE_{41} \vee (a \wedge b)J_n \in M_n(L).$$

Then $A^2 = (a \wedge b)J_n \leqslant A$. This means that A is transitive. Let P be any $n \times n$ permutation matrix. Then there exists a unique permutation σ of the set $\{1, 2, \ldots, n\}$ such that $P = \bigvee_{i=1}^{n} E_{\sigma(i)i}$, and so $P^{\mathrm{T}} = \bigvee_{i=1}^{n} E_{i\sigma(i)}$. Therefore

$$\begin{aligned} F = (f_{ij})_{n\times n} &= PAP^{\mathrm{T}} \\ &= aE_{\sigma(1)\sigma(2)} \vee bE_{\sigma(2)\sigma(3)} \vee aE_{\sigma(3)\sigma(4)} \vee bE_{\sigma(4)\sigma(1)} \vee (a \wedge b)J_n. \end{aligned}$$

It is clear that

$$f_{\sigma(1)\sigma(2)} = f_{\sigma(3)\sigma(4)} = a, \quad f_{\sigma(2)\sigma(3)} = f_{\sigma(4)\sigma(1)} = b$$

and

$$f_{\sigma(2)\sigma(1)} = f_{\sigma(4)\sigma(3)} = f_{\sigma(3)\sigma(2)} = f_{\sigma(1)\sigma(4)} = a \wedge b.$$

Since σ is a permutation, we have $\sigma(i) \neq \sigma(j)(i \neq j)$. If there exists some $n \times n$ permutation matrix P such that $F = PAP^{\mathrm{T}}$ satisfies $f_{ij} \nless f_{ji}$ for $i > j$. Then the corresponding permutation σ satisfies the conditions $\sigma(1) > \sigma(2)$, $\sigma(2) > \sigma(3)$, $\sigma(3) > \sigma(4)$ and $\sigma(4) > \sigma(1)$. This implies $\sigma(1) > \sigma(1)$, which leads to a contradiction. This proves the necessity.

Sufficiency: If L is a linear lattice. Then by using the proof of Theorem 2 in [8], we have that for any $n \times n$ transitive matrix A over L there exists a permutation P such that $F = PAP^{\mathrm{T}}$ satisfies $f_{ij} \geqslant f_{ji}$ for $i > j$. That is, $f_{ij} \nless f_{ji}$ for $i > j$. This proves the sufficiency. □

Theorem 5.3. *Let L be a distributive lattice with $w(L) = 2$. Then for any 3×3 transitive matrix A over L there exists a 3×3 permutation matrix P such that $F = PAP^{\mathrm{T}}$ satisfies $f_{ij} \nless f_{ji}$ for $i > j$.*

In order to give the proof of Theorem 5.3, we need a lemma.

Lemma 5.1. *Let L be a distributive lattice with $\omega(L) = 2$ and $A = (a_{ij}) \in M_3(L)$ be transitive. If $a_{21} \nless a_{12}$, $a_{32} \nless a_{23}$ and $a_{31} < a_{13}$, then $a_{12} \nless a_{21}$ or $a_{23} \nless a_{32}$.*

Proof. Since A is transitive, we have that for any $i, j \in \{1, 2, 3\}$, $a_{ij} \geqslant a_{ij}^{(2)} = \bigvee_{k=1}^{3}(a_{ik} \wedge a_{kj})$, and so $a_{ij} \geqslant a_{ik} \wedge a_{kj}$ for all i, j, k in $\{1, 2, 3\}$.

Assume that the statement $a_{12} \not< a_{21}$ or $a_{23} \not< a_{32}$ is false. Then, we have that $a_{12} < a_{21}$ and $a_{23} < a_{32}$.

Since $w(L) = 2$, there are four cases to consider for the elements a_{21}, a_{32} and a_{13}.

Case I: The elements a_{21}, a_{32} and a_{13} are in the same chain. In this case, we have:

(1) If $a_{21} \leqslant a_{32} \leqslant a_{13}$ or $a_{21} \leqslant a_{13} \leqslant a_{32}$, then $a_{21} \leqslant a_{13} \wedge a_{32} \leqslant a_{12}$. This contradicts the assume $a_{12} < a_{21}$.

(2) If $a_{32} \leqslant a_{21} \leqslant a_{13}$ or $a_{32} \leqslant a_{13} \leqslant a_{21}$, then $a_{32} \leqslant a_{21} \wedge a_{13} \leqslant a_{23}$, which contradicts the assume $a_{23} < a_{32}$.

(3) If $a_{13} \leqslant a_{21} \leqslant a_{32}$ or $a_{13} \leqslant a_{32} \leqslant a_{21}$, then $a_{13} \leqslant a_{32} \wedge a_{21} \leqslant a_{31}$. This contradicts the condition $a_{31} < a_{13}$.

Case II: $a_{21} \| a_{32}$. In this case, we have that a_{13} and a_{21} are comparable or a_{13} and a_{32} are comparable.

(1) If $a_{13} \geqslant a_{21}$, then $a_{21} = a_{21} \wedge a_{13} \leqslant a_{23} < a_{32}$, which contradicts the condition $a_{21} \| a_{32}$.

(2) If $a_{13} \leqslant a_{21}$, then $a_{21} \geqslant a_{13} > a_{31} \geqslant a_{32} \wedge a_{21} \geqslant a_{23} \wedge a_{21} \geqslant (a_{21} \wedge a_{13}) \wedge a_{21} = a_{13}$, and so $a_{13} > a_{13}$, which is impossible.

(3) If $a_{13} \geqslant a_{32}$, then $a_{32} = a_{13} \wedge a_{32} \leqslant a_{12} < a_{21}$. This contradicts the condition $a_{21} \| a_{32}$.

(4) If $a_{13} \leqslant a_{32}$, then $a_{32} \geqslant a_{13} > a_{31} \geqslant a_{32} \wedge a_{21} \geqslant a_{32} \wedge a_{12} \geqslant a_{32} \wedge (a_{13} \wedge a_{32}) = a_{13}$, and so $a_{13} > a_{13}$, which is impossible.

Case III: $a_{21} \| a_{13}$. In this case, we have that a_{32} and a_{21} are comparable or a_{32} and a_{13} are comparable.

(1) If $a_{32} \geqslant a_{21}$, then $a_{21} = a_{32} \wedge a_{21} \leqslant a_{31} < a_{13}$, which contradicts the condition $a_{21} \| a_{13}$.

(2) If $a_{32} \leqslant a_{21}$, then $a_{21} \geqslant a_{32} > a_{23} \geqslant a_{21} \wedge a_{13} \geqslant a_{21} \wedge a_{31} \geqslant a_{21} \wedge (a_{32} \wedge a_{21}) = a_{32}$, and so $a_{32} > a_{32}$, which leads to a contradiction.

(3) If $a_{32} \geqslant a_{13}$, then $a_{13} = a_{13} \wedge a_{32} \leqslant a_{12} < a_{21}$. This contradicts the condition $a_{21} \| a_{13}$.

(4) If $a_{32} \leqslant a_{13}$, then $a_{13} \geqslant a_{32} > a_{23} \geqslant a_{21} \wedge a_{13} \geqslant a_{12} \wedge a_{13} \geqslant (a_{13} \wedge a_{32}) \wedge a_{13} = a_{32}$, and so $a_{32} > a_{32}$, which is impossible.

Case IV: $a_{13} \| a_{32}$. In this case, we have that a_{21} and a_{13} are comparable or a_{21} and a_{32} are comparable.

(1) If $a_{21} \geqslant a_{13}$, then $a_{13} = a_{21} \wedge a_{13} \leqslant a_{23} < a_{32}$, which contradicts the condition $a_{13} \| a_{32}$.

(2) If $a_{21} \leqslant a_{13}$, then $a_{13} \geqslant a_{21} > a_{12} \geqslant a_{13} \wedge a_{32} \geqslant a_{13} \wedge a_{23} \geqslant a_{13} \wedge (a_{21} \wedge a_{13}) = a_{21}$, and so $a_{21} > a_{21}$. This is a contradiction.

(3) If $a_{21} \geqslant a_{32}$, then $a_{32} = a_{32} \wedge a_{21} \leqslant a_{31} < a_{13}$. This contradicts the condition $a_{13} \| a_{32}$.

(4) If $a_{21} \leqslant a_{32}$, then $a_{32} \geqslant a_{21} > a_{12} \geqslant a_{13} \wedge a_{32} \geqslant a_{31} \wedge a_{32} \geqslant (a_{32} \wedge a_{21}) \wedge a_{32} = a_{21}$, and so $a_{21} > a_{21}$. This is a contradiction.

Therefore, we have that $a_{12} \nless a_{21}$ or $a_{23} \nless a_{32}$. This proves the lemma. □

The proof of Theorem 5.3. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in M_3(L)$ be transitive.

(1) If $a_{21} \nless a_{12}$, $a_{31} \nless a_{13}$ and $a_{32} \nless a_{23}$, then, setting $P = I_3$, we have $F = PAP^{\mathrm{T}} = A$. It is clear that $f_{21} \nless f_{12}$, $f_{31} \nless f_{13}$ and $f_{32} \nless f_{23}$.

(2) If $a_{21} < a_{12}$, $a_{31} \nless a_{13}$ and $a_{32} \nless a_{23}$, then, putting $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we have $F = PAP^{\mathrm{T}} = \begin{bmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$, so that $f_{21} = a_{12} \nless a_{21} = f_{12}$, $f_{31} = a_{32} \nless a_{23} = f_{13}$ and $f_{32} = a_{31} \nless a_{13} = f_{23}$.

(3) If $a_{21} \nless a_{12}$, $a_{31} \nless a_{13}$ and $a_{32} < a_{23}$, then, taking $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, we have $F = PAP^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{13} & a_{12} \\ a_{31} & a_{33} & a_{32} \\ a_{21} & a_{23} & a_{22} \end{bmatrix}$. Hence $f_{21} = a_{31} \nless a_{13} = f_{12}$, $f_{31} = a_{21} \nless a_{12} = f_{13}$ and $f_{32} = a_{23} \nless a_{32} = f_{23}$.

(4) If $a_{21} \nless a_{12}$, $a_{31} < a_{13}$ and $a_{32} \nless a_{23}$, then, by Lemma 5.1, we have that $a_{12} \nless a_{21}$ or $a_{23} \nless a_{32}$. If $a_{12} \nless a_{21}$, then, taking $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, we have $F = PAP^{\mathrm{T}} = \begin{bmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{13} & a_{11} \end{bmatrix}$, and so $f_{21} = a_{32} \nless a_{23} = f_{12}$, $f_{31} = a_{12} \nless a_{21} = f_{13}$ and $f_{32} = a_{13} \nless a_{31} = f_{23}$; if $a_{23} \nless a_{32}$, then, setting $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we have $F = PAP^{\mathrm{T}} = \begin{bmatrix} a_{33} & a_{31} & a_{32} \\ a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \end{bmatrix}$, and so $f_{21} = a_{13} \nless a_{31} = f_{12}$, $f_{31} = a_{23} \nless a_{32} = f_{13}$ and $f_{32} = a_{21} \nless a_{12} = f_{23}$.

(5) If $a_{21} \not< a_{12}$, $a_{31} < a_{13}$ and $a_{32} < a_{23}$, then, taking $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we have $F = PAP^{\mathrm{T}} = \begin{bmatrix} a_{33} & a_{31} & a_{32} \\ a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \end{bmatrix}$, and so $f_{21} = a_{13} \not< a_{31} = f_{12}$, $f_{31} = a_{23} \not< a_{32} = f_{13}$ and $f_{32} = a_{21} \not< a_{12} = f_{23}$.

(6) If $a_{21} < a_{12}$, $a_{31} \not< a_{13}$ and $a_{32} < a_{23}$, then, putting $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we have that $B = QAQ^{\mathrm{T}} = \begin{bmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$ and B is transitive. Also, $b_{21} = a_{12} \not< a_{21} = b_{12}$, $b_{32} = a_{31} \not< a_{13} = b_{23}$ and $b_{31} = a_{32} < a_{23} = b_{13}$. By Lemma 5.1, we have that $b_{12} \not< b_{21}$ or $b_{23} \not< b_{32}$. That is, $a_{21} \not< a_{12}$ or $a_{13} \not< a_{31}$. But $a_{21} < a_{12}$, we have $a_{13} \not< a_{31}$. Now put $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then $F = PAP^{\mathrm{T}} = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$, and so $f_{21} = a_{23} \not< a_{32} = f_{12}$, $f_{31} = a_{13} \not< a_{31} = f_{13}$ and $f_{32} = a_{12} \not< a_{21} = f_{23}$.

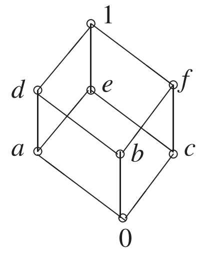
(7) If $a_{21} < a_{12}$, $a_{31} < a_{13}$ and $a_{32} \not< a_{23}$, then taking $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, we have $F = PAP^{\mathrm{T}} = \begin{bmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{13} & a_{11} \end{bmatrix}$, and so $f_{21} = a_{32} \not< a_{23} = f_{12}$, $f_{31} = a_{12} \not< a_{21} = f_{13}$ and $f_{32} = a_{13} \not< a_{31} = f_{23}$.

(8) If $a_{21} < a_{12}$, $a_{31} < a_{13}$ and $a_{32} < a_{23}$, then putting $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we have $F = PAP^{\mathrm{T}} = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$, and so $f_{21} = a_{23} \not< a_{32} = f_{12}$, $f_{31} = a_{13} \not< a_{31} = f_{13}$ and $f_{32} = a_{12} \not< a_{21} = f_{23}$.

This completes the proof. $\square$

Remark 5.1. If the distributive lattice L satisfies $\omega(L) \geqslant 3$, then the result in Theorem 5.3 is not true.

Example 5.1. Consider the lattice $L = \{0, a, b, c, d, e, f, 1\}$ whose diagram is as follows:



It is easy to verify that L is a distributive lattice with $\omega(L) = 3$.

Let $A = \begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix} \in M_3(L)$.

Then $A^2 = O \leqslant A$, which means that A is transitive.

Put now

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is clear that the matrices P_1, P_2, P_3, P_4, P_5 and P_6 are the only permutation matrices in $M_3(L)$. By a short computation, we have

$$F_1 = P_1AP_1^{\mathrm{T}} = A, \quad F_2 = P_2AP_2^{\mathrm{T}} = \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & a \\ c & 0 & 0 \end{bmatrix},$$

$$F_3 = P_3AP_3^{\mathrm{T}} = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ b & 0 & 0 \end{bmatrix}, \quad F_4 = P_4AP_4^{\mathrm{T}} = \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & b \\ a & 0 & 0 \end{bmatrix},$$

$$F_5 = P_5AP_5^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} \quad \text{and} \quad F_6 = P_6AP_6^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & b \\ c & 0 & 0 \\ 0 & a & 0 \end{bmatrix}.$$

It is easy to see that none of the matrices F_1, F_2, F_3, F_4, F_5 and F_6 satisfies the condition $f_{ij} \nless f_{ji}$ for all $i > j$.

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