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The Katětov construction modified for a T_0-quasi-metric space

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ABSTRACT

We discuss the existence and uniqueness of a T_0-quasi-metric space $q\mathbb{U}$ defined by the following three conditions: (i) $q\mathbb{U}$ is bicomplete and supseparable, (ii) every isometry between two finite subspaces of $q\mathbb{U}$ extends to an isometry of $q\mathbb{U}$ onto itself, and (iii) $q\mathbb{U}$ contains an isometric copy of every supseparable T_0-quasi-metric space.

1. Introduction

We establish the existence and uniqueness of an asymmetric analogue of the (universal ultrahomogeneous separable complete) Urysohn metric space $\mathbb{U}$. Indeed we show the existence and uniqueness (up to isometry) of a T_0-quasi-metric space $q\mathbb{U}$ possessing the following properties:

(a) $q\mathbb{U}$ is bicomplete and supseparable;
(b) $q\mathbb{U}$ is ultrahomogeneous, that is, every isometry between two finite subspaces of $q\mathbb{U}$ extends to an isometry of $q\mathbb{U}$ onto itself;
(c) $q\mathbb{U}$ is q-universal, that is, it contains an isometric copy of every supseparable T_0-quasi-metric space.

Naturally our arguments are related to the metric theory (presented for instance in [12,11]), but in general we have to work more carefully to compensate for the possible asymmetry and the nontrivial zero values of the distance function. In some sense our approach replaces functions by function pairs, similarly as the one in [6]. In this way we are able to follow essentially the classical metric theory, as it was for instance developed in [11].

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2. Preliminary remarks

For the basic concepts used from the theory of asymmetric topology we refer the reader to [2] and [8]. For the convenience of the reader and in order to fix our terminology we recall the following concepts.

Definition 1. Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then d is called a *quasi-pseudometric* on X if

(a) $d(x, x) = 0$ whenever $x \in X$,
(b) $d(x, z) \leqslant d(x, y) + d(y, z)$ whenever $x, y, z \in X$.[3]

We shall say that d is a T_0-*quasi-pseudometric* or T_0-*quasi-metric* provided that d also satisfies the following condition: For each $x, y \in X$,

$$d(x, y) = 0 = d(y, x) \quad \text{implies that} \quad x = y.$$

We observe that T_0-quasi-metrics are called *di-metrics* and T_0-quasi-metric spaces are called *di-spaces* in [6].

Remark 1. Let d be a quasi-pseudometric on a set X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate quasi-pseudometric of* d. Other notations instead of d^{-1} have been used, for instance in [6] the quasi-pseudometric conjugate of d is denoted by d^t.

As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric*. Note that for any (T_0-)quasi-pseudometric d, $d^s = \sup\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric).

We observe that for any quasi-pseudometric d on a set X we have that $|d(x, y) - d(a, b)| \leqslant d^s(x, a) + d^s(y, b)$ whenever $x, y, a, b \in X$.

Let (X, d) be a quasi-pseudometric space. For each $x \in X$ and $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ denotes the *open* ϵ*-ball* at x. The collection of all "open" balls yields a base for a topology $\tau(d)$. It is called the *topology induced by* d on X.

Given $a, b \in \mathbb{R}$, we shall put $a \dot{-} b = \max\{a - b, 0\}$.

A map $f : (X, d) \to (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called an *isometric map* provided that $e(f(x), f(y)) = d(x, y)$ whenever $x, y \in X$. A bijective isometric map will be called an *isometry*. Observe that if $f : X \to Y$ is an isometric map between two quasi-pseudometric spaces X and Y and if X is a T_0-quasi-metric space, then f is injective (see [9, Lemma 4]).

Two quasi-pseudometric spaces (X, d) and (Y, e) will be called *isometric* provided that there exists a (*bijective*) isometry $f : (X, d) \to (Y, e)$. A map $f : (X, d) \to (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called *nonexpansive* provided that $e(f(x), f(y)) \leqslant d(x, y)$ whenever $x, y \in X$.

Next we list the four properties of quasi-pseudometric spaces that will turn out to be crucial for our present investigations:

A quasi-pseudometric space is called *bicomplete* provided that the pseudometric space (X, d^s) is complete. It is well known that each T_0-quasi-metric space (X, d) has a unique up to isometry T_0-quasi-metric bicompletion $(\widetilde{X}, \widetilde{d})$, in which X is $\tau((\widetilde{d})^s)$-densely embedded (see for example [8, p. 278]).

A quasi-pseudometric space (X, d) is called *supseparable* provided that the pseudometric space (X, d^s) is separable (equivalently, the topology $\tau(d)$ has a countable base; compare [7, Theorem 4] and use the fact that each quasi-pseudometrizable space with a countable network has a countable base [7, p. 60]). We shall call a set *supdense* in X if it is dense in X with respect to the topology $\tau(d^s)$.

The following result is obvious.

Lemma 1. *Let (X, d_X) and (Y, d_Y) be bicomplete and supseparable T_0-quasi-metric spaces. Let S_X and S_Y be countable supdense subspaces of X resp. Y such that there is an isometry $f : S_X \to S_Y$. Then the unique supcontinuous* (*that is, $d^s_X - d^s_Y$-continuous*) *extension of f to a map $\widehat{f} : X \to Y$ yields a* (*bijective*) *isometry.*

Proof. We sketch a proof (compare [8, p. 278]) of the result, which uses standard arguments. Given $x \in X$ we can extend f from S_X to $\widetilde{f}$ on X by finding a sequence $(x_n)_{n\in\mathbb{N}}$ from S_X converging to x with respect to the topology $\tau(d^s_X)$ and setting $\widetilde{f}(x)$ equal to the $\tau(d^s_Y)$-limit of $(f(x_n))_{n\in\mathbb{N}}$. Then $\widetilde{f}$ is well defined and an isometry from (X, d_X) onto (Y, d_Y). □

As already stated in the introduction, a T_0-quasi-metric space X is called *ultrahomogeneous* if each isometry between two finite subspaces of X can be extended to an isometry of X onto itself.

[3] In some cases we need to replace $[0, \infty)$ by $[0, \infty]$ (where for a d attaining the value ∞, the triangle inequality is interpreted in the obvious way). In such a case we shall speak of an *extended quasi-pseudometric*. In the following we sometimes apply concepts from the theory of quasi-pseudometrics to extended quasi-pseudometrics (without changing the usual definitions of these concepts).

A supseparable T_0-quasi-metric space is called *q-universal* if it contains an isometric copy of each supseparable T_0-quasi-metric space.

In the first part of this article we shall show that there is an up to isometry unique bicomplete supseparable T_0-quasi-metric space that is ultrahomogeneous and q-universal. We shall call it the *Urysohn T_0-quasi-metric space* and denote it by $q\mathbb{U}$.

We shall say that a T_0-quasi-metric space X satisfies the (quasi-metric)[4] *one-point-extension property* if for any finite subspace A of X the isometric embedding $i : A \to X$ can be extended to an isometric embedding $A \cup \{\omega\} \to X$ where $A \cup \{\omega\}$ is an arbitrary T_0-quasi-metric one-point-extension of A.

During the proof of our main result of this article (Theorem 2) we shall establish the following related auxiliary results:

A bicomplete supseparable T_0-quasi-metric space is q-universal and ultrahomogeneous if and only if it satisfies the (quasi-metric) one-point-extension property (Theorem 1).

Two bicomplete supseparable T_0-quasi-metric spaces X and Y that have the (quasi-metric) one-point-extension property are isometric (Lemma 9).

3. The construction due to Katětov modified for a T_0-quasi-metric space

Katětov's construction for a metric space (X,d) is described in [5, Fact 1.3] and [11, p. 139]. Related discussions can be found in [4,10].

For the following, let a function $f : (X,d) \to [0,\infty)$ be called *Katětov* provided that $|f(x) - f(y)| \leqslant d(x,y) \leqslant f(x) + f(y)$ whenever $x, y \in X$. By $K(X,d)$ we shall denote the set of all Katětov functions on (X,d). As usual we shall assume that $K(X,d)$ carries the metric $k(f,g) = \sup_{x\in X} |f(x) - g(x)|$ whenever $f, g \in K(X,d)$. Then $(K(X,d),k)$ is a complete metric space, called the *Katětov space*.

We shall now generalize the concept of a Katětov function on a metric space to the concept of a Katětov function pair on a quasi-pseudometric space.

Let (X,d) be a quasi-pseudometric space. Furthermore let $Q(X,d)$ be the set of all pairs of functions $f = (f_1, f_2)$ where for $i = 1, 2$, $f_i : X \to [0,\infty)$ satisfy the two conditions:

(a) f is *tight*, that is, $d(x,y) \leqslant f_2(x) + f_1(y)$ whenever $x, y \in X$. (Let us note that in [6] instead of "tight" the term "ample" was used.)
(b) f is *nonexpansive*, that is

$$f_1(x) - f_1(y) \leqslant d^{-1}(x,y) \quad \text{and} \quad f_2(x) - f_2(y) \leqslant d(x,y) \quad \text{whenever } x, y \in X.$$

The terminology in (b) is readily explained by equipping $[0,\infty)$ with the T_0-quasi-metric $u(x,y) = x \dot{-} y$ whenever $x, y \in [0,\infty)$.

Each element in $Q(X,d)$ will be called a *Katětov pair* on (X,d). Operations on pairs are applied to each component separately if nothing else is stated. For instance it is readily checked that $Q(X,d)$ is convex and $\max\{f,g\} \in Q(X,d)$ whenever $f, g \in Q(X,d)$.

Remark 2. Note that we do not require that our pairs are minimal tight (that is, extremal) with respect to the pointwise product order on function pairs (compare [6, Lemmas 6 and 3]).

Hence in general our pairs (f_1, f_2) do not have the property that $\sup_{y\in X}(d(x,y) \dot{-} f_1(y)) = f_2(x)$ and $\sup_{y\in X}(d(y,x) \dot{-} f_2(y)) = f_1(x)$ whenever $x \in X$ $(+)$, since these two conditions imply that the pair $f = (f_1, f_2)$ is minimal tight:

Indeed (compare [1]) let $g \leqslant f$, i.e. $g_1 \leqslant f_1$ and $g_2 \leqslant f_2$, where f satisfies the above equalities $(+)$, and g is a tight function pair. Then for each $x \in X$, $f_2(x) \leqslant \sup_{y\in X}(d(x,y) \dot{-} f_1(y)) \leqslant \sup_{y\in X}(d(x,y) \dot{-} g_1(y)) \leqslant g_2(x)$ by tightness of g. So $g_2 = f_2$. Similarly $g_1 = f_1$ and therefore the pair f is minimal tight.

We define an (extended) T_0-quasi-metric D on $Q(X,d)$ as follows:

$$D\big((f_1, f_2), (g_1, g_2)\big) = \sup_{x\in X}\big(f_1(x) \dot{-} g_1(x)\big) \vee \sup_{x\in X}\big(g_2(x) \dot{-} f_2(x)\big) \quad \text{whenever } f, g \in Q(X,d).$$

The definition of D is related to the definition of the Hausdorff quasi-pseudometric on the set $\mathcal{P}_0(X)$ of all nonempty subsets of X (see [6, Remark 4]). It is easy to see that D is an extended T_0-quasi-metric. Indeed it will follow from Corollary 1 below that D is even a T_0-quasi-metric on $Q(X,d)$. (In the following we shall call $(Q(X,d), D)$ *the Katětov pairspace of* (X,d).)

It is readily checked that for each $a \in X$, the function pair $f_a(x) = (d(a,x), d(x,a))$ whenever $x \in X$ satisfies conditions (a) and (b) and hence belongs to $Q(X,d)$. In fact we show next that in an obvious way we can interpret any T_0-quasi-metric space (X,d) as a subspace of its Katětov pairspace $(Q(X,d), D)$. Although the setting is slightly different, the next two

[4] Normally "quasi-metric" will be clear from the context so that we can delete it.

proofs are essentially contained in [6] (compare also for instance [3]). Since they are short, but nevertheless important, we include them here for the convenience of the reader.

Lemma 2. *(Compare [6, Lemma 1].) Let (X,d) be a quasi-pseudometric space. For any $a,b \in X$ we have $d(a,b) = D(f_a, f_b)$.*

Therefore, $e_X : (X,d) \to (Q(X,d), D)$ defined by $e_X(a) = f_a$ whenever $a \in X$ is an isometric map. In the case that (X,d) is a T_0-quasi-metric space, e_X is injective.

Proof. Obviously $\sup_{x\in X}(d(a,x) - d(b,x)) = d(a,b)$, as we see by setting $x = b$ and using the triangle inequality. Similarly $\sup_{x\in X}(d(x,b) - d(x,a)) = d(a,b)$ whenever $a,b \in X$. Hence e_X is an isometric map. If for $a,b \in X$ we have that $(e_X(a))_2 = (e_X(b))_2$, then $0 = d(b,b) = d(b,a)$ and $0 = d(a,a) = d(a,b)$. Consequently $a = b$ by the T_0-property and e_X is injective. We could also use the result [9, Lemma 4] mentioned in Section 2. □

Lemma 3. *(Compare [6, Lemma 8].) Let $f \in Q(X,d)$ and $a \in X$. Then $D(f, f_a) = f_1(a)$ and $D(f_a, f) = f_2(a)$.*

Proof. We have $f_1(a) \leqslant \sup_{x\in X}(f_1(x) - d(a,x))$, because $d(a,a) = 0$. Furthermore for any $x \in X$, $f_1(x) - f_1(a) \leqslant d^{-1}(x,a)$, since f_1 is nonexpansive on (X, d^{-1}). Thus $f_1(x) - d(a,x) \leqslant f_1(a)$ whenever $x \in X$. So $\sup_{x\in X}(f_1(x) - d(a,x)) = f_1(a)$. Furthermore $d(x,a) - f_2(x) \leqslant f_1(a)$ whenever $x \in X$, since f is tight. So $\sup_{x\in X}((f_a)_2(x) \dot{-} f_2(x)) \leqslant f_1(a)$. According to the definition of D, certainly $D(f, f_a) = f_1(a)$.

Similarly one verifies that $f_2(a) = \sup_{x\in X}(f_2(x) - d(x,a))$ and $\sup_{x\in X}(d(a,x) \dot{-} f_1(x)) \leqslant f_2(a)$. In particular $D(f_a, f) = f_2(a)$ according to the definition of D. □

Remark 3. Note that an important application of Lemma 3 yields $D(f_x, f_y) = (f_x)_1(y) = (f_y)_2(x) = d(x,y)$ whenever $x, y \in X$.

Corollary 1. *For any $f, g \in Q(X,d)$ and $a \in X$ we have $D(f,g) \leqslant D(f, f_a) + D(f_a, g) = f_1(a) + g_2(a)$. Indeed D is a bicomplete T_0-quasi-metric on $Q(X,d)$.*

Proof. By Lemma 3 we only need to prove that the space $(Q(X,d), D)$ under consideration is bicomplete. So let us consider a D^s-Cauchy-sequence $((f_n)_1, (f_n)_2)_{n\in\mathbb{N}}$ in $Q(X,d)$. Since for each $x \in X$, $((f_n)_1(x))_{n\in\mathbb{N}}$ and $((f_n)_2(x))_{n\in\mathbb{N}}$ are Cauchy sequences in $([0,\infty), u^s)$, these sequences converge to, say $f_1(x)$, resp. $f_2(x)$. It is readily checked that (f_1, f_2) belongs to $Q(X,d)$ and that the sequence $D^s(((f_n)_1, (f_n)_2), (f_1, f_2))$ (where $n \in \mathbb{N}$) converges to 0. Hence we are done. □

Lemma 4. *Given an arbitrary T_0-quasi-metric space (X,d), for any function pair $f = (f_1, f_2) \in Q(X,d)$ and any $a \in X$ the following conditions are equivalent*:

(a) $f_2(a) = 0$.
(b) *$d(a,x) \leqslant f_1(x)$ and $f_2(x) \leqslant d(x,a)$ whenever $x \in X$.*
(c) $D(f_a, f) = 0$.

Proof. By definition of D, conditions (b) and (c) are obviously equivalent. We still show explicitly the equivalence of (a) and (b), although it would also follow from Lemma 3, which shows that conditions (a) and (c) are equivalent:

(a) → (b): By tightness of f we have $d(a,x) \leqslant f_2(a) + f_1(x)$ whenever $x \in X$. Thus $d(a,x) \leqslant f_1(x)$ whenever $x \in X$, since $f_2(a) = 0$. Furthermore by nonexpansivity of f, $f_2(x) - f_2(a) \leqslant d(x,a)$ whenever $x \in X$. Thus $f_2(x) \leqslant d(x,a)$ whenever $x \in X$, since $f_2(a) = 0$.

(b) → (a): Set $x = a$. Then $f_2(a) \leqslant d(a,a) = 0$ and thus $f_2(a) = 0$. □

Analogously to Lemma 4 we can prove the following result:

Lemma 5. *Given any T_0-quasi-metric space (X,d), for any function pair $f = (f_1, f_2) \in Q(X,d)$ and any $a \in X$ the following conditions are equivalent*:

(a) $f_1(a) = 0$.
(b) *$d(x,a) \leqslant f_2(x)$ and $f_1(x) \leqslant d(a,x)$ whenever $x \in X$.*
(c) $D(f, f_a) = 0$.

Corollary 2. *Given any T_0-quasi-metric space (X,d), any function pair $f \in Q(X,d)$ and any $a \in X$, we have that $f_1(a) = f_2(a) = 0$ if and only if $f = f_a$.*

Proof. The result immediately follows from the two preceding results. □

It is easy to check that for any quasi-pseudometric space (X,d) we have that $(f_1,f_2)\in Q(X,d)$ implies that $(\frac{f_1+f_2}{2},\frac{f_1+f_2}{2})\in Q(X,\frac{d+d^{-1}}{2})$. The proofs of the following results are obvious, too.

Remark 4. Let (X,d) be a T_0-quasi-metric space.

(a) Then the map $s:(Q(X,d),D)\to(Q(X,d^{-1}),D^{-1})$ defined by $s((f_1,f_2))=(f_2,f_1)$ whenever $(f_1,f_2)\in Q(X,d)$ is an isometry.

Hence the Katětov pairspace $(Q(X,d),D)$ of (X,d) is isometric to the conjugate of the Katětov pairspace $(Q(X,d^{-1}),D)$ of (X,d^{-1}).

(b) Let (X,m) be a metric space. Then for each $f:X\to[0,\infty)$ we have $f\in K(X,m)$ if and only if $(f,f)\in Q(X,m)$. Furthermore the subspace $\{(f,f)\colon\ f\in K(X,m)\}$ of $(Q(X,m),D)$ is isometric to $(K(X,m),k)$.

Of course the concept of a Katětov function pair is motivated by the following observation:

Remark 5. Let (X,d) be a T_0-quasi-metric space and $f=(f_1,f_2)\in Q(X,d)$ be such that $f\neq f_x$ whenever $x\in X$. Then we can obtain a T_0-quasi-metric one-point-extension $X^+=X\cup\{f\}$ of X by extending d to X^+ as follows: $d(f,x)=f_1(x)$ and $d(x,f)=f_2(x)$ whenever $x\in X$, and $d(f,f)=0$. [Considering the various cases, we see that the statement that d so defined satisfies the triangle inequality on X^+ is a consequence of the fact that $f\in Q(X,d)$. Hence d is a T_0-quasi-metric on X^+, because it is not possible that $f_1(x)=f_2(x)=0$ for some $x\in X$, since $f\neq f_x$ whenever $x\in X$ (see Corollary 2).]

On the other hand, given a T_0-quasi-metric one-point-extension $X^+=X\cup\{\omega\}$ with quasi-metric d of X^+, we can set $f_1(x)=d(\omega,x)$ and $f_2(x)=d(x,\omega)$ whenever $x\in X$ in order to obtain $(f_1,f_2)\in Q(X,d)$ such that $(f_1,f_2)\neq f_x$ whenever $x\in X$.

Lemma 6. *(Compare [5, Fact 1.4].) If X is a subspace of a quasi-pseudometric space (Y,d), then $(Q(X,d),D)$ can be interpreted as a subspace of $(Q(Y,d),D)$. Indeed for any pair $f\in Q(X,d)$ we define an extension f_Y of f to Y by $(f_Y)_2(y)=\inf\{d(y,x)+f_2(x)\colon x\in X\}$ and $(f_Y)_1(y)=\inf\{f_1(x)+d(x,y)\colon x\in X\}$ whenever $y\in Y$. (We shall say that the pair f_Y is controlled by the subspace X.)*

Proof. We verify that f_Y extends f. This is obvious, since f is nonexpansive on X and since $d(y,y)=0$ whenever $y\in X$. Therefore $(f_Y)_1(y)=f_1(y)$ and $(f_Y)_2(y)=f_2(y)$ whenever $y\in X$ by definition of f_Y.

We next check that f_Y belongs to $Q(Y,d)$. Let $x,y\in Y$. Then for each $r\in X$ we have $d(x,r)+f_2(r)\leqslant d(x,y)+d(y,r)+f_2(r)$. Thus $\inf_{r\in X}(d(x,r)+f_2(r))\leqslant d(x,y)+\inf_{r\in X}(d(y,r)+f_2(r))$. Hence $(f_Y)_2(x)-(f_Y)_2(y)\leqslant d(x,y)$.

Similarly, $f_1(r)+d(r,y)\leqslant f_1(r)+d(r,x)+d(x,y)$ whenever $r\in X$. Hence $\inf_{r\in X}(f_1(r)+d(r,y))-\inf_{r\in X}(f_1(r)+d(r,x))\leqslant d(x,y)$. Consequently $(f_Y)_1(y)-(f_Y)_1(x)\leqslant d^{-1}(y,x)$. Thus $((f_Y)_1,(f_Y)_2)$ is nonexpansive on Y.

Let $x,y\in Y$ and $\epsilon>0$. By definition of $((f_Y)_1,(f_Y)_2)$ there are $a_1,a_2\in X$ such that $f_1(a_1)+d(a_1,y)-\epsilon\leqslant(f_Y)_1(y)$ and $d(x,a_2)+f_2(a_2)-\epsilon\leqslant(f_Y)_2(x)$.

Hence $d(x,y)\leqslant d(x,a_2)+d(a_2,a_1)+d(a_1,y)\leqslant d(x,a_2)+(f_2(a_2)+f_1(a_1))+d(a_1,y)\leqslant(f_Y)_2(x)+(f_Y)_1(y)+2\epsilon$. It follows that $((f_Y)_1,(f_Y)_2)$ is tight on Y.

We finally check that $\Phi(f)=f_Y$ where $f\in Q(X,d)$ is an isometric map: Let $f,g\in Q(X,d)$. Obviously by the first established property, $\sup_{y\in Y}((f_Y)_1(y)\dot{-}(g_Y)_1(y))\geqslant\sup_{x\in X}(f_1(x)\dot{-}g_1(x))$.

Let $y\in Y$ and $\epsilon>0$. Then there is $t\in X$ such that $g_1(t)+d(t,y)<(g_Y)_1(y)+\epsilon$. Moreover $(f_Y)_1(y)\leqslant f_1(t)+d(t,y)$ by definition of f_Y. Thus $(f_Y)_1(y)-(g_Y)_1(y)\leqslant f_1(t)-g_1(t)+\epsilon$ by adding these two inequalities.

Therefore $\sup_{y\in Y}((f_Y)_1(y)\dot{-}(g_Y)_1(y))\leqslant\sup_{x\in X}(f_1(x)\dot{-}g_1(x))$. Similarly

$$\sup_{y\in Y}\big((g_Y)_2(y)\dot{-}(f_Y)_2(y)\big)=\sup_{x\in X}\big(g_2(x)\dot{-}f_2(x)\big).$$

Hence $D(f,g)=D(\Phi(f),\Phi(g))$. We have shown that $\Phi:Q(X,d)\to Q(Y,d)$ is an isometric map. □

Lemma 7. *(Compare [5, Fact 1.6].) Let (X,d) be a T_0-quasi-metric space and let $\phi:X\to X$ be an isometry. Let us now identify (X,d) with the afore-mentioned subspace of $(Q(X,d),D)$ (see Lemma 2).*

Then $\Phi(f)=f\circ\phi^{-1}$ whenever $f\in Q(X,d)$ defines an isometry $\Phi:(Q(X,d),D)\to(Q(X,d),D)$ which extends ϕ and is unique with this property.

Proof. Given $x\in X$, we have that for each $y\in X$, $\Phi(f_x)(y)=(f_x\circ\phi^{-1})(y)=(d(x,\phi^{-1}(y)),d(\phi^{-1}(y),x))=(d(\phi(x),y),d(y,\phi(x)))=f_{\phi(x)}(y)$. Therefore $\Phi(f_x)=f_{\phi(x)}$ and Φ restricted to $\{f_x\colon\ \in X\}$ is equal to ϕ.

It is obvious that Φ is surjective, since for any $g\in Q(X,d)$, we have that $g\circ\phi\in Q(X,d)$ and $g=(g\circ\phi)\circ(\phi^{-1})$. We also get that $D(\Phi(f),\Phi(g))=D(f\circ\phi^{-1},g\circ\phi^{-1})=D(f,g)$ whenever $f,g\in Q(X,d)$. Thus $\Phi:Q(X,d)\to Q(X,d)$ is an isometric map. The map Φ is injective, since $Q(X,d)$ is a T_0-space and Φ is an isometric map (see [9, Lemma 4]). We conclude that Φ is an isometry.

Suppose that $\Psi:Q(X,d)\to Q(X,d)$ is an isometry extending ϕ, which we assume to be defined on $\{f_x\colon\ x\in X\}$.

Then, by Lemma 3, for each $x\in X$ we have $(\Psi(f))_1(\phi(x))=D(\Psi(f),\Psi(f_x))=D(f,f_x)=f_1(x)$. Hence $(\Psi(f))_1\circ\phi=f_1$ and thus $(\Psi(f))_1=f_1\circ\phi^{-1}$. Similarly one shows that $(\Psi(f))_2=f_2\circ\phi^{-1}$. Thus $\Psi(f)=f\circ\phi^{-1}$. □

Lemma 8. *If (X,d) is a supseparable T_0-quasi-metric space (with a countable supdense subset E in X), then the space $(\mathcal{F}(X),D)$, defined in the first lines of the following proof, is also a supseparable T_0-quasi-metric space.*

Proof. Let $\mathcal{F}(X)=\{f_X\in Q(X,d)\colon\ A\subseteq X,\ A$ finite, $f\in Q(A,d)\}$. Hence $\mathcal{F}(X)$ consists of all Katětov function pairs on X controlled by finite subspaces of X (see Lemma 6). Of course, we equip $\mathcal{F}(X)$ with the restriction of the T_0-quasi-metric D of $Q(X,d)$, which for convenience we also denote by D.

Let $E_{\mathbb{Q}}$ be the set of those elements f_X where $f\in Q(A,d)$, A is a finite subset of E. Hence $E_{\mathbb{Q}}$ consists of the Katětov function pairs on X controlled by pairs defined on any finite subspace of E. It is obvious that $E_{\mathbb{Q}}$ is supseparable by Lemma 6, since otherwise there would be a finite subspace F of E such that the metric space $(Q(F),D^s)$ would not be separable, which however contradicts separability of the usual topology on the reals. We shall show that $E_{\mathbb{Q}}$ is supdense in $\mathcal{F}(X)$.

Let $f\in\mathcal{F}(X)$ and $\epsilon>0$. Then there is a finite $A\subseteq X$ that controls f. Since (X,d^s) is a metric (and hence Hausdorff) space there is a bijection $\phi:A\to B$ where B is a finite subset of E and such that $d^s(a,\phi(a))<\epsilon/2$ whenever $a\in A$. For each $a\in A$ and $i\in\{1,2\}$ find a number $g_i(\phi(a))\in[0,\infty)$ such that $|g_i(\phi(a))-f_i(a)|<\epsilon/2$, by setting $g_i(\phi(a))=f_i(\phi(a))$.

For convenience in the following we shall denote the (canonical, controlled) extension g_X of g from B to X by g (compare Lemma 6). Obviously $g\in E_{\mathbb{Q}}$.

Let $x\in X$. *Case* 1: Suppose that $g_2(x)\geqslant f_2(x)$. Then for some $a\in A$, we have $f_2(x)=d(x,a)+f_2(a)$ by definition of f_X. Then $g_2(x)-f_2(x)\leqslant d(x,\phi(a))+g_2(\phi(a))-d(x,a)-f_2(a)\leqslant d^s(a,\phi(a))+|g_2(\phi(a))-f_2(a)|<\epsilon$. Thus $|g_2(x)-f_2(x)|<\epsilon$.

Case 2: Suppose that $f_2(x)>g_2(x)$. Then there is $a\in A$ such that $g_2(x)=d(x,\phi(a))+g_2(\phi(a))$ by definition of g_X. Then $f_2(x)-g_2(x)\leqslant d(x,a)+f_2(a)-d(x,\phi(a))-g_2(\phi(a))\leqslant d^s(a,\phi(a))+|f_2(a)-g_2(\phi(a))|<\epsilon$. Thus $|f_2(x)-g_2(x)|<\epsilon$ in this case, too.

Similarly one shows that $|f_1(x)-g_1(x)|<\epsilon$ whenever $x\in X$. Consequently $D^s(f,g)\leqslant\epsilon$. It follows that the statement holds. □

Remark 6. Let (X,d) be a T_0-quasi-metric space. Note that for each $x\in X$, f_x on X is controlled by the subspace $\{x\}$ of X. So X embeds via $x\mapsto e_X(x)$ indeed isometrically into the subspace $\mathcal{F}(X)$ of $Q(X,d)$ (compare Lemma 2).

Example 1 *(Amalgamation).* Given two finite T_0-quasi-metric spaces (X,d_X) and (Y,d_Y) and an isometry i from a subspace $A\subseteq X$ to a subspace $B\subseteq Y$, there exists a T_0-quasi-metric space $Z=X\cup_i Y$, the *coproduct of X and Y amalgamated along i* or A, such that X and Y are both T_0-quasi-metric subspaces of Z and such that for all $a\in A$, $i(a)$ coincides with a in Z: Making use of this identification between the elements of A and B, for $x\in X\setminus Y$ and $y\in Y\setminus X$ we set that $d_Z(x,y)=\inf_{a\in A}\{d_X(x,a)+d_Y(i(a),y)\}$ and $d_Z(y,x)=\inf_{a\in A}\{d_Y(y,i(a))+d_X(a,x)\}$, while the subspaces X and Y of Z carry their T_0-quasi-metrics d_X and d_Y, respectively.

Proof. The triangle inequality $d_Z(x,z)\leqslant d_Z(x,y)+d_Z(y,z)$ $(x,y,z\in Z)$ on Z is readily verified: Essentially we have to consider two cases, since all other nontrivial cases are analogous: Either the path x,y,z crosses into the other subspace twice in a row, or only once.

Case 1: We are going to show that $d_X(x_1,x_2)\leqslant d_Z(x_1,y')+d_Z(y',x_2)$ with $x_1,x_2\in X\setminus Y$ and $y'\in Y\setminus X$.

Indeed say $d_Z(x_1,y')=d_X(x_1,a)+d_Y(i(a),y')$ and $d_Z(y',x_2)=d_Y(y',i(b))+d_X(b,x_2)$. Then $d_X(x_1,x_2)\leqslant d_X(x_1,a)+d_X(a,b)+d_X(b,x_2)=d_X(x_1,a)+d_Y(i(a),i(b))+d_X(b,x_2)\leqslant d_X(x_1,a)+d_Y(i(a),y')+d_Y(y',i(b))+d_X(b,x_2)$ and the triangle inequality holds.

Case 2: We want to prove that $d_Z(x,y)\leqslant d_Z(x,y')+d_Y(y',y)$ where $x\in X\setminus Y$, $y'\in Y\setminus X$ and $y\in Y$.

Assume that $d_Z(x,y')=d_X(x,x')+d_Y(i(x'),y')$. Then $d_Z(x,y)\leqslant d_X(x,x')+d_Y(i(x'),y)\leqslant d_X(x,x')+d_Y(i(x'),y')+d_Y(y',y)=d_Z(x,y')+d_Y(y',y)$. Hence the inequality is verified in this case, too.

Note that d_Z satisfies the T_0-property: Consider the only interesting case that $x\in X\setminus Y$ and $y\in Y\setminus X$, and that $d_X(x,z_1)+d_Y(i(z_1),y)=0$ and $d_Y(y,i(z_2))+d_X(z_2,x)=0$. Then $0=d_Y(i(z_1),i(z_2))=d_X(z_1,z_2)$ and $0=d_X(z_2,z_1)$. Hence $z_1=z_2$, and then $x=z_2$, $i(z_2)=y$ and consequently $i(x)=y$, which contradicts our choice of x and y. We conclude that this case cannot occur. □

Let us formally repeat a definition from Section 2.

Definition 2. We say that a T_0-quasi-metric space X has the (quasi-metric) *one-point-extension property* $(*)$ if whenever $A\subseteq X$ is a finite subspace of X and $A'=A\cup\{b'\}$ is (an abstract) T_0-quasi-metric one-point-extension of A, the isometric embedding $f:A\to X$ extends to an isometric embedding $\widetilde{f}:A'\to X$.

In the following we essentially modify Pestov's approach [11, p. 140] from the metric setting by stressing Katětov's ideas in order to obtain our main results. We observe that while many arguments follow closely their metric counterparts, some new ideas seem to be required in the proof of Lemma 11.

Lemma 9. *Let X and Y be two supseparable bicomplete T_0-quasi-metric spaces each of which satisfies the one-point-extension property $(*)$. Then X and Y are isometric. Moreover, if A and B are finite subspaces of X and Y, respectively, and $i:A\to B$ is an isometry, then i extends to an isometry $f:X\to Y$.*

Proof. Choose countable supdense subsets S_X of X and S_Y of Y, and enumerate them as follows: $S_X = \{x_i\colon\ i \in \mathbb{N}\}$ and $S_Y = \{y_i\colon\ i \in \mathbb{N}\}$.

Let $f_0 = i$ be the start of the induction. The result of the n-th inductive step will be an isometric map f_n with finite domain, extending the isometric map f_{n-1} and such that $\{x_1, \dots, x_n\} \subseteq \operatorname{dom} f_n$ and $\{y_1, \dots, y_n\} \subseteq \operatorname{im} f_n$. Finally, the map $f(x_n) = f_n(x_n)$ for all $n \in \mathbb{N}$, is an isometry between S_X and S_Y whose restriction to A is i. The unique extension of f by supcontinuity to X establishes an isometry between X and Y with the desired properties (see Lemma 1).

Suppose that for some $n \in \mathbb{N} \cup \{0\}$ an isometric map f_n has already been constructed in such a way that, for n positive, the domain $\operatorname{dom} f_n$ is finite and contains the set $\{x_1, \dots, x_n\} \cup \operatorname{dom} f_{n-1}$, the restriction of f_n to the domain of f_{n-1} coincides with the latter isometric map, and the image $\operatorname{im} f_n$ contains $\{y_1, \dots, y_n\}$.

Step $n + 1$ of the induction is split into two substeps.

If $x_{n+1} \notin \operatorname{dom} f_n$, then, using the one-point-extension property $(*)$, one can find a point $y \in Y \setminus \operatorname{im} f_n$ in such a way that the mapping $\widetilde{f}_n$ defined by $\widetilde{f}_n(a) = f_n(a)$ if $a \in \operatorname{dom} f_n$ and $\widetilde{f}_n(a) = y$ if $a = x_{n+1}$ is an isometric map. If $x_{n+1} \in \operatorname{dom} f_n$, then nothing happens and we set $\widetilde{f}_n = f_n$.

If $y_{n+1} \notin \operatorname{im}(\widetilde{f}_n)$ then, again by the one-point-extension property $(*)$, one can find an $x \in X \setminus \operatorname{dom} \widetilde{f}_n$ so that the bijection $f_{n+1}(a) = \widetilde{f}_n(a)$ if $a \in \operatorname{dom} \widetilde{f}_n$ and $f_{n+1}(a) = y_{n+1}$ if $a = x$ is an isometric map. Again in the case where $y_{n+1} \in \operatorname{im}(\widetilde{f}_n)$ we simply put $f_{n+1} = \widetilde{f}_n$.

Clearly for each n, $\operatorname{dom} f_{n+1}$ is finite, $f_{n+1}|\operatorname{dom} f_n = f_n$, $\{x_1, \dots, x_{n+1}\} \subseteq \operatorname{dom} f_{n+1}$ and $\{y_1, \dots, y_{n+1}\} \subseteq \operatorname{im} f_{n+1}$, which completes the proof. □

Theorem 1. *A bicomplete supseparable T_0-quasi-metric space X is ultrahomogeneous and q-universal if and only if it has the one-point-extension property $(*)$.*

Proof. Suppose that X is a bicomplete supseparable ultrahomogeneous q-universal T_0-quasi-metric space. Let $A \subseteq X$ be a finite subspace, and let $A' = A \cup \{b'\}$ be any abstract T_0-quasi-metric one-point-extension of A. Because of q-universality of X, there is a copy A'' of A' in X. Hence there is an isometry between A and the copy of A contained in A''. Since X is ultrahomogeneous, that map extends to an isometry j of X onto itself. Hence $j^{-1}(A'')$ yields a subspace of X isometric to A' containing A. So X has the one-point-extension property $(*)$.

Assume that the bicomplete supseparable T_0-quasi-metric space X has the one-point-extension property $(*)$. According to Lemma 9 X is ultrahomogeneous. If Y is any countable T_0-quasi-metric space, an isometric embedding of Y into X is constructed by an obvious induction using the one-point-extension property of X. Finally in the case of an arbitrary supseparable T_0-quasi-metric space Y an isometric embedding from a countable supdense subspace of Y into X can be extended to an isometric embedding of Y into the bicomplete space X by a standard argument, and so X is q-universal. □

Theorem 2. *An ultrahomogeneous q-universal subseparable bicomplete T_0-quasi-metric space exists and is unique up to isometry.*

Proof. Let X and Y be two such spaces. They possess the one-point-extension property by Theorem 1 and, starting with the trivial isometry between any two singletons in X resp. Y and using Lemma 9, one concludes that there is an isometry between X and Y. Hence the uniqueness of the space under consideration is proved. In the following we shall deal with the existence of such a space, which will finally be established in Theorem 3. □

The following property generalizes the one-point-extension property $(*)$.

Definition 3. Let us say that a T_0-quasi-metric space (X, d) has the (*quasi-metric*) *approximate one-point-extension property* if for every finite subspace $A \subseteq X$, every abstract T_0-quasi-metric one-point-extension $A' = A \cup \{b'\}$ with quasi-metric $d_{A'}$ and every $\epsilon > 0$ there exists $b \in X$ such that

$$\sup_{a \in A}\bigl(d_{A'}(b', a) \dot{-} d(b, a)\bigr) \vee \sup_{a \in A}\bigl(d(a, b) \dot{-} d_{A'}(a, b')\bigr) < \epsilon$$

and

$$\sup_{a \in A}\bigl(d(b, a) \dot{-} d_{A'}(b', a)\bigr) \vee \sup_{a \in A}\bigl(d_{A'}(a, b') \dot{-} d(a, b)\bigr) < \epsilon.$$

Of course, the preceding definition could be stated more concisely with the help of $|\cdot|$. But the given formulation has the advantage that it stresses the connection with the distance D used throughout this article.

Lemma 10. *The bicompletion $(\widetilde{X}, d)$ of a T_0-quasi-metric space X with the approximate one-point-extension property has the approximate one-point-extension property.*

Proof. Let A be a finite T_0-quasi-metric subspace of the bicompletion $\widetilde{X}$ of X and let $A' = A \cup \{b'\}$ be an abstract T_0-quasi-metric one-point-extension of A. Let $\epsilon > 0$. By density of the set X in the metric space $(\widetilde{X}, d^s)$, we find a bijection $i : A \to X$

such that $d^s(a, i(a)) < \epsilon/3$ for all $a \in A$. Amalgamate the T_0-quasi-metric space A' and $A \cup i(A)$ along A and consider the subspace $B = i(A) \cup \{b'\}$ of the amalgam Z. Since X has the approximate one-point-extension property, there is an element $b \in X$ with

$$\sup_{a\in A}\big(d_Z\big(b', i(a)\big) \dot{-} d\big(b, i(a)\big)\big) \vee \sup_{a\in A}\big(d\big(i(a), b\big) \dot{-} d_Z\big(i(a), b'\big)\big) < \epsilon/3$$

and

$$\sup_{a\in A}\big(d\big(b, i(a)\big) \dot{-} d_Z\big(b', i(a)\big)\big) \vee \sup_{a\in A}\big(d_Z\big(i(a), b'\big) \dot{-} d\big(i(a), b\big)\big) < \epsilon/3.$$

Consequently

$$\sup_{a\in A}\big(d_{A'}\big(b', a\big) \dot{-} d(b, a)\big) \vee \sup_{a\in A}\big(d(a, b) \dot{-} d_{A'}\big(a, b'\big)\big) < \epsilon,$$

since for instance

$$\sup_{a\in A}\big(d_{A'}\big(b', a\big) \dot{-} d(b, a)\big) \leqslant \sup_{a\in A}\big[\big(d_{A'}\big(b', a\big) \dot{-} d_Z\big(b', i(a)\big)\big) + \big(d_Z\big(b', i(a)\big) \dot{-} d\big(b, i(a)\big)\big) + \big(d\big(b, i(a)\big) \dot{-} d(b, a)\big)\big] < \epsilon.$$

Here for the first difference of the sum we can use the following argument: $|d_{A'}(b', a) - d_Z(b', i(a))| = |d_Z(b', a) - d_Z(b', i(a))| \leqslant (d_Z)^s(a, i(a)) = d^s(a, i(a))$.

Similarly we also see that

$$\sup_{a\in A}\big(d(b, a) \dot{-} d_{A'}\big(b', a\big)\big) \vee \sup_{a\in A}\big(d_{A'}\big(a, b'\big) \dot{-} d(a, b)\big) < \epsilon.$$

We conclude that $\widetilde{X}$ has the approximate one-point-extension property. □

Lemma 11. *A bicomplete T_0-quasi-metric space (X, d) with the approximate one-point-extension property possesses the one-point-extension property.*

Proof. Let $A \subseteq X$ be a finite subspace of X and let $A'_1 = A \cup \{b'\}$ be an abstract T_0-quasi-metric one-point-extension of A with T_0-quasi-metric $d_{A'_1}$. For the following induction we also set $A_0 = A$ and $K_1 = 1$.

We shall choose a d^s-Cauchy sequence $(b_n)_{n\in\mathbb{N}}$ of elements of X in such a way that for all $a \in A$ the sequence $(d(a, b_n))_{n\in\mathbb{N}}$ converges to $d_{A'_1}(a, b')$, as well as the sequence $(d(b_n, a))_{n\in\mathbb{N}}$ converges to $d_{A'_1}(b', a)$. Hence, denoting by c the $\tau(d^s)$-limit of $(b_n)_{n\in\mathbb{N}}$ in X, which exists by bicompleteness of X, the subspace $A \cup \{c\}$ of X and A'_1 are isometric T_0-quasi-metric spaces. Hence we shall conclude that (X, d) has the one-point-extension property $(*)$.

The construction is by induction on $i \in \mathbb{N}$. At step $i \geqslant 1$ we are given a positive constant K_i and a T_0-quasi-metric one-point-extension $A'_i = \{b'\} \cup A_{i-1}$ of the subspace A_{i-1} of X where $A_{i-1} = A \cup \{b_1, \ldots, b_{i-1}\}$.

For $i \geqslant 2$ take as constant $K_i = \max\{d_{A'_i}(b', b_{i-1}), d_{A'_i}(b_{i-1}, b')\}$ where we observe that for $i \geqslant 2$, K_i is positive, since $d_{A'_i}$ is a T_0-quasi-metric.

For $i \geqslant 1$ we approximate by $b_i \in X$ such that $|d_{A'_i}(b', a) - d(b_i, a)| < \frac{K_i}{2}$ and $|d(a, b_i) - d_{A'_i}(a, b')| < \frac{K_i}{2}$ whenever $a \in A_{i-1}$. Of course, b_i exists, since X has the approximate one-point-extension property.

For $i \geqslant 1$ we set $A_i = A_{i-1} \cup \{b_i\}$ and take the amalgam Z'_{i+1} (with carrier set A'_{i+1}) of A_i and A'_i along A_{i-1}. Now we use a modification of a trick due to Urysohn that will finally assure convergence of the constructed sequence $(b_i)_{i\in\mathbb{N}}$. A modification of the method of Urysohn seems necessary, since a T_0-quasi-metric may attain nontrivial zero-distances.

For $i \geqslant 1$ we modify the space Z'_{i+1} by setting $d_{A'_{i+1}}(b', b_i) = \min\{d_{Z'_{i+1}}(b', b_i), 2^{-i}\}$ and $d_{A'_{i+1}}(b_i, b') = \min\{d_{Z'_{i+1}}(b_i, b'), 2^{-i}\}$. No other distances are changed, that is, $d_{A'_{i+1}} = d_{Z'_{i+1}}$ otherwise. In particular for each $i \geqslant 1$, $(A'_i, d_{A'_i})$ is a subspace of $(A'_{i+1}, d_{A'_{i+1}})$ and the subset A_i of $(A'_{i+1}, d_{A'_{i+1}})$ carries the T_0-quasi-metric inherited from X.

Note that for each $i \geqslant 2$ we have $K_i \leqslant 2^{-(i-1)}$ by the definition of $d_{A'_i}(b', b_{i-1})$ and $d_{A'_i}(b_{i-1}, b')$. Observe that by definition of K_1 the latter inequality also holds for $i = 1$.

It remains to show that for each $i \geqslant 1$, $d_{A'_{i+1}}$ is a T_0-quasi-metric. The T_0-axiom is obviously satisfied, since $d_{Z'_{i+1}}$ is a T_0-quasi-metric. So we need only to verify the triangle inequality for $d_{A'_{i+1}}$. Note that moving from Z'_{i+1} to A'_{i+1} we have changed at most two distances. Because $d_{Z'_{i+1}}$ itself satisfies the triangle inequality, it suffices to consider the case that we reduce the distances of Z'_{i+1} to the ones given by $d_{A'_{i+1}}$ in exactly one instance in that inequality; of course, if we reduce on the left-hand side of the triangle inequality, it remains satisfied.

Then obviously we have only to consider four nontrivial cases, one of which we deal next in detail:

For $i \geqslant 1$ we have to show for instance that for any $a \in A_{i-1}$, $d(b_i, a) \leqslant d_{A'_{i+1}}(b_i, b') + d_{A'_{i+1}}(b', a)$ where $d_{A'_{i+1}}(b_i, b') = 2^{-i}$.

But this is satisfied, since for $i \geqslant 1$ we have, by the approximation assumption, $2^{-i} \geqslant \frac{K_i}{2} > d(b_i, a) - d_{A'_i}(b', a) = d(b_i, a) - d_{A'_{i+1}}(b', a)$ where $a \in A_{i-1}$.

Similarly, one can show that for any $a \in A_{i-1}$, $d(a, b_i) \leqslant d_{A'_{i+1}}(a, b') + d_{A'_{i+1}}(b', b_i)$ and $d_{A'_{i+1}}(b', a) \leqslant d_{A'_{i+1}}(b', b_i) + d(b_i, a)$ and $d_{A'_{i+1}}(a, b') \leqslant d(a, b_i) + d_{A'_{i+1}}(b_i, b')$. Thus $d_{A'_{i+1}}$ is indeed a T_0-quasi-metric whenever $i \geqslant 1$.

Hence we have finished the description of our induction. It remains to verify that $(b_i)_{i\in\mathbb{N}}$ is a d^s-Cauchy sequence in X.

For each $i \geqslant 1$, by the approximation assumption we have $|d(b_{i+1}, b_i) - d_{A'_{i+1}}(b', b_i)| < \frac{K_{i+1}}{2}$. Thus $d(b_{i+1}, b_i) \leqslant d_{A'_{i+1}}(b', b_i) + \frac{K_{i+1}}{2} \leqslant 2^{-i} + \frac{K_{i+1}}{2} \leqslant 2 \cdot 2^{-(i+1)} + 2^{-(i+1)} = 3 \cdot 2^{-(i+1)}$. Analogously $d(b_i, b_{i+1}) \leqslant 3 \cdot 2^{-(i+1)}$ whenever $i \geqslant 1$. We conclude that $(b_i)_{i\in\mathbb{N}}$ is a Cauchy sequence on (X, d^s).

By the approximation conditions, we obviously have, $d_{A'_1}(b', x) = \lim_{n\to\infty} d(b_n, x)$ and $d_{A'_1}(x, b') = \lim_{n\to\infty} d(x, b_n)$ whenever $x \in A$. Let c be the $\tau(d^s)$-limit of $(b_n)_{n\in\mathbb{N}}$ in X. Then $d_{A'_1}(b', x) = d(c, x)$ and $d_{A'_1}(x, b') = d(x, c)$ whenever $x \in A$. We conclude that X indeed has the one-point-extension property. □

We are now ready to complete the proof of Theorem 2. Starting with an arbitrary (nonempty) T_0-quasi-metric space X, one can form an increasing sequence of iterated T_0-quasi-metric extensions of the form $X, \mathcal{F}(X), \mathcal{F}^2(X) = \mathcal{F}(\mathcal{F}(X)), \ldots, \mathcal{F}^n(X) = \mathcal{F}(\mathcal{F}^{n-1}(X)), \ldots$. Furthermore set $\mathcal{F}^\omega(X) = \bigcup_{i=1}^{\infty} \mathcal{F}^n(X)$ (compare Lemma 8). Then $\mathcal{F}^\omega(X)$ is, in a natural way, a T_0-quasi-metric space, containing an isometric copy of X (compare Remark 6).

Lemma 12. *Let A be a finite subspace of a T_0-quasi-metric space (X, d) and let $A' = A \cup \{b'\}$ be an abstract T_0-quasi-metric one-point-extension of A. Then the isometric embedding $a \mapsto e_X(a)$ of A into $\mathcal{F}(X)$ (see Lemma 2 and Remark 6) extends to an isometric embedding $A' \to \mathcal{F}(X)$.*

Proof. The Katětov function pair $f_{b'}(a) = (d_{A'}(b', a), d_{A'}(a, b'))$ (where $a \in A$) on A controls a function pair $f \in Q(X, d)$ according to Lemma 6. We have $D(f_a, f) = f_2(a) = d_{A'}(a, b')$ and $D(f, f_a) = f_1(a) = d_{A'}(b', a)$ whenever $a \in A$ by Lemma 3 and the definition of f. Hence $\{f_a\colon a \in A\} \cup \{f\}$ is a subspace of $\mathcal{F}(X)$ isometric to A'. □

Theorem 3. *Let X be an arbitrary (nonempty) supseparable T_0-quasi-metric space. The bicompletion of the space $\mathcal{F}^\omega(X)$ yields a copy of $q\mathbb{U}$.*

Proof. Given a finite subspace S of $\mathcal{F}^\omega(X)$, there is $n \in \mathbb{N}$ such that $S \subseteq \mathcal{F}^n(X)$. As a consequence of Lemma 12 the Katětov function pairs on $\mathcal{F}^n(X)$ representing the abstract one-point-extensions of S belong to $\mathcal{F}^{n+1}(X) \subseteq \mathcal{F}^\omega(X)$. Hence $\mathcal{F}^\omega(X)$ has the one-point-extension property, which is also true for its bicompletion by Lemmas 10 and 11. Since that bicompletion is supseparable as a consequence of Lemma 8, the proof of Theorem 2 is finished. □

It is natural to study the conjugate and the supremum space of the Urysohn T_0-quasi-metric space $q\mathbb{U}$. So let us consider the space $q\mathbb{U}$ with its quasi-metric, say, D.

We first note that $(q\mathbb{U}, D^{-1})$ is isometric to $(q\mathbb{U}, D)$, because $(q\mathbb{U}, D^{-1})$ is obviously supseparable, bicomplete, and also has the one-point-extension property, as the following simple argument shows: Let (A, d) be a finite subspace of $(q\mathbb{U}, D^{-1})$ with isometric embedding i and let (A', d') be an abstract T_0-quasi-metric one-point-extension of (A, d). Then by the one-point-extension property of $(q\mathbb{U}, D)$ the isometric embedding $i : (A, d^{-1}) \to (q\mathbb{U}, D)$ extends to an isometric embedding $j : (A', (d')^{-1}) \to (q\mathbb{U}, D)$. Obviously j is also an isometric embedding of (A', d') into $(q\mathbb{U}, D^{-1})$ extending i. Hence $(q\mathbb{U}, D^{-1})$ has the one-point-extension property and we are done.

The supremum space $(q\mathbb{U}, D^s)$ requires a more detailed investigation. Obviously the space $(q\mathbb{U}, D^s)$ is a complete and separable metric space. By q-universality each separable metric space is isometric to a subspace of $(q\mathbb{U}, D)$, hence, as a metric space, to a subspace of $(q\mathbb{U}, D^s)$. In particular $\mathbb{U}$ is isometric to a subspace of $(q\mathbb{U}, D^s)$. On the other hand observe that $(q\mathbb{U}, D^s)$ is a separable metric space and therefore embeds isometrically into $\mathbb{U}$.

However it follows from Example 2 below that $(q\mathbb{U}, D^s)$ is not a (metric) ultrahomogeneous space. In fact we shall show that $(q\mathbb{U}, D^s)$ does not have the (metric) one-point-extension property (compare [11, Theorem 3.4.4]). Hence $(q\mathbb{U}, D^s)$ cannot be isometric to the universal ultrahomogeneous separable complete Urysohn metric space $\mathbb{U}$.

Example 2. Let $A = \{0, 1, 2\}$ be equipped with the T_0-quasi-metric d where $d(x, y) = 0$ if $x \leqslant y$ and $d(x, y) = 1$ otherwise. Then d^s is the discrete metric on A. Extend d^s to a metric m on $A' = A \cup \{\omega\}$ such that $m(\omega, 0) = 1 = m(2, \omega)$ and $m(\omega, 1) = 2$. One readily checks that m is a metric on A'.

In order to reach a contradiction suppose that there is a T_0-quasi-metric d' on A' such that d' restricted to A equals d and $(d')^s = m$. Then $1 + 0 = (d')^s(\omega, 0) + d(0, 1) \geqslant d'(\omega, 0) + d'(0, 1) \geqslant d'(\omega, 1)$. Similarly $0 + 1 = d(1, 2) + (d')^s(2, \omega) \geqslant d'(1, 2) + d'(2, \omega) \geqslant d'(1, \omega)$. Therefore $1 \geqslant (d')^s(\omega, 1) = m(\omega, 1) = 2$. We have reached a contradiction and conclude that d' does not exist.

With the help of Example 2 we can now establish that $(q\mathbb{U}, D^s)$ does not have the (metric) one-point-extension property: By q-universality of $(q\mathbb{U}, D)$, the T_0-quasi-metric space (A, d) of Example 2 embeds isometrically into $(q\mathbb{U}, D)$ by some map i. So (A, d^s) embeds isometrically by that map i into $(q\mathbb{U}, D^s)$. Let (A', m) be the abstract (metric) one-point-extension of (A, d^s) constructed in Example 2. In order to reach a contradiction, suppose that i extends to an isometric embedding j of (A', m) into $(q\mathbb{U}, D^s)$. Then D restricted to $j(A') \times j(A')$ would yield a T_0-quasi-metric such that $(j(A), D)$ is isometric to (A, d) and such that $(j(A'), D^s)$ is isometric to (A', m), but such a D on $j(A') \times j(A')$ does not exist according to the argument given in Example 2.

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