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ScienceDirect
J. Math. Pures Appl. 92 (2009) 323–341
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Pure point spectrum for measure dynamical systems on locally compact Abelian groups

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Received 6 March 2007

Available online 29 May 2009

Abstract

We show equivalence of pure point diffraction and pure point dynamical spectrum for measurable dynamical systems built from locally finite measures on locally compact Abelian groups. This generalizes all earlier results of this type. Our approach is based on a study of almost periodicity in a Hilbert space. It allows us to set up a perturbation theory for arbitrary equivariant measurable perturbations.

Résumé

Dans cet article on établit une équivalence entre le spectre purement ponctuel de diffraction et le spectre purement ponctuel dynamique pour des systèmes dynamiques mesurables sur des groupes abéliens localement compacts de mesures finies. L'approche adoptée utilise la notion de presque périodicité dans un espace de Hilbert. On peut ainsi introduire une théorie des perturbations dans le cas de perturbations mesurables équivariantes arbitraires.

Keywords: Pure point spectrum; Almost periodicity in Hilbert space; Aperiodic order; Quasicrystals

1. Introduction

This paper deals with mathematical diffraction theory and its relationship to dynamical systems. Our main motivation comes from the study of aperiodic order.

The study of (dis)order is a key issue in mathematics and physics today. Various regimes of disorder have attracted particular attention in recent years. A most prominent one is long range aperiodic order or, for short, aperiodic order. There is no axiomatic framework for aperiodic order yet. It is commonly understood to mean a form of (dis)order at the very border between periodicity and disorder. While giving a precise meaning to this remains one of the fundamental

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0021-7824/$ – see front matter
doi:10.1016/j.matpur.2009.05.013

mathematical challenges in the field, a wide range of distinctive feature has been studied in such diverse fields as combinatorics, discrete geometry, harmonic analysis, K-theory and Schrödinger operators (see e.g. the monographs and proceeding volumes [6,19,31,33,38,44]).

Part of this research is certainly triggered by the actual discovery of physical substances exhibiting this form of disorder twenty five years ago [39,20]. These substances were discovered experimentally by their unusual and rather striking diffraction patterns. These exhibit a (large) pure point component (meaning order) with symmetries incompatible with periodicity (meaning aperiodicity). Of course, the discovery of quasicrystals by diffraction experiments lead to a particular interest in diffraction theory of aperiodic order. Besides this externally motivation, there also is a strong intrinsic mathematical interest in diffraction theory.

In order to be more precise on this point, let us shortly and with some grains of salt describe mathematical diffraction theory (see Section 4 for details). In mathematical diffraction theory, the solid in question is modeled by a measure. The diffraction is then described by the Fourier transform of this measure. The basic intuition is now that order in the original measure will show up as a (large) pure point component in its Fourier transform. A particular instance of this intuition is given by the Poisson summation formula. To extend this intuition has been a driving force of the conceptual mathematical study of diffraction for aperiodic order (see e.g. Lagarias' article [22]). Let us emphasize that this conceptual mathematical question has already attracted attention before the dawn of quasicrystals, as can be seen e.g. in Meyer's book [30] or the corresponding chapters in Queffélec's book [35].

As mentioned already, we will be concerned with the connection of diffraction theory and dynamical systems. Recall that (dis)order is commonly modeled by dynamical systems. The elements of the dynamical system then represent the various manifestations of the 'same' form of disorder. This dynamical system then induces a unitary representation of the translation group. The spectrum of the dynamical system is the spectrum of this unitary representation. Starting with the work of Dworkin [11], it has been shown in various degrees of generality [17,18,40,42] that the dynamical spectrum contains the diffraction spectrum (see [35] for a similar statement as well). On the other hand, by the work of van Enter and Międkisz [12] it is clear that in general the dynamical spectrum may be strictly larger than the diffraction spectrum.

In view of the results of [12], it is most remarkable that the two spectra are yet equivalent once it comes to pure point spectrum. More precisely, pure point dynamical spectrum is equivalent to pure point diffraction spectrum. This type of result has been obtained by various groups in recent years [23,2,15]. First Lee/Moody/Solomyak [23] showed the equivalence for uniquely ergodic dynamical systems of point sets in Euclidean space satisfying a strong local regularity condition viz finite local complexity. Their result was then extended by Gouéré [15] and by Baake/Lenz [2] to more general contexts. In particular, it was freed from the assumptions of unique ergodicity and finite local complexity. While there is some overlap between [15] and [2], these works are quite different in terms of models and methods. Gouéré deals with measurable point processes in Euclidean space, using Palm measures and Bohr/Besicovich almost periodicity. Thus, his result is set in the measurable category. Baake/Lenz leave the context of points altogether by dealing with translation bounded measures on locally compact Abelian groups. Their results are then, however, restricted to a topological context. In fact, a key step in their setting is to replace the combinatorial analysis of [23] by a suitable application of the Stone/Weierstrass theorem.

Given this state of affairs it is natural to ask whether the corresponding results of [15] and [2] on dynamical systems can be unified. This amounts to developing a diffraction theory based on measure dynamical systems in the measurable category. This is not only of theoretical interest. It is also relevant for perturbation theory. More precisely, one may well argue that aperiodic order is topological in nature and, hence, a treatment of aperiodic order in the topological category suffices. However, a more realistic treatment should allow for perturbations as well. By their very nature, these perturbations should not be restricted to the topological category. They should rather be as general as possible. In order to accommodate this a measurable framework seems highly desirable. The overall aim of this article is then to provide such a framework. More precisely, the aims are to:

- develop a diffraction theory for measure dynamical systems unifying the corresponding treatments of [15,14] and [2],
- set up a measurable perturbation theory for these systems.

Along our way, we will actually present,

- a new method of proving the equivalence of pure point diffraction and pure point dynamical spectrum based on a stability result for the pure point subspace of a unitary representation.

This stability result may be of independent interest. Its proof is close in spirit to considerations of [15] by relying on almost periodicity on Hilbert space. It also ties in well with other recent work focusing on almost periodicity in the study of pure point diffraction [4,27,32,41,43].

The paper is organized as follows:

In Section 2, we discuss some general facts concerning the point spectrum of a strongly continuous unitary representation. The main abstract result, Theorem 2.3, gives a stability result for the pure point subspace. This result is then applied to measure dynamical systems and gives Corollary 2.5. This corollary establishes that the subspace belonging to the point spectrum is invariant under composition with bounded functions. These results are the main abstract new ingredients in our reasoning. They may be useful in other situations as well. The dynamical systems we are dealing with are introduced in Section 3. They are built from locally finite measures on locally compact Abelian (LCA) groups. We study a dense set of functions on the corresponding L^2-space and use it to obtain strong continuity of the associated representation of G in Theorem 3.6. The setting for diffraction theory is discussed in Section 4. As shown there, the topological approach of [2] can be extended to a measurable setting, once a certain finiteness assumption is made. In particular, there is an abstract way to define the autocorrelation measure, Proposition 4.1. We then come to the relationship between diffraction and the spectral theory of the dynamical systems in Section 5. The crucial link is provided by Theorem 5.3 which states that the Fourier transform of the autocorrelation is a spectral measure for a sub-representation. When combined with the abstract results of Section 2, this gives Theorem 5.5 showing the equivalence of the two notions of pure point spectrum. In Section 6 we use our results to briefly set up a perturbation theory.

2. Point spectrum of strongly continuous unitary representations and measurable dynamical systems

In this section, we discuss the pure point subspace of a strongly continuous unitary representation. We obtain an abstract stability result for this subspace and apply it to dynamical systems.

Let G be a locally compact, σ-compact, Abelian group. The dual group of G is denoted by $\widehat{G}$, and the pairing between a character $\lambda \in \widehat{G}$ and an element $t \in G$ is written as (λ, t), which, of course, is a number on the unit circle (see [7,13,36] for further background on harmonic analysis).

A unitary representation T of G in the Hilbert space $\mathcal{H}$ is a group homomorphism into the group of unitary operators on $\mathcal{H}$. It is called strongly continuous if the map $G \to \mathcal{H}$, $t \mapsto T^t f$, is continuous for each $f \in \mathcal{H}$. As usual, the inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$.

A non-zero $f \in \mathcal{H}$ is called an *eigenfunction* of T if there exists a $\lambda \in \widehat{G}$ with $T^t f = (\lambda, t) f$ for every $t \in G$. The closure of the linear span of all eigenfunctions of T will be denoted by $\mathcal{H}_{pp}(T)$. T is said to have *pure point spectrum*, if $\mathcal{H}_{pp}(T) = \mathcal{H}$.

For a strongly continuous T there exists by Stone's Theorem (compare [29]) a map,

$$E_T : \text{Borel sets on } \widehat{G} \to \text{Projections on } \mathcal{H},$$

with

- $E_T(\emptyset) = 0$, $E_T(G) = \mathit{Identity}$,
- $E_T(A) = \bigoplus E_T(A_j)$ whenever A is the disjoint union of the Borel sets A_j, $j \in \mathbb{N}$,

such that

$$\langle f, T^t f \rangle = \int_{\widehat{G}} (\lambda, t)\, d\langle f, E_T(\lambda) f \rangle =: \int_{\widehat{G}} (\lambda, t)\, d\rho_f(\lambda),$$

where ρ_f is the measure on $\widehat{G}$ defined by $\rho_f(B) := \langle f, E_T(B) f \rangle$. Because of its properties E_T is called a projection valued measure.

A continuous function p on G with values in a Banach space $(X, \|\cdot\|)$ (e.g. $X = \mathbb{C}$ or $X = \mathcal{H}$) is called *almost periodic* if for every $\varepsilon > 0$, the set

$$\{ t \in G \colon \| p(t+s) - p(s) \| < \varepsilon \text{ for all } s \in G \}$$

is relatively dense in G. Here, a subset S of G is called relatively dense if there exists a compact subset K of G with $S + K = G$. If $p(t) = T^t x$ for some $x \in \mathcal{H}$, and a strongly continuous T then the almost periodicity of p is equivalent to the closure $\overline{p(G)}$ of $p(G)$ being compact.

We can now formulate the following characterization of $\mathcal{H}_{pp}(T)$.

Lemma 2.1. *Let T be a strongly continuous unitary representation of G on $\mathcal{H}$. Then, the following assertions are equivalent for $f \in \mathcal{H}$:*

(i) *The map $G \to \mathcal{H}$, $t \mapsto T^t f$, is almost periodic.*
(ii) *The map $G \to \mathbb{C}$, $t \mapsto \langle f, T^t f\rangle$, is almost periodic.*
(iii) *ρ_f is a pure point measure.*
(iv) *f belongs to $\mathcal{H}_{pp}(T)$.*

Proof. The equivalence of (iii) and (iv) is standard. The equivalence of (ii) and (iii) follows by a result of Wiener as $t \mapsto \langle f, T^t f\rangle$ is the Fourier transform of ρ_f. The implication (i) $\Rightarrow$ (ii) is clear. It remains to show (ii) $\Rightarrow$ (i): A direct calculation gives

$$\left\| f - T^t f \right\|^2 = 2\langle f, f\rangle - \langle f, T^t f\rangle - \overline{\langle f, T^t f\rangle} \leqslant 2\left|\langle f, f\rangle - \langle f, T^t f\rangle\right|.$$

Now, the desired result follows. □

For our further analysis, we need some more pieces of notation. A measure ρ on $\widehat{G}$ is said to be supported on the subset S of $\widehat{G}$ if there exists a measurable subset S' of S with $\rho(\widehat{G} \setminus S') = 0$. For a subgroup S of $\widehat{G}$ equipped with the discrete topology (which may not be the topology induced by G!), the dual group $\widehat{S}$ is compact. The injective group homomorphism $S \to \widehat{G}$, $\lambda \mapsto \lambda$, induces the group homomorphism $G \to \widehat{S}$, $t \mapsto (\lambda \mapsto (\lambda, t))$. The latter will be denoted by j. It has dense range. Conversely, if $\mathbb{T}$ is a compact group and $j : G \to \mathbb{T}$ is a continuous group homomorphism with dense range, then $\widehat{\mathbb{T}}$ can naturally be considered to be a subgroup of $\widehat{G}$ with the discrete topology via $i : \widehat{\mathbb{T}} \to \widehat{G}$, $i(\lambda)(t) := (\lambda, j(t))$.

Lemma 2.2. *Let a strongly continuous unitary representation T of G on $\mathcal{H}$ and $f \in \mathcal{H}$ be given.*

(a) *If f belongs to $\mathcal{H}_{pp}(T)$ with ρ_f supported on the subgroup S of $\widehat{G}$ and $j : G \to \widehat{S}$ is the canonical group homomorphism, then $t \mapsto T^t f$ can be lifted to a continuous map on $\widehat{S}$, i.e. there exists a continuous map $u : \widehat{S} \to \mathcal{H}$ with $u \circ j(t) = T^t f$ for every $t \in G$.*
(b) *Let $\mathbb{T}$ be a compact group and $j : G \to \mathbb{T}$ a continuous group homomorphism with dense range. If $u : \mathbb{T} \to \mathcal{H}$ is continuous with $u \circ j(t) = T^t f$ for every $t \in G$, then f belongs to $\mathcal{H}_{pp}(T)$ and ρ_f is supported on $i(\widehat{\mathbb{T}})$.*

Proof. (a) As f belongs to $\mathcal{H}_{pp}(T)$ and ρ_f is supported on S, we have $f = \sum_{\lambda \in S} c_\lambda f_\lambda$ with f_λ which are either 0 or normalized eigenfunctions to λ. Then, $\sum_{\lambda \in S} |c_\lambda|^2 < \infty$ as the f_λ are pairwise orthogonal. As $|(\sigma, \lambda)|$ has modulus one for each $\lambda \in S$ and $\sigma \in \widehat{S}$ and $\sigma \mapsto (\sigma, \lambda)$ is continuous, the map, $u : \widehat{S} \to \mathcal{H}$,

$$u(\sigma) := \sum_{\lambda \in S} (\sigma, \lambda) c_\lambda f_\lambda,$$

can easily be seen to have the desired properties.

(b) By $u \circ j(t) = T^t f$, we infer that $\{T^t f\colon t \in G\}$ is contained in $u(\mathbb{T})$, which is compact as u is continuous and $\mathbb{T}$ is compact. Hence, f belongs to $\mathcal{H}_{pp}(T)$. Moreover, by $\widehat{\rho_f}(t) = \langle f, T^t f\rangle = \langle f, u \circ j(t)\rangle$, we infer that $\widehat{\rho_f}(t)$ can be lifted to a continuous function g on $\mathbb{T}$. Note that g is positive definite as $\widehat{\rho_f}(t)$ is positive definite and j has dense range. Since $\mathbb{T}$ is a compact group and g is positive definite and continuous, we can write:

$$g = \sum_{k=1}^{N} a_k \chi_k.$$

Here, $N = \infty$ or $N \in \mathbb{N}$ and the χ_k belong to $\widehat{\mathbb{T}}$ and the sum on the right converges uniformly. Evaluating g on $j(t)$ for some $t \in G$ we then obtain:

$$\widehat{\rho_f}(t) = \sum_{k=1}^{N} a_k \chi_k \circ j(t).$$

Since the (inverse) Fourier transform is continuous in the uniform topology, we obtain by taking the inverse Fourier transform

$$\rho_f = \sum_{k=1}^{N} a_k \delta_{-i(\chi_k)},$$

with the sum on the right converging in the vague topology. This gives that ρ_f is supported on $i(\widehat{\mathbb{T}}) \subset \widehat{G}$. □

These lemmas yield the following abstract stability result for $\mathcal{H}_{pp}(T)$.

Theorem 2.3. *Let T be a strongly continuous unitary representation of G on $\mathcal{H}$. Let $C: \mathcal{H} \to \mathcal{H}$ be continuous with $T^t C f = C T^t f$ for each $t \in G$ and $f \in \mathcal{H}$. Then, C maps $\mathcal{H}_{pp}(T)$ into $\mathcal{H}_{pp}(T)$. If f belongs to $\mathcal{H}_{pp}(T)$ and ρ_f is supported on the subgroup S of $\widehat{G}$, then so is ρ_{Cf}.*

Remark. Let us emphasize that C is not assumed to be linear.

Proof. Choose $f \in \mathcal{H}_{pp}(T)$ arbitrary. Let $A := \overline{\{T^t C f\colon t \in G\}}$ and $B := \overline{\{T^t f\colon t \in G\}}$. Then, B is compact by the Lemma 2.1. As C is continuous and commutes with T this yields that $A = C(B)$ is compact as well. Then, by Lemma 2.1 again, Cf belongs to $\mathcal{H}_{pp}(T)$.

It remains to show the statement about ρ_{Cf}: Equip S with the discrete topology and denote its compact dual group by $\mathbb{T}$. As ρ_f is supported in S, part (a) of the previous lemma shows that $t \mapsto T^t f$ can be lifted to a continuous map u on $\mathbb{T}$. As C is continuous, $t \mapsto C T^t f = T^t C f$, then lifts to the continuous map $C \circ u: \mathbb{T} \to \mathcal{H}$. By (b) of the previous lemma, ρ_{Cf} is then supported in S as well. □

We now come to an application of these considerations to measurable dynamical systems. Let a measurable space (Ω, Σ_Ω) consisting of a set Ω and a σ-algebra Σ_Ω on it be given. Let

$$\alpha: G \times \Omega \to \Omega,$$

be an action which is measurable in each variable. Then (Ω, α) is called a measurable dynamical system. Let m be a G-invariant probability measure on Ω and denote the set of square integrable functions on Ω, with respect to m, by $L^2(\Omega, m)$. This space is equipped with the inner product $\langle f, g\rangle := \int \overline{f(\omega)} g(\omega)\, dm(\omega)$. The action α induces a unitary representation T of G on $L^2(\Omega, m)$ in the obvious way, namely $T^t h$ is given by $(T^t h)(\omega) := h(\alpha_{-t}(\omega))$.

Let $C_c(\mathbb{C})$ be the set of continuous functions on $\mathbb{C}$ with compact support. Then, the following holds:

Lemma 2.4. *Let (Ω, Σ_Ω) be a measure space with a probability measure m. For each $g \in C_c(\mathbb{C})$, the map $C_g: L^2(\Omega, m) \to L^2(\Omega, m)$, $f \mapsto g \circ f$, is uniformly continuous.*

Proof. Choose $\varepsilon > 0$ arbitrary. As g is uniformly continuous, there exists a $\delta > 0$ such that

$$\bigl|g(x) - g(y)\bigr|^2 \leqslant \frac{\varepsilon}{2} \quad \text{whenever } |x - y| \leqslant \delta.$$

Set $M := \max\{|g(x)|\colon x \in \mathbb{C}\}$. By a direct Tchebycheff type estimate we have for arbitrary $h, h' \in L^2(\Omega, m)$,

$$m(\Omega_{\delta,h,h'}) \leqslant \frac{\|h - h'\|^2}{\delta^2},$$

where

$$\Omega_{\delta,h,h'} = \bigl\{\omega \in \Omega\colon \bigl|h(\omega) - h'(\omega)\bigr| \geqslant \delta\bigr\}.$$

Setting $D_{h,h'} := |g \circ h - g \circ h'|$ we obtain

$$\int_{\Omega} D_{h,h'}^2 \, dm(\omega) = \int_{\Omega_{\delta,h,h'}} D_{h,h'}^2 \, dm + \int_{\Omega \setminus \Omega_{\delta,h,h'}} D_{h,h'}^2 \, dm \leqslant m(\Omega_{\delta,h,h'}) 4M^2 + m(\Omega \setminus \Omega_{\delta,h,h'}) \frac{\varepsilon}{2}$$
$$\leqslant \frac{4M^2 \|h - h'\|^2}{\delta^2} + \frac{\varepsilon}{2}.$$

This finishes the proof. □

Corollary 2.5. *Let (Ω, α) be a measurable dynamical system and m an α-invariant probability measure on Ω such that the associated unitary representation is strongly continuous. Then, for arbitrary $f \in \mathcal{H}_{pp}(T)$ and $g \in C_c(\mathbb{C})$, the function $g \circ f$ belongs to $\mathcal{H}_{pp}(T)$ and if ρ_f is supported on the subgroup S of $\widehat{G}$, so is $\rho_{g \circ f}$.*

Proof. This follows from the previous lemma and Theorem 2.3. □

We also note the following result on compatibility of almost periodicity with products, which is a slight generalization of Lemma 1 in [2] and Lemma 3.7 in [23].

Lemma 2.6. *Let (Ω, α) be a measurable dynamical system and m an α-invariant probability measure on Ω such that the associated unitary representation is strongly continuous. Let f and g be bounded functions in $\mathcal{H}_{pp}(T)$ such that ρ_f and ρ_g are supported on the subgroup S of $\widehat{G}$. Then, fg is a bounded function in $\mathcal{H}_{pp}(T)$ and ρ_{fg} is supported on S as well.*

Proof. It is shown in Lemma 1 of [2] that the product of bounded functions in $\mathcal{H}_{pp}(T)$ belongs again to $\mathcal{H}_{pp}(T)$. Here, we give a different proof, which shows the statement on the support of the spectral measures as well. By (a) of Lemma 2.2, there exist continuous maps $u_f : \widehat{S} \to L^2(\Omega, m)$ and $u_g : \widehat{S} \to L^2(\Omega, m)$ with $u_f \circ j(t) = T^t f$ and $u_g \circ j(t) = T^t g$ for all $t \in G$. Then, using the boundedness of f and g, we can easily infer that

$$u := u_f u_g : \widehat{S} \to L^2(\Omega, m)$$

is continuous. By construction we have $u \circ j(t) = T^t(fg)$ for all $t \in G$. Thus, the desired statement follows from (b) of Lemma 2.2. □

3. Measure dynamical systems

In this section, we introduce the measurable dynamical systems we are dealing with. These will be dynamical systems of measures on groups. These systems are interesting objects in their own right. Moreover, as discussed in the next section, they provide an adequate framework for diffraction theory.

Let G be the fixed σ-compact LCA group. The set of continuous functions on G with compact support is denoted by $C_c(G)$. It is equipped with the locally convex limit topology induced by the canonical embeddings $C_K(G) \hookrightarrow C_c(G)$, where $C_K(G)$ is the space of complex continuous functions with support in $K \subset G$ compact. The support of $\varphi \in C_c(G)$ is denoted by $\operatorname{supp}(\varphi)$. The set $\mathcal{M}(G)$ is then defined to be the dual of the space of $C_c(G)$ i.e. the space of continuous linear functionals on $C_c(G)$. The elements of $\mathcal{M}(G)$ can be considered as complex measures. The total variation $|\mu|$ of an element of $\mathcal{M}(G)$ is again an element of $\mathcal{M}(G)$ and in fact a positive regular Borel measure characterized by

$$|\mu|(\varphi) = \sup\{|\mu(\psi)|: \psi \in C_c(G) \text{ real valued with } |\psi| \leqslant \varphi\},$$

for every nonnegative $\varphi \in C_c(G)$. Moreover, there exist a measurable $u : G \to \mathbb{C}$ with $|u(t)| = 1$ for $|\mu|$-almost every $t \in G$ with

$$\mu(\varphi) = \int u\varphi \, d|\mu|,$$

for every $\varphi \in C_c(G)$. This allows us in particular to define the restriction of μ to subsets of G in the obvious way.

The space $\mathcal{M}(G)$ carries the vague topology. This topology equals the weak-$*$ topology of $C_c(G)^*$, i.e., it is the weakest topology which makes all functionals $\mu \mapsto \mu(\varphi)$, $\varphi \in C_c(G)$, continuous. Thus, if we define:

$$f : C_c(G) \to \{\text{functions on } \mathcal{M}(G)\}, \qquad \varphi \mapsto f_\varphi, \quad \text{by } f_\varphi(\mu) := \int_G \varphi(-s)\, d\mu(s),$$

then the topology is generated by

$$\{f_\varphi^{-1}(O) \colon \varphi \in C_c(G),\ O \subset \mathbb{C} \text{ open}\}.$$

Here, the reader might wonder about the sign in the definition of f_φ. This sign is not necessary. However, it does not matter either as $G \to G$, $s \mapsto s^{-1}$, is a homeomorphism. It will simplify some formulae later on.

We will be concerned with measurable dynamical systems consisting of elements of $\mathcal{M}(G)$. Thus, we need a σ-algebra on $\mathcal{M}(G)$ and an action of G. These will be provided next. We start with the σ-algebra. As discussed above, $\mathcal{M}(G)$ is a topological space. Thus, it carries a natural σ-algebra, namely the Borel σ-algebra generated by the open sets. Denote this algebra by $\Sigma_{\mathcal{M}(G)}$.

Remark. If G has a countable basis of the topology then the restriction of the Borel σ-algebra to the set $\mathcal{M}(G)_+$ of nonnegative measures is the σ-algebra Σ' generated by $\{f_\varphi^{-1}(O) \cap \mathcal{M}(G)_+ \colon \varphi \in C_c(G),\ O \subset \mathbb{C} \text{ open}\}$. This is a consequence of the well-known second countability of the vague topology on $\mathcal{M}(G)_+$ (see Chapter IV, Section 31 in [5]). A proof can be given along the following line: The set $\mathcal{M}(G)_+$ with the vague topology is a second countable metric space and a metric can be given as

$$d(\mu, \nu) := \sum_{n \in \mathbb{N}} \frac{|f_{\varphi_n}(\mu) - f_{\varphi_n}(\nu)|}{2^n(1 + |f_{\varphi_n}(\mu) - f_{\varphi_n}(\nu)|)},$$

with a suitable dense set $\{\varphi_n \colon n \in \mathbb{N}\}$ in $C_c(G)$. The definition of the metric shows that all balls $B_s(\mu) := \{\nu \in \mathcal{M}(G)_+ \colon d(\mu, \nu) < s\}$, $\mu \in \mathcal{M}(G)_+$, $s \geqslant 0$, belong to Σ'. This is then true for countable unions of such balls as well and the statement follows.

Lemma 3.1. *The map $f_\varphi \circ |\cdot|$ is measurable for every $\varphi \in C_c(G)$. In particular, the map $\mathcal{M}(G) \to \mathcal{M}(G)$, $\mu \mapsto |\mu|$, is measurable.*

Proof. It suffices to show that $f_\varphi \circ |\cdot|$ is measurable for every $\varphi \in C_c(G)$ with $\varphi \geqslant 0$. Standard theory (see Chapter 6 in [34] and Proposition 1 in [2]) gives:

$$f_\varphi(|\mu|) = \sup\{|f_{\psi\varphi}(\mu)| \colon \psi \in C_c(G),\ \|\psi\|_\infty \leqslant 1\}.$$

As $\mu \mapsto |f_{\psi\varphi}(\mu)|$ is continuous, $f_\varphi \circ |\cdot|$ is then semicontinuous and hence measurable. □

As for the action of G on $\mathcal{M}(G)$, there is a natural action of G on $\mathcal{M}(G)$ given by:

$$\alpha : G \times \mathcal{M}(G) \to \mathcal{M}(G), \qquad \alpha_t(\mu) := \delta_t * \mu,$$

where δ_t is the unit point measure at $t \in G$. Here, the convolution $\mu * \nu$ of two convolvable elements of $\mathcal{M}(G)$ is the measure defined by $(\mu * \nu)(\varphi) := \int_{G \times G} \varphi(s + t)\, d\mu(t)\, d\nu(s)$.

The map α is measurable in each variable, as shown in the next lemma.

Lemma 3.2.

(a) *For fixed $\mu \in \mathcal{M}(G)$, the map $G \to \mathcal{M}(G)$, $t \mapsto \alpha_t \mu$, is continuous, hence also measurable.*
(b) *For fixed $t \in G$, the map $\mathcal{M}(G) \to \mathcal{M}(G)$, $\mu \mapsto \alpha_t \mu$, is continuous, hence also measurable.*

Proof. This is straightforward. □

Putting this together, we see that $\mathcal{M}(G)$ equipped with the Borel σ-algebra and the natural action of G by shifts is a measurable dynamical system.

As discussed in Section 2, every α-invariant probability measure m on $\mathcal{M}(G)$ induces a unitary representation T of G on $L^2(\mathcal{M}(G), m)$.

For further understanding of this unitary representation, it will be crucial to control it by a suitable set of functions. This set of functions is introduced next.

Definition 3.3. Consider the algebra generated by the set:

$$\{g \circ f_\varphi \colon g \in C_c(\mathbb{C}),\ \varphi \in C_c(G)\}.$$

Let $\mathcal{A}(G)$ be the closure of this algebra in the algebra of all continuous bounded functions on $\mathcal{M}(G)$ equipped with the supremum norm. An α-invariant probability measure m on $\mathcal{M}(G)$ is said to satisfy the denseness assumption (D) if the algebra $\mathcal{A}(G)$ is dense in $L^2(\mathcal{M}(G), m)$.

Note that condition (D) means that the set of finite products of functions of the form $g \circ f_\varphi$, $g \in C_c(\mathbb{C})$, $\varphi \in C_c(G)$, is total in $L^2(\mathcal{M}(G), m)$.

We now discuss two instances in which condition (D) holds.

Proposition 3.4. *Let m be an α-invariant probability measure on $\mathcal{M}(G)$. If the restriction of the σ-algebra of $\mathcal{M}(G)$ to the support of m is generated by the set $\{f_\varphi^{-1}(O)\colon \varphi \in C_c(G), O \subset \mathbb{C}$ open$\}$, then* (D) *holds. In particular,* (D) *holds whenever G has a countable basis of topology and m is supported on the set of nonnegative measures.*

Proof. As $\mathbb{C}$ has a countable basis of the topology, the σ-algebra on $\mathcal{M}(G)$ is then generated by the set $\{f_\varphi^{-1}(K)\colon \varphi \in C_c(G),\ K \subset \mathbb{C} \text{ compact}\}$. In particular, the corresponding set of products of characteristic functions,

$$\{1_K \circ f_\varphi \colon \varphi \in C_c(G),\ K \subset \mathbb{C} \text{ compact}\},$$

is total in $L^2(\mathcal{M}(G), m)$. Here, 1_S denotes the characteristic function of S. Therefore, it suffices to show that all functions of the form $1_K \circ f_\varphi$, $\varphi \in C_c(G)$, $K \subset \mathbb{C}$ compact, can be approximated by functions of the form $g \circ f_\varphi$ with $g \in C_c(\mathbb{C})$. This can be done by choosing, for $K \subset \mathbb{C}$ compact, a compact $L \subset \mathbb{C}$ containing K and a sequence (g_n) of nonnegative functions in $C_c(\mathbb{C})$ such that g_n converge pointwise to 1_K, are all supported in L and are uniformly bounded by, say, 1.

The 'in particular' statement now follows from the last remark. □

Proposition 3.5. *The condition* (D) *is satisfied whenever m is supported on a compact α-invariant subset of $\mathcal{M}(G)$.*

Proof. This follows by a Stone/Weierstrass type argument (see [2] as well): The algebra in question separate the points, does not vanish identically anywhere and is closed under complex conjugation. The algebra is then dense in the set of continuous functions on the compact support of m. Hence, it is dense in $L^2(\mathcal{M}(G), m)$ as well. □

Remark. The previous propositions imply that (D) holds in all the settings considered for diffraction so far. More precisely, the setting of uniformly discrete point sets discussed e.g. in the survey article [22] and its generalization to translation bounded measures [2] deal with compact subsets of $\mathcal{M}(G)$. On the other hand the point process setting first introduced by [15] deals with $\mathbb{R}^d$ and hence admits a countable basis of topology.

Theorem 3.6. *Let m be an α-invariant probability measure on $\mathcal{M}(G)$, which satisfies* (D). *Then, the representation T is strongly continuous.*

Proof. As T^t is unitary for every $t \in G$, it is bounded with norm 1 uniformly in $t \in G$. By (D), it therefore suffices to show continuity of $t \mapsto T^t f$ for f a finite product of functions of the form $g \circ f_\varphi$ with $\varphi \in C_c(G)$ and $g \in C_c(\mathbb{C})$. It suffices to show continuity at $t = 0$. As

$$T^t(f_1 f_2) - f_1 f_2 = (T^t f_1) T^t f_2 - f_1 f_2 = (T^t f_1)(T^t f_2 - f_2) + (T^t f_1 - f_1) f_2,$$

and functions of the form $g \circ f_\varphi$ with $g \in C_c(\mathbb{C})$ and $\varphi \in C_c(G)$ are bounded, it suffices to consider $f = g \circ f_\varphi$. Let K be an arbitrary compact neighborhood of $0 \in G$. Let L be a compact set in G with $L \supset K - \operatorname{supp}\varphi$ and $\psi \in C_c(G)$ nonnegative with $\psi \equiv 1$ on L. Then,

$$\left|\varphi(t-s) - \varphi(-s)\right| \leqslant \psi(-s)\left\|\varphi(t-\cdot) - \varphi(-\cdot)\right\|_\infty,$$

for every $s \in G$ and $t \in K$ and, in particular,

$$\left|f_\varphi(\alpha_{-t}\mu) - f_\varphi(\mu)\right| \leqslant \int_G \left|\varphi(t-s) - \varphi(-s)\right| d|\mu|(s) \leqslant \left\|\varphi(t-\cdot) - \varphi(-\cdot)\right\|_\infty f_\psi\left(|\mu|\right), \tag{$*$}$$

for every $t \in K$. As $\mu \mapsto f_\psi(|\mu|)$ is measurable, the set,

$$\Omega_N := \left\{\mu \in \mathcal{M}(G):\ f_\psi\left(|\mu|\right) \leqslant N\right\},$$

is measurable for each $N \in \mathbb{N}$. Obviously, these sets are increasing and cover $\mathcal{M}(G)$. Thus, for each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ with $m(\mathcal{M}(G) \setminus \Omega_{N(\varepsilon)}) \leqslant \varepsilon$. Invoking $(*)$, we then get:

$$\begin{aligned}\left\|T^t g \circ f_\varphi - g \circ f_\varphi\right\|^2 &= \int_{\mathcal{M}(G)} \left|g \circ f_\varphi(\alpha_{-t}\mu) - g \circ f_\varphi(\mu)\right|^2 dm(\mu) \\ &= \int_{\Omega_{N(\varepsilon)}} |\cdot|^2\, dm(\mu) + \int_{\mathcal{M}(G)\setminus\Omega_{N(\varepsilon)}} |\cdot|^2\, dm(\mu) \\ &\leqslant B(t,\varepsilon) + \varepsilon 4\|g\|_\infty^2,\end{aligned}$$

with

$$B(t,\varepsilon) := \sup\left\{\left|g(x) - g(y)\right|^2:\ |x-y| \leqslant N(\varepsilon)\left\|\varphi(t-\cdot) - \varphi(-\cdot)\right\|_\infty\right\}.$$

As g is uniformly continuous, $B(t,\varepsilon)$ becomes arbitrarily small for t close to 0 and ε fixed. This easily shows the desired continuity. □

For our further consideration, we will need a certain finiteness assumption on the probability measure m. This assumption is well known in the theory of stochastic processes. It is given in the next definition.

Definition 3.7. The α-invariant probability measure m on $\mathcal{M}(G)$ is called square integrable if $f_\varphi \circ |\cdot| : \mathcal{M}(G) \to \mathbb{C}$ belongs to $L^2(\mathcal{M}(G), m)$ for every $\varphi \in C_c(G)$.

Lemma 3.8. *Let m be an α-invariant square integrable probability measure m on $\mathcal{M}(G)$. Then, the map $C_c(G) \to L^2(\mathcal{M}(G), m)$, $\varphi \mapsto f_\varphi$, is continuous.*

Proof. We have to show that $C_K(G) \to L^2(\mathcal{M}(G), m)$, $\varphi \mapsto f_\varphi$, is continuous for every compact K in G. Let $\psi \geqslant 0$ be a function in $C_c(G)$ with $\psi \equiv 1$ on K. Then,

$$\left|\varphi(s)\right| \leqslant \psi(s)\|\varphi\|_\infty$$

for every $\varphi \in C_K(G)$. This gives:

$$\begin{aligned}\|f_\varphi\|^2 &= \int \left|\int \varphi(-s)\, d\omega(s)\right|^2 dm(\omega) \leqslant \int \left|\int \psi(-s)\|\varphi\|_\infty\, d|\omega|(s)\right|^2 dm(\omega) \\ &= \|\varphi\|_\infty^2 \int \left|f_\psi\left(|\omega|\right)\right|^2 dm(\omega),\end{aligned}$$

and the desired continuity follows. □

Remark. Note that the lemma gives another proof for the strong continuity of T in the case of square integrable measures m satisfying (D). More precisely, for $\varphi \in C_c(G)$ and $g \in C_c(\mathbb{C})$, the map $G \to L^2(\mathcal{M}(G), m)$, $t \mapsto T^t(g \circ f_\varphi)$ can be composed into the continuous maps $G \to C_c(G)$, $t \mapsto \varphi(\cdot - t)$, $f : C_c(G) \to L^2(\mathcal{M}(G), m)$, and $C_g : L^2(\mathcal{M}(G), m) \to L^2(\mathcal{M}(G), m)$, $h \mapsto g \circ h$.

Definition 3.9. Let m be square integrable. Then, $\mathcal{U}$ is defined to be the closure of $\{f_\varphi\colon \varphi \in C_c(G)\}$ in $L^2(\mathcal{M}(G), m)$.

Lemma 3.10. *Let m be square integrable. Then, $\mathcal{U}$ is a T-invariant subspace.*

Proof. This is immediate from $T^t f_\varphi = f_{\varphi_t}$ with $\varphi_t(s) = \varphi(t-s)$. □

4. Diffraction theory and the autocorrelation measure

In this section, we present a basic setup for diffraction theory for our measure dynamical systems. Before we actually start with the mathematical formulation, we shortly discuss the physical context of our setting and the relationship of the presented material to earlier work. For recent surveys we refer the reader to [1,25,26].

In the simplest models for diffraction of a solid, the solid in question is modeled by a subset Λ of Euclidean space, which describes the positions of the atoms of the solid. The diffraction of an incoming beam is then governed by interference of beams scattered by different points of the solid. Thus, the relevant set is the set of differences between points of Λ or rather differences averaged according to occurrence. This yields the so-called autocorrelation measure γ_Λ of Λ given by:

$$\gamma_\Lambda := \lim_{n\to\infty} \frac{1}{|B_n|} \sum_{x,y\in B_n\cap\Lambda} \delta_{x-y} = \lim_{n\to\infty} \frac{1}{|B_n|} \delta_{\Lambda\cap B_n} * \delta_{-\Lambda\cap B_n}.$$

Here, B_n is the ball around the origin with radius n, $|B_n|$ is its volume, δ_x denotes the point measure at x and the limit is assumed to exist. The diffraction of Λ is then described by the Fourier transform of γ_Λ (see e.g. [9,16,17] for further details on this approach). In order to obtain existence of the limit, one usually introduces a framework of dynamical systems and uses an ergodic theorem.

In fact, as shown recently [15,14] (see [2] as well), it is possible to express the limit by a closed formula. This opens up the possibility to define the autocorrelation by this closed formula irrespective whether the dynamical system is ergodic or not. This approach has been taken in [2]. In fact, as argued in [2,3], it is more appropriate to work with measures than with point sets. This lead to the notion of measure dynamical system. In the framework of aperiodic order, it is natural to restrict attention to topological dynamical systems and this is what has been analyzed in [2,3]. However, as discussed in the introduction both from the mathematical point of view and from the point of view of perturbation theory, it is natural to leave the topological category and develop diffraction theory in the measurable category. This is done next. While our overall line of reasoning certainly owes to [2], we have to overcome various technical issues. More precisely, as [2] deals with a topological situation and compact spaces, all functions f_φ (defined above) were uniformly bounded there. This is not the case here anymore. To remedy this, we use the assumption of square integrability.

We start by introducing some further notation: For a measure μ on G and a set $B \subset G$, we denote by μ_B the restriction of μ to B. For a function ζ on G we define $\widetilde{\zeta}$ by $\widetilde{\zeta}(s) := \overline{\zeta(-s)}$. For a measure μ on G we define the measure $\widetilde{\mu}$ by $\widetilde{\mu}(\varphi) := \overline{\mu(\widetilde{\varphi})}$.

Lemma 4.1. *Let m be an α-invariant square integrable probability measure m on $\mathcal{M}(G)$. Let a function $\sigma \in C_c(G)$ be given with $\int_G \sigma(t)\,dt = 1$. For $\varphi \in C_c(G)$, define*:

$$\gamma_{\sigma,m}(\varphi) := \int_\Omega \int_G \overline{\int_G \overline{\varphi(s-t)\sigma(t)}\, d\omega(s)}\, d\omega(t)\, dm(\omega).$$

Then, the following holds:

(a) *The map $\gamma_{\sigma,m} : C_c(G) \to \mathbb{C}$ is continuous, i.e., $\gamma_{\sigma,m} \in \mathcal{M}(G)$.*
(b) *For $\varphi, \psi \in C_c(G)$, the equation $(\widetilde{\varphi} * \psi * \gamma_{\sigma,m})(t) = \langle f_\varphi, T^t f_\psi\rangle$ holds.*
(c) *The measure $\gamma_{\sigma,m}$ does not depend on $\sigma \in C_c(G)$, provided $\int_G \sigma\, dt = 1$.*
(d) *The measure $\gamma_{\sigma,m}$ is positive definite.*

Proof. (a) We have to show that $\gamma_{\sigma,m}$ restricted to $C_K(G)$ is continuous for every compact K in G. Choose $\psi \in C_c(G)$ nonnegative with $\psi \equiv 1$ on $\operatorname{supp}(\sigma) + K$. Thus,

$$\left|\varphi(s-t)\sigma(t)\right| \leqslant \psi(s)\|\varphi\|_\infty |\sigma|(t),$$

for all $\varphi \in C_K(G)$ and $s, t \in G$. For $\varphi \in C_K(G)$ we can then estimate:

$$\begin{aligned}\left|\gamma_{\sigma,m}(\varphi)\right| &\leqslant \left|\int_\Omega \int_G \int_G \left|\varphi(s-t)\sigma(t)\right| d|\omega|(t)\, d|\omega|(s)\, dm(\omega)\right| \\ &\leqslant \int_\Omega \int_G \int_G \psi(s)\|\varphi\|_\infty \left|\sigma(t)\right| d|\omega|(t)\, d|\omega|(s)\, dm(\omega) \\ &= \|\varphi\|_\infty \int_\Omega f_{\widetilde{\psi}}\big(|\omega|\big) f_{|\widetilde{\sigma}|}\big(|\omega|\big)\, dm(\omega),\end{aligned}$$

and the statement follows.

(b) This follows by a direct computation (see Proposition 6 of [2] as well).

(c) Fix $\varphi \in C_c(G)$. By α-invariance of m, we find that the map $\sigma \mapsto \gamma_{\sigma,m}(\varphi)$ is α-invariant and hence a multiple of Haar measure on G. This shows the claim.

(d) This is a direct consequence of (b). □

The preceding lemma allows us to associate to any square integrable probability measure an autocorrelation and a diffraction measure. They are defined next.

Definition 4.2. Let m be an α-invariant square integrable probability measure m on $\mathcal{M}(G)$. Then, the measure $\gamma_m := \gamma_{\sigma,m}$ for $\sigma \in C_c(G)$ with $\int_G \sigma(t)\, dt = 1$ is called the autocorrelation. As γ_m is positive definite, its Fourier transform $\widehat{\gamma}$ exists and is a positive measure on $\widehat{G}$. This measure is called the diffraction measure of the dynamical system.

As discussed in the introduction to this section the usual approach to autocorrelation proceeds by an averaging procedure along (models of) the substance in question. In our framework, the substances are modeled by measures. Thus, we will have to average measures. This is discussed in the remainder of this section. It will turn out that averaging is possible once ergodicity is known. This is a consequence of the validity of Birkhoffs ergodic theorem (see Appendix A and in particular Lemma A.3 for further details as well). We will have to exercise quite some care as the functions f_φ are not bounded.

Definition 4.3. A sequence (B_n) of compact subsets of G is called a *van Hove sequence* if,

$$\lim_{n\to\infty} \frac{|\partial^K B_n|}{|B_n|} = 0,$$

for all compact $K \subset G$. Here, for compact B, K, the "K-boundary" $\partial^K B$ of B is defined as

$$\partial^K B := \overline{\big((B+K)\setminus B\big)} \cup \big[(\overline{G\setminus B} - K) \cap B\big],$$

where the bar denotes the closure.

As discussed in Appendix A, in our setting there exists a van Hove sequence (B_n) such that for any compact $K \subset G$ and any α-invariant ergodic probability measure m on $\mathcal{M}(G)$ and any $f \in L^1(\mathcal{M}(G), m)$,

$$\lim_{n\to\infty} \frac{1}{|B_n|} \int_{\partial^K B_n} \left|f\big(\alpha_t(\omega)\big)\right| dt = 0, \tag{$\sharp$}$$

holds for m-almost every $\omega \in \Omega$. Without loss of generality we can assume that $B_n = -B_n$ for all n. Fix such a sequence for the rest of this section.

Lemma 4.4. *Let m be an α-invariant square integrable ergodic probability measure on $\mathcal{M}(G)$. Then for all $\phi, \psi \in C_c(G)$ nonnegative, there exists a $C < \infty$ and a set $\mathcal{M}(G)'$ of full measure in $\mathcal{M}(G)$ such that for all $\omega \in \mathcal{M}(G)'$ we have*:

$$\limsup_{n\to\infty} \frac{|\tilde{\omega}_{B_n}| * |\omega_{B_n}|(\psi * \tilde{\phi})}{|B_n|} \leqslant C.$$

Proof. Let K be a compact subset of G with $K = -K$ and $\operatorname{supp}(\psi), \operatorname{supp}(\phi) \subset K$. As a product of two L^2-functions the function $f_{\tilde{\psi}} \circ |\cdot| f_{\tilde{\phi}} \circ |\cdot|$ belongs to L^1, (♯) implies (see Proposition A.3 as well) that almost surely,

$$\lim_{n\to\infty} \frac{\int_{B_n+K} T^t[f_{\tilde{\psi}} \circ |\cdot| f_{\tilde{\phi}} \circ |\cdot|]\, dt}{|B_n|} = \int_{\mathcal{M}(G)} f_{\tilde{\psi}}\big(|\mu|\big) f_{\tilde{\phi}}\big(|\mu|\big)\, dm(\mu) =: C < \infty.$$

For $v, s \in B_n$ the function $t \mapsto \psi(-t+v)\phi(-t+s)$ is zero outside $B_n + K$ and hence

$$\int_{B_n+K} \psi(-t+v)\phi(-t-s)\, dt = \int_G \psi(-t+v)\phi(-t-s)\, dt = \psi * \tilde{\phi}(s+v).$$

Moreover, since ϕ, ψ are nonnegative a short calculation gives that

$$T^t f_{\tilde{\phi}}\big(|\omega|\big) = \int_G \phi(-s-t)\, d|\tilde{\omega}|(s); \qquad T^t f_{\tilde{\psi}}\big(|\omega|\big) = \int_G \psi(v-t)\, d|\omega|(v).$$

Thus,

$$\begin{aligned} C &= \lim_{n\to\infty} \frac{\int_{B_n+K} T^t f_{\tilde{\psi}}(|\omega|) T^t f_{\tilde{\phi}}(|\omega|)\, dt}{|B_n|} \\ &= \lim_{n\to\infty} \frac{\int_{B_n+K} \int_G \int_G \psi(-t+v)\phi(-t-s)\, d|\omega|(v)\, d|\tilde{\omega}|(s)\, dt}{|B_n|} \\ &\geqslant \limsup_{n\to\infty} \frac{\int_{B_n} \int_{B_n} \int_{B_n+K} \psi(-t+v)\phi(-t-s)\, dt\, d|\omega|(v)\, d|\tilde{\omega}|(s)}{|B_n|} \\ &= \limsup_{n\to\infty} \frac{\int_{B_n} \int_{B_n} \psi * \tilde{\phi}(s+v)\, d|\omega|(v)\, d|\tilde{\omega}|(s)}{|B_n|} \\ &= \limsup_{n\to\infty} \frac{|\tilde{\omega}_{B_n}| * |\omega_{B_n}|(\psi * \tilde{\phi})}{|B_n|}, \end{aligned}$$

and the proof is finished. □

Lemma 4.5. *Let m be an α-invariant square integrable ergodic probability measure on $\mathcal{M}(G)$. Let $\phi, \psi \in C_c(G)$ be given. Then*

$$\lim_{n\to\infty} \frac{\tilde{\omega} * \omega_{B_n}(\tilde{\phi} * \psi) - \tilde{\omega}_{B_n} * \omega_{B_n}(\tilde{\phi} * \psi)}{|B_n|} = 0,$$

almost surely in ω.

Proof. Let K be a compact subset of G with $K = -K$ and $\operatorname{supp}(\phi), \operatorname{supp}(\psi) \subset K$. Then,

$$(\tilde{\omega} * \omega_{B_n} - \tilde{\omega}_{B_n} * \omega_{B_n})(\tilde{\phi} * \psi) = \int_G \int_G \int_G \overline{\phi(-v-s)} \psi(r-v) 1_{B_n}(s)\big(1 - 1_{B_n}(r)\big)\, d\omega(s)\, d\tilde{\omega}(r)\, dv.$$

For the integrand not to vanish we need $s \in B_n$, $r \in G \setminus B_n$ and $v \in (B_n + K) \cap [(G \setminus B_n) + K] \subset \partial^K B_n$. Hence, we can estimate:

$$\begin{aligned}\big|(\tilde{\omega} * \omega_{B_n} - \tilde{\omega}_{B_n} * \omega_{B_n})(\tilde{\phi} * \psi)\big| &\leqslant \int\limits_{\partial^K B_n} \int\limits_G \int\limits_G \big|\overline{\phi(-v-s)}\big|\big|\psi(r-v)\big|\, d|\omega|(s)\, d|\tilde{\omega}|(r)\, dv \\ &= \int\limits_{\partial^K B_n} T^{-v}\big[f_{|\phi|}\big(|\omega|\big) f_{|\psi|}\big(|\omega|\big)\big]\, dv.\end{aligned}$$

The proof follows now from (♯). □

Theorem 4.6. *Assume that G is second countable. Let m be an α-invariant square integrable ergodic probability measure m on $\mathcal{M}(G)$. Then, almost surely in ω,*

$$\lim_{n\to\infty} \frac{\tilde{\omega}_{B_n} * \omega_{B_n}}{|B_n|} = \gamma_m,$$

where the limit is taken in the vague topology.

Proof. The proof proceeds in three steps.

Step 1. Let $\phi, \psi \in C_c(G)$, $t \in G$ be given and set $Z_n := \tilde{\phi} * \psi * \tilde{\omega} * \omega_{B_n}(t)$. Then, $\lim_{n\to\infty} |B_n|^{-1} Z_n = \langle f_\phi, T^t f_\psi\rangle$.

Proof of Step 1. Let a compact set K in G with $K = -K$, $0 \in K$ and $\operatorname{supp}(\psi) \subset K$ be given. We are going to show that $|B_n|^{-1} Z_n$ is of the same size as $|B_n|^{-1} \int_{B_n} \overline{f_\phi(\alpha_{t-v}(\omega))} f_\psi(\alpha_{-v}(\omega))\, dv$, which by Birkhoff's ergodic theorem converges to $\langle f_\phi, T^t f_\psi\rangle$.

A direct calculation (see Theorem 5 in [2] as well) shows,

$$Z_n - \int\limits_{B_n} \overline{f_\phi\big(\alpha_{t-v}(\omega)\big)} f_\psi\big(\alpha_{-v}(\omega)\big)\, dv = \int\limits_G \overline{f_\phi\big(\alpha_{t-v}(\omega)\big)} D(v)\, dv,$$

with $D(v) := \int_G \psi(v-s)(1_{B_n}(s) - 1_{B_n}(v))\, d\omega(s)$. Then $D(v)$ is supported on $\partial^K B_n$ and hence

$$\Delta(n) := \left| Z_n - \int\limits_{B_n} \overline{f_\phi\big(\alpha_{t-v}(\omega)\big)} f_\psi\big(\alpha_{-v}(\omega)\big)\, dv \right| \leqslant \int\limits_{\partial^K B_n} \big|\overline{f_\phi\big(\alpha_{t-v}(\omega)\big)} D(v)\big|\, dv.$$

Now, note that $|D(v)| \leqslant \int_G |\psi(v-s)|\, d|\omega|(s) = f_{|\psi|}(\alpha_{-v}|\omega|)$, thus

$$\Delta(n) \leqslant \int\limits_{\partial^K B_n} \big|\overline{f_\phi\big(\alpha_{t-v}(\omega)\big)} f_{|\psi|}\big(\alpha_{-v}|\omega|\big)\big|\, dv.$$

Application of (♯) now completes the proof.

Step 2. Let D be a countable subset of $C_c(G)$. Then, there exists a set Ω in $\mathcal{M}(G)$ of full measure with $\lim_{n\to\infty} \frac{\tilde{\omega}_{B_n} * \omega_{B_n}(\tilde{\phi} * \psi)}{|B_n|} = \gamma_m(\tilde{\phi} * \psi)$ for all ϕ, ψ in D and $\omega \in \Omega$.

Proof of Step 2. This follows immediately from Step 1, (b) of Lemma 4.1 and Lemma 4.5.

Step 3. There exists a set Ω in $\mathcal{M}(G)$ of full measure with $\lim_{n\to\infty} \frac{\tilde{\omega}_{B_n} * \omega_{B_n}(\sigma)}{|B_n|} = \gamma_m(\sigma)$ for all $\sigma \in C_c(G)$.

Proof of Step 3. Since G is σ compact, we can find a sequence K_j of compact sets so that $G = \bigcup_j K_j$ and $K_j \subset K_{j+1}^\circ$. It follows that $G = \bigcup_j K_j^\circ$ and in particular that each compact subset $K \subset G$ is contained in some K_j.

As G is second countable, there exists a countable dense subset D_j in each $C_{K_j}(G)$.

By second countability again, there exists furthermore an approximate unit given by a sequence (i.e. a sequence (δ_n) in $C_c(G)$ such that $\varphi * \delta_n$ converges to φ with respect to $\|\cdot\|_\infty$ for all $\varphi \in C_c(G)$). Moreover, we can pick this sequence so that there exists a fixed compact set $0 \in K = -K$ such that $\operatorname{supp}(\delta_n) \subset K$, $\forall n$.

Let

$$D := \left(\bigcup_j D_j\right) \cup \{\delta_n \mid n \in \mathbb{N}\}.$$

Then D is countable.

Lets observe that for each $\sigma \in C_c(G)$, there exists j so that $\operatorname{supp}(\sigma) \subset K_j$. Thus, for all $\epsilon > 0$, there exists some $\psi \in D_j$ and $\phi = \delta_n$ so that $\|\sigma - \phi * \tilde{\psi}\|_\infty \leqslant \epsilon$, and $\operatorname{supp}(\sigma), \operatorname{supp}(\phi * \tilde{\psi}) \subset K_j + K$.

For each $j \in \mathbb{N}$ we can chose nonnegative $\phi_j, \psi_j \in C_c(G)$, $j \in \mathbb{N}$, such that $\phi_j * \tilde{\psi}_j \geqslant 1$ on $K_j + K$.

By Lemma 4.4, for each j there exists a constant $C_j \geqslant 0$ and a subset Ω_j of full measure so that for all $\omega \in \Omega_j$ we have:

$$\lim_{n\to\infty} \frac{|\tilde{\omega}_{B_n}| * |\omega_{B_n}|(\phi_j * \tilde{\psi}_j)}{|B_n|} \leqslant C_j.$$

Let Ω' be the set of full measure given by D in Step 2. Then $\Omega := \Omega' \cap (\bigcap_j \Omega_j)$ has full measure.

Let $\sigma \in C_c(G)$ and $\omega \in \Omega$. Then there exists an j so that $\operatorname{supp}(\sigma) \subset K_j$.

Let

$$C := \max\bigl\{C_j + 1,\ |\gamma_m|(\phi_j * \tilde{\psi}_j), 1\bigr\}.$$

Since $\omega \in \Omega_j$, there exists an N_0 so that for all $n > N_0$ we have:

$$\frac{|\tilde{\omega}_{B_n}| * |\omega_{B_n}|(\phi_j * \tilde{\psi}_j)}{|B_n|} \leqslant C_j + 1 \leqslant C.$$

Let $\epsilon > 0$. Then there exists $\psi, \phi \in D$ so that

$$|\sigma - \phi * \tilde{\psi}| \leqslant \frac{\epsilon}{C} \phi_j * \tilde{\psi}_j.$$

When combined with the definition of C, this gives easily,

$$\bigl|\gamma_m(\sigma - \phi * \tilde{\psi})\bigr| \leqslant \epsilon \quad \text{and} \quad \left|\frac{\tilde{\omega}_{B_n} * \omega_{B_n}(\sigma - \phi * \tilde{\psi})}{|B_n|}\right| \leqslant \varepsilon \quad \forall n > N_0.$$

Moreover, since $\omega \in \Omega'$, by Step 2 we have:

$$\left|\frac{\tilde{\omega}_{B_n} * \omega_{B_n}(\phi * \tilde{\psi})}{|B_n|} - \gamma_m(\phi * \tilde{\psi})\right| \leqslant \epsilon \quad \forall n > N_1.$$

Hence, for all $n > N := \max\{N_0, N_1\}$ we have:

$$\left|\frac{\tilde{\omega}_{B_n} * \omega_{B_n}(\sigma)}{|B_n|} - \gamma_m(\sigma)\right| \leqslant 3\epsilon. \qquad \square$$

5. Dynamical systems and pure point diffraction

In this section, we relate spectral theory of measure dynamical systems to diffraction theory. We will assume that we are given an α-invariant square integrable probability measure m on $\mathcal{M}(G)$ such that the associated unitary representation T_m is strongly continuous. The associated autocorrelation will be denoted by $\gamma = \gamma_m$. We will then discuss the relationship between $\widehat{\gamma_m}$ and the spectrum of the unitary representation T_m. Our main result shows that, given (D), pure pointedness of $\widehat{\gamma_m}$ is equivalent to pure pointedness of T_m. This generalizes the corresponding results of [2,15,23].

Proposition 5.1. *The equation $\rho_{f_\varphi} = |\widehat{\varphi}|^2 \widehat{\gamma_m}$ holds for every $\varphi \in C_c(G)$.*

Proof. The proof can be given exactly as in [2]. We include it for completeness reasons. By the very definition of ρ_{f_φ} above, the (inverse) Fourier transform (on $\widehat{G}$) of ρ_{f_φ} is $t \mapsto \langle f_\varphi, T^t f_\varphi\rangle$. By Lemma 4.1, we have $\langle f_\varphi, T^t f_\varphi\rangle = (\widetilde{\varphi} * \varphi * \gamma_m)(t)$. Thus, taking the Fourier transform (on G), we infer $\rho_{f_\varphi} = |\widehat{\varphi}|^2\, \widehat{\gamma_m}$. $\square$

Note that the closed T-invariant subspace $\mathcal{U}$ of $L^2(\mathcal{M}(G), m)$ gives rise to a representation $T_{\mathcal{U}}$ of G on $\mathcal{U}$ by restricting the representation T to $\mathcal{U}$. The spectral family of $T_{\mathcal{U}}$ will be denoted by $E_{T_{\mathcal{U}}}$.

Definition 5.2. Let ρ be a nonnegative measure on $\widehat{G}$ and let S be an arbitrary strongly continuous unitary representation of G on an Hilbert space. Then, ρ is called a *spectral measure* for S if the following holds for all Borel sets B: $E_S(B) = 0$ if and only if $\rho(B) = 0$.

Theorem 5.3. *Let m be a square integrable probability measure on $\mathcal{M}(G)$ with associated autocorrelation $\gamma = \gamma_m$. Then, the measure $\widehat{\gamma}$ is a spectral measure for $T_{\mathcal{U}}$.*

Proof. Given the previous results the proof follows as in [2]. We only sketch the details: Let B be a Borel set in $\widehat{G}$. Then, $E_{T_{\mathcal{U}}}(B) = 0$ if and only if $\langle f_\varphi, E_T(B) f_\varphi\rangle = 0$ for every $\varphi \in C_c(G)$. By Proposition 5.1, we have $\rho_{f_\varphi} = |\widehat{\varphi}|^2 \widehat{\gamma}_m$ and, in particular,

$$\langle f_\varphi, E_T(B) f_\varphi\rangle = \rho_{f_\varphi}(B) = \int_B |\widehat{\varphi}|^2 \, d\widehat{\gamma}_m.$$

These considerations show that $E_{T_{\mathcal{U}}}(B) = 0$ if and only if $0 = \int_B |\widehat{\varphi}|^2 \, d\widehat{\gamma}_m$ for every function $\varphi \in C_c(G)$. By density, this is equivalent to $\widehat{\gamma}_m(B) = 0$. □

The preceding considerations allow us to characterize the eigenvalues of $T_{\mathcal{U}}$. In this context, this type of result seems to be new. It may be useful in other situations as well. For a characterization of continuous eigenvalues we refer to [24].

Corollary 5.4. *Let m be a square integrable probability measure on $\mathcal{M}(G)$ with associated autocorrelation $\gamma = \gamma_m$. For $\varphi \in C_c(G)$ and $\lambda \in \widehat{G}$, the following assertions are equivalent*:

(i) $|\widehat{\varphi}|^2(\lambda)\widehat{\gamma}(\{\lambda\}) > 0$.
(ii) $E(\{\lambda\}) f_\varphi \neq 0$.
(iii) *There exists an $f \neq 0$ with $f = E(\{\lambda\}) f$ in the closed convex hull of $\{\overline{(\lambda, t)} T^t f_\varphi \colon t \in G\}$.*

Proof. Proposition 5.1 gives:

$$\langle E(\{\lambda\}) f_\varphi, E(\{\lambda\}) f_\varphi\rangle = \rho_{f_\varphi}(\{\lambda\}) = |\widehat{\varphi}|^2(\lambda)\widehat{\gamma}(\{\lambda\}),$$

and the equivalence between (i) and (ii) follows. The implication (iii) ⇒ (ii) is immediate as $E(\{\lambda\})T^t f_\varphi = (\lambda, t) E(\{\lambda\}) f_\varphi$ for every $t \in G$. It remains to show (ii) ⇒ (iii). Let (B_n) be a van Hove sequence in G.

As $\varphi \mapsto \frac{1}{|B_n|}\int_{B_n} \varphi(s)\, ds$ is a probability measure on G, the standard theory of vector valued integration (see e.g. Chapter 3 in [37]) shows that the L^2-valued integral,

$$\frac{1}{|B_n|} \int_{B_n} \overline{(\lambda, t)} T^t f_\varphi \, dt,$$

belongs to the closed convex hull of $\{\overline{(\lambda, t)} T^t f_\varphi \colon t \in G\}$ for every $n \in \mathbb{N}$. As von Neumann's ergodic theorem (see [21]) gives:

$$E(\{\lambda\}) f_\varphi = \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} \overline{(\lambda, t)} T^t f_\varphi \, dt,$$

where the limit is in the L^2-sense, the claim follows. □

Our main result reads as follows:

Theorem 5.5. *Let m be an α-invariant square integrable probability measure on $\mathcal{M}(G)$ satisfying* (D) *with associated autocorrelation $\gamma = \gamma_m$. The following assertions are equivalent*:

(i) *The measure $\widehat{\gamma}$ is pure point.*
(ii) *T has pure point spectrum.*

In this case, the group generated by $\{\lambda \in \widehat{G}\colon \widehat{\gamma}(\{\lambda\}) > 0\}$ is the set of eigenvalues of T.

Proof. The implication (ii) $\Rightarrow$ (i) is immediate from Theorem 5.3.

As for (i) $\Rightarrow$ (ii), we note that f_φ belongs to $\mathcal{H}_{pp}(T)$ for every $\varphi \in C_c(G)$ by Proposition 5.1. By Corollary 2.5, this implies that $g \circ f_\varphi$ belongs to $\mathcal{H}_{pp}(T)$ for every $g \in C_c(\mathbb{C})$. By Lemma 2.6, products of functions of the form $g \circ f_\varphi$, $g \in C_c(\mathbb{C})$, $\varphi \in C_c(G)$, then belong to $\mathcal{H}_{pp}(T)$ as well. Now, (ii) follows from (D).

It remains to show the last statement: set $L := \{\lambda \in \widehat{G}\colon \widehat{\gamma}(\{\lambda\}) > 0\}$ and denote the group generated by L in $\widehat{G}$ by S. By Theorem 5.3 every $\lambda \in L$ is an eigenvalue of $T_{\mathcal{U}}$ and hence of T. As the eigenvalues form a group, we infer that S is contained in the group of eigenvalues of T. Moreover, by Proposition 5.1, the spectral measure ρ_{f_φ} is supported on S (and even on L) for every $\varphi \in C_c(G)$. Thus, by Corollary 2.5, the spectral measure ρ_f is supported on S for every f of the form $f = g \circ f_\varphi$ with $g \in C_c(\mathbb{C})$ and $\varphi \in C_c(G)$. By Lemma 2.6 this then holds as well for finite products of such functions. As finite products of such functions are total by (D), we infer that the spectral measure of every f is supported on S. Thus, the set of eigenvalues is contained in S. $\square$

Remark. The implication (ii) $\Rightarrow$ (i) in Theorem 5.5 holds even if m doesn't satisfy (D).

6. Perturbation theory: Abstract setting

In this section, we shortly discuss a stability result for pure point diffraction. In the topological setting, an analogous result is discussed in [3,8]. Our result is more general in two ways: First of all, the map Φ below does not need to be continuous but only measurable. Secondly, the underlying space $\mathcal{M}(G)$ is much bigger than the spaces considered in [3] and hence we obtain quite some additional freedom for perturbations (see [24] for applications).

Definition 6.1. Let m be an α-invariant square integrable probability measure on $\mathcal{M}(G)$. A measurable map $\Phi : \mathcal{M}(G) \to \mathcal{M}(G)$ is said to satisfy condition (C) with respect to m if the following holds:

- $\Phi \circ \alpha_t = \alpha_t \circ \Phi$ for every $t \in G$.
- The measure $\Phi^*(m)$ defined by $\Phi^*(m)(f) := m(f \circ \Phi)$ is square-integrable.

If Φ satisfies (C) the measure $\Phi^*(m)$ inherits many properties of m. For example it can easily be seen to be ergodic if m is ergodic [10]. Moreover, we have the following result on equivariant measurable perturbations.

Theorem 6.2. *Let m be an α-invariant square integrable probability measure on $\mathcal{M}(G)$ satisfying* (D) *such that $\widehat{\gamma_m}$ is a pure point measure supported on the group S. Let $\Phi : \mathcal{M}(G) \to \mathcal{M}(G)$ satisfy condition* (C)*. Then, the dynamical system $(\mathcal{M}(G), \Phi^*(m))$ has pure point spectrum supported in S. In particular, the measure $\widehat{\gamma_{\Phi^*(m)}}$ is a pure point measure supported on S as well.*

Proof. Set $n := \Phi^*(m)$. Denote the unitary representation of G induced by α on $L^2(\mathcal{M}(G), m)$ and on $L^2(\mathcal{M}(G), n)$ by T_m and T_n respectively. Note that T_m is strongly continuous by assumption (D). Thus, T_n is strongly continuous as, by definition of n, $\|T_n^t f - f\|_{L^2(n)} = \|T_m^t(f \circ \Phi) - f \circ \Phi\|_{L^2(m)}$ for all $f \in L^2(\mathcal{M}(G), n)$. The spectral measures associated to $f \in L^2(\mathcal{M}(G), m)$ and $g \in L^2(\mathcal{M}(G), n)$ will be denoted by ρ_f^m and ρ_g^n respectively. The definition of n shows that

$$\widehat{\rho_g^n}(t) = \langle g, T_n^t g\rangle = \langle g \circ \Phi, T_m^t g \circ \Phi\rangle = \widehat{\rho_{g\circ\Phi}^m}(t),$$

for all $t \in G$. This, gives,

$$\rho_g^n = \rho_{g\circ\Phi}^m.$$

As $\widehat{\gamma}_m$ is a pure point measure supported on the group S, we infer from our main result that ρ_f^m is a pure point measure supported on S for every $f \in L^2(\mathcal{M}(G), m)$. Thus, the preceding considerations show that ρ_g^n is a pure point measure supported on S for every $g \in L^2(\mathcal{M}(G), n)$. In particular, T_n has pure point spectrum supported on S. As $\widehat{\gamma}_n$ is a spectral measure for a sub representation of T_n it must then be a pure point measure supported on S as well. □

Acknowledgements

The authors are grateful to Michael Baake for various remarks which improved the manuscript. D.L. would like to thank Jean-Baptiste Gouéré for illuminating discussions. N.S. would like to thank Robert Moody for stimulating interest in this subject and NSERC for ongoing financial support. Part of this work was done while one of the authors (N.S.) was visiting TU Chemnitz. Financial support from DFG for this visit as well as the hospitality of Fakultät für Mathematik is gratefully acknowledged. Finally, the authors would like to thank the anonymous referee for a very careful reading of the manuscript, pointing out some inconsistencies and various further well placed suggestions.

Appendix A. Averaging sequences

In this appendix we consider the following situation. Let X be a set with a σ-algebra $\mathcal{B}$ and a measurable action $\alpha : G \times X \to X$ of the locally compact amenable group G on X. Let μ be an α-invariant ergodic probability measure on X. As usual a sequence (B_n) of compact subsets of G is called a *Følner sequence* if,

$$\frac{|B_n \triangle (B_n K)|}{|B_n|} \xrightarrow{n\to\infty} 0,$$

for all compact $K \subset G$. Here, $\triangle$ denotes the symmetric difference. We say that the Birkhoff ergodic theorem holds along the Følner sequence (B_n) if for any $f \in L^1(X, \mu)$,

$$\lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(\alpha_t x)\, dt = \int_X f(x)\, d\mu,$$

for μ-almost every $x \in X$. The aim of this appendix is to show that any Følner sequence admits a subsequence (B_n) such that Birkhoff ergodic theorem holds along $(B_n K)$ for any compact $K \subset G$ containing 0. This will show that certain "boundary terms" which we meet in our considerations indeed go to zero.

Definition A.1. A Følner sequence B_n is called *tempered* if there exists a constant $C > 0$ so that $|\bigcup_{k<n}(B_k^{-1} B_n)| \leqslant C|B_n|$.

As shown by Lindenstrauss in [28] the following holds:

(A) Every Følner sequence has a tempered subsequence.
(B) The Birkhoff ergodic theorem holds along any tempered Følner sequence.

Here, (B) is one of the main results of [28].

Lemma A.2. *Let $(B_n')_n$ be a Følner sequence and let $(K_l)_l$ be an arbitrary sequence of compact sets in G. Then, there exists a subsequence (B_n) of (B_n') so that the Birkhoff ergodic theorem holds simultaneously along $(B_n K_l)_n$ for any $l \in \mathbb{N}$.*

Proof. Since (B_n') is a Følner sequence, the sequence $(B_n' K_l)_n$ is also a Følner sequence for any fixed l. Hence any subsequence of it is Følner again. By (A), we can then find a subsequence $(B_{k(1,n)})_n$ so that $(B_{k(1,n)} K_1)_n$ is tempered. An inductive argument shows that for each l there exists a subsequence $(B_{k(l,n)})_n$ of $(B_{k(l-1,n)})_n$ so that $(B_{k(l,n)} K_l)_n$

is tempered. Then, by (B), Birkhoff's ergodic theorem holds simultaneously along all $(B_{k(l,n)}K_l)_n$. A simple diagonalization procedure now completes the proof. □

Lemma A.3. *Any Følner sequence contains a subsequence $(B_n)_n$ so that for any compact $K \subset G$ containing* 0 *and any $f \in L^1(\mu)$ we have*:

$$\lim_{n\to\infty} \frac{\int_{B_n K} f(\alpha_t(x))\,dt}{|B_n|} = \int_X f(y)\,d\mu(y),$$

for μ-almost every $x \in X$.

Proof. Since G is σ-compact, we can find an increasing sequence of compact sets K_l, $l \in \mathbb{N}$ so that K_1 contains 0, K_l is contained in the interior of K_{l+1} for each $l \in \mathbb{N}$ and the union over all K_l is just G. Then, any compact $K \subset G$ is a subset of some K_l. Now, let $(B_n)_n$ be the subsequence defined by Lemma A.2. Let $f \in L^1(\mu)$ and a compact $K \subset G$ with $0 \in K$ be given and choose l with $K \subset K_l$. Then,

$$\begin{aligned}\left|\frac{\int_{B_n K} f(\alpha_t(x))\,dt - \int_{B_n} f(\alpha_t(x))\,dt}{|B_n|}\right| &\leqslant \frac{\int_{(B_n K)\setminus B_n} |f(\alpha_t(x))|\,dt}{|B_n|} \leqslant \frac{\int_{(B_n K_l)\setminus B_n} |f(\alpha_t(\omega))|\,dt}{|B_n|}\\ &= \frac{1}{|B_n|}\biggl(\int_{B_n K_l} \bigl|f\bigl(\alpha_t(x)\bigr)\bigr|\,dt - \int_{B_n} \bigl|f\bigl(\alpha_t(x)\bigr)\bigr|\,dt\biggr).\end{aligned}$$

As $|B_n K_l|/|B_n| \to 1$ when $n \to \infty$, and Birkhoff's ergodic theorem holds along both (B_n) and $(B_n K)$ the result follows easily. □

We now come to the desired result on the vanishing of boundary type terms.

Proposition A.4. *Let $(B_n)_n$ be a Følner sequence as in Lemma* A.3. *Then, for all $f \in L^1(\mu)$ and all compacts $K \subset G$,*

$$\lim_{n\to\infty} \frac{\int_{C_n} |f(\alpha_t(x))|\,dt}{|B_n|} = 0,$$

for μ-almost every $x \in X$ along any sequence (C_n) with $C_n \subset B_n K$ for all $n \in \mathbb{N}$ and $|C_n|/|B_n| \to 0$, $n \to \infty$.

Proof. Let $\epsilon > 0$ be arbitrary. Set $\widetilde{K} := K \cup \{0\}$.

For $N \in \mathbb{N}$ we define the function f^N on X by $f^N(x) := f(x)$ if $|f(x)| \leqslant N$ and $f(x) = 0$ otherwise. Then $\lim_{N\to\infty} f^N = f$ in $L^1(X,\mu)$. Therefore, there exists an $N \in \mathbb{N}$ with $\|f - f^N\|_1 \leqslant \epsilon$. By Lemma A.3, for almost every $x \in X$, there exists an $n_1 = n_1(x,\epsilon)$ so that for all $n \geqslant n_1$, we have:

$$\int_{B_n K} \bigl|f\bigl(\alpha_t(x)\bigr) - f^N\bigl(\alpha_t(x)\bigr)\bigr|\,dt \leqslant \int_{B_n \widetilde{K}} \bigl|f\bigl(\alpha_t(x)\bigr) - f^N\bigl(\alpha_t(x)\bigr)\bigr|\,dt \leqslant 2\epsilon|B_n|.$$

Thus, for such x and $n \geqslant n_1$,

$$\begin{aligned}\frac{\int_{C_n} |f(\alpha_t(x))|\,dt}{|B_n|} &\leqslant \frac{\int_{C_n} |f^N(\alpha_t(x))|\,dt}{|B_n|} + \frac{\int_{C_n} |(f(\alpha_t(x)) - f^N(\alpha_t(x)))|\,dt}{|B_n|}\\ &\leqslant \frac{\int_{C_n} N\,dt}{|B_n|} + \frac{\int_{B_n K} |(f(\alpha_t(x)) - f^N(\alpha_t(x)))|\,dt}{|B_n|}\\ &\leqslant \frac{N|C_n|}{|B_n|} + 2\epsilon.\end{aligned}$$

As $|C_n|/|B_n| \to 0$ by assumption, we obtain:

$$\frac{\int_{C_n} |f(\alpha_t(x))|\,dt}{|B_n|} \leqslant 3\epsilon$$

for such x and all large enough n. As $\epsilon > 0$ is arbitrary, the statement follows. □

When dealing with σ-compact, locally compact Abelian groups we can do better than Følner sequences. Namely, in this case, there exists a van Hove sequence as shown in [40, p. 249]. Of course, every van Hove sequence is a Følner sequence. In this case, we can apply the previous proposition with $C_n = \partial^K B_n \subset B_n \widetilde{K}$, $n \in \mathbb{N}$, and $\widetilde{K} := K \cup \{0\}$ compact. This is used in Section 4.

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