Hierarchical open Leontief models

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Abstract

We study the concept of a nonnegative matrix inducing hierarchies and apply it to the nonnegative input–output matrix of the open Leontief model, leading to properties which are stronger than the feasibility of the model. We show some examples of these matrices. The characterization of nonnegative matrices inducing hierarchies for any permutation of indices leads to certain matrices of class $M(x)$.© 2007 Elsevier Inc. All rights reserved.

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1. Introduction

Leontief’s input–output model is an outstanding example of the importance of mathematical modelling in economics (see [1,3,6,10,13]). An open Leontief model is called feasible if, for any nonnegative demand vector $d$, one can always find a nonnegative production vector $x$. Given an open Leontief model, there exists a nonnegative matrix $T$ (called input–output matrix) such that the feasibility of the model is equivalent to the fact that $I - T$ is a nonsingular $M$-matrix (that is, $(I - T)^{-1}$ is nonnegative). The results of this paper allow us to require additional properties to the open Leontief model by assuming additional properties of the matrix $T$. For instance, we characterize matrices $T$ such that for any nonnegative and increasing demand vector $d$, one can always find a nonnegative and increasing production vector $x$.

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We introduce two concepts of induced hierarchies by a nonnegative matrix, and we apply them to the open Leontief model. In Section 2, we analyze these concepts. In Remark 2.8 we provide a method to find all the possible induced hierarchies by a given nonnegative matrix $T$. At the end of Section 2 we find an important and wide family of nonnegative matrices inducing a hierarchy: the matrices whose $2 \times 2$ minors are nonnegative. In Section 3 we interpret the concepts and results of Section 2 in terms of the open Leontief model and its associated price valuation system.

In Section 4, we characterize nonnegative matrices inducing hierarchies for any permutation of indices. The characterization leads to certain matrices of class $M(x)$, the matrices of class $M(x)$ have already been used by other authors in related problems and in [9] a generalization of these matrices to multisector models is provided. We also include the interpretation of these results in terms of the open Leontief model and characterize the matrices such that the fact that the demand of the commodity $i$ increases less than the demand of the commodity $j$ implies that the production of the commodity $i$ also increases less than the production of the commodity $j$.

As usual, we shall denote by $I$ the identity matrix, by $A^T$ the transpose of a matrix $A$ and by $A \geq 0$ (respectively, $v \geq 0$) a nonnegative matrix $A$ (respectively, a nonnegative vector $v$). Finally, given two vectors $u, v \in \mathbb{R}^n$, we write $u \geq v$ if $u - v \geq 0$.

2. Nonnegative matrices and induced hierarchies

We shall use the following notations: $\delta_{kl}$ denotes the delta of Kronecker (1 if $k = l$ and 0 otherwise), $e = (1, 1, \ldots, 1)^T$ and the matrices

$$E = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}, \quad E^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}. \quad (2.1)$$

Finally, given a permutation $(i_1, \ldots, i_n)$ of the indices $(1, \ldots, n)$, let $P$ be the corresponding permutation matrix

$$P = (\delta_{i_j,j})_{1 \leq j \leq n}. \quad (2.2)$$

Let us now introduce the two main concepts of this paper.

**Definition 2.1.** Let $T$ be a nonnegative matrix. We say that $T$ *induces a hierarchy of indices* $(i_1, \ldots, i_n)$ in nonnegative vectors (or simply that $T$ is IHN for $(i_1, \ldots, i_n)$) if for any vector $v = (v_1, \ldots, v_n)^T$ such that $0 \leq v_{i_1} \leq \cdots \leq v_{i_n}$ we have that $(Tv)_{i_1} \leq \cdots \leq (Tv)_{i_n}$. We say that $T$ *induces a hierarchy of indices* $(i_1, \ldots, i_n)$ (or simply that $T$ is IH for $(i_1, \ldots, i_n)$) if for any vector $v = (v_1, \ldots, v_n)^T$ such that $v_{i_1} \leq \cdots \leq v_{i_n}$ we have that $(Tv)_{i_1} \leq \cdots \leq (Tv)_{i_n}$. Finally, we say that $T$ is IHN (resp., IH) if it is IHN (resp., IH) for $(1, \ldots, n)$.

The proofs of the following results are rather simple and straightforward, although we include them for the sake of completeness.
Lemma 2.2

(i) Let \( v = (v_1, \ldots, v_n)^T \) be a vector and let \((i_1, \ldots, i_n)\) be a permutation of the indices \((1, \ldots, n)\). Then \( v \) satisfies
\[
v_{i_1} \leq \cdots \leq v_{i_n}
\]
if and only if the last \( n - 1 \) components of \( E^{-1} P v \) are nonnegative. A nonnegative vector \( v \) satisfies (2.3) if and only if \( E^{-1} P v \) is nonnegative.

(ii) If \( T \) is IH for \((i_1, \ldots, i_n)\), then it is also IH for \((i_n, \ldots, i_1)\).

Proof

(i) It is sufficient to take into account that
\[
E^{-1} P(v_1, v_2, \ldots, v_n)^T = (v_{i_1}, v_{i_2} - v_{i_1}, \ldots, v_{i_n} - v_{i_{n-1}})^T.
\]

(ii) Let \( w = (w_1, \ldots, w_n)^T \) be a vector such that \( w_{i_1} \geq \cdots \geq w_{i_n} \). Since \(-w_{i_1} \leq \cdots \leq -w_{i_n}\), we have \((T(-w))_{i_1} \leq \cdots \leq (T(-w))_{i_n}\) and so \((Tw)_{i_1} \geq \cdots \geq (Tw)_{i_n}\).

The next result characterizes the matrices of Definition 2.1.

Proposition 2.3

(i) A matrix \( T \geq 0 \) is IHN for \((i_1, \ldots, i_n)\) if and only if \( E^{-1} P T P^T E \geq 0 \).

(ii) A matrix \( T \geq 0 \) is IH for \((i_1, \ldots, i_n)\) if and only if \( E^{-1} P T P^T E \geq 0 \) and \( T e = \beta e \) for some nonnegative real number \( \beta \).

Proof

(i) The matrix \( E^{-1} P T P^T E \) is nonnegative if and only if it transforms nonnegative vectors into nonnegative vectors. Using Lemma 2.2(i) (and (2.4)), we can obtain that the set of nonnegative vectors is given by the set of vectors \( E^{-1} P v \), where \( v \) is any vector such that \( 0 \leq v_{i_1} \leq \cdots \leq v_{i_n} \). Thus, \( E^{-1} P T P^T E \geq 0 \) if and only if \( E^{-1} P T P^T E (E^{-1} P v) \geq 0 \) for any vector such that \( 0 \leq v_{i_1} \leq \cdots \leq v_{i_n} \). Therefore \( E^{-1} P T P^T E \geq 0 \) if and only if \( E^{-1} P T P^T v \geq 0 \) for any vector such that \( 0 \leq v_{i_1} \leq \cdots \leq v_{i_n} \), which is equivalent by Lemma 2.2 (i) to saying that \( T v \) satisfies that \( 0 \leq (T v)_{i_1} \leq \cdots \leq (T v)_{i_n} \).

(ii) Let us assume now that \( T \) is IH for \((i_1, \ldots, i_n)\). In particular, it is IHN for \((i_1, \ldots, i_n)\) and, by (i), \( E^{-1} P T P^T E \geq 0 \). Since the components of \( e \) satisfy (2.3) we have that \((T e)_{i_1} \leq \cdots \leq (T e)_{i_n} \). Since the components of \( e \) also satisfy that \( e_{i_n} \leq \cdots \leq e_{i_1} \), by Lemma 2.2 (ii) we also have that \((T e)_{i_1} \geq \cdots \geq (T e)_{i_n} \) and so \( T e = \beta e \), with \( \beta \geq 0 \) because \( T \geq 0 \).

For the converse, let us assume that \( E^{-1} P T P^T E \geq 0 \) and \( T e = \beta e \) for some nonnegative real number \( \beta \) and that we have a vector \( v = (v_1, \ldots, v_n)^T \) satisfying (2.3). By Lemma 2.2(i) we know that the last \( n - 1 \) components of \( E^{-1} P v \) are nonnegative and that we have to prove that the last \( n - 1 \) components of \( E^{-1} P T v \) are nonnegative. From \( E^{-1} P T P^T E \geq 0 \), \((E^{-1} P v)_{i_1} \geq 0 \) for \( i = 2, \ldots, n \) and \( E^{-1} P T v = (E^{-1} P T P^T E)(E^{-1} P v) \), we can conclude that it is sufficient to see that the \( n - 1 \) last components of the first column of \( E^{-1} P T P^T E \) are equal to zero. Since \( T e = \beta e \) and \( P^Te = e \), we have that \( P T P^T e = \beta e \). Therefore \( E^{-1} P T P^T e \), which is the first column of \( E^{-1} P T P^T E \), will be of the form \((\beta, 0, \ldots, 0)^T\), and the result follows.
Taking into account the previous proposition, we can deduce the following remark characterizing nonnegative matrices which are IHN (resp., IH).

**Remark 2.4.** By Proposition 2.3(i), $T \geq 0$ satisfies that $T$ is IHN if and only if $E^{-1}TE \geq 0$. In this case, given any vector $v \geq 0$, we can find a vector $w \geq 0$ such that $T(Ev) = EsT$ with the choice $w := (E^{-1}TE)v(\geq 0)$. Thus, if $P$ denotes the cone of nonnegative vectors and one considers the simplest hierarchy of indices $(1, \ldots, n)$, then the following assertions are equivalent for a nonnegative matrix $T = (t_{ij})_{1 \leq i, j \leq n}$:

(i) $T$ is IHN (resp., IH).
(ii) $T$ leaves the cone $E\mathcal{P}$ invariant, i.e. $T(E\mathcal{P}) \subseteq E\mathcal{P}$ (resp., leaves the cone $E\mathcal{P} + Re$ invariant).
(iii) $E^{-1}TE \geq 0$ (resp., $E^{-1}TE \geq 0$ and $Te = \beta e$ for some $\beta \geq 0$).
(iv) $\sum_{j=k}^{n} t_{ij} \leq \sum_{j=k}^{n} t_{i+1,j}$ for $k = 1, \ldots, n, i = 1, \ldots, n - 1$ (resp., $\sum_{j=1}^{n} t_{ij} = \sum_{j=1}^{n} t_{i+1,j}$ for $i = 1, \ldots, n - 1$ and $\sum_{j=1}^{n} t_{ij} \leq \sum_{j=1}^{n} t_{i+1,j}$ for $k = 2, \ldots, n, i = 1, \ldots, n - 1$).

Let us recall that a nonsingular matrix $A$ with positive diagonal entries and nonpositive off-diagonal entries is called $M$-matrix if $A^{-1} \geq 0$. Given a matrix $A$, $\rho(A)$ denotes the spectral radius of $A$, that is, $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. Let us also recall that, by the well-known Perron–Frobenius theorem, if $A$ is nonnegative then $\rho(A)$ is an eigenvalue of $A$ and we can take a nonnegative associated eigenvector.

It is well known (cf. Lemma (2.1) of Chapter VI of [3]) that if a nonnegative matrix $T$ satisfies $\rho(T) < 1$ then $I - T$ is an $M$-matrix. The next result shows that the properties IHN and IH of $T$ are inherited by the corresponding nonnegative matrix $(I - T)^{-1}$. This fact will be crucial in the applications of the following sections.

**Proposition 2.5.** If $T \geq 0$ with $\rho(T) < 1$ is IHN (resp., IH) for $(i_1, \ldots, i_n)$, then $(I - T)^{-1}$ is again IHN (resp., IH) for $(i_1, \ldots, i_n)$.

**Proof.** By Lemma (2.1) of Chapter VI of [3], $(I - T)^{-1}$ is a nonnegative matrix. Observe that, for all $k \geq 1$, $E^{-1}PTP^TE \geq 0$ implies $E^{-1}PT^kP^TE \geq 0$ and observe that $Te = \beta e$ ($\beta \geq 0$) implies $T^ke = \beta^ke$. Taking into account that, again by Lemma (2.1) of Chapter VI of [3], $(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$, the result follows from Proposition 2.3. In fact, if $T \geq 0$ with $\rho(T) < 1$ is IHN for $(i_1, \ldots, i_n)$, then $E^{-1}P(I - T)^{-1}P^TE = \sum_{k=0}^{\infty} (E^{-1}PT^kP^TE) \geq 0$. In addition, if $T$ is IH for $(i_1, \ldots, i_n)$, then $Te = \beta e$ ($\beta \geq 0$) and so $\beta$ is an eigenvalue of $T$ which must coincide with $\rho(T) < 1$ (use, for instance, the Gerschgorin circles to see that $\rho(T) \leq \beta$). So, $(I - T)^{-1}e = (\sum_{k=0}^{\infty} \beta^k)e = (1/(1 - \beta))e$ and $(1/(1 - \beta)) = (1/(1 - \rho(T))) \geq 0$. □

The following lemma provides a necessary condition for a matrix being IHN for some permutation of indices.

**Lemma 2.6.** If $T = (t_{ij})_{1 \leq i, j \leq n} \geq 0$ is IHN for $(i_1, \ldots, i_n)$, then $t_{i_1, i_n} \geq t_{k, i_n}$ for all $k = 1, \ldots, n$.

**Proof.** Let $P$ be the permutation matrix associated with $(i_1, \ldots, i_n)$. Observe that the $(n, n)$ entry of $PT^kP^TE$ is $t_{i_1, i_n}$ and that the remaining elements of that column are the remaining entries of the column $i_n$ of $T$. By Proposition 2.3 (i), $E^{-1}PTP^TE \geq 0$, which implies $t_{i_1, i_n} \geq t_{k, i_n}$ for all $k = 1, \ldots, n$. □
Remark 2.7. If \( T \geq 0 \) is IH for \((i_1, \ldots, i_n)\), then by Proposition 2.3(ii), \( Te = \beta e \) for some \( \beta \geq 0 \) and so \( T \) is a nonnegative multiple of a stochastic matrix. Then it is well known that \( \rho(T) = \beta \).

Remark 2.8. From Lemma 2.2 and Lemma 2.6 we can deduce a method to find all the possible induced hierarchies in nonnegative vectors by a given nonnegative matrix \( T \). The method consists of the following steps:

1. Choose the columns such that the diagonal entry is greater than or equal to the other elements of its column. If \( T \) has not columns satisfying this property, then it is not IHN for any \((i_1, \ldots, i_n)\) by Lemma 2.6.
2. If \( \mathbf{c} = (c_1, \ldots, c_n)^T \) is a column chosen in the previous step, let \( c_{i_n} \) be its diagonal entry and let \((i_1, \ldots, i_n)\) be a permutation of the indices such that \( c_{i_1} \leq \cdots \leq c_{i_n} \). Then let \( P \) be the permutation matrix of (2.2) and let us form \( PT P^T \) (observe that its last column is \((c_{i_1}, \ldots, c_{i_n})^T\)).
3. Construct the matrix \( B \) obtained from \( PT P^T \) by adding to the \( i \)th column of \( PT P^T \) (for \( i = 1, 2, \ldots, n-1 \)) the columns \( i+1, \ldots, n \) (this means that \( B = PT P^TE \)).
4. If \( r_1, \ldots, r_n \) are the row vectors of \( B \), \( T \) is IHN for \((i_1, \ldots, i_n)\) if and only if \( r_1 \leq \cdots \leq r_n \) (this is equivalent to check if \( E^{-1}B = E^{-1}PT P^TE \geq 0 \)).

To check that \( T \) is IH for \((i_1, \ldots, i_n)\), we first have to check whether all the row sums of \( T \) are equal (this is equivalent to check if \( Te = \beta e \)). If, in addition, we want to check if \((I - T)^{-1} \) is IH, it is sufficient by Proposition 2.5 to check if \( \rho(T) < 1 \), which holds if and only if \( \beta < 1 \) by Remark 2.7.

We conclude this section presenting a class of matrices which are IH (that is, IH for \((1, \ldots, n)\)).

Definition 2.9. A matrix is called TP\(_2\) if it is nonnegative and all its \(2 \times 2\) minors are nonnegative. If all the minors of \( A \) are nonpositive, then \( A \) is called \textit{totally nonnegative}.

Totally nonnegative matrices have many applications in different areas such as statistics, mechanics, economics, computer-aided design or approximation theory (see [5,2,7]). Matrices with all minors nonpositive are called totally nonpositive and are also related with matrices appearing in economy (cf. [4]). Totally nonpositive as well as totally nonnegative matrices are examples of sign-regular matrices and, therefore, of the more general class of SR\(_2\) matrices, matrices with weakly constant sign and whose \(2 \times 2\) minors also have constant sign (see [8]).

Proposition 2.10. Let \( T \) be a TP\(_2\) matrix with \( Te = \beta e \) for some \( \beta \geq 0 \). Then \( T \) is IH.

Proof. We can assume that \( \beta > 0 \) because, otherwise, \( T = 0 \) and the result is trivial. By Theorem 2.3(ii), we have to prove that \( A = (a_{ij})_{1 \leq i, j \leq n} := E^{-1}TE \) is nonnegative. Clearly, the matrix \( E \) is totally nonnegative. Then, by the Cauchy–Binet formula (cf. (1.23) of [2]), we have that the matrix \( B = (b_{ij})_{1 \leq i, j \leq n} := TE \) is TP\(_2\). In particular, the first row of \( B \), which coincides with the first row of \( A \), is nonnegative. Since \( Te = \beta e \), we have that the first column of \( B \) is \((\beta, \ldots, \beta)^T \) and so the first column of \( A \) is \((\beta, 0, \ldots, 0)^T \).

Let us consider now indices \( i, j \in \{2, \ldots, n\} \) and the submatrix of \( B \)

\[
\begin{pmatrix}
  b_{i-1,1} & b_{i-1,j} \\
  b_{i1} & b_{ij}
\end{pmatrix}
= \begin{pmatrix}
  \beta & b_{i-1,j} \\
  \beta & b_{ij}
\end{pmatrix}
\]
If we subtract the first row of this submatrix from the second one we obtain the matrix
\[
\begin{pmatrix}
\beta & b_{i-1,j} \\
0 & a_{ij}
\end{pmatrix}
\].
Since \( B \) is TP2, we can deduce now that
\[
\det \begin{pmatrix}
\beta & b_{i-1,j} \\
\beta & b_{ij}
\end{pmatrix} \geq 0
\]
and so \( A \geq 0 \).
\( \square \)

**Remark 2.11.** Given a TP2 matrix \( T \), we can always obtain a matrix \( K \) which is TP2 and stochastic and so satisfying the hypotheses of Proposition 2.10. In fact, if \( T \neq 0 \), we can express \( T = DK \), where \( D \) is the diagonal matrix whose \( i \)th diagonal entry is the sum of the elements of the \( i \)th row of \( T \) and the resulting matrix \( K \) is a TP2 matrix satisfying \( Ke = e \). Besides, for any \( \beta \geq 0 \), \( \beta K \) is also TP2 and satisfies \( (\beta K)e = \beta e \).

3. Applications to the open Leontief model

Let us recall some basic facts on the open Leontief model (for more details, see Chapter 9 of [3], Chapter 11 of [1], Section 5.3 of [12] and Chapter 4 of [13]). Let us assume that the economy is divided into \( n \) sectors, each producing one commodity to be consumed by itself, by other industries and by the outside sector. Let \( x = (x_1, \ldots, x_n)^T \) be the production (or output) vector (\( x_i \) denotes the gross product of sector \( i \)) and let \( t_{ij} \) be the input coefficient which represents the number of units of commodity \( i \) required to produce one unit of commodity \( j \). The matrix \( T = (t_{ij})_{i \leq j \leq n} \) is called the input matrix for the model. If \( y = (y_1, \ldots, y_n)^T := Tx \), then \( y_i \) represents the sum of the sales of sector \( i \) to the sectors \( 1, \ldots, n \) and we say that \( y \) is the inside sale vector. Finally, let \( d = (d_1, \ldots, d_n)^T \) be the demand vector (\( d_i \) denotes the final demand on sector \( i \) by the outside sector). Now, from \( d = x - y = x - Tx \), we obtain
\[
d = (I - T)x. \tag{3.1}
\]
The model is called feasible if, for any nonnegative demand vector \( d \), the system (3.1) has a nonnegative solution vector \( x \).

The previous economic model also has an associated price valuation system. Let \( p = (p_1, \ldots, p_n)^T \) be the price vector (\( p_j \) denotes the price of the \( j \)th commodity). If \( z = (z_1, \ldots, z_n)^T := T^T p \), then \( z_j \) represents the unit cost of the \( j \)th commodity and we say that \( z \) is the cost vector. Finally, let \( v = (v_1, \ldots, v_n)^T \) be the value added vector (\( v_j \) denotes the value added per unit of the \( j \)th commodity). Now, from \( v^T = p^T - z^T = p^T - p^T T \) we obtain the dual system to (3.1):
\[
v = (I - T^T)p. \tag{3.2}
\]
The model is called profitable if for any nonnegative value added vector \( v \), the system (3.2) has a nonnegative solution vector \( p \). By Theorem (3.9) of Chapter IX of [3], we obtain the well-known equivalence for an open Leontief model with nonnegative input matrix \( T \) of the following three properties: the model is feasible, the model is profitable and \( I - T \) is a nonsingular \( M \)-matrix.

If we apply the definition of matrix inducing a hierarchy in nonnegative vectors to the matrices \( T \) and \( T^T \) associated to an open Leontief model, we obtain the following interpretation:

**Proposition 3.1.** Consider an open Leontief model with input matrix \( T (\geq 0) \). Then

(i) \( T \) is IHN for \((i_1, \ldots, i_n)\) if and only if for any production vector \( x = (x_1, \ldots, x_n)^T \) such that
\[
0 \leq x_{i_1} \leq \cdots \leq x_{i_n}
\]
one has that the inside sale vector \( y \) satisfies that \( 0 \leq y_{i_1} \leq \cdots \leq y_{i_n} \).
Let us remember that we have characterized the matrices of the previous proposition in Proposition 2.3(i) and that in Remark 2.8 we have shown how to find all the possible induced hierarchies in nonnegative vectors by a nonnegative matrix. On the other hand, if \( T \) induces a hierarchy of indices \((i_1, \ldots, i_n)\) in nonnegative vectors we have that \( x_{i_1} \leq \cdots \leq x_{i_n} \) for a nonnegative production vector \( x \) implies that the inside sale vector \( y \) satisfies that \( y_{i_1} \leq \cdots \leq y_{i_n} \). Taking into account that the demand vector \( d \) satisfies that \( d = x - y \), it does not necessarily imply that \( d_{i_1} \leq \cdots \leq d_{i_n} \) implies that \( x_{i_1} \leq \cdots \leq x_{i_n} \) (and so \( y_{i_1} \leq \cdots \leq y_{i_n} \)). However, part (i) of the next proposition shows that this holds for feasible Leontief models when the input matrix \( T \geq 0 \) is IHN for \((i_1, \ldots, i_n)\). Its proof is a consequence of Proposition 2.5. Part (ii) is a consequence of (i) and Proposition 3.1(i).

**Proposition 3.2.** Consider a feasible open Leontief model with input matrix \( T \geq 0 \) which is IHN for \((i_1, \ldots, i_n)\). Let \( d \geq 0 \) be a demand vector such that \( d_{i_1} \leq \cdots \leq d_{i_n} \). Then

(i) The production vector \( x \geq 0 \) solution of (3.1) satisfies that \( x_{i_1} \leq \cdots \leq x_{i_n} \).

(ii) The inside sale vector \( y \) satisfies that \( y_{i_1} \leq \cdots \leq y_{i_n} \).

Analogous comments and conclusions can be applied to the price, cost and value added vectors of the dual system, assuming that \( T^T \) is IHN for \((i_1, \ldots, i_n)\). For brevity, from now on we shall not mention the results for the dual system (3.2).

Sierksma [10] studied the effects of changing the demand vector on the production vector. Given two demand vectors \( d^1, d^2 \), we define \( \Delta d := d^2 - d^1 \). Let \( x^1, x^2 \) be the production vectors which are solution of (3.1) for \( d^1 \) and \( d^2 \), respectively. Now define \( \Delta x \) by \( \Delta x := x^2 - x^1 \). Finally, given the inside sale vectors \( y^1 := Tx^1 \) and \( y^2 := Tx^2 \), we define \( \Delta y := y^2 - y^1 \). It is straightforward to derive from (3.1) the equation

\[
\Delta d = (I - T) \Delta x.
\] (3.3)

If we apply the definition of matrix inducing a hierarchy to the matrix \( T \) (associated to an open Leontief model) and we use Lemma 2.2 (ii), we obtain the following result.

**Proposition 3.3.** Consider an open Leontief model with input matrix \( T \geq 0 \). Then \( T \) is IH for \((i_1, \ldots, i_n)\) if and only if for any change of the production vector \( \Delta x = ((\Delta x)_1, \ldots, (\Delta x)_n) \) such that \((\Delta x)_{i_1} \leq \cdots \leq (\Delta x)_{i_n}\) one has that the change of the inside sale vector \( \Delta y \) satisfies that \((\Delta y)_{i_1} \leq \cdots \leq (\Delta y)_{i_n}\).

The following result follows from (3.3), Proposition 2.5, Remark 2.7 and Proposition 3.3.

**Proposition 3.4.** Consider a feasible open Leontief model with input matrix \( T \geq 0 \) such that \( T \) is IH for \((i_1, \ldots, i_n)\) and \( Te = \beta e \) with \( 0 \leq \beta < 1 \). Let \( \Delta d \) be a change of the demand vector such that \((\Delta d)_{i_1} \leq \cdots \leq (\Delta d)_{i_n}\). Then the change of the production vector \( \Delta x \) satisfies that \((\Delta x)_{i_1} \leq \cdots \leq (\Delta x)_{i_n}\) and the change of the inside sale vector \( \Delta y \) satisfies that \((\Delta y)_{i_1} \leq \cdots \leq (\Delta y)_{i_n}\).
4. Characterizations

In this section we shall characterize several properties, starting with the property that a given matrix is IH for any permutation \((i_1, \ldots, i_n)\). First we introduce the class \(M(x)\), which will play a key role in the characterizations.

**Definition 4.1.** A matrix \(T = (t_{ij})_{1 \leq i,j \leq n}\) belongs to the class \(M(x)\) if \(t_{jj} - t_{ij} = x\) for any \(j \neq i\), i.e. if it is of the form

\[
T = \begin{pmatrix}
  t_{11} & t_{22} - x & \cdots & t_{nn} - x \\
  t_{11} - x & t_{22} & \cdots & t_{nn} - x \\
  \vdots & \vdots & \ddots & \vdots \\
  t_{11} - x & t_{22} - x & \cdots & t_{nn}
\end{pmatrix}.
\]

Equivalently, \(T \in M(x)\) if it can be written in the form \(T = es^T + xI\), where \(s \in \mathbb{R}^n\), \(x \in \mathbb{R}\). Clearly \(T \in M(x)\) is nonnegative if and only if \(x \leq t_{ii}\) and \(t_{ii} \geq 0\) for all \(i\). More details on these matrices can be found in [10] or in Chapter 9 of [3]. In [11] they appeared as example of two-minor stable matrices, which are matrices such that all \(2 \times 2\) minors with no element on the main diagonal are zero. In that paper it was shown that two-minor stable matrices present nice properties in the open Leontief model with taxes and subsidies.

**Theorem 4.2.** Let \(T = (t_{ij})_{1 \leq i,j \leq n}\) be a nonnegative matrix. Then the following properties are equivalent:

(i) \(T\) is IH for any permutation \((i_1, \ldots, i_n)\).

(ii) \(T\) is IHN for any permutation \((i_1, \ldots, i_n)\).

(iii) \(T \in M(x)\) with \(x \geq 0\).

**Proof.**

(i) \(\Rightarrow\) (ii). It is obvious because IH is a stronger property than IHN.

(ii) \(\Rightarrow\) (iii). Let \(P\) be the permutation matrix associated with \((i_1, \ldots, i_n)\). Since \(T\) is IHN for \((i_1, \ldots, i_n)\), \(E^{-1}PTP^T E \geq 0\) by Proposition 2.3(i), which implies that all columns of \(PTP^T E\) are increasing. In particular, its last column (which coincides with the \(n_{th}\) column of \(T\)) satisfies \(t_{i_1,i_n} \leq t_{i_2,i_n} \leq \cdots \leq t_{i_n,i_n}\). If we now consider all permutations of the form \((j_1, \ldots, j_{n-1}, i_n)\) (for any permutation \((j_1, \ldots, j_{n-1})\) of \((1, \ldots, n-1)\)), we conclude that the \(n_{th}\) column of \(T\) satisfies \(t_{k,i_n} = c(\leq t_{i_n,i_n})\) for all \(k \neq i_n\), where \(c\) is a constant. The argument used for \(i_n\) can be used for any column. In particular, the column \(i_{n-1}\)th of \(T\) satisfies \(t_{k,i_{n-1}} = c(\leq t_{i_{n-1},i_{n-1}})\) for all \(k \neq i_{n-1}\), where \(c'\) is a constant. Since the \((n-1)th\) column of \(PTP^T E\) is increasing, we deduce that \(c' + t_{i_n,i_n} \geq t_{i_{n-1},i_{n-1}} + c\). But interchanging the roles of \(i_n\) and \(i_{n-1}\), we derive \(t_{i_{n-1},i_n} + c' \geq t_{i_n,i_n}\) and conclude that \(t_{i_n,i_n} - c = t_{i_{n-1},i_{n-1}} - c' =: x(\geq 0)\). Since any column can play the role of \(i_{n-1}\) or \(i_n\), we have already proved that (iii) holds.

(iii) \(\Rightarrow\) (i). Clearly, \(T \in M(x)\) if and only if \(PTP^T \in M(x)\) for any permutation matrix \(P\). Thus the result follows from

\[
Te = \left(\sum_{i=1}^{n} t_{ii} - (n-1)x\right) e = (s^T e + x)e,
\]

(4.1)
\[
E^{-1}TE = \begin{pmatrix}
\sum_{i=1}^{n} t_{ii} - (n-1)x & \sum_{i=2}^{n} t_{ii} - (n-2)x & \ldots & \ldots & \ldots & t_{nn} - x \\
0 & x & 0 & \ldots & \ldots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & x \\
0 & 0 & \ldots & 0 & x & 0
\end{pmatrix}
\]

(which is nonnegative because \((0 \leq x \leq t_{ii})\) by the nonnegativity of \(T\)) and Proposition 2.3(ii). □

Now we can derive the following consequence of Proposition 3.3 and Theorem 4.2.

**Corollary 4.3.** Consider an open Leontief model such that its input matrix \(T = (t_{ij})_{1 \leq i, j \leq n}\) belongs to \(M(x)\) with \(x \geq 0\) and \(x \leq t_{kk}\) for all \(k\). If \((\Delta x)_i \leq (\Delta x)_j\) then \((\Delta y)_i \leq (\Delta y)_j\).

In terms of the open Leontief model, the previous result means that if the production of the commodity \(i\) increases less than the production of the commodity \(j\) then the inside sales of the commodity \(i\) also increase less than the inside sales of the commodity \(j\).

The following result deals with the changes of demands and the first part is closely related with (i) \(\Rightarrow\) (ii) of Theorem 19 of [10]. In contrast to that result, we do not require here that the matrix \(I - T\) is diagonally dominant by rows.

**Proposition 4.4.** Consider a feasible open Leontief model whose input matrix \(T\) belongs to \(M(x)\) with \(0 \leq x < 1\) and \(x \leq t_{kk}\) for all \(k\). If \((\Delta d)_i \leq (\Delta d)_j\), then \((\Delta x)_i \leq (\Delta x)_j\) and \((\Delta y)_i \leq (\Delta y)_j\).

**Proof.** By Lemma 14 of [10], \(I - T \in M(1 - x)\) and, by Theorem 18 of [10], the adjoint matrix \(\text{adj}(I - T) \in M(\det(I - T)/(1 - x))\). Thus, the nonnegative matrix

\[
(I - T)^{-1} = \frac{\text{adj}(I - T)}{\det(I - T)} \in M\left(\frac{1}{1 - x}\right).
\]

Now the result follows from Proposition 3.7 and Theorem 4.2. □

Let us characterize the matrices associated to open feasible models such that \((\Delta d)_i \leq (\Delta d)_j\) implies \((\Delta x)_i \leq (\Delta x)_j\).

**Theorem 4.5.** Consider an open Leontief model with input matrix \(T\) \((\geq 0)\) such that \(Te = \beta e\) with \(0 \leq \beta < 1\). Then the following properties are equivalent:

(i) \((I - T)^{-1}\) is IH for any permutation \((i_1, \ldots, i_n)\).

(ii) \((\Delta d)_i \leq (\Delta d)_j\) implies \((\Delta x)_i \leq (\Delta x)_j\).

(iii) \(T \in M(x)\) with \(x < 1\).

**Proof.** Observe that, by Remark 2.7, \(\rho(T) < 1\). Hence \(I - T\) is nonsingular and \((I - T)^{-1} \geq 0\) (that is, the model is feasible). From (3.3) we derive

\[
\Delta x = (I - T)^{-1} \Delta d,
\]

(4.3)
(i) \iff (ii). It follows from formula (4.3).

(i) \iff (iii). Observe that we can apply formula (4.2) and so \( T \in M(x) \) with \( x < 1 \) if and only if

\[
(I - T)^{-1} \in M(y), \quad y := (1 - x)^{-1} \geq 0.
\]  

(4.4)

Therefore the equivalence of (i) and (iii) follows from applying Theorem 4.2 to \((I - T)^{-1}\) and using the equivalence of (i) and (iii) of Theorem 4.2. \(\square\)

**Remark 4.6.** Let us observe that, if a nonnegative matrix \( T \in M(x) \) with \( x < 0 \) and \( T e = \beta e \) with \( 0 \leq \beta < 1 \), then \((I - T)^{-1}\) is IH for any permutation \((i_1, \ldots, i_n)\) by Theorem 4.5, although \( T \) does not satisfy the same property by Theorem 4.2. In fact, taking into account theorems 4.2 and 4.5 and formula (4.4), for the values \( x \in (0, 1) \) it happens that \( T \in M(x) \) with \( T e = \beta e \) \((0 \leq \beta < 1)\) if and only if \((I - T)^{-1}\) is IH for any permutation \((i_1, \ldots, i_n)\). Finally, although the characterization of Theorem 4.5 permits \( x < 0 \), let us observe that a strong restriction comes from the fact that \( T e = \beta e \) with \( 0 \leq \beta < 1 \). Observe that the matrices satisfying the properties of Theorem 4.5 are the matrices \( T = (t_{ij})_{1 \leq i,j \leq n} \in M(x) \) with \( x < 1, x \leq t_{ii}, t_{ii} \geq 0 \) and

\[
\sum_{i=1}^{n} t_{ii} - (n-1)x < 1
\]  

(4.5)

because, if (4.2) does not hold, then by (4.1) \( T e = \beta e \) with \( \beta \geq 1 \). Let us remark that, if \( x < 0 \), then (4.2) implies that \( x \geq -1/(n-1) \).

Taking into account the previous remark and Theorem 4.5, we can derive the following characterization:

**Corollary 4.7.** Given a matrix \( T = (t_{ij})_{1 \leq i,j \leq n} \), the following properties are equivalent:

(i) \( T \) is the input matrix of an open Leontief model such that \((\Delta d)_i \leq (\Delta d)_j\) implies \((\Delta x)_i \leq (\Delta x)_j\).

(ii) \( T \in M(x) \) with \( x < 1, x \leq t_{ii}, t_{ii} \geq 0 \) and (4.5) holds.

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**References**


