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Finite identification from the viewpoint of epistemic update

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ABSTRACT

Formal learning theory constitutes an attempt to describe and explain the phenomenon of learning, in particular of language acquisition. The considerations in this domain are also applicable in philosophy of science, where it can be interpreted as a description of the process of scientific inquiry. The theory focuses on various properties of the process of hypothesis change over time. Treating conjectures as informational states, we link the process of conjecture-change to epistemic update. We reconstruct and analyze the temporal aspect of learning in the context of dynamic and temporal logics of epistemic change. We first introduce the basic formal notions of learning theory and basic epistemic logic. We provide a translation of the components of learning scenarios into the domain of epistemic logic. Then, we propose a characterization of finite identifiability in an epistemic temporal language. In the end we discuss consequences and possible extensions of our work.

1. Introduction

Modal logics of epistemic and doxastic change have been developed and applied in the context of multi-agent systems to analyze the process of epistemic and doxastic change. Formal learning theory, on the other hand, is concerned with functions that identify a correct hypothesis from a range of possibilities on the basis of inductively given streams of data. These functions can be viewed as agents that change their beliefs about which hypothesis is correct. In this paper we investigate the connection between formal learning theory and modal logics of belief change, on the example of finite identification. The motivation for bridging learning theory and modal logics of belief change is twofold. By analyzing the epistemic and temporal structure underlying formal learning theory, we provide additional insight into the semantics of inductive learning. On the other hand, importing the ideas, problems and methodology from learning theory enriches logics of epistemic and doxastic change by new concepts and new perspectives (for philosophical and methodological discussion see [19]).

Let us give a short explanation of how the bridging is established (a similar approach can be found in [13,20]). First, we focus on the language learning paradigm, in which languages are treated as sets of positive integers. Learning is viewed as a process in which an agent (let us call her Learner) considers some range of languages; one of the languages is the actual one, and Learner's general aim is to find out which one it is. The elements of the language are given to Learner one by one. The infinite sequence of data that governs this enumeration includes all and only elements of the language. Several success conditions of such a learning process can be defined. For instance, it can be assumed that each time Learner gets a piece of information, she is allowed to make a conjecture. We can define the learning process to be successful if Learner's conjectures stabilize on the proper language. This learnability condition is called *identification in the limit* [22]. Another, more restrictive notion of success requires that Learner, while successively given the data about the language, gives an answer only once, at some finite stage of the procedure. This kind of learnability is known as *finite identification* [27].

Intuitively, our approach to inductive learning in the context of epistemic and epistemic temporal logic is as follows. We take the initial class of languages to be possible worlds in an epistemic model, which mirrors Learner's initial uncertainty

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over the range of languages. The incoming piece of information is taken to be an event that modifies the initial model. We investigate the properties of iterated update situations that correspond to finite identifiability and we describe the emergence of irrevocable knowledge in this setting. The structure resulting from updating the model with a sequence of events can be viewed as an epistemic temporal forest. We explicitly focus on protocols that are associated with set learning scenarios. We give a modal characterization of forests that are generated from a learning situation that satisfies a finite identifiability of languages condition. We observe that a special case of such protocol-based setting, namely when only one stream of events is allowed in each state, can be used to model function-learning paradigm. We show how the simple setting of iterated epistemic update cannot account for all possible learning situation. In the end we conclude our considerations and present possible directions of further work.

2. Formal learning theory

Let U be an infinite recursive set; we call any $S \subseteq U$ a language. Learning theory is mostly concerned with indexed families of recursive languages, i.e., class $\mathcal{C}$ for which a computable function $f : N \times U \to \{0, 1\}$ exists that uniformly decides $\mathcal{C}$, i.e.,

$$f(i, w) = \begin{cases} 1 & \text{if } w \in S_i, \\ 0 & \text{if } w \notin S_i. \end{cases}$$

The global input for Learner is given as an infinite stream of data. In learning theory, such streams are often called *texts* (positive presentations).

Definition 1. By a (positive presentation) ε of S we mean an infinite sequence of elements from S enumerating all and only the elements from S (allowing repetitions).

Definition 2. We will use the following notation:

- ε_n is the nth element of ε;
- $\varepsilon \restriction n$ is the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$;
- $\text{set}(\varepsilon)$ is the set of elements that occur in ε;
- if $\alpha, \beta \in U^*$, then by $\alpha \sqsubset \beta$ we mean that α is a proper initial segment of β.
- L is a learning function—a recursive map from finite data sequences to indices of hypotheses, $L : U^* \to N$. We will sometimes take the learning function to be $L : U^* \to N \cup \{\uparrow\}$. Then the function is allowed to refrain from giving a natural number answer, in which case the output is marked with $\uparrow$.

2.1. Finite identification

Finite identifiability of a class of languages from positive data is defined by the following chain of conditions.

Definition 3. A learning function L:

1. finitely identifies $S_i \in \mathcal{C}$ on ε iff, when inductively given ε, at some point L outputs a single value i;
2. finitely identifies $S_i \in \mathcal{C}$ iff it finitely identifies S_i on every ε for S_i;
3. finitely identifies $\mathcal{C}$ iff it finitely identifies every $S_i \in \mathcal{C}$;
4. a class $\mathcal{C}$ is finitely identifiable iff there is a learning function L that finitely identifies $\mathcal{C}$.

Example 1. Let $\mathcal{C}_1 = \{S_i = \{0, i\} \mid i \in N\}$. $\mathcal{C}_1$ is finitely identifiable by the following function $L : U^* \to N \cup \{\uparrow\}$:

$$L(\varepsilon \restriction n) = \begin{cases} \uparrow & \text{if } \text{set}(\varepsilon \restriction n) = \{0\} \text{ or } \exists k < n\, L(\varepsilon \restriction k) \neq \uparrow, \\ \max(\text{set}(\varepsilon \restriction n)) & \text{otherwise.} \end{cases}$$

In other words, L outputs the correct hypothesis as soon as it receives a number different than 0, and the procedure ends.

To see how restrictive this notion is, we can consider finite class of languages that is not finitely identifiable.

Example 2. Let $\mathcal{C}_2 = \{S_i = \{0, \ldots, i\} \mid i \in \{1, 2, 3\}\}$. $\mathcal{C}_2$ is not finitely identifiable. To see that, assume that $S_2 = \{0, 1, 2\}$ is chosen to be the actual world. Then learning function can never conclusively decide that S_2 is the actual language. As, for all it knows, 3 might appear in the future, so it has to leave the S_3-possibility open.

A necessary and sufficient condition for finite identifiability has already been given [25,27].

Definition 4 [27]**.** A set D_i is a definite finite tell-tale set for $S_i \in \mathcal{C}$ if

1. $D_i \subseteq S_i$,
2. D_i is finite, and
3. for any index j, if $D_i \subseteq S_j$ then $S_i = S_j$.

On the basis of this notion, finite identifiability is characterized in the following way.

Theorem 1 [27]**.** *A class $\mathcal{C}$ is finitely identifiable from positive data iff there is an effective procedure $\mathcal{D} : N \to \mathcal{P}^{<\omega}(N)$, given by $n \mapsto \mathcal{D}_n$, that on input i produces a definite finite tell-tale of S_i.*

In other words, each set in a finitely identifiable class contains a finite subset that distinguishes it from all other sets in the class. Moreover, for the effective identification it is required that there is a *recursive* procedure that provides such definite finite tell-tale set.

2.2. Identification in the limit

To see that a more general notion of identification is possible let us consider again Example 2, i.e., take $\mathcal{C}_2 = \{S_i = \{0, \ldots, i\} \mid i \in \{1, 2, 3\}\}$, and assume that S_2 is the actual language. As we saw Learner cannot conclusively decide that S_2 is the case. There is however a way to deal with this kind of uncertainty. Namely, if we allow Scientist to answer each time he gets a new piece of data, we can define the success of learning using the notion of *convergence* to the right answer. After seeing 1 and 2 Learner can keep conjecturing S_2 indefinitely, because in fact 3 will never appear. This leads to the notion of identification in the limit.

Definition 5 (*Identification in the limit* [22])**.** Learning function L:

1. identifies $S_i \in \mathcal{C}$ in the limit on ε iff for co-finitely many m, $L(\varepsilon{\restriction}m) = i$;
2. identifies $S_i \in \mathcal{C}$ in the limit iff it identifies S_i in the limit on every ε for S_i;
3. identifies $\mathcal{C}$ in the limit iff it identifies in the limit every $S_i \in \mathcal{C}$.
4. $\mathcal{C}$ is identifiable in the limit iff there is a learning function that identifies $\mathcal{C}$ in the limit.

Below we give some examples of classes of languages which are identifiable in the limit. First let us consider an example of a finite class of finite sets.

Example 3. Recall the class $\mathcal{C}_2$ from the previous example. $\mathcal{C}_2$ is identifiable in the limit by the following function $L : U^* \to N$:

$$L(\varepsilon{\restriction}n) = \max(\mathrm{set}(\varepsilon{\restriction}n)).$$

Learning by erasing. Learning by erasing [24] is an epistemologically intuitive modification of the identification in the limit. It has not drawn much attention in the field of formal learning theory but for our purposes (a comparison with the approach of dynamic epistemic logic) it is interesting. Very often the cognitive process of converging to a correct conclusion consists of eliminating those possibilities that are falsified during the inductive inquiry. Accordingly, the outputs of the learning function are negative, i.e., the function each time eliminates a hypothesis, instead of explicitly guessing one that is supposed to be correct. The difference between the definition of this approach and the usual identification is in the interpretation of the conjecture of the learning function. In learning by erasing one assumes an ordering of the initial hypothesis space isomorphic to the natural numbers. This allows one to interpret the actual positive guess of the learning-by-erasing function to be the least hypothesis (in the given ordering) not yet eliminated.

Let us give now the two definitions that shape the notion of learning by erasing.

Definition 6 (*Function stabilization*)**.** In learning by erasing we say that a function stabilizes to number k on environment ε iff for co-finitely many $n \in N$:

$$k = \min(\{N - \{L(\varepsilon{\restriction}1), \ldots, L(\varepsilon{\restriction}n)\}\}).$$

Definition 7 (*Learning by erasing* [24])**.** We say that a learning function L:

1. learns $S_i \in \mathcal{C}$ by erasing on ε iff L stabilizes to k on ε;
2. learns $S_i \in \mathcal{C}$ by erasing iff it learns by erasing S_i from every ε for S_i;
3. learns $\mathcal{C}$ by erasing iff it learns by erasing every $S_i \in \mathcal{C}$.
4. $\mathcal{C}$ is learnable by erasing iff there is a learning function that learns $\mathcal{C}$ by erasing.

It is easy to observe that in this setting learnability heavily depends on the chosen *enumeration* of languages, since the positive conjecture of the learning function is interpreted as the minimal one that has not yet been eliminated.

Several types of learning by erasing have been proposed. They vary in the condition of which hypotheses the learning function is allowed to remove (for details and results on learning by erasing see [24]).

3. Dynamic epistemic logic

Modal logics of epistemic change are used to analyze the information flow in multi-agent systems (see, e.g., [2,7,17,18]). We focus on the approach of *dynamic epistemic logic* (DEL [29], see also [14]), in which it is possible to formalize the principles of such changes.

Let us begin with the notion of epistemic model. To better suit our purposes we will restrict ourselves to the single-agent case, indicating in relevant places how to make the step towards the multi-agent extension. We fix PROP to be a countable set of propositional letters.

Definition 8. A single-agent epistemic model $\mathcal{M}$ is a triple:

$$\langle W, \sim, V\rangle,$$

where $W \neq \emptyset$ is a set of states, $\sim$ is a binary equivalence uncertainty relation on W, and $V : \textsc{Prop} \to \wp(W)$ is a valuation, where PROP is a countable set of propositional letters, and $\wp(W)$ is the power set of W.

For the multi-agent modelling we extend the above definition with the uncertainty relations of all agents. Therefore, instead of $\sim$ we get $(\sim_a)_{a\in\mathcal{A}}$, where $\mathcal{A}$ is the set of agents.

In the above definition the set W stands for the possible states, and $\sim$ for the uncertainty of the agent. In other words, $w \sim v$ means that the agent cannot distinguish (is uncertain) between the possibilities w and v.

Epistemic models are static—they represent the informational state of an agent in temporal isolation. We will now make the setting more dynamic by assuming that the agent observes some incoming data and is allowed to revise his informational state. We will restrict to the simplest method, called *update* (see [4]). Update is a policy that restricts models—each time a piece of data is encountered, it is assumed to be truthful and all worlds of the epistemic model that do not satisfy this new information are eliminated.

The definition below formalizes the notion of update—the revision with formula φ results in removing all states that do not make φ true.

Definition 9. The update of an epistemic model $\mathcal{M} = \langle W, \sim, V\rangle$ with a formula φ, formally $\mathcal{M} \mid \varphi$, results in the new epistemic model $\mathcal{M}' = \langle W', \sim', V'\rangle$, where:

1. $W' = \{w \in W \mid w \models \varphi\}$;
2. $\sim' = \sim \restriction W'$;
3. $V' = V \restriction W'$.

Let us consider two simple examples of single-agent propositional update.

Example 4. Let us take a single-agent epistemic model $\mathcal{M} = \langle W, \sim, V\rangle$, where $W = \{w_1, w_2, w_3\}$, $\sim = W \times W$, $\textsc{Prop} = \{p_1, p_2, p_3, p_4\}$ and the valuation $V : \textsc{Prop} \to \mathcal{P}(W)$ is defined in the following way $V(p_1) = \{w_1, w_2, w_3\}$, $V(p_2) = \{w_1, w_2\}$, $V(p_3) = \{w_2, w_3\}$, $V(p_4) = \{w_3\}$, in other words: $w_1 \models p_1 \wedge p_2 \wedge \neg p_3 \wedge \neg p_4$, $w_2 \models p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4$, and $w_3 \models p_1 \wedge p_3 \wedge p_4 \wedge \neg p_2$. Let us assume that w_2 is the actual world, and that the agent receives propositional information that is consistent with w_2 in the following order: p_1, p_2, p_3. Receiving p_1 does not change anything—every world satisfies p_1. Then p_2 comes in, eliminating w_3, since $w_3 \not\models p_2$. The agent is now uncertain only between w_1 and w_2. The last information p_3 allows deleting w_1 because $w_1 \not\models p_3$. The uncertainty of the agent now disappears—the only possibility left is w_2. Moreover, whatever true (consistent with the actual world w_2) information comes in w_2 cannot be eliminated.

In fact, if any of the worlds is the actual one, and the agent will receive truthful and complete propositional information about it, he will be able to eventually eliminate all other worlds, and therefore gain full certainty about his situation.

Example 5. Let us again take a similar epistemic model, this time with the following valuation $V(p_1) = \{w_1, w_2, w_3\}$, $V(p_2) = \{w_1, w_2\}$, $V(p_3) = \{w_1\}$. Now, only one world, namely w_1, can get identified by receiving truthful and complete propositional information. In case w_2 (or w_3) is the actual world, the agent will never be able to eliminate w_1 (or w_1 and w_2), and therefore the uncertainty will always remain.

3.1. Dynamic epistemic learning scenarios

In Examples 4 and 5 the uncertainty range of the agent is revised as new pieces of information (in the form of propositions) are received. The data comes from a completely trusted source, and as such, causes the agents to eliminate the worlds that do not satisfy it.

In learning theory it is common to assume the truthfulness of incoming data, and therefore, in principle, it is justified to use epistemic update as a way to perform the inquiry (for observational interpretation of update see [5]). In this section we will present single-agent update-based learning scenarios in the framework of epistemic logic.

First, the initial learning model is a simple epistemic model whose worlds correspond to the initial class of languages being learned.

Definition 10. Let $\mathcal{C} = \{S_1, S_2, \ldots\}$ be a class of sets such that for all $i \in N$, $S_i \subseteq N$. Our *initial learning model* $\mathcal{M}_\mathcal{C}$ is a triple:

$$\langle W_\mathcal{C}, \sim, V_\mathcal{C} \rangle,$$

where $W_\mathcal{C} = \mathcal{C}$, $\sim\, = W_\mathcal{C} \times W_\mathcal{C}$, $V_\mathcal{C} : \textsc{Prop} \cup \textsc{nom} \rightarrow \mathcal{P}(W_\mathcal{C})$, such that $S_i \in V_\mathcal{C}(p_n)$ iff $n \in S_i$ and for each set $S_i \in \mathcal{C}$, we take a nominal $i \in \textsc{nom}$ and we set $V_\mathcal{C}(i) = \{S_i\}$.

In words, we identify states of the model with sets, we also assume that our agent does not have any particular initial information or preference over the possibilities. The interpretation of the propositional letters is as follows. Let $\mathcal{C} = \{S_1, S_2, \ldots\}$ be a class of sets, and let $U = \bigcup \mathcal{C}$ be the universal set of $\mathcal{C}$. For every piece of data $n \in U$ we take a propositional letter p_n. The nominals correspond to indices of sets. They can be interpreted as finite descriptions of sets or as theories that describe possible sequences of events.

In the previous section (see Examples 4 and 5) we touched on our central topic of *iterated* update. This leads us to consider *sequences* of updates that are executed on a given epistemic model.

Definition 11. A data stream is an infinite sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ such that for every $i \geq 1$ there is an $p_n \in \textsc{Prop}$ such that $\varepsilon_i = p_n$.

The intuition is that, at stage i, the agent observes the proposition of ε_i. For clarity, we will call finite parts of such data streams *data sequences*.

Definition 12. A data sequence is a finite sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$, where for every $0 < i \leq n$, there is $p_n \in \textsc{Prop}$, such that $\sigma_i = p_n$.

For our data streams and data sequences we will use the same notation as for texts (see Definition 2).

The data streams are not entirely arbitrary, they should reflect the reality, be consistent with the actual world. The analogy with scientific inquiry can be used here: one can base theories on the results of experiments if the results are assumed to be consistent with reality. We will additionally require that every elementary information true in the actual world will eventually appear in the stream.

Definition 13. We will say that ε is a data stream for $w \in W$ iff ε enumerates all and only those propositional letters that are true in w, i.e.: $p \in set(\varepsilon)$ iff $w \in V(p)$.

The 'confrontation' of the epistemic model with a logical formula has been already given in Definition 9. Now we need to define what it means to perform *iterated* update.

Definition 14. The iterated update of model $\mathcal{M}$ with the data sequence $\varepsilon\restriction n = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ can be defined in the following way:

$$\mathcal{M}^{\varepsilon\restriction n} := (((\mathcal{M} \mid \varepsilon_1) \mid \varepsilon_2) \cdots) \mid \varepsilon_n.$$

We will refer to such $\mathcal{M}^{\varepsilon\restriction n}$ by $\varepsilon\restriction n$-generated epistemic model.

In other words, the epistemic model is updated with a data sequence in a step-by-step manner.

3.2. Finite identification and epistemic logic

The research in dynamic epistemic and dynamic doxastic logic often touches the subject of converging to some desired states: (common) knowledge or (joint) true belief (see, e.g. [3]). In this respect it is concerned with multi-agent versions of the belief-revision problem. In this section we will show how to use the notion of finite identification to characterize convergence to irrevocable knowledge. To establish the first connection we will restrict ourselves to the single-agent case.

Definition 15. Iterated epistemic update of model $\mathcal{M}$ with an infinite data stream ε *stabilizes* to $\mathcal{M}'$ iff there is an $n \in N$, such that for all $m \geq n$, $\mathcal{M}^{\varepsilon\restriction m} = \mathcal{M}'$. In such cases we will sometimes write that the generated epistemic model $\mathcal{M}^\varepsilon$ stabilizes to $\mathcal{M}'$.

We are now ready to show that epistemic update performed on finitely identifiable class of sets leads to irrevocable knowledge. We will use the non-computable (general) characterization of finite identifiability of sets from positive data based on Theorem 1.

Theorem 2 [27]. *A class $\mathcal{C}$ is finitely identifiable from positive data if and only if for every set $S_i \in \mathcal{C}$ there is a definite finite tell-tale set D_i.*

Theorem 3. *The following are equivalent:*

1. *$\mathcal{C}$ is finitely identifiable.*
2. *For every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\sim' = \{(S_i, S_i)\}$.*

Proof. $(1 \Rightarrow 2)$ Let us assume that $\mathcal{C}$ is finitely identifiable. Then, by Theorem 2, for every set $S_i \in \mathcal{C}$ there is a finite definite tell-tale set $D_i \subseteq S_i$ such that D_i is not a subset of any other set in $\mathcal{C}$. Let us then take one S_i and the corresponding finite definite tell-tale set D_i. For every data stream ε for S_i there is a finite initial segment, $\varepsilon\restriction m$, such that $D_i \subseteq \text{set}(\varepsilon\restriction m)$. Then by stage m every S_j such that $i \neq j$ has been eliminated by the update.

$(2 \Rightarrow 1)$ Let us assume that for every $S_i \in W_{\mathcal{C}}$ and a data stream ε for S_i, the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = S_i$ and $\sim' = \{(S_i, S_i)\}$. Assume for contradiction that $\mathcal{C}$ is not finitely identifiable. Therefore, by Theorem 2, there is a set $S_i \in \mathcal{C}$ such that every finite subset of S_i is included in some $S_j \in \mathcal{C}$ such that $i \neq j$. Then for all n, if $\mathcal{M}^{\varepsilon\restriction n}_{\mathcal{C}} = \langle W^{\varepsilon\restriction n}_{\mathcal{C}}, \sim^{\varepsilon\restriction n}, V^{\varepsilon\restriction n}_{\mathcal{C}}\rangle$ then $\{S_i, S_j\} \subseteq W^{\varepsilon\restriction n}_{\mathcal{C}}$, so it clearly does not stabilize to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\sim' = \{(S_i, S_i)\}$. Contradiction. □

3.3. Finite identifiability in epistemic language

The previous subsection gives a semantic characterization of finite identifiability. In this part we will move towards explicitly involving the notion of knowledge. We first look at the core language of epistemic logic (see, e.g. [11]).
Syntax. Our epistemic language $\mathcal{L}_{EL}$ is defined as follows:

$$\varphi := i \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid K\varphi$$

where p ranges over PROP—a countable set of proposition letters and i over the set of nominals NOM. For the multi-agent logic we simply add K_a for every $a \in \mathcal{A}$, where $\mathcal{A}$ is the set of agents.
Semantics. Let us show how we interpret $\mathcal{L}_{EL}$ language.

Definition 16 (*Truth definition*). We give the semantics of $\mathcal{L}_{EL}$.

$$\begin{array}{ll}
\mathcal{M}, w \models i & \text{iff } w \in V(i) \\
\mathcal{M}, w \models p & \text{iff } w \in V(p) \\
\mathcal{M}, w \models \neg\varphi & \text{iff } \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \vee \psi & \text{iff } \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models K\varphi & \text{iff for all } v \text{ such that } w \sim v \text{ we have } \mathcal{M}, v \models \varphi
\end{array}$$

In the multi-agent version in the last clause we replace K with K_a and $\sim$ with $\sim_a$, for $a \in \mathcal{A}$ assuming that $\mathcal{A}$ is the set of agents.

With respect to the language of epistemic logic $\mathcal{L}_{EL}$, the following corollary corresponds to the semantic characterization in Theorem 3.

Corollary 1. *The following are equivalent:*

1. *$\mathcal{C}$ is finitely identifiable.*
2. *For every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\mathcal{M}'_{\mathcal{C}}, S_i \models K\, i$.*

Proof. From Theorem 3 we know that 1 is equivalent to:

\# For all $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\sim' = \{(S_i, S_i)\}$.

($\# \Rightarrow 2$) Let us take $S_i \in W_\mathcal{C}$ and data stream ε for S_i and assume that the generated epistemic model $\mathcal{M}_\mathcal{C}^\varepsilon$ stabilizes to $\mathcal{M}'_\mathcal{C} = \langle W'_\mathcal{C}, \sim', V_\mathcal{C} \rangle$, where $W'_\mathcal{C} = \{S_i\}$ and $\sim' = \{(S_i, S_i)\}$. Then by definition of semantics of $\mathcal{L}_{EL}$ $\mathcal{M}', S_i \models K\, i$, since it is true that for all $S_j \in \{S_k \in W'_\mathcal{C} \mid S_i \sim' S_k\}$, we have that $\mathcal{M}'_\mathcal{C}, S_j \models i$.

($2 \Rightarrow \#$) Let us assume that for all $S_i \in W_\mathcal{C}$ and data stream ε for S_i the generated epistemic model $\mathcal{M}_\mathcal{C}^\varepsilon$ stabilizes to $\mathcal{M}'_\mathcal{C} = \langle W'_\mathcal{C}, \sim', V_\mathcal{C} \rangle$, where $W'_\mathcal{C} = \{S_i\}$ and $\mathcal{M}'_\mathcal{C}, S_i \models K\, i$. That means that for all $S_j \in \{S_k \in W'_\mathcal{C} \mid S_i \sim' S_j\}$ we have that $\mathcal{M}'_\mathcal{C}, S_j \models i$. But from definition of the valuation $V_\mathcal{C}$ we know that S_i is the only state in $W_\mathcal{C}$ that validates i. Therefore $\sim' = \{(S_i, S_i)\}$. $\square$

Until now we have shown how to express the outcome of finite identifiability learning scenarios in $\mathcal{L}_{EL}$. The language of epistemic logic can account only for the result of epistemic change. Our aim is to give a logical formula that can express the possibility of convergence. As the latter is a temporal property, we will propose to view the structure generated by iterated epistemic update as a temporal branching model.

4. Learning and temporal logic

Before we make the transition to the temporal setting, we will first discuss a generalization of the update with incoming information—the concepts of *event models* and *product update* introduced by Baltag et al. [2].

4.1. Event models and product update

Iterated update, defined in Section 3 can be placed in a more general perspective. Obviously, the incoming information does not have to be propositional. It does not even have to be purely linguistic. It can be any *event* that itself has an epistemic structure. To consider changes caused by such arbitrary events, we will now introduce the notion of event model, which represents the epistemic and informational content of what 'happens'.

Definition 17. An *event model* is a triple:

$$\mathcal{E} = \langle E, (\sim_a^\mathcal{E})_{a \in \mathcal{A}}, \mathtt{pre} \rangle,$$

where $E \neq \emptyset$ is a set of events; for every agent $a \in \mathcal{A}$, $\sim_a^\mathcal{E}$ is a binary equivalence relation on E, and $\mathtt{pre} : E \to \mathcal{L}_{EL}$, is a precondition function where $\mathcal{L}_{EL}$ is a set of formulas of some epistemic language. A pair $(\mathcal{E}, e)$, where $e \in E$ is called a *pointed event model*.

For every agent $a \in \mathcal{A}$, the relation $\sim_a^\mathcal{E}$ encodes that agent's epistemic information about the event taking place. The precondition function maps events to epistemic formulas. An event will be executable in some state only if that state satisfies the precondition of this event.

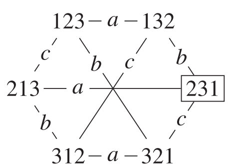
The effect of updating an epistemic model $\mathcal{M}$ with an event model $\mathcal{E}$ can be computed according to the so-called *product update*.

Definition 18. Let $\mathcal{M} = \langle W, (\sim_a)_{a \in \mathcal{A}}, V \rangle$ be an epistemic model and $\mathcal{E} = \langle E, (\sim_a^\mathcal{E})_{a \in \mathcal{A}}, \mathtt{pre} \rangle$ be an event model. The *product update* of $\mathcal{M}$ with $\mathcal{E}$ gives a new epistemic model $\mathcal{M} \otimes \mathcal{E} = \langle W', (\sim'_a)_{a \in \mathcal{A}}, V' \rangle\}$, where:

1. $W' = \{(w, e) \mid w \in W \,\&\, e \in E \,\&\, w \models \mathtt{pre}(e)\}$;
2. $(w, e) \sim'_a (w', e')$ iff $w \sim_a w'$ and $e \sim_a^\mathcal{E} e'$;
3. and the valuation is as follows: $(w, e) \in V'(p)$ iff $w \in V(p)$.

The next example shows the product update caused by an event model of public announcement.

Example 6. Let us consider the following multi-agent scenario. Anne, Bob and Carl are playing a card game. The deck of cards consisting of: 1, 2, 3. Each person gets one card. We can represent the situation after dealing as a triple *xyz*, where x, y, z are cards and the first position in the triple assigns the value to a (Anne), the second to b (Bob), etc. For instance, 231 means that Anne has 2, Bob has 3 and Carl has 1. All the possible situations after a deal are: 123, 132, 213, 231, 312, 321. We assume that all the players witness the fact of dealing but they do not know the distribution of the cards. The epistemic model $\mathcal{M}$ of this situation is illustrated in the figure.



Let us then assume that the actual world is 231. Obviously each player's knowledge does not allow certainty about which is the actual world. In the model the uncertainty of the agent x about the worlds w and w' is symbolized by the following: $w \sim_x w'$ (in the figure this relation is depicted by two states being joined by a line labeled with x, dropping reflexive loops).

Let us now assume that Anne shows her card to all the players publicly, i.e., all the players see her card and all of them know that all of them see it. This event is modeled by $\mathcal{E} = \langle E, \{\sim_i^{\mathcal{E}}\}_{i \in A}, \mathtt{pre} \rangle$, where $E = \{e\}$, for each $x \in A$, $e \sim_x^{\mathcal{E}} e$ and $\mathtt{pre}(e) = 2\_\,\_$ (i.e., 'Anne has 2').

$$\overset{a,b,c}{\boxed{e : 2\_\,\_}}$$

The public announcement of 'Anne has 2' results in the epistemic situation which can be presented as $\mathcal{M}' = \mathcal{M} \otimes \mathcal{E}$ depicted below:

$$213 \overset{a}{\text{———————}} \boxed{231}$$

Event $\mathcal{E}$ of the above example is a public announcement of: 'Anne has 2'. In dynamic epistemic logic the public announcement of φ is represented by '$!\varphi$' and corresponds to the elimination of all those possible worlds that do not satisfy φ. In other words, public announcement works as relativization of the model to those worlds that satisfy the content of the announcement. [1]

4.2. Dynamic epistemic logic protocols

By making a step from dynamic epistemic logic into epistemic temporal logic we can analyze the temporal aspects of update. Redefining the iterated epistemic update in terms of protocols (see [16,28]) will bring us closer to the temporal setting. A protocol specifies sequences of events that are admissible in certain epistemic situations. In this section, following [8], we will give the definition of local protocols, and epistemic models generated with respect to a protocol. By doing this we prepare the grounds for our learning-theoretic setting.

The admissible runs of some informational process are defined by a protocol P that maps states in an epistemic model to sets of finite and infinite sequences of event models closed under taking prefixes. In general not every sequence of events may be possible at a given state.

Let $\mathbb{E}$ be the class of all event models. Accordingly, every state of the epistemic model is assigned a set of sequences (infinite and finite) of event models closed under taking finite prefixes, an element of $\mathrm{Prot}(\mathbb{E}) = \{\mathtt{P} \subseteq \mathcal{P}(\mathbb{E}^* \cup \mathbb{E}^\omega) \mid \mathtt{P}$ is closed under finite prefixes$\}$.

Definition 19. Let us take an epistemic model $\mathcal{M} = \langle W, (\sim_a)_{a \in \mathcal{A}}, V \rangle$. A *local protocol* for $\mathcal{M}$ is a function $P : W \to \mathrm{Prot}(\mathbb{E})$.

Until now we have been concerned with the $\varepsilon\restriction n$-generated epistemic model $\mathcal{M}$, where $\varepsilon\restriction n$ is some sequence of propositions. We will now provide an analogous notion of a model generated from a sequence of event models but according to some specific local protocol.

Definition 20. Let $\mathcal{M} = \langle W, (\sim_a)_{a \in \mathcal{A}}, V \rangle$ be an epistemic model. We define the *$(P, \varepsilon\restriction n)$-generated epistemic model* $\mathcal{M}^{P,\varepsilon\restriction n}$ inductively, as follows:

$$\mathcal{M}^{P,\varepsilon\restriction 0} = \mathcal{M}$$
$$\mathcal{M}^{P,\varepsilon\restriction n+1} = \langle W^{P,\varepsilon\restriction n+1}, \sim^{P,\varepsilon\restriction n+1}, V^{P,\varepsilon\restriction n+1} \rangle, \text{ where:}$$
$$W^{P,\varepsilon\restriction n+1} := \{s \mid s \in W^{P,\varepsilon\restriction n};\ s \models \mathtt{pre}(\varepsilon_{n+1})\ \&\ \varepsilon\restriction n+1 \in P(s)\};$$
$$\sim^{P,\varepsilon\restriction n+1} := \sim^{P,\varepsilon\restriction n}\restriction W^{P,\varepsilon\restriction n+1};$$
$$V^{P,\varepsilon\restriction n+1} := V^{P,\varepsilon\restriction n}\restriction W^{P,\varepsilon\restriction n+1}.$$

The protocol-based approach to update has a straightforward temporal interpretation. The question of how iterated product update can be interpreted in temporal logics is interesting because the latter are widely used to study the evolution of a system over time. Moreover, epistemic extensions of temporal logics offer a global view of the evolution of a multi-agent system as events take place, focusing on the information that agents possess. Obviously, all of these aspects are crucial for inductive inference.

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[1] Other illustrations of the strength of product update can be found in [1,6,12].

4.3. Dynamic epistemic and epistemic temporal logic

Epistemic temporal logics are interpreted on epistemic temporal models (see, e.g. [28]).

Definition 21. An epistemic temporal model $\mathcal{H}$ is a tuple:

$$\langle W, \Sigma, H, (\sim_a)_{a \in \mathcal{A}}, V \rangle,$$

where $W \neq \emptyset$ is a countable set of initial states; Σ is a countable set of events; $H \subseteq W(\Sigma^* \cup \Sigma^\omega)$ is a set of histories (sequences of events starting at states from W) closed under non-empty finite prefixes; for each $a \in A$, $\sim_a \subseteq H \times H$ is an equivalence relation; and $V : \textsc{Prop} \to \mathcal{P}(H)$ is a valuation. We write wh to denote some finite history starting in the state w, and $w\varepsilon$ analogously for an ω-history.

We sometimes refer to the $\langle W, \Sigma, H \rangle$-part of an ETL model as the *temporal protocol* this model is based on. We refer to the information of an agent a at h with $\mathcal{K}_a[wh] = \{vh' \in H \mid wh \sim_a vh'\}$.

The relation between the dynamics of epistemic update and epistemic temporal logic has already been studied (see, e.g. [9,10]). In particular, it has been observed that iterated epistemic update in dynamic epistemic logic generates epistemic temporal forests satisfying certain properties (see [8]). We will refer to this construction by $\texttt{For}(\mathcal{M}, P)$ and define it below.

We construct the forest by induction, starting with the epistemic model and then checking which events can be executed according to the precondition function and to the protocol. Finally, the new information partition is updated at each stage according to the product update. Since product update describes purely epistemic change, the valuation stays the same as in the initial model.

Definition 22. An epistemic model $\mathcal{M} = \langle W, (\sim_a^{\mathcal{M}})_{a \in \mathcal{A}}, V^{\mathcal{M}} \rangle$ and a local protocol $P : W \to \text{Prot}(\mathbb{E})$ generates an ETL forest $\texttt{For}(\mathcal{M}, P)$ of the form:

$$\mathcal{H} = \langle W^{\mathcal{H}}, \mathbb{E}, H, (\sim_a)_{a \in \mathcal{A}}, V \rangle, \text{ where:}$$

1. $W^{\mathcal{H}} := W$;
2. H is defined inductively as follows:

$$H_0 := W^{\mathcal{H}};$$

$$H_{n+1} := \{(we_1 \ldots e_{n+1}) \mid (we_1 \ldots e_n) \in H_n; \mathcal{M}^{\varepsilon\restriction n}, w \models \texttt{pre}(e_{n+1}) \text{and } (e_1 \ldots e_{n+1}) \in P(w)\};$$

$$H := \textstyle\bigcup_{0 \le k < \omega} H_k;$$

3. if $w, v \in W^{\mathcal{H}}$, then $w \sim_a v$ iff $w \sim_a^{\mathcal{M}} v$;
4. $whe \sim_a vh'e'$ iff $whe, vh'e' \in H_k$, $wh \sim_a vh'$, e and e' are states in an event model $\mathcal{E}$ and $e \sim_a^{\mathcal{E}} e'$;
5. Finally, $wh \in V(p)$ iff $w \in V^{\mathcal{M}}(p)$.

The correspondence between the iterated product update and an epistemic temporal forest relies on some properties of epistemic temporal agents. To be precise, it has been shown that the structures of iterated DEL update are in fact epistemic temporal frames that satisfy the following conditions: perfect recall, synchronicity, uniform no miracles and propositional stability. Let us introduce those epistemic multi-agent assumptions.

Definition 23. Let us take $\mathcal{H} = \langle W, \Sigma, H, (\sim_a)_{a \in \mathcal{A}}, V \rangle$ to be an epistemic temporal model.

Perfect recall $\mathcal{H}$ satisfies perfect recall iff

$$\text{for all } whe, vh'f \in H \text{ if } \mathcal{K}_a[whe] = \mathcal{K}_a[vh'f], \text{ then } \mathcal{K}_a[wh] = \mathcal{K}_a[vh'].$$

The condition of perfect recall expresses that agents do not forget past information as further events take place.

Synchronicity $\mathcal{H}$ satisfies synchronicity iff

$$\text{for all } wh, vh' \in H \text{ if } \mathcal{K}_a[wh] = \mathcal{K}_b[vh'], \text{ then length}[wh] = \text{length}[vh'].$$

Synchronicity is satisfied if the agents have access to some external discrete clock and thus can keep track of the time.

Uniform-no-miracles $\mathcal{H}$ satisfies uniform no miracles iff

$$\text{for all } wh, vh' \in H \text{ such that } wh \sim_a vh'$$
$$\text{and for all } e_1, e_2 \in \Sigma \text{ with } whe_1, vh'e_2 \in H \text{if there are } sh'', th''' \in H \text{ such that } sh''e_1 \sim_a th'''e_2,$$
$$\text{then } whe_1 \sim_a vh'e_2.$$

Uniform-no-miracles means that if an agent cannot distinguish between a history terminating with e_1 and a history whose last event is e_2, then at any time if he is unable to distinguish between two histories wh and vh' then he is still unable to distinguish between whe_1 and vhe_2. This property characterizes local 'updaters' that do not take into account the whole history but that proceed in a step-by-step manner.

Propositional Stability $\mathcal{H}$ satisfies propositional stability iff for all $wh, whe \in H$ we have $p \in V(whe)$ iff $p \in V(wh)$.

The following result indicates that the iterated product update of an epistemic model $\mathcal{M}$ according to a protocol P generates an epistemic temporal forest that validates the above-mentioned epistemic properties.

Theorem 4 [8]. *An ETL-model $\mathcal{H}$ is isomorphic to the forest generated by the sequential product update of an epistemic model according to some state-dependent DEL-protocol iff it satisfies perfect recall, synchronicity, uniform-no-miracles and propositional stability.*

4.4. Learning in a temporal perspective

Let us now see how the above construction can be used to analyze finite identifiability.

Learning event models. In our learning setting the incoming information has a purely propositional character. A simple *event learning model* can be associated with every such piece of data in the following way.

Definition 24. Let $\mathcal{C} = \{S_1, S_2, \ldots\}$ be a class of sets and, as before, $U = \bigcup \mathcal{C}$ is the universal set of $\mathcal{C}$. Let $\mathtt{E} : N \rightarrow \mathbb{E}$ be a function that transforms an integer into an event model in the following way: for each $n \in N$, $\mathtt{E}(n) = \mathcal{E}_n = \langle \{e\}, \sim^{\mathcal{E}_n}, \mathtt{pre}_{\mathcal{E}} \rangle$, where $\sim\, = \{(e, e)\}$ and $\mathtt{pre}_{\mathcal{E}}(e) = p_n$. Similarly, if $S \subseteq N$, $\mathtt{E}(S) = \{\mathtt{E}(n) \mid n \in S\}$.

In other words, for every piece of data n from U we take a propositional letter p_n, and since we consider simple announcement scenarios, we construct an event model that consists of one state. For simplicity, we will sometimes refer to such $\mathcal{E}_n$ with p_n. By making the conceptual transition from the simple propositional update to the event models we want to show to what extent our framework conforms to the general setting described in the previous section.

Set-learning local protocol. Intuitively, given a state $S_i \in W_{\mathcal{C}}$, our protocol P should authorize at S_i any ω-sequence that enumerates S_i and nothing more. Our set-learning scenarios allow any enumeration of elements of a given set. Therefore the corresponding local protocol can be defined in the following way.

Definition 25. Let $\mathcal{C} = \{S_1, S_2, \ldots\}$ be a class of sets and $U = \bigcup \mathcal{C}$ be the universal set of $\mathcal{C}$. For every $S_i \in W_{\mathcal{C}}$, the *set-learning local protocol*, $P(S_i)$, is the smallest subset of $(\mathtt{E}(U))^{\omega}$ that contains:

$$\{f : \omega \rightarrow \mathtt{E}(S_i) \mid f \text{ is surjective}\},$$

and that is closed under non-empty finite prefixes. Accordingly, $P(W_{\mathcal{C}}) := \bigcup_{S_i \in W_{\mathcal{C}}} P(S_i)$.

Set-learning local protocols restrict the admissible sequences of events only in terms of content and not in terms of ordering. It is easy to observe that such a local protocol can replace the sets in learning scenarios. In principle we can then skip the precondition check and instead decide whether an event can take place just on the basis of the protocols. We will return to this issue in the end of this paper.

Initial learning model with local protocol. Let us now complement our definition of the initial learning model (Definition 10) with the local set-learning protocol.

Definition 26. Let $\mathcal{C} = \{S_1, S_2, \ldots\}$ be a class of sets such that for all $i \in N$, $S_i \subseteq N$. The *initial learning model with local protocol* consists of:

1. an epistemic model $\mathcal{M}_{\mathcal{C}} = \langle W_{\mathcal{C}}, \sim, V_{\mathcal{C}} \rangle$, where $W_{\mathcal{C}} = \mathcal{C}$, $\sim\, = W_{\mathcal{C}} \times W_{\mathcal{C}}$, $V_{\mathcal{C}} : \textsc{Prop} \cup \textsc{nom} \rightarrow \mathcal{P}(W_{\mathcal{C}})$, such that $S_i \in V_{\mathcal{C}}(p_n)$ iff $n \in S_i$ and for each set $S_i \in \mathcal{C}$, we take a nominal i and we set $V_{\mathcal{C}}(i) = \{S_i\}$.
2. for each $S_i \in W_{\mathcal{C}}$, a set-learning local protocol $P(S_i)$.

Now we are ready to define how our initial learning model and a set-learning protocol generate an epistemic temporal forest. We define the additional set of designated propositional letters based on the previously used set of nominals $\textsc{nom}$, $\textsc{Prop}_{\textsc{nom}} := \{q_i \mid i \in \textsc{nom}\}$.

Definition 27 (*Epistemic temporal learning forest*). A learning model $\mathcal{M}_{\mathcal{C}} = \langle W_{\mathcal{C}}, \sim^{\mathcal{M}}, V_{\mathcal{C}}^{\mathcal{M}} \rangle$ together with the local set-learning protocol P generates an ETL forest $\mathtt{For}(\mathcal{M}, P)$ of the form:

$$\mathcal{H} = \langle W^{\mathcal{H}}, \mathbb{E}, H, P, \sim, V \rangle, \text{ where:}$$

1. $W^{\mathcal{H}} := W_{\mathcal{C}}$,
2. H is defined inductively as follows:

$$H_0 := W^{\mathcal{H}};$$
$$H_{n+1} := \{(we_1 \ldots e_{n+1}) \mid (we_1 \ldots e_n) \in H_n;\ \mathcal{M}_C^{P,\varepsilon\restriction n}, w \models \mathtt{pre}(e_{n+1})\text{and } (e_1 \ldots e_{n+1}) \in P(w)\};$$
$$H := \bigcup_{0 \le k < \omega} H_k;$$

3. If $w, v \in W^{\mathcal{H}}$, then $w \sim v$ iff $w \sim^{\mathcal{M}} v$;
4. $whe \sim_a vh'e'$ iff $whe, vh'e' \in H_k$, $wh \sim vh'$, and $e = e'$;
5. Finally, the valuation $V : \textsc{Prop} \cup \textsc{Prop}_{\textsc{nom}} \to \mathcal{P}(H)$ is defined in the following way:
   - for every $p \in \textsc{Prop}$, $wh \in V(p)$ iff $w \in V_{\mathcal{C}}^{\mathcal{M}}(p)$;
   - for every $q_i \in \textsc{Prop}_{\textsc{nom}}$, $wh \in V(q_i)$ iff $w \in V_{\mathcal{C}}^{\mathcal{M}}(i)$.

The above construction is in the strict correspondence with the general case of generated epistemic temporal forest of Definition 22. Our concept allows a slight simplification in point 4 because of the very simple structure of our public announcement events. At this point we have the temporal structures that correspond to the learning situation. The next step is to give a temporal characterization of forests that satisfy the identifiability condition.

4.5. Finite identifiability in ETL

In this section we will give a general characterization of finite identification in the epistemic extension of CTL* (see [15]), $\mathcal{L}_{\mathrm{ETL}^*}$. The aim of this section is to give a formula of epistemic temporal logic that characterizes classes of sets that are finitely identifiable.

4.5.1. Epistemic temporal language

Syntax. The syntax of our epistemic temporal language of $\mathcal{L}_{\mathrm{ETL}^*}$ is defined in the following way.

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid K\varphi \mid F\varphi \mid A\varphi$$

where p ranges over a countable set of proposition letters $\textsc{Prop} \cup \textsc{Prop}_{\textsc{nom}}$. $K\varphi$ reads: 'the agent knows that φ'. $F\phi$ means 'at some point in the future on the current infinite sequence ϕ will be true', and we define G to mean $\neg F \neg$. $A\varphi$ means: 'in all infinite continuations conforming to the protocol, φ holds'.
Semantics. $\mathcal{L}_{\mathrm{ETL}^*}$ is interpreted over epistemic temporal frames, $\mathcal{H}$, and pairs of the form (ε, h), the former being a maximal, infinite history in our trees, and the latter a finite prefix of ε (see [26,28]).

Definition 28. We give the semantics of $\mathcal{L}_{\mathrm{ETL}^*}$. We skip the boolean clauses. We take $h \sqsubseteq h'$ to mean that h is an initial segment of h', and $p \in \textsc{Prop} \cup \textsc{Prop}_{\textsc{nom}}$.

$$\begin{array}{lll}
\mathcal{H}, \varepsilon, wh \models p & \text{iff} & wh \in V(p) \\
\mathcal{H}, \varepsilon, wh \models K\varphi & \text{iff} & \text{for all } \varepsilon',\ vh' \text{ if } vh' \sqsubseteq \varepsilon' \text{and} vh' \in K[wh] \text{ then } \mathcal{H}, \varepsilon', vh' \models \varphi \\
\mathcal{H}, \varepsilon, wh \models F\varphi & \text{iff} & \text{there are } \sigma \in \Sigma^* \text{ and } wh' \sqsubseteq \varepsilon \text{ with } wh' = wh\sigma \text{ and } \mathcal{H}, \varepsilon, wh' \models \varphi \\
\mathcal{H}, \varepsilon, wh \models A\varphi & \text{iff} & \text{for all } \varepsilon' \in P(w) \text{ such that } wh \sqsubset \varepsilon' \text{ we have } \mathcal{H}, \varepsilon', wh \models \varphi
\end{array}$$

To give a temporal characterization of finite identifiability we need to express the following idea. In our epistemic temporal forest, at any state, it is the case that on all infinite branches there will be a point in the future after which the agent will keep knowing q_i, which means that he will remain certain about the partition of the tree he is in. The special set of propositional letters $\textsc{Prop}_{\textsc{nom}}$ corresponds to the underlying theory that allows predicting further events. In this sense, knowing that q_i is equivalent to having a complete knowledge about the state, i.e., for any $p \in \textsc{Prop}$, if p holds in the state, then the agent knows that p. Formally, with respect to finite identifiability of sets, the following theorem holds.

Theorem 5. *The following are equivalent:*

1. *$\mathcal{C}$ is finitely identifiable.*
2. *For all $S_i \in W_{\mathcal{C}}$ and $\varepsilon \in P(S_i)$ the learner's knowledge about the initial state stabilizes to S_i on $S_i\varepsilon$ in the generated forest* $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P)$.
3. $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P) \models q_i \to AFGK\, q_i$.

Proof. $(1 \Rightarrow 2)$ Assume that $\mathcal{C}$ is finitely identifiable. Assume, for contradiction, that there is a state $S_i \in W_{\mathcal{C}}$ and ω-sequence $\varepsilon \in P(S_i)$ such that the agent's knowledge does not stabilize to S_i on ε. There are two cases.

Case 1 The learner stabilizes to another state, but then by construction of $P(S_i)$ and definition of the generated epistemic temporal forest, for every finite prefix $h \sqsubset \varepsilon$, $S_i h \in \mathcal{K}[S_i h]$. Contradiction.

Case 2 After each finite prefix $h \sqsubset \varepsilon$, there is at least one state different from S_i that remains epistemically possible. But by construction of $P(S_i)$ this is only possible if $S_i \subset S_j$. Then every finite subset of S_i is a subset of S_j, and therefore $S_i \in \mathcal{C}$ does not have a finite definite tell-tale set. Therefore, from Theorem 2, $\mathcal{C}$ is not finitely identifiable.

$(2 \Rightarrow 3)$ Assume for contradiction that there is ε and $h \sqsubset \varepsilon$ such that:

1. $\texttt{For}(\mathcal{M}_{\mathcal{C}}, P), \varepsilon, h \models q_i$
2. $\texttt{For}(\mathcal{M}_{\mathcal{C}}, P), \varepsilon, h \not\models AFGK\, q_i$

From (1), by the definition of $\texttt{For}(\mathcal{M}_{\mathcal{C}}, P)$, it is the case that for all h' such that $h' \sqsubset \varepsilon$ $\texttt{For}(\mathcal{M}_{\mathcal{C}}, P), \varepsilon, h' \models q_i$ and that $\varepsilon \in P(S_i)$.

From the above and (2), we have that there is $\varepsilon' \in P(S_i)$ such that $h \sqsubset \varepsilon'$ and for all h', if $h \sqsubset h' \sqsubset \varepsilon$, $\texttt{For}(\mathcal{M}_{\mathcal{C}}, P), \varepsilon', h' \not\models K\, q_i$. But then it means that there is $S_i \in W_{\mathcal{C}}$ and $\varepsilon \in P(S_i)$ such that for all $h \sqsubset \varepsilon$, $\mathcal{K}[S_i h'] \neq \{S_i h'\}$. Contradiction.

$(3 \Rightarrow 1)$ Assume that $\mathcal{C} = \{S_1, S_2, \ldots\}$ is a class of sets and that $\texttt{For}(\mathcal{M}_{\mathcal{C}}, P) \models q_i \rightarrow AFGK\, q_i$. Hence, for every $S_i \in W_{\mathcal{C}}$ and $\varepsilon \in P(S_i)$ there is $h \sqsubset \varepsilon$ such that for all h' with $h \sqsubseteq h' \sqsubset \varepsilon$, $\texttt{For}(\mathcal{M}_{\mathcal{C}}, P), \varepsilon, h' \models K\, q_i$. In other words, $\mathcal{K}[S_i h'] = \{S_i h\}$. We claim that the content of h, i.e., $E^{-1}(\mathrm{set}(h))$ is a definite finite tell tale set for S_i. First, observe that by definition of P we have $E^{-1}(\mathrm{set}(h)) \subseteq S_i$. Second, $E^{-1}(\mathrm{set}(h))$ is finite because h is finite. Finally, we have to show that for all $j \neq i$, $E^{-1}(\mathrm{set}(h)) \nsubseteq S_j$. Assume for contradiction that there is an S_j such that $E^{-1}(\mathrm{set}(h)) \subseteq S_j$. But then $S_j h \in \mathcal{K}[S_i h]$. Contradiction. $\square$

5. Identification of sets and uniform-no-miracles

The rule of uniform-no-miracles states that any two histories that are not distinguishable from an agent's perspective cannot get distinguished by extending them with the same event (or two indistinguishable event states). In our learnability context a strengthening of this rule seems interesting.

Let us consider the problem of identification in a more general perspective. Objects to be learned do not have to be sets, in particular their protocols do not have to be order-independent. Except for sets, formal learning theory is also concerned, for example, with learnability of functions. Possible realities can even be more general, they can be classes of functions—scenarios of this kind are at heart of many inductive inference games, e.g., card game *Eleusis* (see, e.g. [30]). Then, the worlds can be identified with protocols that allow certain sequences of events that can be defined by some logical formula. In particular, events might be assumed to occur in a certain order. Let us consider the following example.

Example 7. Let us take two possible worlds: w_1 and w_2 such that:

1. the protocol for w_1 allows all infinite sequences that contain all even numbers, and additionally require that whenever a number is 8 then the successor should be 10;
2. the protocol for w_2 allows all infinite sequences that contain all even numbers, and additionally require that whenever a number is 8 then the successor should be 6.

As long as the learner receives even numbers different than 10 he cannot distinguish between the two states, e.g., the two sequences, h, h', are in both protocols:

- h : 2, 4, 6, 8
- h' : 4, 2, 6, 8

Therefore, we can say that whichever of the two is enumerated, $w_1 \sim w_2$. However, complementing both of them with the same event, 10, leads to 'a miracle'—two hypotheses get to be distinguished.

In principle, there is no reason why such 'miraculous' classes of hypotheses should be excluded from learnability considerations. Such cases show a strength of the protocol based temporal approach over the one-step simple DEL update. The latter is well-suited for set-learning, because set-learning protocols are permutation closed and in this sense they are reducible to the precondition check. This is why we turned to a more liberal setting of epistemic temporal logic in which the 'miracle' of order-dependence is possible.

In general, thinking about learnability in terms of protocols leads to a setting in which the possible realities are identified with sets of scenarios of what should be expected to happen in the future. In this sense, the most general realities are sets—they allow any possible enumeration of their content. Functions allow only one particular sequence of events.[2] In between there are a variety of possibilities for defining protocols that can be characterized in an arbitrary way.

6. Conclusions and perspectives

Our work provides a translation of scenarios from formal learning theory into the domain of dynamic epistemic logic and epistemic temporal logic. In particular, we show that the finite identification of sets can be performed by means of epistemic update, and that its outcome can be characterized in epistemic logic. Moreover, in the more general context of learnability of protocols, we show the characterization of finite identifiability in an epistemic temporal language.

Our results indicate that the two prominent approaches, learning theory and epistemic temporal logic, can be joined in order to describe the notions of belief and knowledge involved in inductive inference. Also, our representation of initial classes of sets and environments gives an interesting application for the theory of protocols. As we indicated in the previous section the temporal logic based approach to inductive inference gives a straightforward framework for analyzing various domains of learning on common ground. In terms of protocols, sets can be seen as classes of specific histories—their permutation closed complete enumerations. Functions, on the other hand, can be seen as 'realities' that allow only one particular infinite sequence of events. We can think of many intermediate concepts that can be the object of learning. The property that distinguishes set- and function-learning protocols is their order-(in)dependence. This feature of protocols corresponds to a property of temporal models that is structurally similar to that of uniform-no-miracles. We perceive its temporal characterization as an interesting direction of future research.

A natural question arises whether it is possible to use the tools of epistemic logic to analyze other conditions for learnability. The answer is affirmative (see [21]). Limiting learning can be modeled in *doxastic* extensions of DEL and ETL, that are enriched by a plausibility relation and therefore also by the notion of belief. Recall that under the condition of finite identifiability, Learner succeeds if at some finite stage he gives a correct answer, i.e., if at some point he can be sure about the conjecture. On the other hand, in limiting conditions for learnability Learner gives an answer at infinitely many stages of the procedure. There is no natural way to represent the current, non-final guess in epistemic frameworks. However, if we add a doxastic plausibility relation over the set of possible states, the current conjecture can be viewed as the minimal state according to the plausibility relation at a given stage. Then, the limiting condition of success can be defined as the stabilization of belief in the generated doxastic epistemic model. This idea resembles learning by erasing, where the actual positive guess is interpreted to be the least index (according to some ordering) not yet eliminated. The difference is that in learning by erasing Learner each time removes only one possibility from the initial range of languages.

Directions of further work include extending our approach to other types of identification, e.g., identification of functions; finding modal framework for learning from both positive and negative information; studying systematically the effects of different restrictions on protocols. We are also interested in investigating various constraints one can enforce on learning functions (e.g., consistency, conservatism or set-drivenness) and comparing them to those of epistemic and doxastic agents in doxastic epistemic temporal logics. Modal logics of belief change are a natural framework to study a variety of notions that underlay certain concepts of learnability. Moreover, they offer the right tools to analyze multi-agent learning situations, in particular the interaction and communication and their effect on learnability.

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[2] Formal learning theory usually takes the functions to be learned from texts that enumerate their graphs. Such texts consist of pairs (n, m), such that $f(n) = m$. This allows transforming functions into sets, and therefore the corresponding strings of data are permutation closed—the 'real' position of the value with respect to the function is explicitly mentioned in the event. We suggest here a more 'observational' perspective, where the stage at which the agent receives an information is important (for a similar approach see [23]).

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