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A PROMETHEE-based approach to portfolio selection problems ☆

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ABSTRACT

In this paper, we study the use of PROMETHEE outranking methods for portfolio selection problems. Starting from a new formulation of the PROMETHEE V method, we develop several alternative approaches based on the concepts of boundary portfolios and *c*-optimal portfolios. The proposed methods are compared in an extensive computational study. Results of these experiments show that methods based on the concept of *c*-optimal portfolios provide a good approximation to the (often computationally untractable) PROMETHEE ranking of all portfolios, while requiring only small computational effort even for large problems. For smaller problems, a PROMETHEE ranking of all boundary portfolios can be performed and provides a close approximation of the total ranking.

1. Introduction

Portfolio selection is a distinct decision problematic, different from others indicated by Roy [1]: choice, ranking and classification. In general terms, a portfolio problem may be defined as a problem which involves:

- the selection of one or several out of a set of possible items;
- under some constraints, which limit the possibility to select items;
- where outcomes are determined by some aggregation of properties of the selected items.

Portfolio decision problems occur in many application areas, well-known examples include financial portfolio problems, in which combinations of several financial assets as stocks are formed [2], and project portfolios [3], in which different projects to be undertaken by an organization are combined. However, other situations exhibit a similar structure such as plant allocation [4], combination problems in chemistry [5], fleet composition in transport [6] or land usage planning [7].

As a general term, we will therefore use the term "items" to indicate objects to be combined in a portfolio. Item is a most general term and, depending on the context analyzed, it may represent one of the following alternatives: a project, stocks (or shares in a company) in the financial market, a plant allocation, a chemistry element, and so on. Depending on the problem context, items can be divisible or indivisible. In some portfolio problems, it is possible to have several copies of identical items in a portfolio (e.g. several shares in the same company), or items can be selected once at most. The problems we are considering here deal with indivisible items which can be selected only once (like projects and plant allocation, for instance).

Recently, the multidimensional nature of portfolio problems has been pointed out. Therefore, the paradigm of multicriteria decision analysis (MCDA) provides adequate support for this type of problem. Portfolio problems often involve multiple attributes, for example, risk and return of financial portfolios, or different kinds of benefits to be achieved by project portfolios. A comprehensive literature review related to MCDA in portfolio problems is given in the next section. Our review emphasizes outranking methods for portfolio selection. In this review, the PROMETHEE method has been found as one of the most widely used outranking methods for applications involving the portfolio problematic. Although relatively few publications directly related to portfolio selection based on outranking methods have been found, this kind of method is relevant depending on the context analyzed and considering its non-compensatory nature.

The present paper proposes an approach to deal with portfolio selection based on the PROMETHEE method. PROMETHEE is, to our knowledge, the only outranking method for which a specific variant for portfolio problems has already been introduced in the literature in the form of the PROMETHEE V method [4]. The main problem in applying outranking methods to portfolio problems is that these methods require a pairwise comparison of alternatives, which limit the number of alternatives which can be considered. However, in portfolio problems, each combination of items fulfilling certain constraints is a potential alternative. This leads to a huge

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number of potential alternatives (portfolios). Typical methods for portfolio selection therefore do not explicitly generate all possible portfolios, but try to directly build the optimal portfolio from the set of available items.

This is also the approach taken in PROMETHEE V, which builds an optimal portfolio based on a PROMETHEE ranking of individual items (rather than entire portfolios). One aim of our paper is to compare the solution obtained by this approach to the solution that would be obtained by applying the PROMETHEE method directly to the set of all possible portfolios. The analysis of PROMETHEE V, on which this study is based, also led to the development of an alternative formulation of PROMETHEE V and several alternative approaches, which involve a computational effort not much larger than PROMETHEE V, but could generate solutions which more closely reflect the ranking of all portfolios. Our study therefore has two main contributions: the development of different, computationally "light" methods for portfolio selection based on the PROMETHEE method, and a computational study comparing the quality of solutions obtained with these methods as well as PROMETHEE V.

The paper is organized as follows. The next section presents a literature review on portfolio decision problems using MCDA methods, emphasizing outranking methods for portfolio selection. Section 3 presents the model formulation and derives some analytical results. In Section 4, a comprehensive analysis is conducted based on computational experiments. In Section 5, we discuss the applicability of the methods and the results obtained, and draw some conclusions.

2. Literature review

A review on MCDA methods in financial context is given by Zopounidis and Doumpos [8], including portfolio performance assessment. They point out that the multidimensional nature of this problem has been emphasized by financial researchers, where the MCDA paradigm can provide an appropriate support. This statement is reinforced by Xidonas et al. [9] and by Zopounidis [2], who put a specific focus on the MCDA perspective of portfolio problems. A similar argument is made by Anagnostopoulos and Mamanis [10], who modeled the selection of financial portfolios. Zopounidis [2] deals with the contribution of MCDA in financial decision problems, including an extensive bibliography on the subject.

A second important application of multicriteria portfolio problems is the selection of project portfolios. Stummer and Heidenberger [11] provide a comprehensive list of different criteria and developed an interactive multiobjective model for project portfolio selection, which was later extended to a group decision procedure [12]. A metaheuristic approach to solve this problem was developed by Carazo et al. [13].

Not much literature has been identified regarding to outranking methods for portfolio selection. A comprehensive literature review on PROMETHEE, including methodologies and applications issues [14], analyses 217 contributions related to PROMETHEE methods from 100 journals and categorizes them. According to that study, only five papers are related to PROMETHEE methods for portfolio selection.

The family of PROMETHEE methods [15] is a set of outranking based approaches for multicriteria problems. The use of PROMETHEE for portfolio problems has been proposed by Brans and Mareschal [4], which is known as the PROMETHEE V method.

A number of applications using PROMETHEE V can be found in the literature. Abu-Taleb and Mareschal [3] conducted an application for the PROMETHEE V method to evaluate water resource projects. Al-Kloub and Abu-Taleb [16] use PROMETHEE V for project portfolios in the water resource context. A leakage management strategy of water distribution networks has been analyzed by Morais and de Almeida [17] using PROMETHEE V in a group decision context.

Some studies have applied multicriteria methods to portfolio problems in the context of financial markets. Bouri et al. [18] have conducted a study using PROMETHEE II and V to select attractive portfolios under the investors' constraints. Hababou and Martel [19] dealt with the context of pension funds. They analyzed the ELECTRE family and the PROMETHEE method, selecting the latter, once discordance indices and veto thresholds were not necessary. Another study [20] deals with financial portfolios, analyzing performance and risk of private pension funds, using the PROMETHEE method.

Some studies are focused on methodological issues, rather than applications. Fernandez-Castro and Jimenez [21] present a methodological contribution related to PROMETHEE V, suggesting that some constraints are soft and propose that some coefficients are estimated by fuzzy numbers. Mavrotas et al. [22] use an alternative approach to PROMETHEE V in a financial context. Instead of solving the knapsack problem as in PROMETHEE V, they conduct a parametric solution procedure.

Although PROMETHEE methods contribute a specific approach to deal with portfolios, other outranking methods have also been considered for portfolio problems. Martel et al. [23] applied ELECTRE methods to portfolio comparisons. Gladish et al. [24] proposed a fuzzy model with ELECTRE I method. Xidonas et al. [9] employ the ELECTRE Tri method, considering that the amount to be invested in each item may change, representing a decision variable. This kind of problem is also pointed out by Hurson and Zopounidis [25] and Zopounidis and Doumpos [8].

Some further studies are not directly related to the most common portfolio problems (projects and financial), instead considering another kind of items, for instance plant allocation [4]. In another study, Nikolic et al. [5] applied PROMETHEE V in the environmental context, concerned with the ranking of copper concentrates according to their quality. Araz et al. [26] used PROMETHEE to approach a portfolio problem in the context of supplier selection for outsourcing.

3. Model formulation

In this section, we develop our model and derive some general results before proceeding to a more comprehensive analysis based on computational experiments in the next section. First, we introduce the necessary notation.

3.1. Definitions and notation

We consider the problem of creating a portfolio of items. The set of available items is denoted by $\mathbf{A} = \{A_1, \ldots, A_n\}$, where each individual item A_i is characterized by two vectors:

- resource usage $(r_{i1}, \ldots, r_{iQ})$ where r_{iq} is the consumption of resource q by item i;
- other attributes $(a_{i1}, \ldots, a_{iK})$, which represent the benefits of the item.

In evaluating a portfolio, the decision maker is only interested in its benefits. Resource usage is only considered as a constraint on the construction of portfolios.

A *portfolio* S is a subset $S \subseteq \mathbf{A}$ of several items. We assume that in terms of resource usage, there are no synergies between items. Therefore, resource consumption $R_q(S)$ of a portfolio S is the sum of resource requirements of all items included in the portfolio:

$$R_q(S) = \sum_{i \in S} r_{iq} \tag{1}$$

For each resource q, the decision maker has a budget limit R_q. A portfolio S is *feasible* if

$$\sum_{i \in S} r_{iq} \le R_q \quad \forall q \tag{2}$$

The set of all feasible portfolios is denoted by F. We call a portfolio a *boundary portfolio* if it is feasible and it is not possible to add any item to the portfolio without violating a resource constraint:

$$\forall i \notin S : \exists q : R_q(S) + r_{iq} > R_q \tag{3}$$

The benefits of a portfolio depend on the benefits of the items it contains. The relationship between benefits of individual items and the benefits of the entire portfolio can be quite complex. In general, we can distinguish two cases:

- *monotonic* attributes, in which the addition of another item to the portfolio will lead to an increase (or at least not to a decrease) in the benefit of the entire portfolio; and
- *non-monotonic* attributes, for which adding another item might decrease the benefit of the entire portfolio.

As a specific case of monotonic attributes, we consider additive aggregation, where the benefit of the portfolio is the sum of benefits of its items:

$$V_k(S) = \sum_{i \in S} a_{ik}, \quad a_{ik} \ge 0 \tag{4}$$

As an example for non-monotonic attributes, we consider the minimum operator (a chain is as strong as its weakest link):

$$V_k(S) = \min_{i \in S} a_{ik} \tag{5}$$

We base our evaluation of portfolios on the PROMETHEE method for multiattribute decision making. Since PROMETHEE is already extensively documented in the literature [27–29], we do not describe it here in detail.

For simplicity, we only consider criteria with a linearly increasing preference function between an indifference threshold t_{ind} and a strict preference threshold t_{pref}, i.e. type V according to the classification by Brans et al. [27]. For the following models, we slightly extend the usual terminology of PROMETHEE. We define the *flow* between two alternatives A_i and A_j in the usual way:

$$\Phi_{ij} = \sum_k w_k f(a_{ik} - a_{jk}) \tag{6}$$

where f is a linear preference function defined as follows:

$$f(d) = \begin{cases} 0 & d \le t_{ind} \\ (d - t_{ind})/(t_{pref} - t_{ind}) & t_{ind} < d \le t_{pref} \\ 1 & d > t_{pref} \end{cases} \tag{7}$$

and w_k is the weight assigned to attribute k by the decision maker. PROMETHEE then defines the *inflow, outflow* and *net flow* of an alternative as

Outflow: $\Phi_i^+ = \sum_j \Phi_{ij}$
Inflow: $\Phi_i^- = \sum_j \Phi_{ji}$
Net flow: $\Phi_i = \Phi_i^+ - \Phi_i^-$.

We extend these definitions and also define a net flow between two alternatives (rather than between one alternative and all the other alternatives) as

$$v_{ij} = \Phi_{ij} - \Phi_{ji} \tag{8}$$

Note that $v_{ij} = -v_{ji}$.

In theory, the PROMETHEE method could be applied to the entire set F of feasible portfolios. In practical problems, the number of feasible portfolios might be too large for such an analysis. Nevertheless, we consider the ranking obtained in this manner as the "true" ranking of all portfolios, and use it as a benchmark to which we compare all other models.

To create a ranking of portfolios, we have to apply the PROMETHEE method to portfolios rather than to alternatives. We define the flow between two portfolios n and m as

$$P_{nm} = \sum_k w_k f(V_{nk} - V_{mk}) \tag{9}$$

where V_{nk} is the aggregate value of portfolio n in attribute k computed according to (4) or (5), depending on the type of attribute k.

The portfolios are then ranked according to their net flows, which are calculated as

$$P_n = \sum_{m \in F} P_{nm} - \sum_{m \in F} P_{mn} \tag{10}$$

Proposition 1. *If all attributes are aggregated by monotonic aggregation, the best portfolio according to* (10) *will be a boundary portfolio. If there are several optimal portfolios, at least one of them is a boundary portfolio.*

Proof. Assume that portfolio n is not a boundary portfolio. That means a portfolio m exists, which contains all items in n, plus at least one more item. Since all attributes are monotonic, portfolio m will not be worse than portfolio n in any attribute. Thus two cases are possible: (i) portfolio m dominates portfolio n, or (ii) they have identical values in all attributes. Since in PROMETHEE a dominating alternative is always preferred over the alternative it dominates, n cannot be optimal if it is dominated by m. If the two portfolios have the same values in all attributes, and n is optimal, then m is optimal, too, and we have at least one optimal portfolio which is also a boundary portfolio. □

Proposition 1 hints at a possible simplification of the portfolio selection problem. Instead of considering all portfolios, it could be sufficient to only consider the set of boundary portfolios. However, the evaluation according to (10) is based on the net flow of a portfolio with respect to all other portfolios, not just with respect to other boundary portfolios. It has been shown [30] that a PROMETHEE ranking can change when additional alternatives are considered. Therefore, an interesting question for our computational study is whether a ranking that only compares boundary portfolios to each other will be similar to a ranking of boundary portfolios based on comparison to all portfolios.

3.2. Item-based optimization models

Although it might be possible to restrict the search for the optimal portfolio to the set of boundary portfolios, the number of portfolios that need to be compared can still be huge. Therefore, we consider models which attempt to directly construct the optimal – or at least a very good – portfolio, based only on data from individual items in this section.

The starting point for this model is a PROMETHEE analysis of the individual items. We denote the net flow of item i with respect to the other items by Φ_i.

Maximizing the net flow of all items contained in the portfolio is a quite natural approach to constructing the optimal portfolio. This approach is taken in the PROMETHEE V method [4]. Let x_i be a binary variable indicating whether item i is included in the portfolio. The optimal portfolio can then be determined by solving the following binary optimization problem:

$$\max \quad \sum_i \Phi_i x_i$$
$$\text{s.t.} \quad \sum_i r_{iq} x_i \le R_q \quad \forall q$$
$$x_i \in \{0,1\} \tag{11}$$

The net flows Φ_i used in model (11) are based on a pairwise comparison of all items. Therefore, an item is considered to contribute much to the objective function of model (11) even if it has a large net flow (i.e. performs very well) only with respect to the other items already contained in the portfolio, but has a very low (or even negative) net flow with respect to the items not contained in the portfolio. Therefore, it might seem to be more appropriate to only consider the flow from items in the portfolio to items outside the portfolio in determining the optimal portfolio.

This problem can also be formulated as a binary linear optimization problem. Let y_{ij} denote a binary variable which is set to one if $A_i \in S$ and $A_j \notin S$. Otherwise, y_{ij} is set to zero.

The objective is to maximize the net flow from items in S to those not in S, therefore the objective function can be formulated as

$$\max \sum_i \sum_j v_{ij} y_{ij} \tag{12}$$

To ensure consistency of the binary variables in the model, $y_{ij}=1$ if and only if $x_i=1$ and $x_j=0$. Since v_{ij} may be positive or negative, y_{ij} must be controlled using lower and upper bounds. If v_{ij} had only positive values, upper bounds would be sufficient.

This linkage between y_{ij}, x_i and x_j can be established via the following constraints:

$$y_{ij} \le x_i \tag{13}$$

$$y_{ij} \le 1-x_j \tag{14}$$

$$y_{ij} \ge x_i - x_j \tag{15}$$

$$y_{ij} \ge 0 \tag{16}$$

Eqs. (13) and (14) together provide an upper bound $y_{ij} \le \min(x_i, 1-x_j)$, which guarantees that $y_{ij}=1$ is only possible if $x_i=1$ and $x_j=0$. However, y_{ij} could still be zero in that case, which would happen if $v_{ij}<0$. This is avoided by the third condition, which provides a lower bound. Since y_{ij} is always forced to be zero or one by constraints (13)–(15), it is not necessary to define it explicitly as a binary variable.

The entire model then becomes

$$\max \quad \sum_i \sum_j v_{ij} y_{ij}$$
$$\text{s.t.} \quad y_{ij} \le x_i$$
$$y_{ij} \le 1-x_j$$
$$y_{ij} \ge x_i - x_j$$
$$\sum_i x_i r_{iq} \le R_q \quad \forall q$$
$$x_i \in \{0,1\} \tag{17}$$

Although models (11) and (17) are based on different concepts, they will lead to the same solution, as shown in the following proposition:

Proposition 2. *Models* (11) *and* (17) *will generate the same solution.*

Proof. We can rewrite the net flow of item i as

$$\Phi_i = \Phi_i^+ - \Phi_i^- = \sum_j \Phi_{ij} - \sum_j \Phi_{ji} = \sum_j v_{ij} \tag{18}$$

Consider an arbitrary portfolio S. Model (11) evaluates this portfolio as

$$\sum_{i \in S} \Phi_i \tag{19}$$

and model (17) evaluates it as

$$\sum_{i \in S} \sum_{j \notin S} v_{ij} \tag{20}$$

The values of these two objective functions are identical. Using (18), we can rewrite (19) as

$$\sum_{i \in S} \sum_j v_{ij} = \sum_{i \in S} \left[\sum_{j \notin S} v_{ij} + \sum_{j \in S} v_{ij} \right] \tag{21}$$

The first term of this equation is equal to (20). The second term is zero. Both i and j enumerate the same items. The term thus represents the sum across a matrix, in which the lower diagonal matrix is the negative of the upper diagonal matrix. For each v_{ij} in the upper diagonal matrix there is the corresponding $v_{ji} = -v_{ij}$ in the lower diagonal matrix and vice versa. Since v_{ii} is also zero for all i, the total sum is zero.

The constraints of both problems describe the same set of feasible portfolios. Since they maximize the same function over the same feasible set, the two models are equivalent. □

Programs (11) and (17) do not necessarily generate boundary portfolios. The reason for this can best be explained considering model (17): if the net flow of the items already contained in a portfolio to another item is larger than the net flow of that item to the items not in the portfolio, the total effect of adding this item would be negative. In model (11), the same situation is reflected by a negative coefficient of the corresponding variable in the objective function, which causes the variable to remain at zero and the item not to be included in the portfolio. Therefore it is possible that the inclusion of items into the portfolio stops before a boundary portfolio is reached. This situation cannot be corrected by altering the objective function coefficient (e.g. omitting the flow from items in the portfolio to the additional item), since this depends on the set of items already contained in the portfolio, which is not known in advance.

To overcome this problem, we propose to solve model (11) (or alternatively, problem (17)) for a fixed number of items c, and systematically vary c. Model (11) is thus extended to

$$\max \quad \sum_i \Phi_i x_i$$
$$\text{s.t.} \quad \sum_i r_{iq} x_i \le R_q \quad \forall q$$
$$\sum_i x_i = c$$
$$x_i \in \{0,1\} \tag{22}$$

We call a portfolio which provides the optimal solution to (22) a *c-optimal* portfolio for a given value of c. Obviously, since portfolios are constrained by available resources and only a finite number of items is available, there is an upper limit to c. This upper limit is denoted by $c_{\max} \le |\mathbf{A}|$.

An interesting question is how the c-optimal portfolios change when c is increased.

Conjecture 1. *Consider the set of c-optimal solutions. There is a critical value c^* so that portfolios for $c < c^*$ are not boundary portfolios and the following condition holds for all $c < c^*$: $S^*_{c-1} \subset S^*_c$. (i.e. when increasing c by one, one item is added, all other items remain in the portfolio).*

It is quite obvious that a value c^* must exist. It could be one, if the first c-optimal portfolio (for $c=1$) is already a boundary portfolio. But it is not possible that the set of c-optimal portfolios does not contain any boundary portfolio. From the definition of a boundary portfolio, if the c-optimal portfolio for a given c is not a boundary portfolio, it is possible to add at least one more item to the portfolio, and thus a c-optimal portfolio for $c+1$ must exist. Since c is bound by c_{max}, at least the last portfolio must then be a boundary portfolio. It is quite plausible that in such cases, the next solution is obtained by simply adding one item to the portfolio.

The c-optimal portfolio for c^* could be considered a good solution to the portfolio problem. It can also be obtained with little effort by solving only a few binary linear programming problems.

However, it is still possible that adding more items to the portfolio beyond c^* improves the portfolio further. Therefore, we consider two more solutions in our computational analysis. One is the portfolio obtained for c_{max}, i.e. the solution to problem (22) containing the largest number of items. Furthermore, we consider the best portfolio according to a PROMETHEE ranking of the entire set of c-optimal portfolios, also including the solutions for $c > c^*$. While these two solutions require to solve a few more optimization models, the total effort is still very small compared to an analysis of all portfolios, and even to a complete PROMETHEE analysis on the set of all boundary portfolios.

4. Computational model

4.1. Research questions

In the preceding section, we have proposed several approaches how the PROMETHEE method could be used for portfolio selection problems without performing a PROMETHEE ranking of the entire set of feasible portfolios. In increasing order of computational effort, these methods are:

1. the PROMETHEE V method (BestFlow), which requires solving model (11) once;
2. the c-optimal portfolio for $c = c^*$ as defined in Conjecture 1 (cStar), which requires solving model (22) for increasing values of c until a boundary portfolio is found;
3. the c-optimal portfolio containing the largest number of items, i.e. the maximum value of c (cMax), which requires solving model (22) for all possible values of c;
4. the best portfolio in a PROMETHEE ranking of all c-optimal portfolios (cRank), which in addition to 3, requires a PROMETHEE ranking of all c-optimal portfolios;
5. the best portfolio in a PROMETHEE ranking of all boundary portfolios (BestBound).

We developed a computational model to compare those five solutions. As a benchmark, we used the PROMETHEE ranking of all feasible portfolios. Since this requires a pairwise comparison of all feasible portfolios, the size of problems analyzed in this study was necessarily limited, even though in a study like this one, much longer computation times can be accepted than in actual applications in a decision support system. However, the results obtained in moderately sized problems used for this computational study can guide the choice of models to be used in actual decision support for larger problems.

As an additional benchmark, we use the best portfolio in the set of c-optimal portfolios according to the ranking of all portfolios (cBest). Although this portfolio cannot be determined without establishing a ranking of all portfolios, it is a useful indicator about how good the portfolios in the set of c-optimal portfolios can become, and whether it might make sense to present the entire set of c-optimal portfolios to the decision maker.

Therefore, the main research question to be studied using our model is:

RQ1: How are the solutions identified in Section 3, and the best c-optimal portfolio (as a benchmark), ranked in a PROMETHEE ranking of all portfolios?

Since the model allows us to study the portfolio selection problem in detail, it can also be used to answer additional questions. The next two questions concern the set of boundary portfolios and its relationship to the entire set of feasible portfolios.

RQ2: What is the fraction of boundary portfolios in the set of all feasible portfolios, and how well does a PROMETHEE ranking of all boundary portfolios agree with their ranking in the set of all feasible portfolios?

The first part of RQ2 is obviously interesting if we want to use the best boundary portfolio as an approximation to the best portfolio selection problem. Selecting the best boundary portfolio is computationally tractable, only if the set of boundary portfolios is considerably smaller than the entire set of feasible portfolios.

It has been shown [30,31] that rank reversals can occur in the PROMETHEE method when additional alternatives are considered. Therefore, the ranking of boundary portfolios within the entire set of feasible portfolios is not necessarily the same as a ranking established only among the boundary portfolios. The second part of RQ2 studies whether this could become a severe problem.

RQ3: In the case of non-monotonic attributes, how likely is it that the optimal portfolio is not a boundary portfolio?

As we have already shown in Proposition 1, the optimal portfolio will always be a boundary portfolio in the case of monotonic attributes. But since it is not necessarily the best boundary portfolio, RQ1 asks by how far the optimal portfolio is missed when only a ranking of boundary portfolios is performed. RQ3 complements RQ1 by looking at the problem from a different perspective. Proposition 1 is valid only for monotonic attributes. If attributes are not monotonic, the optimal portfolio is no longer necessarily contained in the set of boundary portfolios, and RQ3 asks how often the true optimal portfolio will be missed by considering only boundary portfolios.

Our last two research questions deal with the set of c-optimal portfolios.

RQ4: Is Conjecture 1 true, i.e. are c-optimal portfolios for $c \leq c^*$ obtained by adding one item to the preceding c-optimal portfolio?

While c-optimal portfolios for $c < c^*$ are by definition not boundary portfolios, the situation for $c > c^*$ is less obvious. Since boundary portfolios are likely to be good portfolios, it is to be expected that these portfolios will often be boundary portfolios, but there is no guarantee that this will always be the case. Therefore we formulate another research question:

RQ5: How likely is it that c-optimal portfolios for $c > c^*$ are boundary portfolios?

4.2. Model design and parameters

The computational model operated on randomly created portfolio problems, which were parameterized according to the number of items, the number and types of attributes involved and the number and limits of resources. For a given problem dimension, the model first generates the outcome values and resource usage of items by drawing random values from a uniform distribution in the range from zero to one. The model allows for both equal weights and randomly generated weights of attributes. Both variants were tested, but no significant differences in results were found (Table 1).

After generating a problem instance, all feasible portfolios for the problem instance were generated. Portfolio generation used a search tree similar to the approach of Vetschera [32]. At each level of the tree, a decision about adding one item to the portfolio is made. Since resource requirements of a portfolio are assumed to be additive, the tree can be truncated as soon as a resource limit is reached, thereby eliminating most infeasible portfolios. To keep the model computationally tractable, a limit of 300,000 feasible portfolios was imposed and a problem was dropped if this limit was exceeded. The limit was reached only in a few instances in the largest class of problems analyzed (cf. Table 2).

Once all feasible portfolios were generated, the boundary portfolios were identified. This analysis requires an effort which is linear in the number of portfolios and the number of items, since for each portfolio it has to be checked whether it would be possible to add any other item without violating constraints.

The model then simultaneously performed a PROMETHEE ranking of all feasible portfolios and the ranking among boundary portfolios. This was the most computationally demanding part of the model, since PROMETHEE requires a pairwise comparison of portfolios. Thus, for 300,000 portfolios, 90,000,000,000 pairwise comparisons had to be performed. Because of the amount of the data, each portfolio was evaluated individually and only the net flow of each portfolio was stored. Intermediate results could not be stored, since a flow matrix between all portfolios would have required to store 90 billion elements. However, the net flow among boundary portfolios could simultaneously be calculated by adding only the flows from/to other boundary portfolios.

In the next step, all portfolios were tested for efficiency by pairwise comparison among portfolios. Finally, a PROMETHEE ranking of all individual items was performed to calculate their net flow, and the objective value of model (17) was calculated to simultaneously determine the set of c-optimal portfolios. This corresponds to solving model (17) by complete enumeration. Finally, a PROMETHEE ranking of the c-optimal portfolios was performed.

The model was implemented in Object Pascal using the Open Source FreePascal compiler (http://www.freepascal.org). The program can be obtained upon request from the authors.

Computational experiments were performed in a full factorial design using parameter settings in Table 1.

All problems contained five attributes and one resource. Three hundred experiments were performed for each setting involving 10 and 15 items. For the settings involving 20 items, only 100 experiments were performed since they used excessive computation time. In some rare cases, not all 100 experiments could be performed because the number of portfolios would have exceeded the limit. The actual number of experiments that could be carried out in these cases is shown in Table 2.

To obtain information about the computational effort needed to generate the different solutions, we measured the times needed for the following four tasks separately: (i) generating the portfolios (including identification of boundary portfolios), (ii) the PROMETHEE ranking of all portfolios, (iii) determining the efficient portfolios, and (iv) calculating all the c-optimal portfolios by complete enumeration and the PROMETHEE ranking of these portfolios. All experiments were run on a PC with Intel E6550 processor running at 2.33 GHz. Timing results are shown in Fig. 1.

Generation of portfolios (represented as circles in the left part of Fig. 1) usually was completed in less than one-tenth of a second and never exceeded half a second. With the exception of 8 out of 7700 cases, finding the portfolio maximizing the sum of net flows of items (represented by triangles in the same figure) was also completed in less than half a second. Longer running times in the eight exceptional cases, which took up to 50 s, probably were caused by some hard disk activity not related to the experiment. Running times for both activities increased approximately linear in the number of portfolios. In contrast, both the PROMETHEE analysis of portfolios (circles in the right part of Fig. 1), as well as determination of efficient portfolios (triangles in the same graph), showed the expected quadratic increase in running times, which ranged in thousands of seconds for larger problems. The quadratic increase in running times for the PROMETHEE analysis also clearly indicates that time can drastically be decreased by considering only boundary portfolios, which can be generated very rapidly.

Table 1
Parameter settings for computational experiments.

Parameter	Settings
Items	10, 15, 20
Resource limit	2, 2.5, 3
Attribute types	All additives, three additives +2 min
Weights	Equal, random

Table 2
Cases with less than 100 experiments.

Items	Weights	Attributes	Resource limit	Actual experiments
20	Random	Mixed	3	97
20	Equal	Mixed	3	95
20	Random	Additive	3	97
20	Equal	Additive	3	95

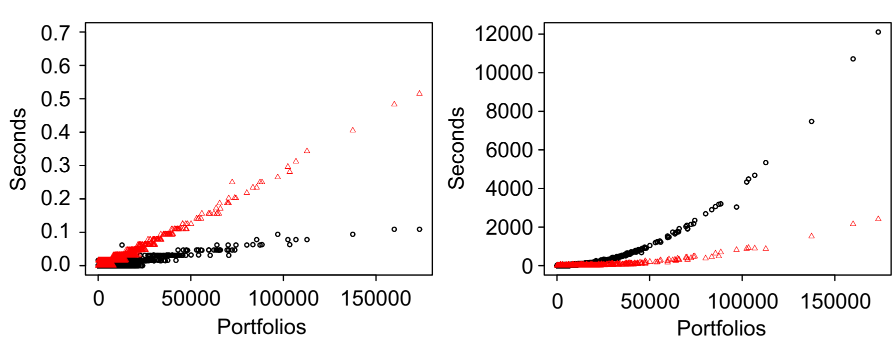

Fig. 1. Timing results for portfolio generation and item models (left), and PROMETHEE ranking and efficiency test of portfolios (right).

4.3. Results

4.3.1. Solutions

Research question RQ1 deals with the performance of the different solutions proposed in Section 3. To evaluate performance, we consider the relative rank of the portfolios indicated by each solution in the PROMETHEE ranking of all feasible portfolios. To correct for differences in the number of feasible portfolios, we use the relative rank, i.e. the rank divided by the total number of feasible portfolios. A relative rank of 1% would therefore indicate that a solution is located in the top 1% of all feasible portfolios.

Table 3 shows the ranks of the portfolios generated by the different solution methods for the cases of additive and mixed attributes. A non-parametric Wilcoxon test indicates that all differences between methods shown in this table are significant at $p < 0.1\%$. Figs. 2 and 3 further differentiate these results according to equal vs. random weighs, the number of items and the resource limits used in each run.

In the case of additive attributes, there is a clear and consistent ranking of methods. The PROMETHEE V portfolio, which maximizes the net flow of items, on average fails to reach the top 10% of all feasible portfolios. Considering the median, which is more robust against outliers, it still does not reach the top 5%. In contrast, the portfolio which is obtained for c^* is within the top 3%, and the portfolio which is best according to a PROMETHEE ranking of the c-optimal portfolios even reaches the top 1% of all feasible portfolios. The c-optimal portfolio containing the largest number of items almost reaches the best among the c-optimal portfolios. These portfolios even come close to the best boundary portfolio, which takes considerably more effort to identify.

For the case of mixed attributes, the picture is less clear. Except for cMax, all solutions considered are on average within the best 5% of all portfolios, and the median value is below 3%. The overall results in Table 3 still show a similar ranking of solution methods. However, the more detailed analysis in Fig. 3 indicates that for several problem types, the c^* portfolio performs worse than the PROMETHEE V portfolio. The best-ranked among the c-optimal portfolios in most cases still outperforms these two solutions, but is considerably worse than both the best boundary portfolio, and the best among the c-optimal portfolios. The biggest change occurs for the largest c-optimal portfolio, which now consistently performs considerably worse than all the other solutions. This is not surprising, since in the case of non-monotonic attributes, adding more items to the portfolio is no longer guaranteed to improve the portfolio. These results show that in the case of mixed monotonic and non-monotonic attributes, the set of c-optimal portfolios still contains very good solutions, but it can be difficult to determine which of the c-optimal solutions is actually best.

Table 3
Ranks of the different solutions (% of all feasible portfolios).

Statistic	BestFlow	cBound	cRank	cMax	BestBound	cBest
Additive attributes						
Mean	11.21	2.85	0.39	0.31	0.14	0.23
Median	6.67	0.52	0.12	0.11	0.07	0.10
SD	12.78	6.71	0.80	0.73	0.23	0.47
Mixed attributes						
Mean	4.84	3.91	2.50	6.35	0.95	0.73
Median	1.68	1.33	0.99	2.97	0.21	0.29
SD	8.13	7.16	4.01	10.11	3.02	1.38

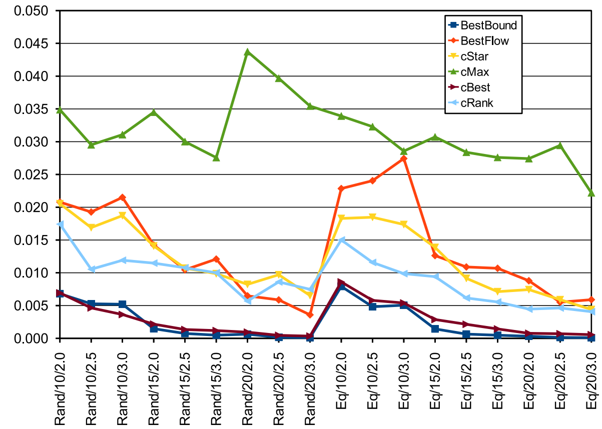

Fig. 3. Median ranks of solutions, mixed attributes.

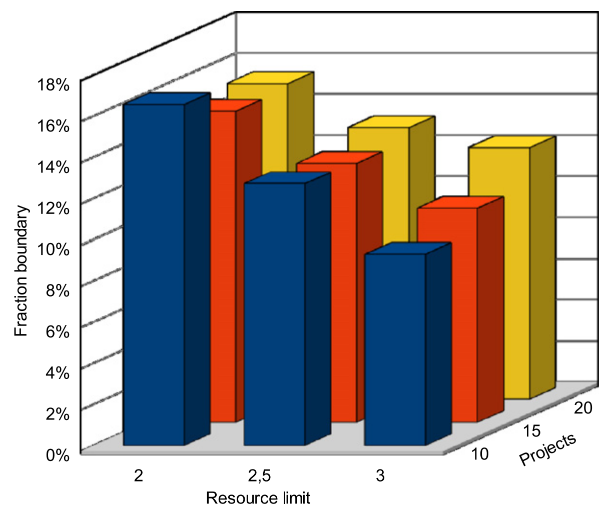

Fig. 4. Fraction of boundary portfolios.

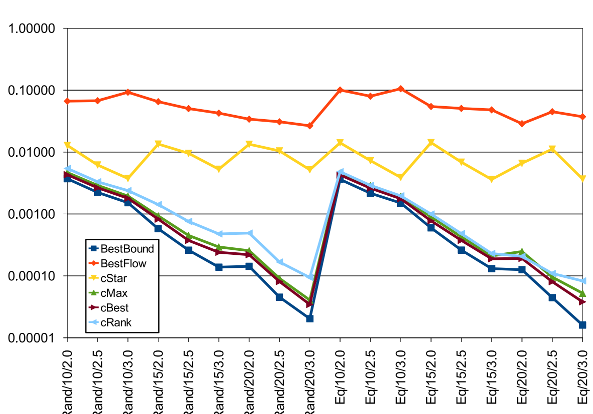

Fig. 2. Median ranks of solutions, additive attributes.

4.3.2. Boundary portfolios

Research questions RQ2 and RQ3 consider the boundary portfolios. Across all experiments, 12.68% of all feasible portfolios were boundary portfolios. As Fig. 4 shows, this fraction varies for the different parameter settings. It generally decreases when resource limits are relaxed. The impact of the number of items varies for different resource limits. In the more constrained settings, there are fewer boundary portfolios in the larger problems. This effect is reversed for the less constrained problems. The non-parametric Kruskal–Wallis test indicates that the impact of resource limits on the fraction of boundary portfolios is highly

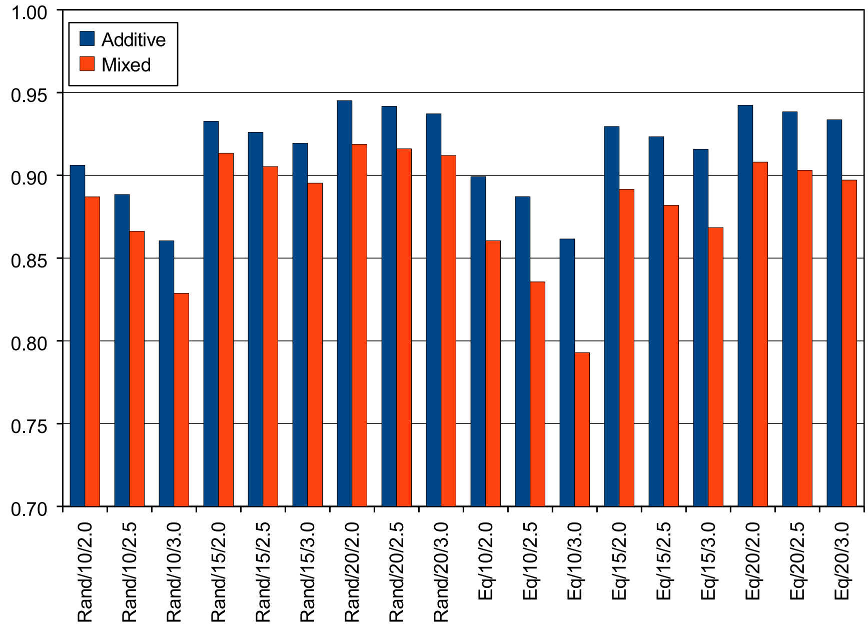

Fig. 5. Correlation coefficients.

significant ($p < 0.0001$) for all problem sizes. Problem size has no significant effect for the tightly constrained problems, for a resource limit of 2.5, the effect is weakly significant ($p = 0.0022$) and for the least constrained problem with a resource limit of 3, it is highly significant ($p < 0.0001$).

The second part of RQ2 concerned the ranking among boundary portfolios and its consistency with the ranking of all portfolios. Consistency could be measured using two approaches: since net flows are measured on a cardinal scale, Pearson correlation coefficients could be calculated. But since we are mainly concerned with the ranking of portfolios, we used the Kendall Tau coefficient of rank correlation between portfolios instead.

Fig. 5 shows the average values of this coefficient across all parameter settings. In most cases, the average is above 0.8, indicating a rather strong positive correlation. Thus we can conclude that on average, the ranking of boundary portfolios obtained from comparing only the boundary portfolios to each other is rather similar to the ranking of boundary portfolios within the entire set of feasible portfolios. This confirms the results of Verly and de Smet [33], who also found that rank reversals in PROMETHEE are very infrequent.

To test the statistical significance of the correlation coefficients, we used the normal approximation of the distribution of Kendall correlation coefficients [34] and calculated the corresponding p values. The correlation coefficients failed to reach significance at $p < 0.001$ only in few rare cases for small problems containing 10 items, as shown in Table 4.

Since we want to determine whether a ranking of boundary portfolios can be used to identify the best portfolio, it is also important to consider extreme cases in which the rankings most strongly disagree. Fig. 6 shows the distribution of correlation coefficients, grouped by resource limit and number of items. Very low correlations are obtained only in problems involving 10 items. This is quite plausible, since these problems also involved a smaller number of portfolios, so a deviation in the ranking of a few portfolios could have a considerable negative impact on the correlation. However, the opposite effect can be observed with respect to the resource limit. Problems with less restricted resources, in which there are more portfolios, do exhibit instances of very low correlations, while problems with tight resource constraints do not.

Table 4
Number of cases (out of 300 experiments) in which Kendall correlation coefficients did not reach significance at $p < 0.1\%$.

Weights	Unequal		Equal	
Attributes	Additive	Mixed	Additive	Mixed
Res. limit				
2	0	0	0	0
2.5	0	3	0	1
3	1	4	1	7

Our analysis of RQ1 has already shown that the best boundary portfolio performs on average very well in terms of the ranking of all portfolios. RQ3 now studies whether the best overall portfolio is likely to be a boundary portfolio in cases with mixed attributes, where this is not true in general. Surprisingly, across all experiments in this setting, the best overall portfolio was a boundary portfolio only in 58.9% of all experiments. This means that in over a third of the cases, the best portfolio could not be found by considering only boundary portfolios. However, as we have already seen in RQ1, the best boundary portfolio usually was not far from the optimum, even when the true optimum was missed.

To test whether this fraction is significantly influenced by problem parameters, we performed proportional tests between subgroups of the data formed by the number of items, and the resource limit, respectively. The results of these tests are summarized in Table 5. The fraction of cases in which the optimal portfolio was a boundary portfolio remains almost constant when the number of items is varied, while increasing the resource limit leads to a weakly significant decline.

4.3.3. *c-optimal portfolios*

Our last two research questions deal with the set of c-optimal portfolios. First, we consider portfolios obtained for $c \le c^*$

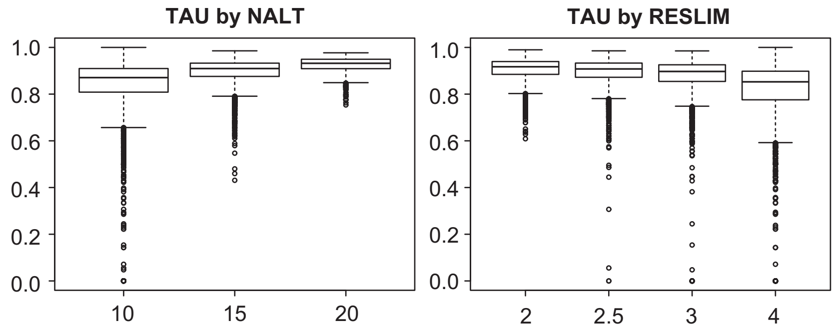

Fig. 6. Correlation between rankings of boundary portfolios and all portfolios, by the number of items (left) and resource limit (right).

Table 5
Effect of resource limit and number of items on the number of cases in which optimal portfolio is a boundary portfolio.

Resource limit	2.0	2.5	3.0	
Fraction (%)	61.57	59.86	55.24	$p=0.0021$
Items	10	15	20	
Fraction (%)	58.67	59.33	58.28	$p=0.8715$

and analyze whether they are actually formed according to Conjecture 1.

As Fig. 7 shows, the conjecture is not fulfilled in all cases. On average across all experiments, it is violated in roughly 20% of all cases. Finding a c-optimal portfolio is still a combinatorial optimization problem. In particular for higher values of c, the optimal solution might be close to being a boundary portfolio for $c+1$. Thus the optimal solution for $c+1$ can involve a change in the composition of the entire portfolio. A proportion test indicates that significant influences exist of both the number of items and the resource limit on the fraction of cases in which Conjecture 1 is fulfilled. It should be noted that Fig. 7 does not differentiate between experiments with additive attributes and experiments with mixed attributes. Both cases must lead to the same results concerning Conjecture 1. Since model (22) uses only the net flow between items, and not flows between portfolios, its optimal solution is not influenced by the way in which evaluations of portfolios are calculated. This could explain the relatively bad performance of this model when compared to the actual ranking of all portfolios, in which the aggregation method plays an important role.

Our last research question deals with the other part of the set of c-optimal portfolios, the subset for which $c > c^*$. As Fig. 8 shows, it is not always the case that these portfolios are all boundary portfolios, although this is true in a large majority of cases.

5. Conclusions

Multicriteria portfolio problems are often approached using compensatory methods, such as additive aggregation procedures. However, there are several practical situations in which non-compensatory approaches based on outranking relations seem more appropriate. This is the case if the decision maker has a preference structure which by nature is non-compensatory, or is unable or unwilling to establish trade-offs required to specify the parameters for compensatory methods. Such situations have been reported in several applications. Comparisons of the use of outranking procedure with multiattribute utility theory have been done for example in contexts such as risk analysis of natural gas pipelines [35] and the selection of outsourcing contracts [36].

In the present study, we have considered the case of non-compensatory preferences in the context of portfolio problems, with a specific focus on the PROMETHEE family of methods. In this context, PROMETHEE V has been proposed as a technique for portfolio evaluation, which involves only moderate computational requirements. The aim of this study was to elaborate more on the trade-off between computational requirements on the one hand, and the quality of the solutions obtained in relation to the (often computationally impossible) PROMETHEE ranking of all portfolios on the other hand. Our study can therefore also be seen as an evaluation of the benefits of spending some additional effort above that required by PROMETHEE V.

We have used a computational model for this comparison. The use of computational models can always be viewed as a limitation on the generalizability of such studies, since only specific settings can be analyzed. To overcome this limitation, we have considered both cases of equal and random weights, where we found no significant differences. We can therefore consider our results to be quite robust with respect to the structure of weights involved.

In terms of other problem characteristics, our study is more focused. In particular, we have concentrated on situations in which benefits on the one hand, and resource requirements on the other hand are strictly separated. Costs, or other resource requirements of portfolios, are not considered as evaluation attributes by the decision maker. The cost for each project is applied only for the budget constraint. In many practical situations involving project portfolio selection, the DM is willing to spend the whole amount (the limit of the cost constraint), in order to implement more alternatives. This seems realistic, when considering that once the DM has a budget, he wants to get the most value of benefit criteria by spending the entire budget.

In such a situation, the idea of a boundary portfolio is particularly appealing. It has led us to introduce the concept of c-optimal portfolios, which in a systematic way allows us to analyze the effects of increasing the size of portfolios by adding more items, which then can lead to the exploration of different boundary portfolios.

The concept of boundary portfolios originated from the consideration of monotonic attributes, where it is particularly natural. However, there are also situations where non-monotonic attributes might be more adequate for a specific criterion. In this case, adding another project might decrease the advantage of the entire portfolio. This kind of criteria do not represent benefits for which the DM has a cumulative preference. However, this is not the most usual situation found in practical problems related to project portfolios. It may be considered for other kind of portfolio, such as chemistry combination. Other practical situation where this could be found is risk evaluation with regard to financial portfolios. To allow for consideration of such portfolio types, we have also included the case of non-monotonic attributes in our computational study.

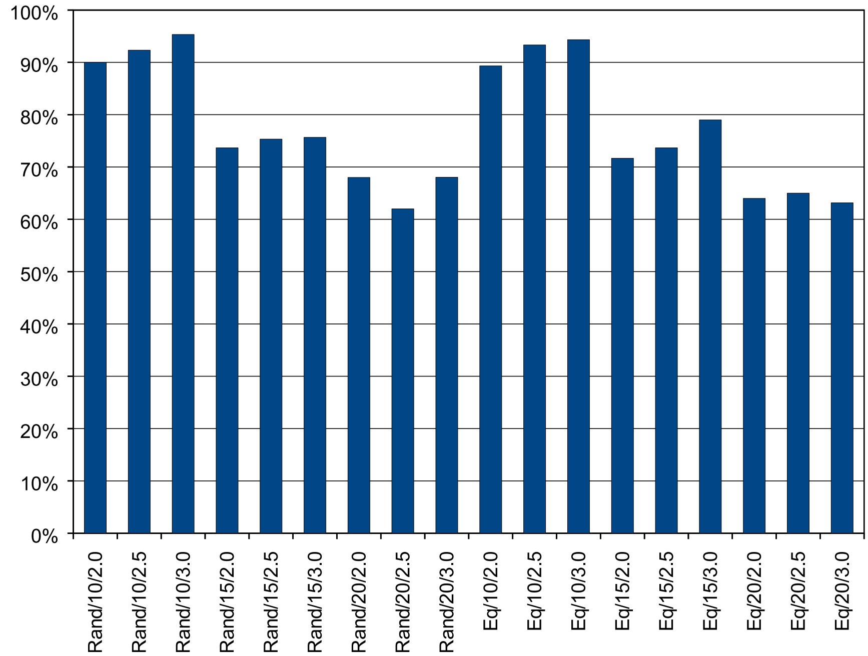

Fig. 7. Fulfillment of Conjecture 1.

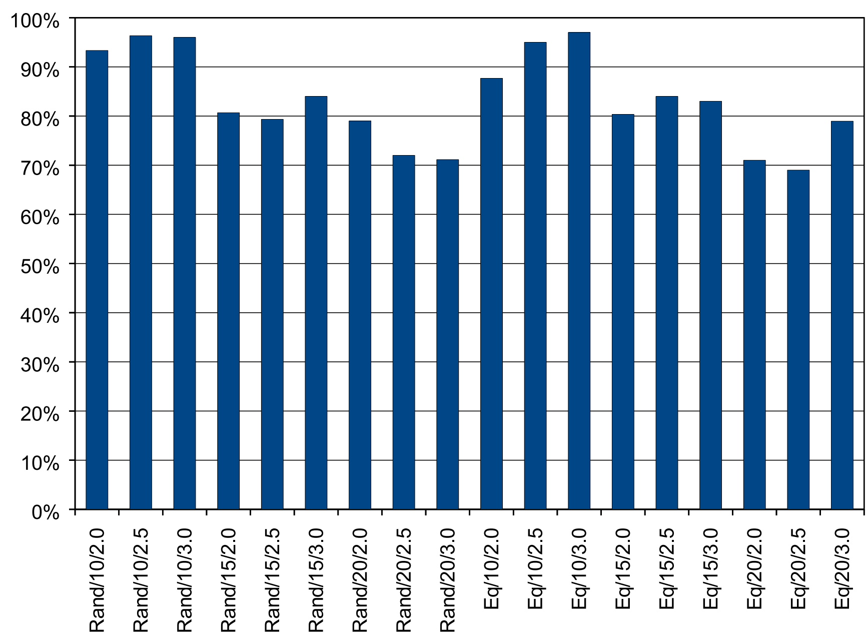

Fig. 8. Fraction of boundary portfolios among c-optimal portfolios for $c > c^*$.

According to our results, the methods we have studied perform best for the case of monotonic attributes. Thus it is reassuring that this is the predominant type of attributes in most practical applications. Nevertheless, we have also found that although the best overall portfolio was not found in the set of boundary portfolio in about 40% of the cases involving non-monotonic attributes, the best boundary portfolio is still not far from the optimum.

Based on the results of our computational study, we can therefore propose a two-level approach how to apply PROMETHEE methods to portfolio problems:

1. If the number of items is not too large, all boundary portfolios should be generated and ranked. Generating all boundary portfolios requires only a minimal computational effort for problems of the size we studied. Since the number of boundary portfolios was found to be approximately one order of magnitude smaller than the total number of feasible portfolios, and PROMETHEE involves a pairwise comparison of all alternatives to each other, this would reduce the computational requirements to about 1% of the effort required to rank all portfolios and still provide a very good approximation of the actual best solution.
2. If the problem is too large to generate and compare all boundary portfolios, at least the set of all c-optimal portfolios should be generated, and a ranking of these portfolios should be performed. While this approach involves a somewhat larger effort than PROMETHEE V, it is still very low compared to the

evaluation of all – or even all boundary – portfolios, and provides a considerably better approximation to the ranking of all portfolios.

This recommendation leads to some important topics for future research: our results have also indicated that often the set of c-optimal portfolios contains portfolios which are better in terms of the overall ranking of portfolios than the one which is best in a ranking just of the c-optimal portfolios. Identifying these "true best" c-optimal portfolios without resorting to the ranking of all portfolios is a challenging topic for future research.

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