Annual oscillation criteria for second-order nonlinear elliptic differential equations

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Abstract

In this paper, we present some new oscillation criteria for the second-order nonlinear elliptic differential equation

\[ \nabla \cdot (A(x) \nabla y) + q(x)f(y) = e(x), \quad x \in \Omega, \]

where \( \Omega \) is an exterior domain in \( \mathbb{R}^N \). These criteria are different from most known ones in the sense that they are based on the information only on a sequence of annuals of \( \Omega \) in \( \mathbb{R}^N \), rather than on the whole exterior domain \( \Omega \). Also information about the distribution of the zero of solutions is obtained.

Keywords: Nonlinear elliptic differential equation; Second order; Oscillation; Annual criteria

1. Introduction

In this paper, we consider the oscillation behavior of solutions to second-order elliptic differential equations of the form

\[ \nabla \cdot (A(x) \nabla y) + q(x)f(y) = e(x), \]

where \( x \in \Omega \), an exterior domain in \( \mathbb{R}^N \). The following notations will be adopted throughout: \( \mathbb{R} \) and \( \mathbb{R}^+ \) denote the intervals \(( -\infty, +\infty )\), \(( 0, +\infty )\), respectively. The norm of \( x \) is denoted by \( |x| = [\sum_{i=1}^{N} x_i^2]^{1/2} \). For a positive constant \( a > 0 \), let \( S_a = \{ x \in \mathbb{R}^N : |x| = a \} \), \( G(a, +\infty) = \{ x \in \mathbb{R}^N : |x| > a \} \), \( G[a, b] = \{ x \in \mathbb{R}^N : a \leq |x| \leq b \} \), \( G(a, b) = \{ x \in \mathbb{R}^N : a < |x| < b \} \). For the exterior domain \( \Omega \) in \( \mathbb{R}^N \), there exists a positive number \( a_0 \) such that \( G(a_0, +\infty) \subset \Omega \).

In what follows, we always assume that:

\[ (C_1) \quad A(x) = (A_{ij}(x))_{N \times N} \text{ is a real symmetric positive definite matrix function (ellipticity condition) with } A_{ij} \in C^{1+\mu}_{\text{loc}}(\Omega, \mathbb{R}), \mu \in (0, 1), i, j = 1, \ldots, N, \lambda_{\text{max}}(x) \text{ denotes the largest (necessarily positive) eigenvalue of the matrix } A(x); \text{ there exists a function } \lambda \in C^1(\mathbb{R}^+, \mathbb{R}^+) \text{ such that } \lambda(r) \geq \max_{|x|=r} \lambda_{\text{max}}(x) \text{ for } r > 0; \]

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(C2) $q, e \in C^2_{\text{loc}}(\Omega, \mathbb{R})$, $\mu \in (0, 1)$ and $q(x) \neq 0$ for $|x| \geq a_0$;
(C3) $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(y)/y \geq K|y|^{v-1}$ for all $y \neq 0$, where $K > 0$ and $v \geq 1$.

A function $y \in C^{2+\mu}_{\text{loc}}(\Omega, \mathbb{R})$, $\mu \in (0, 1)$ is said to be a solution of Eq. (1) in $\Omega$, if $y(x)$ satisfies Eq. (1) for all $x \in \Omega$. For the existence of solutions of Eq. (1), we refer the reader to the monograph [3]. We restrict our attention only to the nontrivial solution $y(x)$ of Eq. (1), i.e., for any $a > a_0$, $\sup\{|y(x)| : |x| > a\} > 0$. A nontrivial solution $y(x)$ of Eq. (1) is called oscillatory if the zero set $\{x : y(x) = 0\}$ of $y(x)$ is unbounded, otherwise it is called nonoscillatory. Eq. (1) is called oscillatory if all its nontrivial solutions are oscillatory.

In the qualitative theory of nonlinear partial differential equations, one of the important problems is to determine whether or not solutions of the equation under consideration are oscillatory. For the semilinear elliptic equation

$$\nabla \cdot (A(x)\nabla y) + q(x)f(y) = 0,$$

the oscillation theory is fully developed by many authors. Noussair and Swanson [7] first extended the Wintner theorem by using the following partial Riccati type transformation:

$$W(x) = -\frac{z(|x|)}{f(y(x))} (A\nabla y)(x),$$

where $z \in C^2$ is an arbitrary positive function. Swanson [9] summarized the oscillation results for Eq. (2) up to 1979. For recent contributions, we refer the reader to [12,13] in which a classical Kamenev theorem [4] is to be extended to Eq. (2). However, as we know Eq. (1) has never been the subject of systematic investigations.

In the case when $N = 1$, Eq. (1) reduces to the following second-order ordinary differential equations

$$r(t)y'' + q(t)y(t) = e(t),$$

$$r(t)y'' + q(t)f(y) = e(t).$$

There are a great number of papers (see, for example, [5,10,2] and the references quoted therein) devoted to Eqs. (4) and (5). The most known oscillation criteria involve $f$ and integral of $q$ and hence require the information of $q(t)$ on the entire half-line $[a_0, +\infty)$. It is difficult to apply them to the cases where $q$ has a “bad” behavior on a big part of $[a_0, +\infty)$.

In 1999, Wong [11] and Kong [6] have, respectively, noted that interval criteria which Ei-Sayed [1] established for oscillation of Eq. (4) are not very sharp, because a comparison with a equation of constant coefficients is used in Ei-Sayed’s proof. Therefore, some other interval criteria for oscillation, that is, criteria given by the behavior of Eqs. (4) and (5) with $e(t) = 0$ only a sequence of subintervals of $[a_0, +\infty)$ are obtained by Wong [11] and Kong [6], respectively.

In 2003, Yang [14] employed the technique in the work of Philos [8] and Kong [6] for Eq. (5), and presented several Interval oscillation criteria for Eq. (5). One of the oscillation criteria of Kamenev’s type in [14] is as follows.

**Theorem A.** Suppose that (C3) hold. Then Eq. (5) with $r(t) \equiv 1$ is oscillatory provided that for each $t \geq a_0$ and for some $\lambda > 1$, the following conditions hold:

(A1) for any $T \geq a_0$, there exist $T \leq a_1 < b_1 \leq a_2 < b_2$ such that

$$e(t) \begin{cases} < 0, & t \in [a_1, b_1], \\ \geq 0, & t \in [a_2, b_2] \end{cases}$$

and $q(t) \not\equiv 0(\neq 0)$, $t \in (a_1, b_1) \cup (a_2, b_2)$

(A2) there exist $c_i \in (a_i, b_i)$ for $i = 1, 2$, such that $T \leq a_1 < b_1 \leq a_2 < b_2$ and the following inequalities hold for $i = 1, 2,$

$$\frac{1}{(c_i - a_i)^{\lambda - 1}} \int_{a_i}^{c_i} (s - a_i)^{\lambda} |e(s)|^{1 -(1/\nu)} |Kq(s)|^{1/\nu} ds \geq \frac{\lambda^2}{4(\lambda - 1)},$$

$$\frac{1}{(b_i - c_i)^{\lambda - 1}} \int_{c_i}^{b_i} (b_i - s)^{\lambda} |e(s)|^{1 -(1/\nu)} |Kq(s)|^{1/\nu} ds \geq \frac{\lambda^2}{4(\lambda - 1)}.$$
Motivated by the ideas of Philos [8], Kong [6], and Yang [14], in this paper we obtain, by using generalized Riccati techniques which are introduced by Noussair [7], several annual criteria for oscillation, that is, criteria given by the behavior of Eq. (1) (or of $A, q, f$ and $e$) only on a sequence of annuals of $\Omega$ in $\mathbb{R}^N$. Our results improve and extend the results of Ei-Sayed [1], Kong [6] and Yang [14]. Also information about the distribution of the zero of solutions for Eq. (1) is obtained.

2. Main results

For convenience, we let

$$Q(r) = \int_{S_r} [Kq(x)]^{1/y}|e(x)|^{1-1/y} \, d\sigma,$$

$$g(r) = \omega \lambda(r)r^{N-1},$$

where $S_r = \{x \in \mathbb{R}^N : |x| = r\}, r > 0$, $d\sigma$ denotes the spherical integral element in $\mathbb{R}^N$, $\omega$ is the area of unit sphere in $\mathbb{R}^N$ and $K$ is defined in $(C_3)$.

**Theorem 1.** Suppose that for any $T \geq a_0$, there exist $T \leq a_1 < b_1 < b_2$ such that

$$e(x) \begin{cases} \leq 0, & x \in G[a_1, b_1], \\ \geq 0, & x \in G[a_2, b_2] \end{cases}$$

and $q(x) \geq 0(\neq 0), x \in G(a_1, b_1) \cup G(a_2, b_2)$.

Denote $\Psi(a_i, b_i) = \{H \in C^1[a_i, b_i], H(r) \geq 0(\neq 0), H(a_i) = H(b_i) = 0, H'_i = 2h(r)\sqrt{H(r)}\}, i = 1, 2$. If there exist $H \in \Psi(a_i, b_i)$ such that

$$M_i(H) = \int_{a_i}^{b_i} \{g(s)h^2(s) - Q(s)H(s)\} \, ds < 0,$$

for $i = 1, 2$, then Eq. (1) is oscillatory.

**Proof.** Suppose to the contrary that there exists a solution $y(x)$ of Eq. (1) such that $y(x) > 0$ for $|x| \geq a_1 \geq a_0$. Define

$$W(x) = \frac{1}{y} (A \nabla y)(x), \quad x \in G[a_1, +\infty)$$

$$V(r) = \int_{S_r} W(x) \cdot \gamma(x) \, d\sigma, \quad x \in G[a_1, +\infty),$$

where $\nabla y$ denotes the gradient of $y(x), \gamma(x) = x/|x|, |x| \neq 0$ is the outward unit normal to $S_r$.

From Eqs. (1) and (8), it follows that

$$\nabla \cdot W(x) = -q(x) \frac{f(y)}{y} - (W^T A^{-1} W)(x) + \frac{e(x)}{y},$$

$$\leq -Kq(x)y^{\nu-1}(x) - (W^T A^{-1} W)(x) + \frac{e(x)}{y}, \quad (10)$$

where $W^T$ denotes the transpose of $W$.

Using Green’s formula in (9), we get

$$V'(r) = \int_{S_r} \nabla \cdot W(x) \, d\sigma$$

$$\leq - \int_{S_r} \left[ Kq(x)y^{\nu-1}(x) - \frac{e(x)}{y(x)} \right] \, d\sigma - \int_{S_r} (W^T A^{-1} W)(x) \, d\sigma. \quad (11)$$
In view of (C₁), we have \((W^TA^{-1}W)(x) \geq \lambda_{\text{max}}^{-1}(x)|W(x)|^2\). Then, by Cauchy–Schwartz inequality, we obtain

\[
\int_{S_r} |W(x)|^2 \, \text{d}\sigma \geq \frac{r^{1-N}}{\omega} \left[ \int_{S_r} W(x) \cdot \gamma(x) \, \text{d}\sigma \right]^2.
\]

Moreover, by (11) and (9), we get

\[
V'(r) \leq - \int_{S_r} \left[ Kq(x)y^{v-1}(x) - \frac{e(x)}{y(x)} \right] \, \text{d}\sigma - \frac{1}{g(r)} V^2(r) \tag{12}
\]

we consider the two following cases: (i) \(v > 1\); (ii) \(v = 1\).

(i) \(v > 1\): By the assumption, we can choose \(a_1, b_1 \geq T_0(a_1 < b_1)\) such that \(e(x) \leq 0, x \in G[a_1, b_1]\). From Young inequality we see that for \(x \in G[a_1, b_1]\),

\[
-\frac{e(x)}{y(x)} + Kq(x)y^{v-1}(x) = \frac{|e(x)|}{y(x)} + Kq(x)y^{v-1}(x) \geq \frac{v-1}{v} \left\{ \left[ \frac{|e(x)|}{y(x)} \right]^{1-1/v} \right\}^{v/(v-1)} + \frac{1}{v} ([Kq(x)]^{1/v}y^{1-1/v}(x))^{v}
\]

hence, we have for \(x \in G[a_1, b_1]\),

\[
V'(r) \leq - \int_{S_r} [Kq(x)]^{1/v}|e(x)|^{1-1/v} \, \text{d}\sigma - \frac{1}{g(r)} V^2(r),
\]

\[
\leq - Q(r) - \frac{1}{g(r)} V^2(r). \tag{14}
\]

Let \(H(r) \in \Psi(a_1, b_1)\) be given as in the hypothesis, Multiplying \(H(r)\) throughout (14) and integrating from \(a_1\) to \(b_1\), we obtain

\[
\int_{a_1}^{b_1} H(s)V'(s) \, \text{d}s \leq - \int_{a_1}^{b_1} Q(s)H(s) \, \text{d}s - \int_{a_1}^{b_1} H(s) \frac{1}{g(s)} V^2(s) \, \text{d}s. \tag{15}
\]

Integrating (15) by parts and using the fact \(H(a_1) = H(b_1) = 0\), we find

\[
- \int_{a_1}^{b_1} 2h(s)\sqrt{H(s)}V(s) \, \text{d}s \leq - \int_{a_1}^{b_1} Q(s)H(s) \, \text{d}s - \int_{a_1}^{b_1} H(s) \frac{1}{g(s)} V^2(s) \, \text{d}s \tag{16}
\]

which is equivalent to

\[
0 \leq - \int_{a_1}^{b_1} Q(s)H(s) \, \text{d}s + \int_{a_1}^{b_1} \left[ 2h(s)\sqrt{H(s)}V(s) - \frac{H(s)}{g(s)} V^2(s) \right] \, \text{d}s
\]

\[
= \int_{a_1}^{b_1} [g(s)h^2(s) - Q(s)H(s)] \, \text{d}s - \int_{a_1}^{b_1} \left[ \frac{H(s)}{g(s)} V(s) - \sqrt{g(s)}h(s) \right]^2 \, \text{d}s
\]

\[
= M_1(H) - \int_{a_1}^{b_1} \left[ \frac{H(s)}{g(s)} V(s) - \sqrt{g(s)}h(s) \right]^2 \, \text{d}s. \tag{17}
\]

Because \(M_1(H) < 0\), (17) is incompatible. This contradiction proves that \(y(x)\) must be oscillatory.

When \(y(x)\) is eventually negative, we use \(H(r) \in \Psi(a_2, b_2)\) and \(e(x) \geq 0, x \in G[a_2, b_2]\) to reach a similar contradiction.
Lemma 1. Assume that there exist $c_1 < b_1 < c_2 < b_2$ such that $q(x) \geq 0$ for $x \in G[c_1, b_1] \cup G[c_2, b_2]$ and

$$e(x) \begin{cases} \leq 0, & x \in G[c_1, b_1], \\ \geq 0, & x \in G[c_2, b_2]. \end{cases}$$

$y(x)$ is a solution of Eq. (1) such that $y(x) > 0$ for $x \in G[c_1, b_1]$ and $y(x) < 0$ for $x \in G[c_2, b_2]$. Then for any $H \in \mathcal{R}$

$$\frac{1}{H(b_i, c_i)} \int_{c_i}^{b_i} H(b_i, s) Q(s) \, ds \leq V(c_i) + \frac{1}{H(b_i, c_i)} \int_{c_i}^{b_i} g(s) h^2_2(b_i, s) \, ds, \quad i = 1, 2.$$  \hspace{1cm} (19)

Proof. Suppose that $y(x)$ is a solution of Eq. (1) such that $y(x) > 0$ for $x \in G[c_1, b_1]$ and $y(x) < 0$ for $x \in G[c_2, b_2]$. Then, similar to the proof of Theorem 1, we multiply (14) by $H(r, s)$, integrate it with respect to $s$ from $r$ to $c_i$, we get for $s \in [c_i, r)$

$$\int_{c_i}^{r} H(r, s) Q(s) \, ds \leq - \int_{c_i}^{r} H(r, s) V'(s) \, ds - \int_{c_i}^{r} H(r, s) \frac{1}{g(s)} V^2(s) \, ds$$

$$= H(r, c_i) V(c_i) - \int_{c_i}^{r} 2 h_2(r, s) \sqrt{H(r, s)} V(s) \, ds - \int_{c_i}^{r} H(r, s) \frac{1}{g(s)} V^2(s) \, ds$$

$$= H(r, c_i) V(c_i) + \int_{c_i}^{r} g(s) h^2_2(r, s) \, ds - \int_{c_i}^{r} \left[ \frac{H(r, s)}{g(s)} V(s) + \sqrt{g(s) h^2_2(r, s)} \right]^2 \, ds$$

$$\leq H(r, c_i) V(c_i) + \int_{c_i}^{r} g(s) h^2_2(r, s) \, ds. \quad (20)$$

Letting $r \to b_i^-$ in (20). Dividing both sides by $H(b_i, c_i)$ we obtain (19).

Lemma 2. Assume that there exist $a_1 < c_1 < a_2 < c_2$ such that $q(x) \geq 0$ for $x \in G(a_1, c_1] \cup G(a_2, c_2]$ and

$$e(x) \begin{cases} \leq 0, & x \in G(a_1, c_1], \\ \geq 0, & x \in G(a_2, c_2]. \end{cases}$$

$y(x)$ is a solution of Eq. (1) such that $y(x) > 0$ for $x \in G(a_1, c_1]$ and $y(x) < 0$ for $x \in G(a_2, c_2]$. Then for any $H \in \mathcal{R}$

$$\frac{1}{H(c_i, a_i)} \int_{a_i}^{c_i} H(s, a_i) Q(s) \, ds \leq - V(c_i) + \frac{1}{H(c_i, a_i)} \int_{a_i}^{c_i} g(s) h^2_1(s, a_i) \, ds, \quad i = 1, 2. \quad (21)$$
Proof. Similar to the proof of Lemma 1, we multiply (14) by \(H(s, r)\) and integrate it with respect to \(s\) from \(r\) to \(c_i\). We have
\[
\int_r^{c_i} H(s, r) Q(s) \, ds \leq - \int_r^{c_i} H(s, r) V'(s) \, ds - \int_r^{c_i} H(r, s) \frac{1}{g(s)} V^2(s) \, ds
\]
\[
= - H(c_i, r) V(c_i) + \int_r^{c_i} 2 h_1(s, r) \sqrt{H(s, r)} V(s) \, ds - \int_r^{c_i} H(s, r) \frac{1}{g(s)} V^2(s) \, ds
\]
\[
= - H(c_i, r) V(c_i) + \int_r^{c_i} g(s) h_1^2(s, r) \, ds - \int_r^{c_i} \left[ \frac{H(s, r)}{g(s)} V(s) - \sqrt{g(s) h_2^2(r, s)} \right]^2 \, ds
\]
\[
\leq - H(c_i, r) V(c_i) + \int_r^{c_i} g(s) h_1^2(s, r) \, ds. \tag{22}
\]
Letting \(r \to a_i^+\) in (22). Dividing both sides by \(H(c_i, a_i)\) we obtain (21). \(\square\)

The following theorem is an immediate result from Lemma 1 and Lemma 2.

**Theorem 2.** Suppose that there exist \(a_1 < b_1 \leq a_2 < b_2\) such that \(q(x) \geq 0\) for \(x \in G(a_1, b_1) \cup G(a_2, b_2)\) and
\[
e(x) \begin{cases} 
\leq 0, & x \in G(a_1, b_1), \\
\geq 0, & x \in G(a_2, b_2)
\end{cases}
\]
further, there exist some \(c_i \in (a_i, b_i)\) and some \(H \in \mathbb{R}\) such that
\[
\frac{1}{H(c_i, a_i)} \int_{a_i}^{c_i} [H(s, a_i) Q(s) - g(s) h_1^2(s, a_i)] \, ds
\]
\[
+ \frac{1}{H(b_i, c_i)} \int_{c_i}^{b_i} [H(b_i, s) Q(s) - g(s) h_2^2(b_i, s)] \, ds > 0, \quad i = 1, 2, \tag{23}
\]
holds, then every nontrivial solution of Eq. (1) has at least one zero either in \(G(a_1, b_1)\) or in \(G(a_2, b_2)\).

Proof. Suppose to the contrary that there exists a solution \(y(x)\) of Eq. (1) such that \(y(x) > 0\) for \(x \in G(T_0, +\infty) (T_0 \geq a_0)\), by the assumption, we can choose \(a_1, b_1 \geq T_0 (a_1 < b_1)\) such that \(e(x) > 0, x \in G(a_1, b_1)\), then from Lemmas 1 and 2 we see that (19) and (21) with \(i = 1\) hold. Adding (19) and (21), we have that
\[
\frac{1}{H(c_i, a_i)} \int_{a_i}^{c_i} [H(s, a_1) Q(s) - g(s) h_1^2(s, a_1)] \, ds
\]
\[
+ \frac{1}{H(b_i, c_i)} \int_{c_i}^{b_i} [H(b_i, s) Q(s) - g(s) h_2^2(b_i, s)] \, ds \leq 0, \tag{24}
\]
which contradicts the assumption (23) with \(i = 1\).

When \(y(x)\) is eventually negative, we choose \(a_2, b_2 \geq T_0\) such that \(e(x) \leq 0, x \in G(a_2, b_2)\) to reach a similar contradiction and hence completes the proof. \(\square\)

**Theorem 3.** Suppose that for any \(T \geq a_0\), the following conditions hold:

(A1) there exist \(T \leq a_1 < b_1 \leq a_2 < b_2\) such that
\[
\begin{cases}
e(x) \leq 0, & x \in G(a_1, b_1), \\
\geq 0, & x \in G(a_2, b_2)
\end{cases}
\]
and \(q(x) \geq 0(\neq 0)\) for \(x \in G(a_1, b_1) \cup G(a_2, b_2)\)

(A2) there exist some \(c_i \in (a_i, b_i), i = 1, 2,\) and some \(H \in \mathbb{R}\) such that \(T \leq a_1 < b_1 \leq a_2 < b_2\) and (23) holds. Then Eq. (1) is oscillatory.
Proof. Pick up a sequence \( \{ T_j \} \subset [a_0, +\infty) \), such that \( j \to \infty, T_j \to \infty \). By the assumption, for each \( j \in \mathbb{N} \), there exist \( a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R} \) such that \( T_j \leq a_1 < b_1 \leq a_2 < c_2 < b_2 \) and (23) holds. From Theorem 2, every solution \( y(x) \) has at least one zero on \( G(a_1, b_1) \) or \( G(a_2, b_2) \). Noting that \( |x| > a_1 \geq T_j, j \in \mathbb{N} \), we see that the zero set \( \{ x \in \Omega : y(x) = 0 \} \) of \( y(x) \) is unbounded. Thus, every nontrivial solution of Eq. (1) is oscillatory. The proof is complete. \( \square \)

Remark 1. With an appropriate choice of function \( H \) one can derive a number of oscillation criteria for Eq. (1).

As an immediate consequence of Theorem 3 we get the following oscillation criteria for Eq. (1).

Corollary 4. Suppose that for any \( T \geq a_0 \), the following conditions hold:

\[
\text{(A1)} \quad \text{there exist } T \leq a_1 < b_1 \leq a_2 < b_2 \text{ such that } e(x) \begin{cases} \< 0, & x \in G(a_1, b_1), \\ > 0, & x \in G(a_2, b_2), \end{cases}
\]

and \( q(x) \geq 0(\neq 0) \) for \( x \in G(a_1, b_1) \cup G(a_2, b_2) \).

\[
\text{(A2)} \quad \text{there exist some } c_i \in (a_i, b_i), \ i = 1, 2, \text{ and some } H \in \mathfrak{R} \text{ such that } T \leq a_1 < b_1 \leq a_2 < b_2 \text{ and the following two inequalities hold for } i = 1, 2,
\]

\[
\int_{a_i}^{b_i} [H(s, a_i) Q(s) - g(s) h_1^2(s, a_i)] ds > 0, \tag{25}
\]

\[
\int_{c_i}^{b_i} [H(b_1, s) Q(s) - g(s) h_2^2(b_1, s)] ds > 0. \tag{26}
\]

Then Eq. (1) is oscillatory.

Moreover, let \( H = H(r - s) \in \mathfrak{R} \), we have \( \partial H(r - s)/\partial r = -\partial H(r - s)/\partial s \), and denote them by \( h(r - s) \). The subclass of \( \mathfrak{R} \) containing such \( H(r - s) \) is denoted by \( \mathfrak{R}_0 \). Applying Theorem 3 to \( \mathfrak{R}_0 \), we obtain the following result.

Corollary 5. Suppose that for any \( T \geq a_0 \), the following conditions hold:

\[
\text{(A1)} \quad \text{there exist } T \leq a_1 < 2c_1 - a_1 \leq a_2 < 2c_2 - a_2 \text{ such that } e(x) \begin{cases} \< 0, & x \in G(a_1, 2c_1 - a_1), \\ > 0, & x \in G(a_2, 2c_2 - a_2), \end{cases}
\]

and \( q(x) \geq 0(\neq 0) \) for \( x \in G(a_1, 2c_1 - a_1) \cup G(a_2, 2c_2 - a_2) \),

\[
\text{(A2)} \quad \text{there exist some } H \in \mathfrak{R}_0 \text{ such that } T \leq a_1 < c_i \text{ for } i = 1, 2 \text{ and the following inequality holds}
\]

\[
\int_{a_i}^{c_i} [H(s - a_i)(Q(s) + Q(2c_i - s)] - [g(s) + g(2c_i - s)] h_2^2(s - a_i)] ds > 0. \tag{27}
\]

Then Eq. (1) is oscillatory.

Proof. Let \( b_i = 2c_i - a_i \), then \( H(b_i - c_i) = H(c_i - a_i) = H((b_i - a_i)/2) \), and for any \( f \in L[a, b] \), we have

\[
\int_{c_i}^{b_i} H(b_i - s) f(s) ds = \int_{a_i}^{c_i} H(s - a_i) f(2c_i - s) ds.
\]

Thus that (27) holds implies that (23) holds for \( H \in \Phi_0 \) and therefore Eq. (1) is oscillatory by Theorem 2.

Define

\[
R(r) = \int_{a_0}^{r} \frac{1}{g(s)} ds, \quad r \geq a_0, \tag{28}
\]
and let
\[ H(r, s) = [R(r) - R(s)]^2, \quad r \geq s \geq a_0, \] (29)
where \( \alpha > 1 \) is a constant. Based on the above results, we obtain the following oscillation criteria of Kamenev’s type. \( \square \)

**Theorem 6.** Assume that \( \lim_{r \to \infty} R(r) = \infty \). If for each \( T \geq a_0 \), the following conditions hold:

(A1) there exist \( T \leq a_1 < b_1 \leq a_2 < b_2 \) such that
\[
e(x) \begin{cases} 
\leq 0, & x \in G(a_1, b_1), \\
\geq 0, & x \in G(a_2, b_2), 
\end{cases}
\]
and \( q(x) \geq 0 (\ne 0) \) for \( x \in G(a_1, b_1) \cup G(a_2, b_2) \)

(A2) there exist \( c_i \in (a_i, b_i) \) for \( i = 1, 2 \), such that \( T \leq a_1 < b_1 \leq a_2 < b_2 \) and the following inequalities hold for \( i = 1, 2 \),
\[
\frac{1}{[R(c_i) - R(a_i)]^2} \int_{a_i}^{c_i} [R(s) - R(a_i)]^2 Q(s) \, ds \geq \frac{\alpha^2}{4(\alpha - 1)}, \tag{30}
\]
\[
\frac{1}{[R(b_i) - R(c_i)]^2} \int_{c_i}^{b_i} [R(b_i) - R(s)]^2 Q(s) \, ds \geq \frac{\alpha^2}{4(\alpha - 1)}. \tag{31}
\]
Then Eq. (1) is oscillatory.

**Proof.** It is easy to see that
\[
h_1(r, s) = \alpha[R(r) - R(s)]^{(\alpha - 2)/2} \frac{1}{2g(r)}, \quad h_2(r, s) = \alpha[R(r) - R(s)]^{(\alpha - 2)/2} \frac{1}{2g(s)}.
\]
Hence we have
\[
\int_{a_i}^{c_i} g(s) h_1^2(s, a_i) \, ds = \int_{a_i}^{c_i} g(s)x^2[R(s) - R(a_i)]^{2-2} \frac{1}{4g^2(s)} \, ds
\]
\[
= \int_{a_i}^{c_i} [R(s) - R(a_i)]^{2-2} \frac{x^2}{4g(s)} \, ds
\]
\[
= \frac{\alpha^2}{4(\alpha - 1)} [R(c_i) - R(a_i)]^{2-1}.
\] (32)

From (30) and (32) we have
\[
\frac{1}{[R(c_i) - R(a_i)]^2} \int_{a_i}^{c_i} [H(s, a_i)Q(s) - g(s)h_1^2(s, a_i)] \, ds
\]
\[
= \frac{1}{[R(c_i) - R(a_i)]^2} \int_{a_i}^{c_i} [R(s) - R(a_i)]^2 Q(s) \, ds - \frac{\alpha^2}{4(\alpha - 1)} > 0
\] (33)
i.e., (25) holds. Similarly, (31) implies (26) holds. From Corollary 4, Eq. (1) is oscillatory. \( \square \)

**Example 1.** Let \( a \geq 0, \gamma > 1 \), then the following nonlinear elliptic differential equation
\[
\frac{\partial}{\partial x_1} \left[ 1 + a \sin^2 r \frac{\partial}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[ 1 + a \sin^2 r \frac{\partial}{\partial x_2} \right] + (\beta \cos r)(y(r))^\gamma[1 + y^2(r)] \text{sgn } y(r) = \sin r,
\] (34)
is oscillatory if
\[
\beta^{1/\gamma} \geq \pi \left[ 1 + \frac{\alpha^2}{4} \frac{\Gamma(7/2)}{\Gamma(3/2 + 1/2\gamma)\Gamma(2 - 1/2\gamma)} \right].
\]
where $r = \sqrt{x_1^2 + x_2^2}$, $r \geq 1$, $N = 2$, and $\Gamma$ is the usual gamma function. In fact

$$\lambda(r) = \frac{1 + a \sin^2 r}{r}, \quad q(x) = \beta \cos r, \quad f(y) = |y(r)|^\gamma [1 + y^2(r)] \text{sgn } y(r), \quad \omega = 2\pi.$$ 

Let $K = 1$, hence

$$Q(r) = \int_{R} [Kq(x)]^{1/\gamma} |e(x)|^{1-1/\gamma} \, dx = 2\pi (\beta \cos r)^{1/\gamma} |\sin r|^{1-1/\gamma},$$

$$g(r) = \omega \lambda(r) r^{N-1} = 2\pi (1 + a \sin^2 r).$$

Choose $a_1 = 2n\pi - \pi/2$, $b_1 = a_2 = 2n\pi$, $b_2 = 2n\pi + \pi/2$ and $H(r) = \sin^2 2r$. It is easy to see that

$$M_1(H) = \int_{a_1}^{b_1} [g(s)h^2(s) - Q(s)H(s)] \, ds$$

$$= 8\pi \int_{2n\pi - \pi/2}^{2n\pi} (1 + a \sin^2 s) \cos^2 2s \, ds - 2\pi \int_{2n\pi - \pi/2}^{2n\pi} (\beta \cos s)^{1/\gamma} |\sin s|^{1-1/\gamma} \sin^2 2s \, ds$$

$$= 8\pi \int_{0}^{2n\pi} (1 + a \sin^2 s) \cos^2 2s \, ds - 8\pi \beta^{1/\gamma} \int_{0}^{(2\pi)/2} (\cos s)^{2+(1/\gamma)} |\sin s|^{3-1/\gamma} \, ds$$

$$= \pi^2 (4 + a) - 4\pi \beta^{1/\gamma} \frac{\Gamma\left(\frac{3}{2} + \frac{1}{2\gamma}\right) \Gamma\left(2 - \frac{1}{2\gamma}\right)}{\Gamma\left(\frac{7}{2}\right)} \leq 0.$$ 

Similarly, for $a_2, b_2$ we can show that $M_2(H) \leq 0$. It follows from Theorem 1 that Eq. (1) is oscillatory.

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**References**


