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A Combinatorial Algorithm to Establish a Fair Border

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A finite algorithm is given for the following problem: a piece of land bordered by n countries is to be divided equally among these n countries in such a way that each country's share is connected and adjacent to its original border.

INTRODUCTION

Since it was introduced by Steinhaus [4], the problem of dividing an object (often thought of as a cake) fairly among n people has generated a number of interesting solutions and variations. There are, for example, different possible definitions for the concept of fairness. We will use the original and most common definition, namely that each person receives at least a $1/n$ share according to his own evaluation.

Also, the idea of what constitutes an allowable cut can vary. The first type considered was where at each stage one person is instructed to cut a piece into two parts of specified sizes according to his own evaluation only. The 'you cut in half and I choose' solution for $n = 2$ and its generalizations are of this type. The solution desired under this interpretation is a discrete type of algorithm (usually recursive), which specifies at each stage who cuts which piece and what the sizes of resulting parts should be. This is the interpretation of an allowable cut that we will use.

An example of a more powerful notion of a cut is exemplified by the well known 'moving knife' solution. Here, a knife moves continuously across the cake, all of the participants continuously evaluate the two parts as the knife moves, and are instructed to say 'stop' when one part first reaches a specified size. Since this type of cut allows all participants to simultaneously judge sizes it may not be considered to be a finite algorithm by everyone [6]. Also, since it is a more powerful interpretation of an allowable cut, it often admits much simpler algorithms, at least in terms of the number of cuts.

Also, in some of the more difficult variations, solutions of an existence type are sometimes given [2, 5, 6]. The solutions are now more measure theoretic, rather than combinatorial.

THE FAIR BORDER PROBLEM

A variation of the fair division problem was suggested by Hill [3], in which the object to be divided is a geographical region, the boundary of which includes borders with the countries which are participating in the division. Each country's share must be a connected region adjacent to that country's border.

In [3] Hill gave a existence proof for this problem, and in [1] Beck gave a solution using a procedure similar to the moving knife type of cut.

In the next section we will give an algorithm where at this stage a country makes a cut subject to only its own evaluation.

The region R to be divided equally among the n countries $C_1, C_2, \ldots, C_n$ will be a simply connected open set in the plane, and the countries correspond to n open sets, the boundaries of which intersect the boundary of R. Let ∂C_i, ∂R and B_i denote the

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boundary of C_i, the boundary of R and the common border $\partial C_i \cap \partial R$ respectively. For convenience we may imagine R drawn on a map, and think of a country selecting a subregion by drawing it on the map.

Each country C_i has its own evaluation of the value of a piece of land, which we will indicate by assigning to C_i a value function or measure v_i. We will leave the discussion of the complete definition of this measure and justification of the assumptions made until the last section. In any combinatorial algorithm of the type described in the introduction, it is clear that at each stage the participant must at least be able to cut a piece of a specified size, and all other participants must be able to evaluate any piece cut by someone else. Thus we assume only that each country can draw a region of the type specified by the algorithm.

The Algorithm

The algorithm will be recursive in nature. It suffices to construct a simply connected open region R_j, for some j, such that:
(i) $v_j(R_j) \geq 1/n$.
(ii) $v_i(R_j) \leq 1/n$ for $i \neq j$.
(iii) $\partial R_j \cap \partial C_j \neq 0$.
(iv) The interior of $R - R_j$ satisfies the original hypotheses about R with C_j removed from consideration.
The region R_j is given to C_j and the other countries divide $R - R_j$ in a similar way.

Step 1. C_1 draws a region R_1 adjacent only to ∂C_1 on ∂R such that $v_1(R_1) = 1/n$ (see Figure 1). At each stage in the procedure, each country C_j will evaluate a region R_i and its complement R_i^* in R. C_j will place its initial j on R_i or R_i^* if $v_j(R_i) > 1/n$ or $v_j(R_i^*) > (n-1)/n$ respectively. Thus, after step 1, some countries may initial R_1, some may initial R_1^*, and those (including C_1) for whom $v_j(R_1) = 1/n$ and $v_j(R_1^*) = (n-1)/n$ initial neither region. A typical case is shown in Figure 1.

An essential feature of the algorithm is that at each step the countries initialing R_i^* will never decrease, and will ultimately increase until no country initials R_i. At this time the region R_i can be given to the country C_i, which will be adjacent to R_i and for which $v_i(R_i) = 1/n$.

Step 2. If no country initials R_1 then R_1 can be given to C_1. Otherwise, we may assume without loss of generality that C_2 initials R_1. C_2 now cuts off a piece S_1 from R_1 such that $v_2(R_1 - S_1) > 1/n$ and $\overline{R_1 - S_1} \cap \partial R = \varnothing$. ($\overline{R_1 - S_1}$ denotes the closure of $R_1 - S_1$. Cutting off S_1 merely disconnects R_1 from the boundary of R_1; see Figure 2). Clearly, every country which initialed R_1^* will not initial $(R_1 - S_1)^*$, and other countries may also.

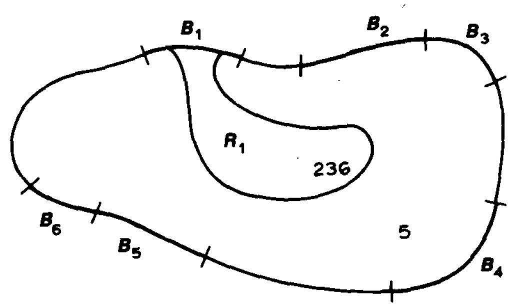

FIGURE 1.

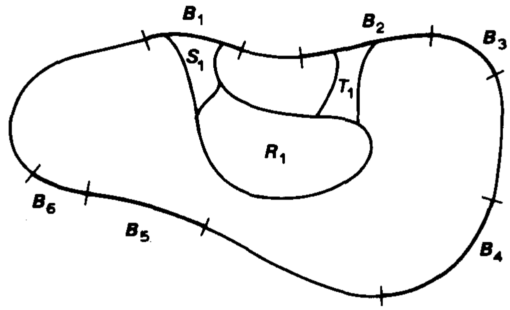

FIGURE 2.

Step 3. Under many natural assumptions about the measures v_j we have $v_1(R_1 - S_1) < 1/n$, and Step 3 may be omitted. For example, if the measures are compatable with Lebesgue measure we may draw S_1 so that $v_1(S_1) > 0$.

However, even if we assume that $v_1(S_1)$ may be 0, so that $v_1(R_1 - S_1) = (n-1)/n$, we need only reduce $R_1 - S_1$ to a subregion R_1' such that $v_2(R_1') > 1/n$ and $v_1(R_1') < 1/n$. This can be done, for example, by choosing a point Q in the interior of $R_1 - S_1$ and having C_2 divide $R_1 - S_1$ into small wedges, with Q as center. If $v_2(R_1 - S_1) = 1/n + \varepsilon$, for some $\varepsilon > 0$, require that $v_2(W) \leqslant \varepsilon/2$ for each such wedge W. Since $v_1(W) > 0$ for at least one such wedge W, letting $R_1' = R_1 - S_1 - W$ suffices.

Step 4. We now have the region R_1' such that every country which initialed R_1^* also initials $(R_1')^*$ and some countries, including C_1, who did not initial R_1^* now initial $(R_1')^*$. Any one country C_j initialing $(R_1')^*$ now draws a region T_1 which connects R_1' to B_2 such that $\bar{T}_1 \cap B_i = \varnothing$ for all $i \neq 2$ and $v_j(R_1' \cup T_1)^* > (n-1)/n$ (see Figure 2).

At this point other countries reduce T_1 in turn, but at all times the resulting region remains simply connected and adjacent to B_2. First, if necessary, each country C_k initialing $(R_1')^*$ reduces so that $v_k(R_1' \cup T_1)^* > (n-1)/n$. This is possible since for each such C_k, $v_k(R_1') < 1/n$. Finally, C_2 reduces $R_1' \cup T_1$ to a region R_2 such that $v_2(R_2) = 1/n$, which is possible since $v_2(R_1' \cup T_1) \geqslant v_2(R_1') > 1/n$.

By this construction we now have a region R_2 such that:

(i) Every C_j initialing R_1^* also initials R_2^*.

(ii) C_1 initials R_2^*.

(iii) C_2 initials neither R_2 nor R_2^*.

(iv) R_2 is adjacent only to ∂C_2 on ∂R, and the interior of $R - R_2$ satisfies the same conditions as R with respect to the countries $C_1, C_3, \ldots, C_n$.

Continuing in this manner we eventually obtain a region R_j as described at the beginning of the algorithm.

THE MEASURES v_i

Although the question of the most general type of measures v_i one can use is not really essential to the algorithm, certain requirements are clear. For any algorithm each v_i must be non-atomic, and the regions constructed by one country must be measurable to all. A simple argument also shows that the boundary of R must have measure zero. Since no country can control how another draws its subregions, and since the boundary of a subregion may recursively become part of the boundary of R at the next stage, the boundaries of any allowable subregion must have zero measure. Thus each v_i must be continuous as regions expand or contract continuously. This is sufficient to assure that all of the regions described in the algorithm can be drawn as described.

For example, one might assume that the v_i measurable sets are the Lebesgue measureable sets and that each v_i is absolutely continuous with respect to Lebesgue measure, as in [5]. Indeed, it seems unlikely that countries would wish to draw non-Lebesgue measurable regions.

Finally, we note that the condition that R be simply connected is not necessary. Merely connect the 'holes' in R to ∂R by arcs in such a way that the interior of the resulting region is simply connected. Consider these new arcs as part of the boundary of R.

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