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CONSTRUCTION OF STEINER QUADRUPLE SYSTEMS HAVING A PRESCRIBED NUMBER OF BLOCKS IN COMMON

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Let $q_v = v(v-1)(v-2)/24$ and let $I_v = \{0, 1, 2, \ldots, q_v - 14\} \cup \{q_v - 12, q_v - 8, q_v\}$, for $v \geq 8$. Further, let $J[v]$ denote the set of all k such that there exists a pair of Steiner quadruple systems of order v having exactly k blocks in common. We determine $J[v]$ for all $v = 2^n$, $n \geq 2$, with the possible exception of 7 cases for $v = 16$ and of 5 cases for each $v \geq 32$. In particular we show: $J[v] \subseteq I_v$ for all $v \equiv 2$ or 4 (mod 6) and $v \geq 8$, $J[4] = \{1\}$, $J[8] = I_8 = \{0, 2, 6, 14\}$, $I_{16} \setminus \{103, 111, 115, 119, 121, 122, 123\} \subseteq J[16]$, and $I_v \setminus \{q_v - h : h = 17, 18, 19, 21, 25\} \subseteq J[v]$ for all $v = 2^n$, $n \geq 5$.

1. Introduction

A *Steiner quadruple system* (SQS) is a pair (Q, q) where Q is a finite set and q is a collection of 4-subsets of Q (called *blocks*) such that every 3-subset of Q is contained in exactly one block of q. The number $|Q|$ is called the *order* of the quadruple system (Q, q) and in 1960 Hanani [4] proved that a necessary and sufficient condition for the existence of a Steiner quadruple system of order v (SQS(v)) is $v \equiv 2$ or 4 (mod 6). It is easy to see that if (Q, q) is an SQS(v), then $|q| = v(v-1)(v-2)/24$. A very interesting question naturally arises: *For a given* $v \equiv 2$ *or* 4 (mod 6), *for which* $k \leq v(v-1)(v-2)/24$ *is it possible to construct a pair of* SQS(v) *having exactly* k *blocks in common*? [14]. To date, the only known results are $J[4] = \{1\}$, $J[8] = \{0, 2, 6, 14\}$, and $J[10] = \{0, 2, 4, 6, 8, 12, 14, 30\}$ (Kramer and Messner [5]). The similar problem for Steiner triple systems has been completely settled by Lindner and Rosa in [11]. This paper is the first general attack on settling the (much more difficult) block intersection problem for SQS.

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Let $q_v = v(v-1)(v-2)/24$ and let

$$I_v = \{0, 1, 2, \ldots, q_v - 14\} \cup \{q_v - 12, q_v - 8, q_v\},$$

for $v \geq 8$. Further, let $J[v]$ denote the set of all k such that there exists a pair of SQS(v) having exactly k blocks in common. We determine $J[v]$ for all $v = 2^n$, $n \geq 2$, with the possible exception of 7 cases for $v = 16$ and of 5 cases for each $v \geq 32$. In particular we show:

$$J[v] \subseteq I_v \quad \text{for all } v \equiv 2 \text{ or } 4 \pmod 6 \text{ and } v \geq 8,$$

$$J[4] = \{1\}, \qquad J[8] = I_8 = \{0, 2, 6, 14\},$$

$$I_{16} \backslash \{103, 111, 115, 119, 121, 122, 123\} \subseteq J[16],$$

$$I_v \backslash \{q_v - h : h = 17, 18, 19, 21, 25\} \subseteq J[v] \quad \text{for all } v = 2^n, n \geq 5.$$

Although it is surely true that $J[v] = I_v$ for all $v = 2^n$, $n \geq 3$, the authors have not as yet been able to handle the few exceptions listed above. Nevertheless we do not hesitate to make the following conjecture.

Conjecture. *$J[v] = I_v$ for all $v = 2^n$ and $v \geq 8$.*

2. Disjoint and mutually balanced PQS

A *partial quadruple system* (PQS) is a pair (P, q) where P is a finite set and q is a collection of 4-subsets of P (called blocks) such that every 3-subset of P is contained in *at most* one block of q. Using graph theoretic terminology we will say that an element x of P has *degree* $d(x) = h$, if x belongs to exactly h blocks of q. Clearly $\sum_{x \in P} d(x) = 4|q|$. Finally, if $x \neq y \in P$ we will write $(x, y)_r$ to indicate that x and y belong to exactly r blocks of q.

Two partial quadruple systems (P, q_1) and (P, q_2) are said to be *mutually balanced*, if any given triple of distinct elements of P is contained in a block of q_1 if and only if it is contained in a block of q_2. Two mutually balanced PQSs are said to be *disjoint* if they have no block in common. It is easy to see that if (P, q_1) and (P, q_2) are any two mutually balanced PQSs, then $|q_1| = |q_2|$.

In this section we will determine some useful properties of *disjoint* and *mutually balanced* (DMB) PQSs. In what follows (P, q_1) and (P, q_2) will be two DMB PQSs with $P = \{1, 2, \ldots, n\}$ and $|q_1| = |q_2| = m$.

Property 2.1. *$n \geq 8$, $m \geq 8$, and $d(x) \geq 4$ for every $x \in P$.*

Proof. If $\{1, 2, 3, 4\} \in q_1$, then necessarily $\{1, 2, 3, x\}$, $\{1, 2, 4, y\}$, $\{1, 3, 4, z\}$ and $\{2, 3, 4, t\}$ belong to q_2. Since $\{x, y, z, t\} \cap \{1, 2, 3, 4\} = \emptyset$, clearly $n \geq 8$. Let now $x \in P$ and let $\{x, 1, 2, 3\} \in q_1$. If $\{x, 1, 2, y_3\}$, $\{x, 1, 3, y_2\}$ and $\{x, 2, 3, y_1\}$ belong to q_2, then $\{y_1, y_2, y_3\} \cap \{x, 1, 2, 3\} = \emptyset$ and there exists in q_1 three distinct blocks containing $\{x, 1, y_3\}$, $\{x, 3, y_2\}$ and $\{x, 2, y_1\}$, respectively. Hence $d(x) \geq 4$. Finally, since $4m = \sum_{x \in P} d(x)$ we must have $m \geq 8$.

Property 2.2. *If* $h = \max\{d(x) : x \in P\}$, *then* $m \geq 2h$.

Proof. It is easy to see that if $\{x, a_i, b_i, c_i\} \in q_1$, $i = 1, \ldots, h$, there are in q_2 h distinct blocks containing x and h distinct blocks containing the triples $\{a_i, b_i, c_i\}$ but not x.

Property 2.3. *If* $x \neq y \in P$ *and* $(x, y)_h$ *then* $d(x) \geq 2h$, $d(y) \geq 2h$.

Proof. If $\{x, y, a_i, b_i\} \in q_1$, $i = 1, 2, \ldots, h$, then there are in q_2 $2h$ distinct blocks containing the triples $\{x, a_i, b_i\}$ and $\{y, a_i, b_i\}$ and at least h additional blocks containing the triples $\{x, y, a_i\}$ and $\{x, y, b_i\}$. Hence $d(x) \geq 2h$, $d(y) \geq 2h$.

Property 2.4. *If* $d(x) > 4$, *then* $d(x) \geq 6$.

Proof. Let $R_i = \{x, a_i, b_i, c_i\}$ $(i = 1, \ldots, 5, \ldots)$ be all of the blocks of q_1 containing x. For every $i \neq j$ we have $|R_i \cap R_j| = 1$ or 2. If there exist two indices $i \neq j$ such that $R_i \cap R_j = \{x\}$, then there are in q_2 (at least) six distinct blocks containing $\{x, a_i, b_i\}$, $\{x, a_i, c_i\}$, $\{x, b_i, c_i\}$, $\{x, a_j, b_j\}$, $\{x, a_j, c_j\}$ and $\{x, b_j, c_j\}$ respectively. Hence $d(x) \geq 6$. If for every $i \neq j$ we have $|R_i \cap R_j| = 2$, then since $d(x) > 4$ for some $y \in P$ there are $r \geq 3$ blocks containing $\{x, y\}$. Clearly $d(x) \geq 6$.

Property 2.5. *If* $m > 8$ *and there exists a block* R *such that* $d(x) = 4$ *for every* $x \in R$, *then* $m \geq 14$.

Proof. Let $\{1, 2, 3, 4\} \in q_1$ and let $\{1, 2, 3, 5\}$, $\{1, 2, 4, 6\}$, $\{1, 3, 4, 7\}$ and $\{2, 3, 4, 8\}$ belong to q_2. If $d(1) = d(2) = d(3) = d(4) = 4$, then we have (necessarily)

in q_1:	$R_1 = \{1, 2, 3, 4\}$,	in q_2:	$T_1 = \{1, 2, 3, 5\}$,
	$R_2 = \{1, 2, 5, 6\}$,		$T_2 = \{1, 2, 4, 6\}$,
	$R_3 = \{1, 3, 5, 7\}$,		$T_3 = \{1, 3, 4, 7\}$,
	$R_4 = \{2, 3, 5, 8\}$,		$T_4 = \{2, 3, 4, 8\}$,
	$R_5 = \{1, 4, 6, 7\}$,		$T_5 = \{1, 5, 6, 7\}$,
	$R_6 = \{2, 4, 6, 8\}$,		$T_6 = \{2, 5, 6, 8\}$,
	$R_7 = \{3, 4, 7, 8\}$		$T_7 = \{3, 5, 7, 8\}$,
			$T_8 = \{4, 6, 7, 8\}$.

If $\{5, 6, 7, 8\} \notin q_1$, then $\{5, 6, 7, x\}$, $\{5, 6, 8, y\}$, $\{5, 7, 8, z\}$ and $\{6, 7, 8, t\}$ belong to q_1 (where x, y, z, t are distinct elements of P) and $\{5, 6, x\}$, $\{5, 7, x\}$, $\{6, 7, x\}$,

$\{7,8,z\}$, $\{6,8,t\}$ and $\{5,8,y\}$ are contained in six distinct blocks of q_2, respectively. Clearly $m \geq 14$. If $R_8 = \{5,6,7,8\} \in q_1$, then (A, τ_1) and (A, τ_2), where $A = \{1,2,\ldots,8\}$, $\tau_1 = \{R_1,\ldots,R_8\}$ and $\tau_2 = \{T_1,\ldots,T_8\}$ are two DMB PQSs. Since

$$\left(\bigcup_{B \in q_1 - \tau_1} B, q_1 - \tau_1\right) \quad \text{and} \quad \left(\bigcup_{B \in q_2 - \tau_2} B, q_2 - \tau_2\right)$$

are two DMB PQSs, it follows that $m \geq 16$.

Property 2.6. *If* $m > 8$, *then* $m = 12$ *or* $m \geq 14$.

Proof. We need consider only the case in which every block B contains an $x \in P$ with $d(x) \geq 6$ (in the other cases we have immediately our statement from Properties 2.4 and 2.5). From Property 2.2 we have $m \geq 12$. Suppose $m = 13$. It follows that $\sum_{x \in P} d(x) = 52$ and $d(x) = 4$ or 6 for every $x \in P$. Further we must have $8 < n < 12$. Under these conditions, for every $x \in P$ such that $d(x) = 6$ there exists a $y \in P$ such that $(x, y)_3$, so that $d(y) = 6$. Necessarily, we have the following blocks, in q_1:

$$\begin{array}{llll} R_1 = \{1,2,3,4\}, & R_4 = \{1,3,5,x_1\}, & R_7 = \{2,3,5,y_1\}, & R_{10} = \{3,4,v_1,t_1\}, \\ R_2 = \{1,2,5,6\}, & R_5 = \{1,4,7,x_2\}, & R_8 = \{2,4,7,y_2\}, & R_{11} = \{5,6,v_2,t_2\}, \\ R_3 = \{1,2,7,8\}, & R_6 = \{1,6,8,x_3\}, & R_9 = \{2,6,8,y_3\}, & R_{12} = \{7,8,v_3,t_3\}; \end{array}$$

in q_2:

$$\begin{array}{llll} T_1 = \{1,2,3,5\}, & T_4 = \{1,3,4,v_1\}, & T_7 = \{2,3,4,w_1\}, & T_{10} = \{3,5,x_1,u_1\}, \\ T_2 = \{1,2,4,7\}, & T_5 = \{1,5,6,v_2\}, & T_8 = \{2,5,6,w_2\}, & T_{11} = \{4,7,x_2,u_2\}, \\ T_3 = \{1,2,6,8\}, & T_6 = \{1.7,8,v_3\}, & T_9 = \{2,7,8,w_3\}, & T_{12} = \{6,8,x_3,u_3\}. \end{array}$$

where $\{x_1,x_2,x_3\} = \{v_1,v_2,v_3\}$, $\{y_1,y_2,y_3\} = \{w_1,w_2,w_3\}$, $\{x_1,x_2,x_3\} \cap \{y_1,y_2,y_3\} = \emptyset$ and

$$\begin{array}{l} (x_1,x_2,x_3) \text{ and } (v_1,v_2,v_3) = \\ (y_1,y_2,y_3) \text{ and } (w_1,w_2,w_3) = \end{array} \begin{cases} (7,6,3) \text{ and } (6,7,3) \text{ respectively, or} \\ (8,5,4) \text{ and } (8,4,5) \text{ respectively, or} \\ (a,a,a) \text{ with } a \notin \{1,2,\ldots,8\}. \end{cases}$$

If $\{x_1, y_1\} = \{7,8\}$, then we can see that for every $i = 1,2,3$ $\{2i+1, 2i+2, v_i, w_i\} \notin q_1$. Hence $m \geq 15$. Suppose, therefore, $\{x_i, y_i\} \cap \{9,10\} \neq \emptyset$. Let $x_i = 9$ (or, likewise, $y_i = 9$). If $w_i = t_i$ and $y_i = u_i$ for every $i = 1,2,3$, then for $A = \{1,2,\ldots,8,x_i,y_i,t_i;\ i = 1,2,3\}$, $R' = \{R_i : i = 1,\ldots,12\}$ and $T = \{T_i : i = 1,\ldots,12\}$, (A, R') and (A, T) are two DMB PQSs. Since $(\bigcup_{B \in q_1 - R'} B, q_1 - R')$ and $(\bigcup_{B \in q_2 - T} B, q_2 - T)$ are also a pair of DMB PQSs, it follows that $m \geq 20$. If there exist at least two indices $i, j \in \{1,2,3\}$ such that $w_i \neq t_i$, $w_j \neq t_j$ (or, likewise, $y_i \neq u_i$, $y_j \neq u_j$), then $m \geq 14$. Suppose, therefore, that

$w_i \neq t_i$ (or $y_i \neq u_i$) for exactly one index $i \in \{1, 2, 3\}$. Necessarily, $t_{i+1} = w_{i+1}$, $t_{i+2} = w_{i+2}$ ($\{i, i+1, i+2\} = \{1, 2, 3\}$). It follows that $\{d(2i+1), d(2i+2)\} \subseteq \{5, 7\}$. Hence $m \geq 14$. It follows that DMB PQSs with $m = 13$ do not exist.

Theorem 2.7. *For every* $v \equiv 2$ *or* 4 (mod 6) *and* $v \geq 8$, $J[v] \subseteq I_v$.

Proof. If two SQS(v) (Q, q_1) and (Q, q_2) have k blocks in common, then there exists a pair of DMB PQSs (P, s_1) and (P, s_2) such that $P \subseteq Q$, $s_1 \subseteq q_1$, $s_2 \subseteq q_2$, and $|s_1| = |s_2| = q_v - k$. The statement follows immediately from Properties 2.1 and 2.6.

3. SQS with blocks in common

In this section we will determine $J[v]$ for all $v = 2^n$, $n \geq 2$, with the possible exception of 7 cases for $v = 16$ and of 5 cases for each $v \geq 32$. Observe that for every $v \equiv 2$ or 4 (mod 6) $q_v \in J[v]$ and, since $D(2v) \geq v$ for $v > 2$ [7] (where $D(2v)$ is the number of pairwise disjoint SQS($2v$) on the same set with $2v$ elements), $0 \in J[v]$ for $v > 4$. The following well-known doubling construction for quadruple systems is the main tool used in what follows.

Let (X, A) and (Y, B) be any two SQS(v) with $X \cap Y = \emptyset$. Let $F = \{F_1, \ldots, F_{v-1}\}$ and $G = \{G_1, \ldots, G_{v-1}\}$ be any two 1-factorizations of K_v (the complete graph on v vertices) on X and Y, respectively, and let α be any permutation on the set $\{1, 2, \ldots, v-1\}$. Define a collection q of blocks of $Q = X \cup Y$, as follows:

(1) Any block belonging to A or B belongs to q;

(2) If $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then $\{x_1, x_2, y_1, y_2\} \in q$ if and only if $\{x_1, x_2\} \in F_i$, $\{y_1, y_2\} \in G_j$, $i\alpha = j$.

It is a routine matter to see that (Q, q) is a SQS($2v$) (cf. [13, 16]). We will denote (Q, q) by $[X \cup Y](A, B, F, G, \alpha)$ and for every F, G, α by $\Gamma(F, G, \alpha)$ the collection of all of the blocks $\{x_1, x_2, y_1, y_2\}$ such that $\{x_1, x_2\} \in F_i$, $\{y_1, y_2\} \in G_j$, and $i\alpha = j$.

Further, if w is a positive integer, we define α_i; $i = 0, 1, 2, \ldots, w-2, w$; to be the permutation on $\{1, 2, \ldots, w\}$ given by

$$\alpha_i = \begin{pmatrix} 1 & 2 & 3 \cdots w \\ x_1 & x_2 & x_3 \cdots x_w \end{pmatrix}, \quad \text{where } x_j = \begin{cases} w, & j = i+1, \\ j, & j < i+1, \\ j-1, & j > i+1. \end{cases}$$

Theorem 3.1. $J[4] = \{1\}$; $J[8] = \{0, 2, 6, 14\}$.

Proof. $J[4] = \{1\}$ is trivial. Since $0, 14 \in J[8]$ and $J[8] \subseteq I_8$ we need show only that $2, 6 \in J[8]$. Let $X = \{1, 2, 3, 4\}$, $Y = \{5, 6, 7, 8\}$, $A = \{X\}$, $B = \{Y\}$; and let $F = \{F_1, F_2, F_3\}$ and $G = \{G_1, G_2, G_3\}$ be two 1-factorizations of K_4 on X and Y respectively. If α_0, α_1 and α_3 are the permutations defined on $\{1, 2, 3\}$

in the remarks preceding Theorem 3.1, then $[X\cup Y](A,B,F,G,\alpha_3)$ and $[X\cup Y](A,B,F,G,\alpha_0)$ are two SQS(3) with 2 blocks in common and $[X\cup Y](A,B,F,G,\alpha_1)$ and $[X\cup Y](A,B,F,G,\alpha_3)$ are two SQS(8) with 6 blocks in common.

Theorem 3.2. *For* $k=105$, 113, 117, 125 *there are* SQS(16) *having exactly k blocks in common.*

Proof. Let $A=\{1,2,3,4\}$, $B=\{5,6,7,8\}$, $C=\{9,10,11,12\}$, $D=\{13,14,15,16\}$. $Q=A\cup B\cup C\cup D$ and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be the following 1-factorizations of K_4 on A, B, C, and D respectively:

	A_1	A_2	A_3
$\mathcal{A}=$	1, 2 3, 4	1, 3 2, 4	1, 4 2, 3

	B_1	B_2	B_3
$\mathcal{B}=$	5, 6 7, 8	5, 7 6, 8	5, 8 6, 7

	C_1	C_2	C_3
$\mathcal{C}=$	9, 10 11, 12	9, 11 10, 12	9, 12 10, 11

	D_1	D_2	D_3
$\mathcal{D}=$	13, 14 15, 16	13, 15 14, 16	13, 16 14, 15

Let α_0, α_1 and α_3 be the permutations defined on $\{1,2,3\}$ as above and consider the collections of blocks of Q, as shown in Fig. 1.

Observe that $(A\cup B\cup C, X_1)$ and $(A\cup B\cup C, X_2)$ are two DMB PQSs with $m=15$ blocks. Further we can prove that $(Q, X_1\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_3)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ is an SQS(16). It follows that

(i) $(Q, X_1\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_3)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ and $(Q, X_2\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_3)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ are two SQS(16) with $k=125$ blocks in common;

(ii) $(Q, X_1\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_3)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ and $(Q, X_2\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_1)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ are two SQS(16) with $k=117$ blocks in common;

(iii) $(Q, X_1\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_3)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ and $(Q, X_2\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_3)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ are two SQS(16) with 113 blocks in common;

(iv) $(Q, X_1\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_3)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ and $(Q, X_2\cup Y\cup Z\cup\Gamma(\mathcal{A},\mathcal{C},\alpha_1)\cup\Gamma(\mathcal{B},\mathcal{D},\alpha_0)\cup\Gamma(\mathcal{A},\mathcal{D},\alpha_3)\cup\Gamma(\mathcal{C},\mathcal{D},\alpha_3))$ are two SQS(16) with 105 blocks in common.

Theorem 3.3. *If* $k\in I_{16}\backslash\{103, 111, 115, 119, 121, 122, 123\}$, *then* $k\in J[16]$.

X_1	X_2	Y
1, 2, 3, 4	1, 2, 3, 5	1, 2, 7, 8
1, 2, 5, 6	1, 2, 4, 6	1, 3, 6, 8
1, 3, 5, 7	1, 3, 4, 7	1, 4, 5, 8
1, 4, 6, 7	1, 5, 6, 7	2, 4, 5, 7
2, 3, 5, 8	2, 3, 4, 8	2, 3, 6, 7
2, 4, 6, 8	2, 5, 6, 8	3, 4, 5, 6
3, 4, 7, 8	3, 5, 7, 8	5, 6, 11, 12
5, 6, 7, 9	4, 6, 7, 8	5, 7, 10, 12
5, 6, 8, 10	5, 6, 9, 10	5, 8, 9, 12
5, 7, 8, 11	5, 7, 9, 11	6, 7, 10, 11
5, 9, 10, 11	5, 8, 10, 11	6, 8, 9, 11
6, 7, 8, 12	6, 7, 9, 12	7, 8, 9, 10
6, 9, 10, 12	6, 8, 10, 12	
7, 9, 11, 12	7, 8, 11, 12	
8, 10, 11, 12	9, 10, 11, 12	

$Z = \{13, 14, 15, 16\} \cup$			
13, 1, 5, 9	14, 1, 5, 10	15, 1, 5, 11	16, 1, 5, 12
13, 1, 6, 12	14, 1, 6, 9	15, 1, 6, 10	16, 1, 6, 11
13, 1, 7, 11	14, 1, 7, 12	15, 1, 7, 9	16, 1, 7, 10
13, 1, 8, 10	14, 1, 8, 11	15, 1, 8, 12	16, 1, 8, 9
13, 2, 5, 12	14, 2, 5, 9	15, 2, 5, 10	16, 2, 5, 11
13, 2, 6, 11	14, 2, 6, 12	15, 2, 6, 9	16, 2, 6, 10
13, 2, 7, 10	14, 2, 7, 11	15, 2, 7, 12	16, 2, 7, 9
13, 2, 8, 9	14, 2, 8, 10	15, 2, 8, 11	16, 2, 8, 12
13, 3, 5, 11	14, 3, 5, 12	15, 3, 5, 9	16, 3, 5, 10
13, 3, 6, 10	14, 3, 6, 11	15, 3, 6, 12	16, 3, 6, 9
13, 3, 7, 9	14, 3, 7, 10	15, 3, 7, 11	16, 3, 7, 12
13, 3, 8, 12	14, 3, 8, 9	15, 3, 8, 10	16, 3, 8, 11
13, 4, 5, 10	14, 4, 5, 11	15, 4, 5, 12	16, 4, 5, 9
13, 4, 6, 9	14, 4, 6, 10	15, 4, 6, 11	16, 4, 6, 12
13, 4, 7, 12	14, 4, 7, 9	15, 4, 7, 10	16, 4, 7, 11
13, 4, 8, 11	14, 4, 8, 12	15, 4, 8, 9	16, 4, 8, 10

Fig. 1.

Proof. Let $X = \{a, b, c, d, e, f, g, h\}$, $Y = \{1, 2, \ldots, 8\}$.

(1) *First, we prove the statement for k even.* Let F, H be the 1-factorizations on X given by Fig. 2, and let G, L and M be the 1-factorizations on Y given by Fig. 3.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
$F=$	a, b	a, d	a, f	a, h	a, c	a, e	a, g
	c, d	c, h	b, c	b, g	b, d	b, f	b, h
	e, f	b, e	c, f	e, g	e, g	c, g	c, e
	g, h	f, g	d, g	d, e	f, h	d, h	d, f

	H_1	H_2	H_3	
$H=$	a, b	a, d	a, f	$H_4 = F_4$
	c, d	b, c	b, e	$H_5 = F_5$
	f, g	e, f	c, h	$H_6 = F_6$
	e, h	g, h	d, g	$H_7 = F_7$

Fig. 2.

$G=$

G_1	G_2	G_3	G_4	G_5	G_6	G_7
1, 2	1, 3	1, 4	1, 5	1, 6	1, 7	1, 8
3, 5	2, 5	2, 6	2, 3	2, 4	2, 8	2, 7
4, 7	4, 6	3, 7	4, 8	3, 8	3, 4	3, 6
6, 8	7, 8	5, 8	6, 7	5, 7	5, 6	4, 5

$L=$

L_1	L_2	L_3	
1, 2	1, 3	1, 4	$L_4=G_4$
3, 5	2, 6	2, 5	$L_5=G_5$
4, 6	4, 7	3, 7	$L_6=G_6$
7, 8	5, 8	6, 8	$L_7=G_7$

$M=$

M_2	$M_1=L_1$
	$M_3=G_3$
1, 3	$M_4=G_4=L_4$
2, 5	$M_5=G_5=L_5$
4, 7	$M_6=G_6=L_6$
6, 8	$M_7=G_7=L_7$

Fig. 3.

Further, let β_i and γ be the following permutations on $\{1,2,\ldots,7\}$:
For $i=0,1,2,4$

$$\beta_i=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 3 & 1 & 2 & y_1 & y_2 & y_3 & y_4\end{pmatrix},\quad \text{where}\quad y_j=\begin{cases}7, & j=i+1,\\ j+3, & j<i+1,\\ j+2, & j>i+1\end{cases}$$

$$\gamma=\begin{pmatrix}1 & 2 & 3 & 4 & 6 & 6 & 7\\ 1 & 4 & 5 & 6 & 7 & 2 & 3\end{pmatrix}.$$

Let $\rho(8)=\{0,2,4,6,8,12,14,16,20,28\}$. If $h\in\rho(8)$, then it is possible to construct four SQS(8) (X,A_j) and (Y,B_j), $j=1,2$, such that $|A_1\cap A_2|+|B_1\cap B_2|=h$. Let α_i be defined on $\{1,2,\ldots,7\}$ as in the remarks preceding Theorem 3.1. Consider $k=16i+h$, for $i=0$, 1, 2, 3, 4, 5, 7, and $h\in\rho(8)$, and let (X,A_j), (Y,B_j), $j=1,2$, be SQS(8) with $|A_1\cap A_2|+|B_1\cap B_2|=h$. Since for $i\neq 6$ $|\Gamma(F,G,\alpha_7)\cap\Gamma(F,G,\alpha_i)|=16i$, it follows that $[X\cup Y](A_1,B_1,F,G,\alpha_7)$ and $[X\cup Y](A_2,B_2,F,G,\alpha_i)$ are two SQS(16) with $16i+h$ blocks in common. For $i=6$, consider the SQS(16) $[X\cup Y](A_1,B_1,F,G,\alpha_7)$ and $[X\cup Y](A_2,B_2,F,M,\alpha_7)$. Since $|\Gamma(F,G,\alpha_7)\cap\Gamma(F,M,\alpha_7)|=96$, these SQS(16) have $96+h$ blocks in common. It follows that if $k\in I_{16}$, $k\neq 16i+10$ $(i=0,1,\ldots,7)$, then $k\in J[16]$.

Let $k=16i+10$ for $i=0,1,\ldots,6$. For $i=0$ we consider two SQS(8) (X,A_j) and (Y,B_j), $j=1,2$, such that $|A_1\cap A_2|=|B_1\cap B_2|=2$. Since $|\Gamma(F,G,\alpha_1)\cap\Gamma(H,L,\gamma)|=6$, clearly $[X\cup Y](A_1,B_1,F,G,\alpha_1)$ and $[X\cup Y](A_2,B_2,H,L,\gamma)$ have 10 blocks in common. For $i=1,2,\ldots,6$, let (X,A_j) and (Y,B_j), $j=0,1,2,3$, be SQS(8) such that $|A_u\cap A_{u+2}|+|B_u\cap B_{u+2}|=16u+12$ for $u=0,1$. Since $|\Gamma(F,G,\beta_4)\cap\Gamma(H,L,\beta_i)|=16i+14$ for $i=0,2,4$, it follows that $[X\cup Y](A_u,B_u,F,G,\beta_4)$ and $[X\cup Y](A_{u+2},B_{u+2},H,L,\beta_i)$ for every $(u,i)\in\{0,1\}\times\{0,2,4\}$ are SQS(16) with $16(i+u)+26$ blocks in common.

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
$J=$	a,b	a,c	a,d	a,e	a,f	a,g	a,h
	c,f	e,c	b,e	b,d	b,h	b,c	b,g
	d,g	b,f	c,g	c,h	c,d	d,h	c,e
	e,h	g,h	f,h	f,g	e,g	e,f	d,f

	N_1	N_2	N_3	
$N=$	a,b	a,c	a,d	$N_4=J_4$
	c,g	b,f	b,e	$N_5=J_5$
	d,e	d,g	c,f	$N_6=J_6$
	f,h	e,h	g,h	$N_7=J_7$

	O_1	O_2	O_3	O_4	O_5	O_6	O_7
$O=$	a,b	a,c	a,d	a,e	a,f	a,g	a,h
	c,h	b,g	b,h	b,c	b,d	b,e	b,f
	d,e	d,h	c,f	d,f	c,g	c,d	c,e
	f,g	e,f	e,g	g,h	e,h	f,h	d,g

	R_1	R_2	R_3	
$R=$	a,b	a,c	a,d	$R_4=O_4$
	c,f	b,h	b,g	$R_5=O_5$
	d,h	d,e	c,h	$R_6=O_6$
	e,g	f,g	e,f	$R_7=O_7$

Fig. 4.

(2) *Now, we prove the statement for k odd.* From Theorem 3.2 there exist SQS(16) with $k=105$, 113, 117, 125 blocks in common. Let J, N, O, R be the 1-factorizations on X given by Fig. 4, and let S, T, U, V be the 1-factorizations on Y given by Fig. 5.

Further, let δ_1 and δ_2 be the permutations on $\{1, 2, \ldots, 7\}$ given by

$$\delta_1=\begin{pmatrix}1&2&3&4&5&6&7\\1&4&5&6&7&2&3\end{pmatrix} \quad\text{and}\quad \delta_2=\begin{pmatrix}1&2&3&4&5&6&7\\1&5&6&7&2&3&4\end{pmatrix}.$$

Let $k=79+2h$; $h=0, 1, 2, 3, \ldots, 10, 11, 14, 15$; and let (X, A_j) and (Y, B_j), $j=1, 2$, be SQS(8) such that for every h

$$|A_1\cap A_2|+|B_1\cap B_2|=\begin{cases}2h & \text{if } h \text{ is even,}\\ 2(h-1) & \text{if } h \text{ is odd.}\end{cases}$$

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
$S=$	1,2	1,3	1,4	1,5	1,6	1,7	1,8
	6,8	2,6	2,5	2,4	2,8	2,3	2,7
	3,7	4,7	3,6	3,8	3,4	4,8	3,5
	4,5	5,8	7,8	6,7	5,7	5,6	4,6

	T_1	T_2	T_3	
$T=$	1,2	1,3	1,4	$T_4=S_4$
	3,6	2,6	2,5	$T_5=S_5$
	4,7	4,5	3,7	$T_6=S_6$
	5,8	7,8	6,8	$T_7=S_7$

	U_1	U_2	U_3	U_4	U_5	U_6	U_7
$U=$	1,2	1,3	1,4	1,5	1,6	1,7	1,8
	3,6	2,8	2,7	2,3	2,4	2,5	2,6
	4,8	4,5	3,8	4,6	3,7	3,4	3,5
	5,7	6,7	5,6	7,8	5,8	6,8	4,7

	V_1	V_2	V_3	
$V=$	1,2	1,3	1,4	$V_4=U_4$
	3,8	2,7	2,8	$V_5=U_5$
	4,5	4,8	3,6	$V_6=U_6$
	6,7	5,6	5,7	$V_7=U_7$

Fig. 5.

Since $|\Gamma(O, U, \alpha_7) \cap \Gamma(R, V, \alpha_7)| = 79$ and $|\Gamma(J, S, \alpha_7) \cap \Gamma(N, T, \alpha_7)| = 81$, it follows that for h even $[X \cup Y](A_1, B_1, O, U, \alpha_7)$ and $[X \cup Y](A_2, B_2, R, V, \alpha_7)$ are two SQS(16) with $k = 79 + 2h$ blocks in common, and for h odd $[X \cup Y](A_1, B_1, J, S, \alpha_7)$ and $[X \cup Y](A_2, B_2, N, T, \alpha_7)$ are SQS(16) with $k = 81 + 2(h-1)$ blocks in common.

Now, let $k = 1 + 16i + r$ for $i = 0, 1, 2, 3$ and $r \in \rho(8) = \{0, 2, 4, 6, 8, 12, 14, 16, 20, 28\}$. Consider two SQS(8) (X, A_j) and (Y, B_j), $j = 1, 2$, such that $|A_1 \cap A_2| + |B_1 \cap B_2| = r$. For $i = 0$, since $|\Gamma(J, S, \delta_1) \cap \Gamma(N, T, \delta_2)| = 1$, the SQS(16) $[X \cup Y](A_1, B_1, J, S, \delta_1)$ and $[X \cup Y](A_2, B_2, N, T, \delta_2)$ have $1 + r$ blocks in common. For $i \neq 0$, we have $|\Gamma(J, S, \delta_7) \cap \Gamma(N, T, \alpha_{i+2})| = 16i + 1$ and so $[X \cup Y](A_1, B_1, J, S, \alpha_7)$ and $[X \cup Y](A_2, B_2, N, T, \alpha_{i+2})$ have $1 + 16i + r$ blocks in common.

Finally, let $k = 67, 71, 73,$ or $16i + 11$ for $i = 0, 1, 2, 3, 4$. Let (X, A_j) and (Y, B_j), $j = 1, 2, \ldots, 10$, be SQS(8) such that

$$|A_u \cap A_{u+5}| + |B_u \cap B_{u+5}| = r_u = \begin{cases} 16(u-1) & \text{if } u = 1, 2, \\ 3u - 1 & \text{if } u = 3, 5, \\ 12 & \text{if } u = 4. \end{cases}$$

Since

$$|\Gamma(J, S, \alpha_2) \cap \Gamma(N, T, \alpha_v)| = s_v = \begin{cases} 59 & \text{if } v = 3, \\ 43 & \text{if } v = 4, \\ 11 & \text{if } v = 7, \end{cases}$$

it follows that $[X \cup Y](A_u, B_u, J, S, \alpha_2)$ and $[X \cup Y](A_{u+5}, B_{u+5}, N, T, \alpha_v)$ are two SQS(16) with $k = r_u + s_v$ blocks in common, for every $(u, v) \in \{1, 2, 3, 4, 5\} \times \{3, 4, 7\}$.

Theorem 3.4. *Let $v = 2^n$, $n \geq 5$. If $k \in I_v \setminus \{q_v - h : h = 17, 18, 19, 21, 25\}$, then $k \in J[v]$.*

Proof. Let $v = 2^n$, $n \geq 5$, $w = 2^{n-1}$, $X = \{1, 2, \ldots, w\}$ and $Y = \{1', 2', \ldots, w'\}$ with $X \cap Y = \emptyset$. Let $\rho(w)$ be the set of all h such that there exist four SQS(w) (X, A_j), (Y, B_j), $j = 1, 2$, with $|A_1 \cap A_2| + |B_1 \cap B_2| = h$. Let F, G, be two 1-factorizations on X and Y respectively and let α_i be defined on $\{1, 2, \ldots, w-1\}$ in the usual way.

Assume $n = 5$. If $k \in I_{32} \setminus \{1215, 1219, 1221, 1222, 1223\}$, it is easy to show that there exists an $(r, u) \in \{0, 1, 2, \ldots, 13, 15\} \times \rho(16)$ such that $k = 64r + u$. It follows that, if $(X, A_j), (Y, B_j), j = 1, 2$, are SQS(16) such that $|A_1 \cap A_2| + |B_1 \cap B_2| = u$, then $[X \cup Y](A_1, B_1, F, G, \alpha_{15})$ and $[X \cup Y](A_2, B_2, F, G, \alpha_r)$ are two SQS(32) with k blocks in common. This finishes the proof for $n = 5$. Assume therefore $n \geq 6$ and assume that for all $m < n$ ($m \geq 5$) if $u = 2^m$ and $k \in I_u \setminus \{q_u - h : h = 17, 18, 19, 21, 25\}$ that $k \in J[u]$. Let $k \in I_v \setminus \{q_v - h : h = 17, 18, 19, 21, 25\}$. Observe that if $k > (w-3)w^2/4 + 2q_w - 26$, since $w^2/4 < q_w - 13$ and $q_v = 2q_w + (w-1)w^2/4$, then there exists an $r \in \rho(w)$ such that $k=(w-1)w^2/4+r$; and if $k \leq (w-3)w^2/4+2q_w-26$, then there exists an

$$(r, u) \in \{0, 1, 2, \ldots, w-3\} \times \{0, 1, 2, \ldots, 2q_w-26\}$$

such that $k=rw^2/4+u$. In every case, therefore, we have $k=rw^2/4+u$ for $r=0, 1, 2, \ldots, w-3, w-1$ and $r \in \rho(w)$. Since for every

$$(r, u) \in \{0, 1, 2, \ldots, w-3, w-1\} \times \rho(w)\ |\Gamma(F, G, \alpha_{w-1}) \cap \Gamma(F, G, \alpha_r)| = rw^2/4,$$

and it is possible to construct four SQS(w) (X, A_j) and (Y, B_j), $j=1, 2$, such that $|A_1 \cap A_2|+|B_1 \cap B_2|=u$, our statement follows from the doubling construction.

Collecting together Theorems 2.7, 3.1, 3.2, 3.3 and 3.4 gives the following theorem (which is, of course, the main result).

Theorem 3.5.

$$J[v] \subseteq I_v \quad \textit{for all } v \equiv 2 \textit{ or } 4 \pmod 6 \textit{ and } v \geq 8,$$

$$J[4]=\{1\}, \qquad J[8]=I_8=\{0, 2, 6, 14\},$$

$$I_{16} \backslash \{103, 111, 115, 119, 121, 122, 123\} \subseteq J[16],$$

and

$$I_v \backslash \{q_v-h : h=17, 18, 19, 21, 25\} \subseteq J[v] \quad \textit{for all } v=2^n, n \geq 5.$$

References

[1] J. Doyen and A. Rosa, An extended bibliography and survey of Steiner systems, Proc. 7th Manitoba Conference on Numerical Math. and Computing (1977) 297–361.

[2] B. Ganter, Finite partial quadruple systems can be finitely embedded, Discrete Math. 10 (1974) 397–400.

[3] P.B. Gibbons, R. Mathon and D.G. Corneil, Steiner quadruple systems on 16 symbols, Proc. 6th Southeastern Conf. Combinatorics, Graph Theory and Computing, Boca Raton (1975) 345–365.

[4] H. Hanani, On quadruple systems, Canad. J. Math. 12 (1960) 145–157.

[5] E.S. Kramer and D.M. Mesner, Intersections among Steiner systems, J. Combinatorial Theory 16(A) (1974) 273–285.

[6] C.C. Lindner, On the construction of nonisomorphic Steiner quadruple systems, Colloq. Math. 29 (1974) 303–306.

[7] C.C. Lindner, A note on disjoint Steiner quadruple systems, Ars Combinatoria 3 (1977) 271–276.

[8] C.C. Lindner, Construction of Steiner triple systems having exactly one triple in common, Canad. J. Math. 26 (1974) 225–232.

[9] C.C. Lindner, E. Mendelsohn and A. Rosa, On the number of 1-factorizations of the complete graph, J. Combinatorial Theory 20(B) (1976) 265–282.

[10] C.C. Lindner and A. Rosa, Finite embedding theorems for partial Steiner quadruple systems, Bull. Soc. Math. Belg. 27 (1975) 315–323.

[11] C.C. Lindner and A. Rosa, Steiner triple systems having a prescribed number of triples in common, Canad. J. Math. 27 (1975) 1166–1175.

[12] C.C. Lindner and A. Rosa, Construction of large sets of almost disjoint Steiner triple systems, Canad. J. Math. 27 (1975) 256–260.

[13] C.C. Lindner and A. Rosa, Steiner quadruple systems all of whose derived Steiner triple systems are nonisomorphic, J. Combinatorial Theory 21(A) (1976) 35–43.

[14] C.C. Lindner and A. Rosa, Steiner quadruple systems—a survey, Discrete Math. 21 (1978) 147–181.

[15] A. Rosa, A theorem on the maximum number of disjoint Steiner triple systems, J. Combinatorial Theory 18(A) (1975) 305–312.

[16] A. Rosa, Algebraic properties of designs and recursive constructions, Proc. Conf. Algebraic Aspects of Combinatorics, University of Toronto (1975) 183–200.