Asymptotic smoothing effect of solutions to weakly dissipative Klein–Gordon–Schrödinger equations

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Abstract
In this paper the authors consider the Cauchy problem of weakly dissipative Klein–Gordon–Schrödinger equations through Yukawa coupling in \(\mathbb{R}^3\). Making use of a Strichartz type inequality and a suitable decomposition of the solution semigroup they prove the asymptotic smoothing effect of the solutions.

Keywords: Weakly dissipative Klein–Gordon–Schrödinger equations; Strichartz type inequality; Asymptotic smoothing effect; Global attractor

1. Introduction
In this paper we study the Cauchy problem of the following Klein–Gordon–Schrödinger equations (KGS)

\[ i\psi_t + \Delta \psi + i\alpha \psi + \phi \psi = f, \]
\[ \phi_{tt} - \Delta \phi + \mu^2 \phi + \beta \phi_t = |\psi|^2 + g, \]
\[ (\psi, \phi, \phi_t)|_{t=0} = (\psi_0, \phi_0, \phi_1)(x). \]
which describe the interaction of scalar nucleons interacting with neutral scalar mesons through Yukawa coupling [16]. $\psi, \phi$ represent a complex scalar nucleon field and a real meson field respectively, $\mu$ describes the mass of the meson which is normalized to be 1 in this paper. $\alpha, \beta$ are positive constants, which stand for the dissipative mechanism of the model; while the complex-valued function $f$ and the real-valued function $g$ are the external sources. When $\alpha = \beta = 0$ and $f = g = 0$, (1.1)–(1.2) are a Hamiltonian system and have many interesting properties, such as soliton solutions in one dimensional case [16]. The Cauchy problem and the initial boundary value problem of this conserved KGS have been quite extensively studied ([2,6,7,12] etc.).

In [4] Biler studied the long time behavior of (1.1)–(1.2) on a bounded domain $\Omega \subset \mathbb{R}^3$ and proved that the global attractor exists in the weak topologies of phase spaces $H^1_0 \times H^1_0 \times L^2(\Omega)$ and $H^2 \cap H^1_0 \times H^2 \cap H^1_0 \times H^1_0(\Omega)$ with finite Hausdorff dimension. Later the result was improved by [14] and [21]. In [14] the author proved that the weak global attractor in $H^2 \cap H^1_0 \times H^2 \cap H^1_0 \times H^1_0(\Omega)$ is in fact a strong one through a suitable decomposition, while in [21] the authors proved that the weak global attractor in $H^1_0 \times H^1_0 \times L^2(\Omega)$ is strong via the energy equality due to Ball [3]. Guo and Li [10] studied the Cauchy problem and proved the existence of a global attractor $H^2 \times H^2 \times H^1(\mathbb{R}^3)$ which attracts bounded sets of $H^3 \times H^3 \times H^2(\mathbb{R}^3)$. Again by applying the energy method Lu and Wang [15] improved this results and showed that the (strong) global attractor is in $H^k \times H^{k-1}(\mathbb{R}^3)$ when $f, g \in H^{k-2}(\mathbb{R}^3)$. A special effort was made by applying the energy equality to prove the existence of global attractor in $H^1 \times H^1 \times L^2(\mathbb{R}^3)$ when $f, g \in L^2(\mathbb{R}^3)$.

The energy method is very useful in the study of the long time behavior of solutions of dissipative partial differential equations, especially when the problems lack the compact embedding and smoothing properties. For applications of this method to other problem see [8,9,22] etc. Besides of this, the Strichartz type inequalities of space–time estimates are also helpful to prove the existence of attractors. See [13] for the damped semilinear wave equation and [9] for the damped nonlinear Schrödinger equation.

In this paper we utilize a Strichartz type inequality and some suitable decomposition of the semigroup generated by (1.1)–(1.2) to prove the asymptotic smoothing effect. We note that our decomposition is much simpler than that of [9], where the decomposition into low and high frequency parts are used. We can do so because the real potential $\phi$ of the complex field $\psi$ in the (KGS) possesses a little better regularity than the potential in the pure Schrödinger equation in [9]. The purpose of this paper is to prove

**Main theorem.** Let $f, g \in L^2(\mathbb{R}^3)$. Then the global attractors in $E_0 = H^1 \times H^1 \times L^2(\mathbb{R}^3)$ and $E_1 = H^2 \times H^2 \times H^1(\mathbb{R}^3)$ are the same.

This result improves the result obtained in [15]. It implies that the attractor in $E_0$ is strongly compact in $E_1$ and thus the asymptotic smoothing effect of the solutions, despite of the fact that the Schrödinger and the wave equations have no smoothing effects. This effect is due to the damping mechanism. We note that, for either Schrödinger equations or wave equations, the damping mechanism may not prevent the occurrence of the singularity if the nonlinearity is strong enough (see [19,20]). The theorem is sharp in the sense that, if $f, g$ are only in $L^2(\mathbb{R}^3)$ and not in $H^\varepsilon(\mathbb{R}^3)$ for any $\varepsilon > 0$, then the stationary solutions
$(ψ, φ)$ are only in $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ and not in $H^{2+r}(\mathbb{R}^3) \times H^{2+r}(\mathbb{R}^3)$, thus neither $ψ$-nor $φ$-components of the attractor can be included in $H^s(\mathbb{R}^3)$ for any $s > 2$.

We arrange this paper as follows. First, in Section 2, following the idea of [9] we obtain some Strichartz type inequality for $ψ$. In Section 3 we decompose the solution semigroup into two parts, one uniformly decaying in $E_0$ and the other uniformly bounded in $E_1$ with the aid of the Strichartz type inequality for $ψ$. Finally, in Section 4 we prove the main theorem.

We denote the spaces of complex-valued and real-valued functions by the same symbols.

\[ W_{m,p}(\mathbb{R}^3) \] is the usual Sobolev space with the norm $\| \cdot \|_{m,p}$.

\[ H_m(\mathbb{R}^3) = W_{m,2}(\mathbb{R}^3), \]

$\| \cdot \|_p = \| \cdot \|_{0,p}$ and $\| \cdot \| = \| \cdot \|_2$. $C$ is a generic constant and may assume various values from line to line.

2. A Strichartz type inequality

In this section we establish some Strichartz type inequality for $ψ$, the complex scalar nucleon component.

We let $θ = ϕ_t + δϕ$ and change (1.1)–(1.3) equivalently to the following

\[ iψ_t + Δψ + iαψ + ϕψ = f, \quad t ∈ \mathbb{R}^+, x ∈ \mathbb{R}^3, \tag{2.1} \]

\[ ϕ_t + δϕ = θ, \quad t ∈ \mathbb{R}^+, x ∈ \mathbb{R}^3, \tag{2.2} \]

\[ θ_t + (β − δ)θ + Aϕ = |ψ|^2 + g, \quad t ∈ \mathbb{R}^+, x ∈ \mathbb{R}^3, \tag{2.3} \]

\[ (ψ, φ, θ)(0, x) = (ψ_0, φ_0, θ_0)(x), \quad x ∈ \mathbb{R}^3, \tag{2.4} \]

where $θ_0 = Δφ_0 + φ_1$, $0 < δ ≤ \min(β/2, 1/(2β))$. $A = 1 − δ(β − δ) − Δ$ is a positive, self-adjoint and elliptic operator of order 2. Let

\[ E_0 = H^1 \times H^1 \times L^2(\mathbb{R}^3), \quad E_1 = H^2 \times H^2 \times H^1(\mathbb{R}^3). \]

In [15] the following lemma was proved:

**Lemma 2.1.** Let $(ψ_0, φ_0, θ_0) ∈ E_0$. Then (2.1)–(2.4) possess a unique solution $(ψ(t), φ(t), θ(t))$, $S(t) : (ψ_0, φ_0, θ_0) ↦ (ψ(t), φ(t), θ(t))$ is continuous from $E_0$ into itself. Moreover, $S(t)$ has a bounded absorbing set in $E_0$ whose $ω$-limit set is the global attractor in $E_0$.

Let $B ⊂ E_0$ be a bounded subset, $(ψ_0, φ_0, θ_0) ∈ B$ and $(ψ, φ, θ) = S(t)(ψ_0, φ_0, θ_0)$ be the solution of (2.1)–(2.4). By the above lemma, there are constants $ρ_B$, $ρ > 0$ and $t_B ≥ 0$ ($ρ$ being independent of $B$) such that

\[ \|ψ\|_{1,2} + \|φ\|_{1,2} + \|φ_t\| + \|θ\| ≤ ρ_B, \quad ∀(ψ_0, φ_0, θ_0) ∈ B, t ≥ 0, \tag{2.5} \]

\[ \|ψ\|_{1,2} + \|φ\|_{1,2} + \|φ_t\| + \|θ\| ≤ ρ, \quad ∀(ψ_0, φ_0, θ_0) ∈ B, t ≥ t_B. \tag{2.6} \]

$B = B(0, ρ)$, the ball centered at 0 of radius $ρ$, is a bounded absorbing set in $E_0$. 
Let $U(t)$ be the unitary group on $L^2(\mathbb{R}^3)$ generated by $i\Delta$. Then we have (see [5,17])

**Lemma 2.2.** (a) The $L^p-L^q$ estimate
\[
\|U(t)u_0\|_{L^10/7}\leq C|t|^{-3/5}\|u_0\|_{L^{10/7}}, \quad \forall u_0 \in L^{10/7}(\mathbb{R}^3), \quad t \neq 0; \quad (2.7)
\]

(b) The Strichartz type estimate
\[
\|U(\cdot)u_0\|_{L^{10/3}(\mathbb{R} \times \mathbb{R}^3)} \leq C\|u_0\|_{L^2(\mathbb{R}^3)}. \quad (2.8)
\]

We write the Schrödinger equation (2.1) into an integral form
\[
e^{\alpha t}\psi(t) = U(t)\psi_0 - i \int_0^t U(t-s)(e^{\alpha s}f)ds + i \int_0^t U(t-s)(e^{\alpha s}(\phi\psi(s)))ds. \quad (2.9)
\]

Applying the above two lemmas we have

**Lemma 2.3.** Let $B \subseteq E_0$ be bounded, $(\psi_0, \phi_0, \theta_0) \in B$, $(\psi, \phi, \theta) = S(t)(\psi_0, \phi_0, \theta_0)$ be the solution of (2.1)–(2.4), and $T > 0$. Then there is a constant $K > 0$ such that
\[
\|\nabla\psi\|_{L^{10/3}(t_0, t_0+T) \times \mathbb{R}^3} \leq K, \quad \forall t_0 \geq 0. \quad (2.10)
\]

**Proof.** Without loss of generality we may assume that $t_0 = 0$. Denote the three terms on the right hand side of (2.9) by $I_1$, $I_2$ and $I_3$, respectively. By (2.8) and Lemma 2.1,
\[
\|\partial I_1\|_{L^{10/3}(0, T) \times \mathbb{R}^3} = \|U(\cdot)\partial\psi_0\|_{L^{10/3}(0, T) \times \mathbb{R}^3} \leq C\|\partial\psi_0\| \leq C, \quad (2.11)
\]

where $\partial = \partial/\partial x_j$, $j = 1, 2$ or $3$. Noting that
\[
\partial I_2 = -i\left[\int_0^t e^{\alpha s}U(t-s)ds\right]\partial f = -ie^{\alpha t}(\alpha - i\Delta)^{-1}\partial f + iU(t)(\alpha - i\Delta)^{-1}\partial f, \quad (2.12)
\]

where $(\alpha - i\Delta)^{-1}\partial$ is a bounded operator from $L^2(\mathbb{R}^3)$ into $H^1(\mathbb{R}^3) \hookrightarrow L^{10/3}(\mathbb{R}^3)$, we have
\[
\left\|e^{\alpha t}(\alpha - i\Delta)^{-1}\partial f\right\|_{L^{10/3}(0, T) \times \mathbb{R}^3} \leq C\left\|e^{\alpha t}\right\|_{L^{10/3}(0, T)}\left\|(\alpha - i\Delta)^{-1}\partial f\right\|_{L^{10/3}} \leq Ce^{\alpha T}, \quad (2.13)
\]
\[
\left\|U(t)(\alpha - i\Delta)^{-1}\partial f\right\|_{L^{10/3}(0, T) \times \mathbb{R}^3} \leq C\left\|(\alpha - i\Delta)^{-1}\partial f\right\| \leq C, \quad (2.14)
\]

where the last inequality follows from the Strichartz type inequality (2.8). Thus
\[
\|\partial I_2\|_{L^{10/3}(0, T) \times \mathbb{R}^3} \leq Ce^{\alpha T}. \quad (2.15)
\]

By Lemma 2.1,
\[
\left\|\partial(\phi\psi)\right\|_{L^{10/7}} \leq \left\|\partial\phi\right\|\left\|\psi\right\|_s + \left\|\phi\right\|s\left\|\partial\psi\right\| \leq C. \quad (2.16)
\]
Therefore, applying (2.7) we have
\[ \| \partial I_3 \|_{L^{10/3}} \leq C \int_0^t \frac{e^{\alpha s}}{|t-s|^{5/2}} \| \partial (\phi \psi)(s) \|_{L^{10/3}} ds \leq C e^{\alpha t}, \]  
(2.17)
and thus
\[ \| \partial I_3 \|_{L^{10/3}(0,T) \times \mathbb{R}^3} \leq C T^{7/10} e^{\alpha T}. \]  
(2.18)
Combining (2.11), (2.15) and (2.18) we have
\[ \| e^{\alpha s} \partial \psi(s) \|_{L^{10/3}(0,T) \times \mathbb{R}^3} \leq C (1 + T^{7/10}) e^{\alpha T}. \]  
(2.19)
The proof of the lemma is completed.

3. Decomposition

Now we decompose \( S(t) \) into two parts, \( S_1(t) \) and \( S_2(t) \), with \( S_1(t) \) decaying uniformly on bounded subset of \( E_0 \), and \( S_2(t) \) being uniformly bounded in \( E_1 \).

Let \( B \subset E_0 \) be a bounded subset, \((\psi_0, \phi_0, \theta_0) \in B \) and \((\psi, \phi, \theta) = S(t)(\psi_0, \phi_0, \theta_0) \) be the solution of (2.1)–(2.4). Define \((\psi_1, \phi_1, \theta_1) = S_1(t)(\psi_0, \phi_0, \theta_0) \) to be the solution of the following problem

\[ i \psi_{1t} + \Delta \psi_{1} + i \alpha \psi_1 + \phi \psi_1 = 0, \]  
(3.1)
\[ \phi_{1t} + \delta \phi_1 = \theta_1, \quad t \in \mathbb{R}^+, \ x \in \mathbb{R}^3, \]  
(3.2)
\[ \theta_{1t} + (\beta - \delta) \theta_1 + A \phi_1 = 0, \]  
(3.3)
\[ (\psi_1, \phi_1, \theta_1)|_{t=0} = (\psi_0, \phi_0, \theta_0)(x), \]  
(3.4)
where \( A = 1 - \delta(\beta - \delta) - \Delta \). Define \( S_2(t) = S(t) - S_1(t) \), i.e., \((\psi_2, \phi_2, \theta_2) = S_2(\psi_0, \phi_0, \theta_0) \) is the solution of the problem

\[ i \psi_{2t} + \Delta \psi_2 + i \alpha \psi_2 + \phi \psi_2 = f, \]  
(3.5)
\[ \phi_{2t} + \delta \phi_2 = \theta_2, \quad t \in \mathbb{R}^+, \ x \in \mathbb{R}^3, \]  
(3.6)
\[ \theta_{2t} + (\beta - \delta) \theta_2 + A \phi_2 = |\psi|^2 + g, \]  
(3.7)
with the homogeneous initial data.

**Lemma 3.1.** Let \( f, g \in L^2(\mathbb{R}^3) \) and \((\psi_0, \phi_0, \theta_0) \in B \). Then

(a) \[ \| \psi_1(t) \|^2 = \| \psi_0 \|^2 e^{-2\alpha t}; \]
(b) \[ \| \nabla \psi_1(t) \|^2 \leq C e^{-\alpha t}; \]
(c) \[ \| \theta_1 \|^2 + (1 - \delta(\beta - \delta)) \| \phi_1 \|^2 + \| \nabla \phi_1 \|^2 \leq (\| \theta_0 \|^2 + (1 - \delta(\beta - \delta)) \| \phi_0 \|^2 + \| \nabla \phi_0 \|^2) e^{-\delta t}. \]

Therefore \( S_1(t) \) decays exponentially and uniformly on bounded sets in \( E_0 \).
Proof. Note that (3.1) is a damped linear Schrödinger equation of $\psi_1$ with a real-valued potential function $\phi$ satisfying (2.5) and (2.6), while (3.2) and (3.3) are the system form of a weakly damped wave equation with constant coefficients. The unique existence of the solutions to (3.1)–(3.4) is clear. Recall that $\beta - \delta < 1/2$ by the choice of $\delta$, (c) is a classical result on damped wave equations (see [1,11,18]). So we need only to prove (a) and (b).

Multiplying (3.1) by $\bar{\psi}_1$, integrating over $\mathbb{R}^3$ and then taking imaginary parts we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\psi_1\|^2 + \alpha \|\psi_1\|^2 = 0,
\]
so we have (a).

Multiplying (3.1) by $-(\bar{\psi}_1t + \alpha \bar{\psi}_1)$, integrating over $\mathbb{R}^3$ and then taking real parts we obtain
\[
\frac{d}{dt} H_0(t) + \alpha H_0(t) = I_0(t),
\]
where
\[
H_0(t) = \|\nabla \psi_1\|^2 - \int \phi |\psi_1|^2,
\]
\[
I_0(t) = -\alpha \|\nabla \psi_1\|^2 + \alpha \int \phi |\psi_1|^2 - 2 \text{Re} \int \phi_t |\psi_1|^2.
\]
By Gagliardo–Nirenberg inequality and (2.5),
\[
\left| \text{Im} \int \phi |\psi_1|^2 \right| \leq C\|\phi\|_6 \|\psi_1\|_{12/5}^2 \leq C\|\phi\|_{1.2} \|\psi_1\|^{3/2} \|\nabla \psi_1\|^{1/2}
\]
\[
\leq \varepsilon \|\nabla \psi_1\|^2 + C(\varepsilon) \rho_B^{4/3} \|\psi_1\|^2,
\]
\[
\left| \text{Im} \int \phi_t |\psi_1|^2 \right| \leq C\|\phi\| \|\psi_1\|^2 \leq C\|\phi_t\| \|\psi_1\|^{1/2} \|\nabla \psi_1\|^{3/2}
\]
\[
\leq \frac{\alpha}{4} \|\nabla \psi_1\|^2 + C\rho_B^4 \|\psi_1\|^2.
\]
(3.9)
Here we assume $\varepsilon = \frac{1}{4} \alpha$. Then, by the above estimates and (a) there exists a $C > 0$ such that
\[
I_0(t) \leq C\left(\rho_B^{4/3} + \rho_B^4\right) \|\psi_1\|^2 \leq C\left(\rho_B^{4/3} + \rho_B^4\right) \|\psi_0\| e^{-2\alpha t}.
\]
(3.10)
By Gronwall inequality we obtain
\[
H_0(t) \leq H_0(0)e^{-\alpha t} + \frac{1}{\alpha} C\left(\rho_B^{4/3} + \rho_B^4\right) \|\psi_0\| e^{-\alpha t} (1 - e^{-\alpha t}) \leq Ce^{-\alpha t}.
\]
(3.11)
In (3.9) we take $\varepsilon = \frac{1}{2}$. By (a) and (3.11),
\[
\|\nabla \psi_1\|^2 \leq 2H_0(t) + C(\varepsilon) \rho_B^{4/3} \|\psi_1\|^2 \leq Ce^{-\alpha t},
\]
so we have (b).  $\square$
Since \((\psi, \phi, \theta)\) and \((\psi_1, \phi_1, \theta_1)\) are uniformly bounded in \(E_0\) with respect to \(t \in \mathbb{R}^+\) and \((\psi_0, \phi_0, \theta_0) \in B\), so is \((\psi_2, \phi_2, \theta_2) = (\psi, \phi, \theta) - (\psi_1, \phi_1, \theta_1)\). Now we show that \((\psi_2, \phi_2, \theta_2)\) is also uniformly bounded in \(E_1\).

**Lemma 3.2.** Let \(f, g \in L^2(\mathbb{R}^3)\). Then there exists a \(C\) such that
\[
\|(\psi_2, \phi_2, \theta_2)\|_{E_1} \leq C, \quad \forall (\psi_0, \phi_0, \theta_0) \in B, \quad t \geq 0.
\]

**Proof.** We need only to show that
\[
\|\Delta \psi_2\|^2 + \|\nabla \theta_2\|^2 + \|\Delta \phi_2\|^2 \leq C.
\]
We differentiate (3.5) in \(t\) to get
\[
i \psi_{2t} + \Delta \psi_2 + i\alpha \psi_2 + \phi \psi_2 + \phi_t \psi_2 = 0, \quad t > 0, \quad x \in \mathbb{R}^3, \quad (3.12)
\]
\[
\psi_{2t}(0, x) = if(x), \quad x \in \mathbb{R}^3. \quad (3.13)
\]
We multiply (3.12) by \(2\bar{\psi}_{2t}\), integrate over \(\mathbb{R}^3\) and then take the imaginary parts to get
\[
\frac{d}{dt}\|\psi_{2t}\|^2 + 2\alpha \|\psi_{2t}\|^2 = -2 \text{Im} \int \phi_1 \bar{\psi}_{2t} \leq 2\|\phi_1\| \|\psi_{2t}\|_{\infty} \|\psi_{2t}\|.
\]
Recall that \(\|\phi_1\| \leq C, \|\phi\|_{1,2} \leq C, \|\psi_{2t}\|_{1,2} \leq C\). From (3.5) we have
\[
\|\Delta \psi_2\| \leq \|\psi_{2t}\| + \alpha \|\psi_2\| + \|\phi\|_{4} \|\psi_{2t}\|_{4} + \|f\| \leq \|\psi_{2t}\| + C.
\]
Thus, by Gagliardo–Nirenberg inequality,
\[
\|\psi_{2t}\|_{\infty} \leq C \|\psi_{2t}\|^{1/4} \|\Delta \psi_2\|^{3/4} \leq C \|\psi_{2t}\|^{3/4} + C.
\]
Therefore the right hand side of (3.14) can be controlled as
\[
2\|\phi_1\| \|\psi_{2t}\| \|\psi_{2t}\|_{\infty} \|\psi_{2t}\| \leq C \|\psi_{2t}\|^{7/4} + C \leq \alpha \|\psi_{2t}\|^2 + C.
\]
Inserting the above inequality into (3.14) we obtain
\[
\frac{d}{dt}\|\psi_{2t}\|^2 + \alpha \|\psi_{2t}\|^2 \leq C.
\]
Applying Gronwall inequality we have
\[
\|\psi_{2t}\|^2 \leq \|f\|^2 e^{-\alpha t} + C \alpha (1 - e^{-\alpha t}) \leq C.
\]
By (3.15) we get
\[
\|\Delta \psi_2\| \leq C.
\]
Multiplying (3.7) by \(-\Delta \theta_2\), integrating over \(\mathbb{R}^3\), noting that by (3.6)
\[
\int g \Delta \theta_2 = \frac{d}{dt} \int g \Delta \phi_2 + \delta \int g \Delta \phi_2,
\]
we get
\[
\frac{d}{dt} H_2(t) + \delta H_2(t) = I_2(t), \quad (3.16)
\]
where
\[ H_2(t) = \| \nabla \theta_2 \|^2 + \| \Delta \phi_2 \|^2 + (1 - \delta(\beta - \delta)) \| \nabla \phi_2 \|^2 + 2 \int g \Delta \phi_2. \]
\[ I_2(t) = - (2\beta - 3\delta) \| \nabla \theta_2 \|^2 - \delta \| \Delta \phi_2 \|^2 + \delta (1 - \delta(\beta - \delta)) \| \nabla \phi_2 \|^2 + 2 \int \nabla |\psi|^2 \nabla \theta_2. \]

By Young inequalities,
\[ \left| \int \nabla |\psi|^2 \nabla \theta_2 \right| \leq (2\beta - 3\delta) \| \nabla \theta_2 \|^2 + \| \nabla \psi \|^2_{10/3} + C \| \psi \|^5_5. \]

From Lemmas 2.1 and 2.3,
\[ I_2 \leq C + \| \nabla \psi \|^2_{10/3} \in L^1_{\text{loc}}(0, \infty). \]
Applying Gronwall inequality we obtain
\[ H_2(t) \leq \frac{C}{\delta} \left( 1 - e^{-\delta t} \right) + \frac{t}{e^{\delta t}} \left( 1 - \delta(t - s) \right) \| \nabla \phi(s) \|^2_{10/3} \int_0^t \| \nabla \psi(s) \|^2_{10/3} ds, \quad t \geq 0. \]

Let \( m \) be the integer such that \( mT < t \leq (m + 1)T \). Then, by Lemma 2.3,
\[ \int_0^t e^{\delta s} \| \nabla \psi(s) \|^2_{10/3} ds \leq \sum_{k=0}^m e^{\delta(k+1)T} \left[ \int_{kT}^{(k+1)T} \| \nabla \psi(s) \|^2_{10/3} ds \right] \]
\[ \leq K e^{\delta T} \sum_{k=0}^m e^{kT} \leq K e^{\delta T} e^{\delta(m+1)T} \frac{e^{\delta T} - 1}{e^{\delta T} - 1} \leq \frac{K e^{2\delta T}}{e^{\delta T} - 1} e^{\delta t}. \]

We obtain
\[ H_2(t) \leq \frac{C}{\delta} + \frac{K e^{2\delta T}}{e^{\delta T} - 1}, \quad t \geq 0, \]
from which we complete the proof of the lemma. \( \square \)

4. Asymptotic smoothing effect

Let \( S(t) \) be the semigroup associated to (2.1)–(2.3), \( \mathcal{A} \) and \( \mathcal{A}_1 \) be the global attractors of \( S(t) \) in \( E_0 = H^1 \times H^1 \times L^2(\mathbb{R}^3) \) and \( E_1 = H^2 \times H^2 \times H^1(\mathbb{R}^3) \), respectively. The existence of \( \mathcal{A} \) and \( \mathcal{A}_1 \) has been proven in [15]. To prove the main theorem we shall prove that \( \mathcal{A} = \mathcal{A}_1 \). That is, we shall prove that \( \mathcal{A} \) satisfies

(a) Compactness: \( \mathcal{A} \) is compact in \( E_1 \);
(b) Invariance: \( S(t)\mathcal{A} = \mathcal{A} \) for all \( t \geq 0 \);
(c) Attractivity: for any \( B \subset E_1 \) bounded,
\[ \lim_{t \to \infty} \text{dist}_{E_1} \left( S(t)B, \mathcal{A} \right) = \lim_{t \to \infty} \sup_{\xi \in B} \text{dist}_{E_1} \left( S(t)\xi, \mathcal{A} \right) = 0, \]
where \( \text{dist}_{E_1} \) stands for the distance in \( E_1 \).
Note that $\mathcal{A}_1$ satisfies (a)–(c). As a global attractor of $S(t)$ in $E_0$, $\mathcal{A}$ is the $\omega$-limit of the absorbing set $B(0, \rho) \subset E_0$ (see (2.6)) under the action of $S(t)$, and satisfies (a)–(c) with $E_1$ being replaced by $E_0$.

Assume that $a \in \mathcal{A} = S(t)A$. Then for any positive integer $m$ there exists an $a_m \in \mathcal{A}$ such that $S(m)a_m = a$. We remark that $B = \{a_m \mid m \geq 1\}$ is bounded in $E_0$.

Let $\xi_m(t) = S_1(t + m)a_m$ and $\eta_m(t) = S_2(t + m)a_m$ be the solutions of (3.1)–(3.3) and (3.5)–(3.7), respectively, with initial time $t_0 = -m$ and initial data $\xi_m(-m) = a_m$, $\eta_m(-m) = 0$. Then $\xi_m(t) + \eta_m(t) = S(t + m)a_m$ is the solution of (2.1)–(2.3) with initial data $(\psi, \phi, \theta)|_{t_0=-m} = a_m$. In particular, we have $\xi_m(0) + \eta_m(0) = a$.

By Lemma 3.2,
\[ \|\eta_m(0)\|_{E_1} \leq C \]
and implies that $\eta_m(0)$ converges weakly to some $\eta$ in $E_1$ (extracting a subsequence if necessary). By Lemma 3.1,
\[ \|\eta_m(0) - a\|_{E_0} = \|\xi_m(0)\|_{E_0} \leq Ce^{-\nu m}, \]
where $\nu = \min\{\delta, \alpha\} > 0$. Thus $\eta_m(0)$ converges strongly to $a$ in $E_0$. Therefore $a = \eta \in E_1$ and $\mathcal{A} \subset E_1$ is bounded. Since $\mathcal{A}_1$ attracts bounded sets in $E_1$,
\[ \text{dist}_{E_1}(S(t)A, \mathcal{A}_1) \to 0 \quad \text{as} \quad t \to \infty. \]
Recalling that $S(t)A = \mathcal{A}$, we have $\mathcal{A} \subset \mathcal{A}_1$. It is obvious that $\mathcal{A}_1 \subset \mathcal{A}$, thus $\mathcal{A} = \mathcal{A}_1$. The proof of the main theorem is completed.

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References


