On the 1-Factors of a Non-separable Graph*

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ABSTRACT

The main result presented is that every 2-connected graph with a 1-factor has more than one. Furthermore, any graph with a 1-factor has more than one if and only if there is a cycle of even length whose edges are alternately in and not in the given 1-factor. A simple extension of the main result to n-connected graphs is also provided.

1. INTRODUCTION

A 1-factor of a graph G is a spanning subgraph in which each vertex has degree 1. In Figure 1, graph G₁ has a 1-factor (the heavy edges) whereas G₂ does not. Tutte [1, 2] has given the following characterization of graphs with 1-factors.

THEOREM (TUTTE).

A necessary and sufficient condition for a graph G to have a 1-factor is that for every set S of vertices of G, the order of S is at least as large as the number of components of G – S having an odd number of vertices.

Again in Figure 1 it may be seen that graph G₂ has a set of two vertices whose removal leaves three components of one vertex each. On the other hand, G₁ has a 1-factor in addition to the one shown. The main purpose

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of this note is to show that, if a non-separable graph (a block) with more than two points has a 1-factor, then it has more than one.

2. TERMINOLOGY

A graph $G$ consists of a finite non-empty set of vertices $V(G)$ and a set of edges $E(G)$ each of which is an unordered pair of vertices. The edge $uv$ is incident with each of its vertices. Two vertices (edges) joined by an edge (vertex) are said to be adjacent. The degree of a vertex $v$ is the number of edges incident with it. An end vertex $v$ has degree one.

A path $P$ of $G$ is an alternating sequence of distinct vertices and edges beginning and ending with vertices (said to be joined by $P$) such that each edge is incident with the vertices before and after it. If the first vertex of $P$ is $u$ and the last is $v$, we shall call $P$ a $u$-$v$ path. We shall also have occasion to denote a path by the sequence of its vertices, e.g., $P = v_1v_2v_3 \cdots v_n$. A path is said to be non-trivial if it contains at least one edge. A cycle consists of a path with at least two edges together with an additional edge joining the first and last vertices.

The graph $G$ is connected if every two vertices are joined by a path. A vertex $v$ is a cut vertex of the connected graph $G$ if the deletion of $v$ (together with all edges incident with $v$) results in a disconnected graph. A connected graph is called non-separable (or a block) if it has no cut vertices. A subgraph $B$ of a graph $G$ is a block of $G$ if it is a maximal connected subgraph of $G$ containing no cut vertices of itself. If $B$ is a block of the graph $G$ and further, if it contains exactly one cut vertex of $G$, it is called an end block of $G$. A graph is $n$-connected if the removal of no set of $n - 1$ or fewer vertices results in a disconnected graph.

3. THE MAIN THEOREM

Let $G$ be a graph with a 1-factor $F$. A cycle (of even length) of $G$ is called $F$-alternating if its edges are alternately in and not in $F$. Obviously,
if $G$ has an alternating cycle, it has another 1-factor, obtainable from $F$ by taking the other edges of this cycle. Our theorem will result from the following lemma.

**Lemma.** A nonseparable graph with more than two points and a 1-factor has an alternating cycle.

**Proof:** The proof is by contradiction, so we assume that $G$ is a non-separable graph which has a 1-factor $F$, but no alternating cycles. For convenience, the edges which are in $F$ will be called red and the other edges blue. Furthermore, in this proof, which is quite lengthy, a connected subgraph $H$ of $G$ is called $t$-admissible if it has the following properties:

1. $H$ has a 1-factor whose edges are in $F$.
2. $t$ is an end vertex of $H$.
3. $H$ has precisely two end blocks.
4. For any vertex $v$ of $H$, there is an alternating $t-v$ path.
5. If $v$ is on a cycle or is an end vertex in $H$, some alternating $t-v$ path has both end edges red.

Figure 2 gives some indication of what a $t$-admissible subgraph looks like. There is a "string" of blocks (perhaps consisting of single edges). Clearly, any non-trivial path from $t$ has a red edge at $t$, since $F$ is a 1-factor, and we speak of an alternating $t-v$ path as red-terminal if there is a red edge at both ends. Finally, we note the following: If $w$ is in the end block which does not contain $t$ and if $v$ is not a cut vertex of $H$, then every cut vertex of $H$ lies on every $t-w$ path.

![Figure 2](image)

Now, using induction on the number of edges, we will in effect show that $G$ itself is $t$-admissible, which is of course impossible. To begin with, $G$ obviously has a path $v_0v_1v_2v_3$ containing two red lines $v_0v_1$ and $v_2v_3$. Taking the vertex $v_0$ as $t$, this is clearly a $t$-admissible subgraph with three edges. Assume that $H$ is $t$-admissible and has $m$ edges. Let $B$ denote the end block of $H$ not containing $t$. Then, since $G$ is non-separable, there is a vertex $u$ in $B$, not a cut vertex of $H$, from which there is an edge $uw$ not in $H$. There are two cases to consider:
(a) If \( v \) is not in \( H \), since \( F \) is a 1-factor of \( G \), there is a red edge \( vw \) with \( w \) not in \( H \). By adding path \( uvw \) to \( H \), we form a new graph having \( m + 2 \) edges. That this new graph is also \( t \)-admissible is readily verified.

(b) The second case is somewhat more complicated. This time \( v \) is already in \( H \). Let \( H' \) be the graph obtained from \( H \) by adding the edge \( uv \). Since \( v \neq t \) (otherwise there would be an alternating cycle), it is clear that properties (2) and (3) hold for \( H' \). Furthermore, \( H' \) satisfies (1) and (4) since \( H \) did. Therefore, to show that \( H' \) is admissible, all that remains is to show that any vertex \( w \) which was not, but now is, on a cycle is joined to \( t \) by a red-terminal alternating path. Of course, we only need consider the case in which there was no red-terminal alternating \( t-w \) path in \( H \).

Since \( w \) is not on a cycle in \( H \), it is on the red-terminal alternating \( t-u \) path, and since by assumption there is no red-terminal alternating \( t-w \) path, there must be a red-terminal alternating \( w-u \) path.

We also claim that there is a red-terminal alternating \( t-v \) path. If \( v \) is on a cycle of \( H \), this follows from the fact that \( H \) is \( t \)-admissible. If \( v \) is not on a cycle and there is no such path, then (as for \( w \)) there would be a red-terminal alternating \( v-u \) path, which together with the edge \( uv \) gives an alternating cycle, which is impossible.

Next we observe that this \( t-v \) path must be disjoint from the \( w-u \) path. This is because \( w \) was put on a cycle by the addition of edge \( uv \) and was not on a cycle before.

This \( t-v \) path followed by the edge \( uv \) and the \( u-w \) path thus gives an alternating red-terminal \( t-w \) path. This proves that \( H' \) satisfies property (5) also and is therefore a \( t \)-admissible subgraph of \( G \) having \( m + 1 \) edges.

This shows that edges can always be added to \( t \)-admissible subgraphs until \( G \) is formed. However, since \( G \) is non-separable, it is not \( t \)-admissible. This contradiction completes the proof of the lemma. This lemma is essentially all that is needed for the main theorem.

**Theorem 1.** If a non-separable graph with more than two points has a 1-factor, it has more than one.

Now consider any graph with two 1-factors. Form the subgraph induced by the edges which lie in one or the other of these 1-factors, but not in both. In this subgraph, each vertex has degree 2 (1 from each 1-factor), and there is therefore a cycle with its edges alternately in the
two 1-factors. Taken together with the lemma, this observation proves the following result, which remains valid when the graph even possesses cut vertices.

**Theorem 2.** Any graph with a 1-factor $F$ has more than one if and only if it has an $F$-alternating cycle.

From the same sort of observation, the following corollary is obtained.

**Corollary.** Let $G$ be a graph with a 1-factor $F$ and let $x$ be an edge of $G$:

1. $x$ is in every 1-factor of $G$ if and only if $x$ is in $F$ and is in no $F$-alternating cycle;
2. $x$ is in no 1-factor of $G$ if and only if it is not in $F$ and is in no $F$-alternating cycle;
3. $x$ is in some, but not all, 1-factors of $G$ if and only if it is in an $F$-alternating cycle.

Can our main result be generalized to $n$-connected graphs? Our final theorem does this, but the result is not thought to be the best possible.

**Theorem 3.** If an $n$-connected graph has a 1-factor, then it has at least $n$ of them.

The proof is by induction, and the result is true for $n = 2$ by Theorem 1. Assume it is true for $n = k$, and suppose that $G$ is $(k + 1)$-connected and has a 1-factor $F$. Then there is an edge $x$ not in $F$ which lies in an $F$-alternating cycle. The removal of $x$ results in a graph $G'$ which is $k$-connected and has $F$ as a 1-factor, and therefore has at least $k$ 1-factors. But $G$ has at least one more, obtainable from $F$ by changing the edges of the $F$-alternating cycle containing $x$. This completes the proof.

**References**