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Socle degrees, resolutions, and Frobenius powers

Andrew R. Kustin^{a,*}, Bernd Ulrich^{b,2}

^a Mathematics Department, University of South Carolina, Columbia, SC 29208, United States

^b Mathematics Department, Purdue University, West Lafayette, IN 47907, United States

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ABSTRACT

We first describe a situation in which every graded Betti number in the tail of the resolution of R/J may be read from the socle degrees of R/J . Then we apply the above result to the ideals J and $J^{[q]}$; and thereby describe a situation in which the graded Betti numbers in the tail of the resolution of $R/J^{[q]}$ are equal to the graded Betti numbers in the tail of a shift of the resolution of R/J .

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Let (R, \mathfrak{m}) be a Noetherian graded algebra over a field of positive characteristic p , with irrelevant ideal \mathfrak{m} . Let J be an \mathfrak{m} -primary homogeneous ideal in R . Recall that if $q = p^e$, then the e th Frobenius power of J is the ideal $J^{[q]}$ generated by all j^q with $j \in J$. Recall, also, that the socle of $\frac{R}{J}$ is the ideal $(J:\mathfrak{m})$ of $\frac{R}{J}$. The *socle degrees* of $\frac{R}{J}$ are the degrees of any homogeneous basis for the graded vector space $\text{soc } \frac{R}{J}$. The basic question is:

Question 0.1. How do the socle degrees of $\frac{R}{J^{[q]}}$ vary with q ?

The question of finding a linear bound for the top socle degree of $R/J^{[q]}$ has been considered by Brenner in [1] from the point of view of finding inclusion–exclusion criteria for tight closure.

* Corresponding author.

E-mail addresses: kustin@math.sc.edu (A.R. Kustin), ulrich@math.purdue.edu (B. Ulrich).

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An answer to Question 0.1 would provide insight into the tight closure of J , and possibly a handle on Hilbert–Kunz functions.

We are particularly interested in how the socle degrees of Frobenius powers encode homological information about the ideal J . For example, the answer to Question 0.1 is well known in the case when J has finite projective dimension: if the socle degrees of $\frac{R}{J}$ are $\{\sigma_i \mid 1 \leq i \leq s\}$, then the socle degrees of $\frac{R}{J^{[q]}}$ are $\{q\sigma_i - (q - 1)a \mid 1 \leq i \leq s\}$, where a is the a -invariant of R . When R is a complete intersection, the converse is established in [6].

In the course of studying Question 0.1 we found some ideals J for which it appeared that the tail of the minimal R -resolution of $R/J^{[p^e]}$ did not depend on e . In other words, there exist a complex \mathbb{G}_\bullet , depending only on J , such that for each exponent e , there exists a twist n_e and a complex

$$\mathbb{F}_{e,d-1} \rightarrow \mathbb{F}_{e,d-2} \rightarrow \cdots \rightarrow \mathbb{F}_{e,0}$$

so that the minimal R -resolution of $R/J^{[p^e]}$ looks like

$$\mathbb{G}_\bullet(-n_e) \rightarrow \mathbb{F}_{e,d-1} \rightarrow \mathbb{F}_{e,d-2} \rightarrow \cdots \rightarrow \mathbb{F}_{e,0}, \tag{0.2}$$

where d is the Krull dimension of R . Our examples were computer calculations made using Macaulay2. The Betti number charts showed that (0.2) might be possible. We performed row and column operations on the matrices in position $d + 1$ to see that these matrices all presented the same d th-syzygy, up to shift; hence, confirming (0.2) for the examples under consideration. To emphasize how special this phenomenon is, we point out that we do not know the matrix in position d . Typically, the entries of this matrix have huge degrees (successive Frobenius powers grow very quickly); but magically, the degrees of the entries in the matrix in position $d + 1$ drop back down to the degrees that appeared in the resolution of R/J . Here is an example. Let P be the polynomial ring $\frac{\mathbb{Z}}{(5)}[x, y, z]$, f be the element $x^3 + y^3 + z^3$ of P , R be the hypersurface ring $P/(f)$, and J be the ideal (x^5, y^5, z^5) of R . The graded Betti numbers in the R -resolution of $R/J^{[p^e]}$ are:

e	pos 0	pos 1	pos 2	pos 3
0	0 : 1	5 : 3	8 : 3 9 : 1	9 : 1 10 : 3
1	0 : 1	25 : 3	38 : 3 39 : 1	39 : 1 40 : 3
2	0 : 1	125 : 3	188 : 3 189 : 1	189 : 1 190 : 3
3	0 : 1	625 : 3	938 : 3 939 : 1	939 : 1 940 : 3
4	0 : 1	3125 : 3	4688 : 3 4689 : 1	4689 : 1 4690 : 3

The notation 5 : 3 means that the module in position 1 in the minimal R -resolution of R/J is $R(-5)^3$. Every entry in the matrix in position 2 in the resolution of $R/J^{[p^4]}$ has degree 1563 or 1564; but every entry of each matrix in position 3 is linear or quadratic. Row and column operations applied to the matrices calculated by Macaulay2 show that in each resolution the matrix in position 3 may be taken to be

$$\begin{bmatrix} 0 & -x^2 & -y^2 & -2z \\ x^2 & 0 & -z^2 & 2y \\ y^2 & z^2 & 0 & -2x \\ 2z & -2y & 2x & 0 \end{bmatrix}. \tag{0.3}$$

The present paper is a first attempt at identifying growth conditions on the socle degrees of $R/J^{[p^e]}$ which force the tail of the minimal R -resolution of $R/J^{[p^e]}$ to be independent of e .

The main result in this paper is Theorem 1.1 which describes a situation in which every graded Betti number in the tail of the resolution of $\frac{R}{J}$ may be read from the socle degrees of $\frac{R}{J}$. In Corollary 2.1, we apply Theorem 1.1 to the ideals J and $J^{[q]}$; and thereby describe a situation in which the tail of the resolution of $R/J^{[q]}$ is isomorphic to a shift of the tail of the resolution of R/J as **graded modules**. The relationship between the differentials in the two resolution tails remains a project for future study.

Empirical evidence (see, for example, (0.3)) indicated that the map in position 3 in the resolution of R/J in Theorem 1.1 might be represented by an alternating matrix. In Section 3, we prove that this does indeed happen. Section 4 consists of a few examples and questions. In particular, we study the case where R has small multiplicity and the socle of R/J is pure. We find a situation where there is only one possibility for the dimension of the socle of R/J . We also compare the R -resolutions of R/J and the canonical module of R/J .

The present paper has inspired further investigation of the phenomenon

Phenomenon 0.4. *Sometimes, if the socle degrees of $R/J^{[q]}$ and R/J are related “correctly”, then the resolutions of $R/J^{[q]}$ and R/J share the same tail.*

One very interesting result along these lines is found in [5]. Let R be the hypersurface ring $k[x, y]/(f)$, where k is a field of characteristic p and $f = x^n + y^n$, and let J be the ideal $(x^N, y^N)R$. Conclusions (a), (b), and (c) are established in [5].

- (a) *The resolutions of R/J and $R/J^{[q]}$ have a common tail, in the sense of (0.2), if and only if $\text{soc}(R/J^{[q]})$ and $[\text{soc}(R/J)](-q-1)N$ are isomorphic as graded vector spaces.*

We emphasize that the common tail \mathbb{G}_\bullet in the setting of [5] is a complex **with differential**. In other words, the first syzygy module $\text{syz}_1^R(R/J^{[q]})$ is isomorphic to a shift of $\text{syz}_1^R(R/J)$. Conclusion (a) confirms Phenomenon 0.4 – including even the differential in the common tail – at least in the set-up of [5].

- (b) *Once N, n , and p are fixed, then there is a finite set of modules $\{M_i\}$ such that for each q there exists an i with $\text{syz}_1^R(R/J^{[q]})$ isomorphic to a shift of M_i .*

Conclusion (b) is astounding. It leads to the natural question:

Question. *What other rings R and ideals J have the property that the set of syzygy modules $\{\text{syz}_d^R(R/J^{[p^e]}) \mid 0 \leq e\}$ is finite?*

In this question, d is the Krull dimension of the ring R and two syzygy modules correspond to the same element of the finite set if one of the modules is isomorphic to a shift of the other module.

- (c) *Furthermore, given e , there exist n, p , and N such that $\text{syz}_1^R(R/J^{[p^i]})$ are distinct for $0 \leq i \leq e-1$, even after shifting; but $\text{syz}_1^R(R/J^{[p^k]})$ is isomorphic to a shift of $\text{syz}_1^R(R/J^{[p^{k+e}]})$, for all k .*

Conclusion (c) is a total surprise. We had been spoiled by the finite projective dimension case and the inability of the computer to compute more than a few successive Frobenius powers and we had become used to having “good behavior” become visible fairly quickly. In fact, we now see that even when “good behavior” is guaranteed to occur, one might have to wait arbitrarily long until it is visible.

We return to the present paper. Our situation is significantly more general than the situation of [5]; and of course, our conclusions are not as explicit. In particular, we usually think of the common tail \mathbb{G} as a graded module, not as a complex with differential.

We write ω_R for the graded canonical module of the graded ring R . See [6] for information about canonical modules and a -invariants. If m is a homogeneous element of a graded module M , then we write $|m|$ for the degree of M . We use $\mathfrak{s}^n(_)$ to indicate that the degree of an element has been

shifted by n . In other words, if m is an element of the graded module M , and n is an integer, then $s^n(m)$ is the element of $M(-n)$ which corresponds to m . In particular,

$$|s^n(m)| = |m| + n. \tag{0.5}$$

So,

$$m \in M_{|m|} \Rightarrow s^n(m) \in M(-n)_{|m|+n}.$$

Let $\mu(M)$ denote the minimal number of generators of the graded R -module M . We always use $(_)*$ to mean the functor $\text{Hom}_R(_, R)$; and we always use $\bar{_}$ to mean the functor $_ \otimes_P R$.

If I is a homogeneous Gorenstein ideal in a standard graded polynomial ring P , over a field, then the last non-zero module in a minimal homogeneous resolution of P/I by free P -modules has rank one and is equal to $P(-b)$ for some twist b . We refer to b as the back twist in the P -resolution of P/I and we observe that the a -invariant of P/I is equal to $b - \dim P$.

1. The main result

In this section k is a field of arbitrary characteristic. The main result in this paper is Theorem 1.1 which describes a situation in which every graded Betti number in the tail of the resolution of $\frac{R}{J}$ may be read from the socle degrees of $\frac{R}{I}$. Remark 1.14 contains alternate versions of hypotheses (b) and (c).

Theorem 1.1. *Let P be a polynomial ring in three variables over a field k . Each variable has degree 1. Let f be a non-zero homogeneous element of P , R be the hypersurface ring $R = P/(f)$, and a be the a -invariant of R . Let I be a homogeneous grade three Gorenstein ideal in P , b be the back twist in the P -resolution of $\frac{P}{I}$, and J be the ideal IR . Let*

$$\mathbb{F}_\bullet: \dots \xrightarrow{d_4} \mathbb{F}_3 \xrightarrow{d_3} \mathbb{F}_2 \xrightarrow{d_2} \mathbb{F}_1 \xrightarrow{d_1} R \rightarrow R/J \rightarrow 0 \tag{1.2}$$

be the graded minimal R -resolution of R/J , and $\{\sigma_i \mid 1 \leq i \leq s\}$ be the socle degrees of $\frac{R}{I}$. Assume that

- (a) I and J have the same number of minimal generators,
- (b) $\text{rank } \mathbb{F}_2 = \dim_k \text{soc } \frac{R}{I}$, and
- (c) $\sigma_i + \sigma_j \neq b + 2a$ for any pair (i, j) .

Then

- (A) $\mathbb{F}_2 = \bigoplus_{i=1}^s R(-b + a - \sigma_i)$,
- (B) $\mathbb{F}_3 = \bigoplus_{i=1}^s R(-(\sigma_i + 3))$, and
- (C) $\mathbb{F}_{i+2} = \mathbb{F}_i(-|f|)$, for all $i \geq 2$.

Proof. Let $Z = \text{im } d_2$.

In the first part of the argument we identify a submodule Z_1 of Z and prove that

$$\omega_{\frac{R}{J}}(-b - a) \cong \frac{I : f}{I}(-|f|) \cong \frac{Z}{Z_1} \tag{1.3}$$

are isomorphic as graded R -modules.

There is no difficulty finishing the proof of (A) once (1.3) has been established. Indeed, the definition of Z yields that \mathbb{F}_2 and Z have the same generator degrees. We know (see, for example, [6, Proposition 1.5]) that $\mu(\omega_{\frac{R}{J}}) = \dim \text{soc } \frac{R}{J}$; furthermore,

$$\text{the generator degrees of } \omega_{\frac{R}{J}} \text{ are } \{-\sigma_i \mid 1 \leq i \leq s\}. \tag{1.4}$$

Use hypothesis (b) to see that

$$\mu(Z) = \mu(\mathbb{F}_2) = \dim \text{soc } \frac{R}{J} = \mu(\omega_{\frac{R}{J}});$$

and therefore, (1.3) shows that $\omega_{\frac{R}{J}}(-b-a)$, Z , and \mathbb{F}_2 all have the same generator degrees. Conclusion (A) now follows immediately.

We establish the left-most isomorphism of (1.3). Observe that the surjection $\frac{P}{I} \rightarrow \frac{R}{J}$ induces the equality

$$\omega_{\frac{R}{J}} = \text{Hom}_{\frac{P}{I}}\left(\frac{R}{J}, \omega_{\frac{P}{I}}\right).$$

The a -invariant of $\frac{P}{I}$ is $b-3$; so, the canonical module of $\frac{P}{I}$ is $\frac{P}{I}(b-3)$ and

$$\omega_{\frac{R}{J}} = \text{Hom}_{\frac{P}{I}}\left(\frac{P}{(I, f)}, \frac{P}{I}(b-3)\right) = \frac{I:f}{I}(b-3).$$

Use the fact that

$$a = |f| - 3 \tag{1.5}$$

to see that the left-most isomorphism of (1.3) holds.

We define the submodule Z_1 of Z . Let $\text{syz}_2^P(\frac{P}{I})$ be the second syzygy of $\frac{P}{I}$ as a P -module and let

$$\mathbb{T}_\bullet: \quad 0 \rightarrow \mathbb{T}_3 \xrightarrow{t_3} \mathbb{T}_2 \xrightarrow{t_2} \mathbb{T}_1 \xrightarrow{t_1} P \rightarrow \frac{P}{I} \rightarrow 0 \tag{1.6}$$

be the minimal P -resolution of P/I . Due to the hypothesis $\mu(J) = \mu(I)$, we may choose \mathbb{T}_1 and t_1 so that $\mathbb{T}_1 \otimes_P R = \mathbb{F}_1$ and $t_1 \otimes_P R = d_1$. Apply $_ \otimes_P R$ to the exact sequence

$$0 \rightarrow \text{syz}_2^P\left(\frac{P}{I}\right) \rightarrow \mathbb{T}_1 \xrightarrow{t_1} P \rightarrow \frac{P}{I} \rightarrow 0$$

and compare the resulting complex to the exact sequence that defines Z :

$$\begin{array}{ccccccc} \text{syz}_2^P\left(\frac{P}{I}\right) \otimes_P R & \longrightarrow & \mathbb{T}_1 \otimes_P R & \xrightarrow{t_1 \otimes_P R} & P \otimes_P R & \longrightarrow & \frac{P}{I} \otimes_P R \\ & & \downarrow \alpha_1 & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{d_1} & R & \longrightarrow & \frac{R}{J} \end{array} \tag{1.7}$$

The bottom row is exact, so α_1 induces a map $\alpha_2: \text{syz}_2^P(\frac{P}{I}) \otimes_P R \rightarrow Z$. Let $Z_1 = \alpha_2(\text{syz}_2^P(\frac{P}{I}) \otimes_P R)$.

We establish the right-most isomorphism of (1.3). If $u \in (I : f)(-|f|)$, then $uf = t_1 \tau_1$ for some $\tau_1 \in \mathbb{T}_1$; and therefore, $\tau_1 \otimes R$ is in $\ker(t_1 \otimes R)$ and $\alpha_1(\tau_1 \otimes R) \in Z$. There is no difficulty in seeing that

$$h : (I : f)(-|f|) \rightarrow \frac{Z}{Z_1},$$

given by

$$h(u) = \text{the class of } \alpha_1(\tau_1 \otimes R), \text{ where } uf = t_1 \tau_1 \text{ for some } \tau_1 \in \mathbb{T}_1,$$

is a well-defined homomorphism of graded P -modules. It is clear that h is surjective and that $I(-|f|)$ is contained in the kernel of h .

Now we show that the kernel of h is contained in $I(-|f|)$. If $u \in (I : f)(-|f|)$ and $h(u)$ is zero in $\frac{Z}{Z_1}$, then there exists $\tau_1 \in \mathbb{T}_1$ with $uf = t_1 \tau_1$ and $\alpha_1(\tau_1 \otimes R) = \alpha_1(\tau'_1 \otimes R)$ in $Z \subseteq \mathbb{F}_1$, for some $\tau'_1 \in \text{syzygy}_2^P(\frac{P}{I})$. The hypothesis $\mu(I) = \mu(J)$ ensures that α_1 is an isomorphism; hence, $(\tau_1 - \tau'_1) \otimes R$ is zero in $\mathbb{T}_1 \otimes R$. In other words, there exists τ''_1 in \mathbb{T}_1 with $\tau_1 - \tau'_1 = f\tau''_1$. Apply t_1 to see that $uf = ft_1(\tau''_1)$ in P . We know that f is regular on P ; so we conclude that $u = t_1(\tau''_1) \in I(-|f|)$. Both isomorphisms of (1.3) have been established, and the first part of the argument is complete.

The second syzygy module Z is a maximal Cohen–Macaulay module over the two-dimensional ring R . A straightforward calculation allows us to decompose

$$\mathbb{F}_3 \xrightarrow{d_3} \mathbb{F}_2 \xrightarrow{d_2} Z$$

into

$$\mathbb{F}_3 \xrightarrow{\begin{bmatrix} d'_3 \\ 0 \end{bmatrix}} \mathbb{F}'_2 \oplus \mathbb{F}''_2 \xrightarrow{\begin{bmatrix} d'_2 & 0 \\ 0 & d''_2 \end{bmatrix}} \begin{matrix} Z' \\ \oplus \\ Z'' \end{matrix},$$

where Z'' is a free R -module, Z' is a maximal Cohen–Macaulay R -module with no free summands, and d''_2 is an isomorphism. Eisenbud’s ground breaking paper [3] guarantees that the minimal resolution

$$\dots \xrightarrow{d_4} \mathbb{F}_3 \xrightarrow{d'_3} \mathbb{F}'_2 \xrightarrow{d'_2} Z' \rightarrow 0 \tag{1.8}$$

of Z' by free R -modules is periodic of period 2. Furthermore, [3] guarantees that the maps d_4 and d'_3 may be pulled back to P to give a matrix factorization of f times the identity matrix.

In the second part of the argument, we show that

$$Z^*(a) \text{ and } \omega_{R/J} \text{ have the same generator degrees} \tag{1.9}$$

and

$$\mathbb{F}_3^*(-3) \text{ and } Z'^*(a) \text{ have the same generator degrees.} \tag{1.10}$$

Apply $\text{Hom}_R(_, R(a))$ to the short exact sequence

$$0 \rightarrow Z \rightarrow \mathbb{F}_1 \rightarrow J \rightarrow 0$$

to obtain the exact sequence

$$0 \rightarrow J^*(a) \rightarrow \mathbb{F}_1^*(a) \rightarrow Z^*(a) \rightarrow \text{Ext}_R^1(J, R(a)) \rightarrow 0.$$

Index shifting gives $\text{Ext}_R^1(J, R(a)) = \text{Ext}_R^2(\frac{R}{J}, R(a))$. The canonical module of R is equal to $\omega_R = R(a)$ and the canonical module of $\frac{R}{J}$ is $\omega_{\frac{R}{J}} = \text{Ext}_R^2(\frac{R}{J}, \omega_R)$; see, for example, [6, Proposition 1.2]. We have produced a homogeneous degree zero surjection

$$Z^*(a) \rightarrow \omega_{\frac{R}{J}} \rightarrow 0. \tag{1.11}$$

Apply $\text{Hom}_R(_, R)$ to (1.8) to see that $Z'^* = \ker d'_3$. Extend the periodic resolution (1.8) to the left to obtain the homogeneous minimal resolution

$$\dots \xrightarrow{d_4} \mathbb{F}_3 \xrightarrow{d'_3} \mathbb{F}'_2 \xrightarrow{d_4(|f|)} \mathbb{F}_3(|f|) \xrightarrow{d'_3(|f|)} \mathbb{F}'_2(|f|) \rightarrow Z'(|f|) \rightarrow 0. \tag{1.12}$$

The module Z' is a maximal Cohen–Macaulay R -module; hence, $\text{Ext}_R^i(Z', R)$ is zero for all positive i ; and therefore, the dual of (1.12), which is

$$0 \rightarrow (Z'(|f|))^* \rightarrow (\mathbb{F}'_2(|f|))^* \xrightarrow{(d'_3(|f|))^*} (\mathbb{F}_3(|f|))^* \xrightarrow{(d_4(|f|))^*} (\mathbb{F}'_2)^* \xrightarrow{d'_3} \mathbb{F}_3 \rightarrow \dots,$$

is exact. We have produced an isomorphism of graded R -modules:

$$\frac{(\mathbb{F}_3(|f|))^*}{\text{im}(d'_3(|f|))^*} \cong Z'^*.$$

The resolution (1.8) is minimal; hence, $\text{im}(d'_3(|f|))^*$ is contained in $\mathfrak{m}(\mathbb{F}_3(|f|))^*$ and $(\mathbb{F}_3(|f|))^*$ and Z'^* have the same generator degrees. Use (1.5) to see that (1.10) holds. Furthermore, we also see that

$$\begin{aligned} \mu(Z^*) &= \mu(Z'^*) + \mu(Z''^*) = \text{rank } \mathbb{F}_3^* + \text{rank } \mathbb{F}''_2 = \text{rank } \mathbb{F}'_2 + \text{rank } \mathbb{F}''_2 \\ &= \text{rank } \mathbb{F}_2 = \dim \text{soc } \frac{R}{J} = \mu(\omega_{\frac{R}{J}}). \end{aligned} \tag{1.13}$$

Combine (1.13) and (1.11) to see that (1.9) holds.

In the third part of the argument we prove that $Z'' = 0$. Suppose that \mathfrak{z} generates a free summand of Z . Let ζ be a homogeneous element of Z^* with $\zeta(\mathfrak{z}) = 1$. On the one hand, $\mathfrak{s}^{-a}(\zeta)$ is a minimal generator of $Z^*(a)$ of degree $-a - |\mathfrak{z}|$. (The shift function is explained in (0.5).) So (1.9), together with (1.4), yields that there exists i with

$$-a - |\mathfrak{z}| = -\sigma_i.$$

On the other hand, in the first part of the argument, we already calculated that

$$|\mathfrak{z}| = b + a - \sigma_j,$$

for some j . Hypothesis (c) guarantees that \mathfrak{z} does not exist. It follows that $Z'' = 0$, $Z = Z'$, and $\mathbb{F}'_2 = \mathbb{F}_2$. Use (1.10), (1.9), and (1.4) to establish conclusion (B) and (1.8) to establish conclusion (C). \square

Remark 1.14. The isomorphism

$$\omega_{\frac{R}{J}}(-b - a) \cong \frac{Z}{\text{im}(\text{syz}_2^P(\frac{P}{I}) \otimes_P R)}$$

of (1.3) would continue to hold even if the hypothesis $\text{rank } \mathbb{F}_2 = \dim_k \text{soc } \frac{R}{J}$ had not been imposed. In other words, in the context of Theorem 1.1, one always has

$$\dim \text{soc } \frac{R}{J} \leq \text{rank } \mathbb{F}_2,$$

and equality holds if and only if every P -syzygy of I is inside the maximal ideal times the module of R -syzygies of J .

We use hypothesis (c) to prove

(c') *the second syzygy of the R -module R/J does not have any free summands.*

Theorem 1.1 remains valid if one replaces hypothesis (c) with hypothesis (c'). However, there are two advantages to (c) over (c'). First, (c) is given in terms of the data $\{\sigma_i\}$, a , and b of Theorem 1.1. Second, in the proof of Corollary 2.1 we apply Theorem 1.1 to both R/J and $R/J^{[q]}$. If we assume that (c) holds for R/J , then the corresponding formula automatically holds for $R/J^{[q]}$, without making any further assumption.

2. The application to Frobenius powers

The present paper was motivated by the observation of Phenomenon 0.4: sometimes, if the socle degrees of $R/J^{[q]}$ and R/J are related “correctly”, then the resolutions of $R/J^{[q]}$ and R/J share the same tail, after a shift. In Corollary 2.1, we apply Theorem 1.1 twice and obtain a situation where the tail of the resolution of $R/J^{[q]}$ is a shift of the tail of the resolution of R/J , as **graded modules**.

Corollary 2.1. *Let P be a polynomial ring in three variables over a field k of positive characteristic p . Each variable has degree 1. Let f be a non-zero homogeneous element of P , R be the hypersurface ring $R = P/(f)$, and a be the a -invariant of R . Let I be a homogeneous grade three Gorenstein ideal in P , b be the back twist in the P -resolution of $\frac{P}{I}$, J be the ideal IR , $\{\sigma_i \mid 1 \leq i \leq s\}$ be the socle degrees of $\frac{R}{J}$, and \mathbb{F}_\bullet , as given in (1.2), be the graded minimal R -resolution of R/J . Let e be a fixed exponent, $q = p^e$, and*

$$\mathbb{F}_{e,\bullet}: \dots \xrightarrow{d_{e,4}} \mathbb{F}_{e,3} \xrightarrow{d_{e,3}} \mathbb{F}_{e,2} \xrightarrow{d_{e,2}} \mathbb{F}_{e,1} \xrightarrow{d_{e,1}} R \rightarrow R/J^{[q]} \rightarrow 0$$

be the graded minimal R -resolution of $R/J^{[q]}$. Assume that

- (a) I , J , and $J^{[q]}$ have the same number of minimal generators,
- (b) $\text{rank } \mathbb{F}_2 = \dim_k \text{soc } \frac{R}{J}$, and
- (c) $\sigma_i + \sigma_j \neq b + 2a$ for any pair (i, j) .

If

$$\text{soc } \frac{R}{J^{[q]}} \quad \text{and} \quad \left(\text{soc } \frac{R}{J} \right) \left(-\frac{b(q-1)}{2} \right) \tag{2.2}$$

are isomorphic as graded vector spaces, then

$$\mathbb{F}_{e,i} \quad \text{and} \quad \mathbb{F}_i\left(-\frac{b(q-1)}{2}\right)$$

are isomorphic as graded modules for all integers $i \geq 2$.

Proof. We first show that $\text{rank } \mathbb{F}_{e,2} = \dim_k \text{soc } \frac{R}{J^{[q]}}$. In light of Remark 1.14, this amounts to showing that every P -syzygy of $I^{[q]}$ is inside the maximal ideal times the module of R -syzygies of $J^{[q]}$. This does happen because the hypothesis tells us that every P -syzygy of I is inside the maximal ideal times the module of R -syzygies of J , and the Frobenius homomorphism is flat on P -modules so every P -syzygy of $I^{[q]}$ is the q th-Frobenius power of a P -syzygy of I .

Notice also that the analogue of hypothesis (c) holds for the ideal $J^{[q]}$ of R . One consequence of hypothesis (2.2) is that the socles of R/J and $R/J^{[q]}$ have the same dimension. Let $\{\sigma_{e,i} \mid 1 \leq i \leq s\}$ be the socle degrees of $R/J^{[q]}$. Hypothesis (2.2) yields that $\sigma_{e,i} + \sigma_{e,j} = \sigma_i + \sigma_j + b(q-1)$. We know that the back twist in the P -resolution of $\frac{P}{I^{[q]}}$ is

$$b_e = qb; \tag{2.3}$$

therefore, hypothesis (c) guarantees that $\sigma_{e,i} + \sigma_{e,j} \neq b_e + 2a$ for any pair (i, j) .

We apply Theorem 1.1 to I and to $I^{[q]}$. It follows from (2.3) that

$$b_e - \frac{b(q-1)}{2} = b + \frac{b(q-1)}{2};$$

and therefore

$$\mathbb{F}_{e,2} = \bigoplus_{i=1}^s R\left(-\left(b_e + a - \sigma_i - \frac{b(q-1)}{2}\right)\right) = \mathbb{F}_2\left(-\frac{b(q-1)}{2}\right).$$

The calculation $\mathbb{F}_{e,3} = \mathbb{F}_3\left(-\frac{b(q-1)}{2}\right)$ is even easier. Both resolutions $\mathbb{F}_{e,\bullet}$ and \mathbb{F}_\bullet are eventually periodic with $\mathbb{F}_{e,i+2} = \mathbb{F}_{e,i}(-|f|)$ and $\mathbb{F}_{i+2} = \mathbb{F}_i(-|f|)$ for $i \geq 2$. \square

Remark. If the polynomial f of Corollary 2.1 is irreducible and all of the minimal generators of I have the same degree, then the hypothesis $\mu(J^{[q]}) = \mu(J)$ automatically holds. Indeed, inflation of the base field $k \rightarrow K$ gives rise to faithfully flat extensions $P \rightarrow P \otimes_k K$ and $R \rightarrow R \otimes_k K$. Consequently, we may assume that k is a perfect field. If g is a minimal generator of $I^{[q]}$, then the hypothesis that all of the minimal generators of I have the same degree ensures that $g = \sum_{i=1}^n \alpha_i g_i^{[q]}$, where (g_1, \dots, g_n) is a minimal generating set for I , and each α_i is in k . The field k is perfect; so each α_i has a q th-root and $g = g_0^q$, for some minimal generator g_0 of I . If g were in (f) , then g_0 would also be in (f) , since (f) is a prime ideal.

Remark. The hypothesis of Corollary 2.1 is far from arbitrary; that is, the only possible number n with

$$\mathbb{F}_{e,3} = \mathbb{F}_3(-n) \quad \text{and} \quad \mathbb{F}_{e,2} = \mathbb{F}_2(-n) \tag{2.4}$$

is $n = \frac{b(q-1)}{2}$. Indeed, if (2.4) occurs, then Theorem 1.1 shows that

$$-\sigma_{e,i} - 3 = -\sigma_i - 3 - n \quad \text{and} \quad -b_e - a + \sigma_{e,i} = -b - a + \sigma_i - n, \tag{2.5}$$

where $\{\sigma_{e,i}\}$ and $\{\sigma_i\}$ are the socle degrees of $R/J^{[q]}$ and R/J , respectively. One may solve (2.5) to see that n must equal $\frac{b(q-1)}{2}$ and $\text{soc} \frac{R}{J^{[q]}}$ must equal $\text{soc} \frac{R}{J}(-n)$.

3. Alternating maps in the resolution of R/J

This section is a continuation of Section 1; the field k may have any characteristic. Corollary 3.1, which is the main result in the section, establishes that, for any ideal as in Theorem 1.1, the maps in the tail of the resolution are alternating. The proof appears at the end of the section.

Corollary 3.1. *If the notation and hypotheses of Theorem 1.1 are in effect, then there exist homogeneous alternating $s \times s$ matrices Φ and Ψ with entries in P such that $\Phi\Psi = fI = \Psi\Phi$ and*

$$\dots \xrightarrow{\psi} \mathbb{F}_3(-|f|) \xrightarrow{\varphi} \mathbb{F}_2(-|f|) \xrightarrow{\psi} \mathbb{F}_3 \xrightarrow{\varphi} \mathbb{F}_2 \xrightarrow{d_2} \mathbb{F}_1 \xrightarrow{d_1} R \tag{3.2}$$

is the minimal homogeneous resolution of R/J by free R -modules, where $\varphi = \overline{\Phi}$ and $\psi = \overline{\Psi}$.

The determinant of Φ cannot be zero; so one consequence of Corollary 3.1 is that s must be even.

There are three steps in the proof of Corollary 3.1. The proof of Theorem 1.1 depended on very careful analysis of Z and Z^* , where Z is the second syzygy module of R/J . Now that the proof of Theorem 1.1 is complete, we are able to prove that Z^* is isomorphic to a shift of Z , and this is the first step in the proof of Corollary 3.1. The isomorphism that we produce in Corollary 3.3 appears to be fairly abstract; however, at least part of it is induced by the alternating map in the Buchsbaum–Eisenbud resolution of the grade three Gorenstein ideal I . Furthermore, there exist a non-zerodivisor c in R with cZ contained in the part of Z on which the isomorphism is known to be alternating. This calculation appears in Proposition 3.10. The final step uses ideas from [4].

Corollary 3.3. *Retain the notation and hypotheses of Theorem 1.1 and its proof. Then the graded R -modules*

$$Z(b), \quad \left(\text{syz}_2^P \left(\frac{P}{I} \right) \otimes_P R \right)^*, \quad \text{and} \quad Z^*$$

are isomorphic.

Proof. Recall the P -resolution \mathbb{T}_\bullet of the codimension three Gorenstein ring P/I which is given in (1.6). The module \mathbb{T}_3 is equal to $P(-b)$ and the entries of the matrix t_3 generate the ideal I . The P -resolution of $\text{syz}_2^P(P/I)$ is

$$0 \rightarrow \mathbb{T}_3 \rightarrow \mathbb{T}_2 \rightarrow \text{syz}_2^P(P/I) \rightarrow 0,$$

and

$$\text{Tor}_1^P(\text{syz}_2^P(P/I), R) = \text{Tor}_3^P(P/I, R) = 0;$$

hence, the R -resolution of $\text{syz}_2^P(P/I) \otimes_P R$ is

$$0 \rightarrow \mathbb{T}_3 \otimes_P R \rightarrow \mathbb{T}_2 \otimes_P R \rightarrow \text{syz}_2^P(P/I) \otimes_P R \rightarrow 0. \tag{3.4}$$

Apply $(_)^* = \text{Hom}_R(_, R)$ to the short exact sequence (3.4) to see that

$$0 \rightarrow \left(\text{syz}_2^P \left(\frac{P}{I} \right) \otimes_P R \right)^* \rightarrow (\mathbb{T}_2 \otimes_P R)^* \xrightarrow{(t_3 \otimes 1)^*} (\mathbb{T}_3 \otimes_P R)^* \rightarrow \frac{R}{J}(b) \rightarrow 0$$

is exact. We have established that

$$Z(b) \cong \left(\text{syz}_2^P \left(\frac{P}{I} \right) \otimes_P R \right)^* \tag{3.5}$$

The isomorphism of (1.3) may be reconfigured as a short exact sequence

$$0 \rightarrow \text{syz}_2^P(P/I) \otimes_P R \xrightarrow{\alpha_2} Z \rightarrow \omega_{R/J}(-b-a) \rightarrow 0. \tag{3.6}$$

(The map $\text{syz}_2^P(P/I) \otimes_P R \rightarrow \mathbb{T}_1 \otimes_P R$ of (1.7) is injective because $\text{Tor}_1^P(I, R) = \text{Tor}_2^P(P/I, R) = 0$.) The grade of the annihilator of the R -module $\omega_{R/J}$ is two; hence, $\text{Ext}_R^i(\omega_{R/J}, R) = 0$ for $0 \leq i \leq 1$. Apply $(_)*$ to the exact sequence (3.6) to conclude that

$$\alpha_2^* : Z^* \rightarrow \left(\text{syz}_2^P \left(\frac{P}{I} \right) \otimes_P R \right)^* \tag{3.7}$$

is an isomorphism. \square

By carefully analyzing the isomorphisms (3.5) and (3.7), we are now able to learn much more about the isomorphism $Z^* \rightarrow Z(b)$ from Corollary 3.3. Keep the minimal P -resolution (\mathbb{T}, t) of P/I from (1.6). Fix an orientation isomorphism $[_]: \mathbb{T}_3 \rightarrow P(-b)$. Buchsbaum and Eisenbud [2] proved that \mathbb{T} has the structure of a $DG\Gamma$ -algebra. In particular, the maps $\eta_i : \mathbb{T}_i \rightarrow \text{Hom}_P(\mathbb{T}_{3-i}, P(-b))$, which are given by

$$\eta_i(\tau_i) = [\tau_i \cdot _], \tag{3.8}$$

for all $\tau_i \in \mathbb{T}_i$, give rise to an isomorphism of complexes $\mathbb{T}_\bullet \cong \text{Hom}_P(\mathbb{T}_\bullet, P(-b))$. Furthermore, if $\tau_2 \in \mathbb{T}_2$, then

$$t_2(\tau_2) \cdot \tau_2 = t_4(\tau_2^{(2)}) = 0. \tag{3.9}$$

Recall that $\bar{_}$ is the functor $_ \otimes_P R$. The $DG\Gamma$ -structure on \mathbb{T} induces a $DG\Gamma$ -structure on $\bar{\mathbb{T}}$.

In this discussion we take Z to be $\frac{\mathbb{F}_2}{\text{im } d_3}$. Every element of Z is equal to $q(y)$, for some element y of \mathbb{F}_2 , where $q : \mathbb{F}_2 \rightarrow \frac{\mathbb{F}_2}{\text{im } d_3} = Z$ is the natural quotient map. The shift function s^n is explained in (0.5).

Proposition 3.10. *Let $\ell : Z^* \rightarrow Z(b)$ be one of the isomorphisms from Corollary 3.3.*

(1) *One may choose ℓ so that for all $\zeta \in Z^*$ and $y \in \mathbb{F}_2$, the following statements are equivalent:*

- (a) $s^b(\ell(\zeta)) = q(y)$ in Z ,
- (b) $s^{-b}([d_2(y) \cdot \bar{\tau}_2]) = \zeta(q(y'))$ in R for all pairs $(\bar{\tau}_2, y') \in \bar{\mathbb{T}}_2 \times \mathbb{F}_2$ with $\bar{t}_2(\bar{\tau}_2) = d_2 y'$ in $\bar{\mathbb{T}}_1 = \mathbb{F}_1$.

(2) *The map ℓ from (1) is an alternating map in the sense that $\zeta(s^b \ell(\zeta))$ is equal to zero for all ζ in Z^* .*

Proof. We first prove (1). The map ℓ is the composition of isomorphisms

$$Z^* \xrightarrow{\alpha_2^*} \left(\text{syz}_2^P \left(\frac{P}{I} \right) \otimes_P R \right)^* \xrightarrow{\gamma} Z(b),$$

where α_2 is defined in (1.7) and γ is one the isomorphisms of (3.5). The map α_2^* is shown to be an isomorphism in (3.7). Eventually, we will make a particular choice for γ . Fix $\zeta \in Z^*$. Every element

of $Z(b)$ has the form $\mathbf{s}^{-b}(q(y))$ for some $y \in \mathbb{F}_2$. Fix one such y . We compare the elements $\alpha_2^*(\zeta)$ and $\gamma^{-1}(\mathbf{s}^{-b}(q(y)))$ of $(\text{syz}_2^P(\frac{P}{I}) \otimes_P R)^*$. For R -modules M and N , we let

$$\langle \cdot, \cdot \rangle : \text{Hom}_R(M, N) \otimes M \rightarrow N$$

represent the evaluation map. We write $\text{syz}_2^P(\frac{P}{I})$ as $\frac{\mathbb{T}_2}{\text{im } t_3}$. Let

$$q' : \mathbb{T}_2 \rightarrow \frac{\mathbb{T}_2}{\text{im } t_3} = \text{syz}_2^P\left(\frac{P}{I}\right)$$

be the natural quotient map. A typical element of $\text{syz}_2^P(\frac{P}{I}) \otimes R$ has the form $\overline{q'(\tau_2)}$ for some $\tau_2 \in \mathbb{T}_2$. We see that

$$\begin{aligned} \text{condition (a) holds} &\Leftrightarrow \ell(\zeta) = \mathbf{s}^{-b}(q(y)) \\ &\Leftrightarrow \alpha_2^*(\zeta) = \gamma^{-1}(\mathbf{s}^{-b}(q(y))) \\ &\Leftrightarrow \langle \alpha_2^*(\zeta), \overline{q'(\tau_2)} \rangle = \langle \gamma^{-1}(\mathbf{s}^{-b}q(y)), \overline{q'(\tau_2)} \rangle, \quad \text{for all } \tau_2 \in \mathbb{T}_2. \end{aligned}$$

We next compute $\langle \alpha_2^*(\zeta), \overline{q'(\tau_2)} \rangle$ and $\langle \gamma^{-1}(\mathbf{s}^{-b}q(y)), \overline{q'(\tau_2)} \rangle$.

If $\tau_2 \in \mathbb{T}_2$, then $t_2(\tau_2)$ is in the kernel of t_1 and $\overline{t_2(\tau_2)}$ is in the kernel of $\bar{t}_1 = d_1$, see (1.7); hence, there is an element y' in \mathbb{F}_2 with $d_2(y') = \overline{t_2(\tau_2)}$. It follows that $\alpha_2(\overline{q'(\tau_2)}) = q(y')$; and therefore,

$$\langle \alpha_2^*(\zeta), \overline{q'(\tau_2)} \rangle = \zeta(q(y')).$$

The argument of (3.5) shows that there exists an isomorphism ξ for which the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{d_2} & \mathbb{F}_1 = \bar{\mathbb{T}}_1 & \xrightarrow{\bar{t}_1} & \bar{\mathbb{T}}_0 \\ \xi \downarrow & & \eta_1 \downarrow & & \downarrow \\ \text{Hom}_R(\text{syz}_2^P(\frac{P}{I}) \otimes R, R(-b)) & \xrightarrow{\bar{q}^*} & \text{Hom}_R(\bar{\mathbb{T}}_2, R(-b)) & \xrightarrow{\bar{t}_3^*} & \text{Hom}_R(\bar{\mathbb{T}}_3, R(-b)) \end{array} \tag{3.11}$$

commutes, where η_1 is defined in (3.8). Twist the isomorphism

$$\xi : Z \rightarrow \text{Hom}_R\left(\text{syz}_2^P\left(\frac{P}{I}\right) \otimes R, R(-b)\right) = \left(\text{syz}_2^P\left(\frac{P}{I}\right) \otimes R\right)^*(-b)$$

by b to obtain the isomorphism

$$\xi(b) : Z(b) \rightarrow \left(\text{syz}_2^P\left(\frac{P}{I}\right) \otimes R\right)^*.$$

At this point we choose γ to be the inverse of $\xi(b)$. Therefore,

$$\gamma^{-1}(\mathbf{s}^{-b}q(y)) = (\xi(b))(\mathbf{s}^{-b}q(y)) = \mathbf{s}^{-b}(\xi(q(y))).$$

It follows that

$$\langle \gamma^{-1}(\mathbf{s}^{-b}q(y)), \overline{q'(\tau_2)} \rangle = \langle \mathbf{s}^{-b}(\xi(q(y))), \overline{q'(\tau_2)} \rangle = \mathbf{s}^{-b}(\langle \xi(q(y)), \overline{q'(\tau_2)} \rangle).$$

Follow the commutative square in (3.11) which defines ξ to see that

$$\langle \xi(q(y)), \overline{q'(\tau_2)} \rangle = [d_2(y) \cdot \bar{\tau}_2];$$

thus,

$$\langle \gamma^{-1}(\mathbf{s}^{-b}q(y)), \overline{q'(\tau_2)} \rangle = \mathbf{s}^{-b}([d_2(y) \cdot \bar{\tau}_2]),$$

and the proof of (1) is complete.

The proof of (2) has two parts. We first show that $\zeta(\mathbf{s}^b\ell(\zeta)) = 0$ whenever ζ is in Z^* and $\mathbf{s}^b\ell(\zeta)$ is an element of the submodule Z_1 (see (1.7)) of Z ; then we show that there is a non-zerodivisor c in R with $cZ \subseteq Z_1$. Let $\zeta \in Z^*$ and let y be an element of \mathbb{F}_2 with $\mathbf{s}^b\ell(\zeta) = q(y)$. Suppose first that there exists an element $\bar{\tau}_2$ of $\bar{\mathbb{T}}_2$ with $\bar{t}_2(\bar{\tau}_2) = d_2(y)$. In this case, we use (1) to see that

$$\zeta(\mathbf{s}^b\ell(\zeta)) = \zeta(q(y)) = \mathbf{s}^{-b}([d_2(y) \cdot \bar{\tau}_2]) = \mathbf{s}^{-b}([\bar{t}_2(\bar{\tau}_2) \cdot \bar{\tau}_2]) = 0.$$

The final equality follows from (3.9). We turn to the second part of the argument. The grade two ideal $J = IR$ is not contained in any associated prime ideal of the hypersurface ring R . We show that if c is any element of I and y is any element of \mathbb{F}_2 , then $cd_2(y)$ is in the image of \bar{t}_2 . Let τ_1 be any lifting of the element $d_2(y)$ from $\mathbb{F}_2 = \mathbb{T}_1 \otimes_P R$ back to \mathbb{T}_1 . The fact that $d_2(y)$ is in the kernel of $d_1 = \bar{t}_1$ ensures that there is an element u in P with $t_1(\tau_1) = uf$. There is an element τ'_1 in \mathbb{T}_1 with $t_1(\tau'_1) = c$. We see that

$$ct_1(\tau_1) = t_1(\tau'_1)uf;$$

and therefore, $c\tau_1 - uf\tau'_1$ is in $\ker t_1 = \text{im } t_2$. Apply $\bar{} = _ \otimes_P R$ to see that $cd_2(y) = c\bar{\tau}_1 \in \text{im } \bar{t}_2$. \square

Proof of Corollary 3.1. We saw at the end of the proof of Theorem 1.1 that Z is a maximal Cohen-Macaulay R -module with no free summands and therefore Z has a periodic resolution of period two which is induced by a matrix factorization of f . Indeed, the proof of [3, Theorem 6.1] shows that there exist matrices D_3 and D_4 over P so that

(1)

$$\begin{aligned} D_3 : \bigoplus_{i=1}^s P(-\sigma_i - 3) &\rightarrow \bigoplus_{i=1}^s P(\sigma_i - b - a) \quad \text{and} \\ D_4 : \bigoplus_{i=1}^s P(\sigma_i - b - a - |f|) &\rightarrow \bigoplus_{i=1}^s P(-\sigma_i - 3) \end{aligned} \tag{3.12}$$

are homogeneous maps,

(2) the map D_3 of (3.12) is a lift of $d_3 : \mathbb{F}_3 \rightarrow \mathbb{F}_2$ to P ,

(3) both product matrices D_3D_4 and D_4D_3 are equal to f times an identity matrix, and

(4) the complex

$$\dots \rightarrow \mathbb{F}_3(-|f|) \xrightarrow{\bar{D}_3} \mathbb{F}_2(-|f|) \xrightarrow{\bar{D}_4} \mathbb{F}_3 \xrightarrow{\bar{D}_3} \mathbb{F}_2 \rightarrow Z \rightarrow 0$$

is the minimal homogeneous resolution of Z .

It follows immediately, that

$$\dots \rightarrow \mathbb{F}_3(-|f|) \xrightarrow{\bar{D}_3} \mathbb{F}_2(-|f|) \xrightarrow{\bar{D}_4} \mathbb{F}_3 \xrightarrow{\bar{D}_3} \mathbb{F}_2 \xrightarrow{d_2} \mathbb{F}_1 \xrightarrow{d_1} \mathbb{F}_0 \rightarrow R/J \rightarrow 0$$

is the minimal homogeneous resolution of R/J . We will modify D_3 and D_4 to produce the desired matrices Φ and Ψ .

We next identify an alternating matrix M , with entries in R , so that

$$M : \bigoplus_{j=1}^s R(\sigma_j + 3 - |f|) \rightarrow \bigoplus_{i=1}^s R(b + a - \sigma_i) \tag{3.13}$$

is a homogeneous map and the maps

$$\bigoplus_{j=1}^s R(\sigma_j + 3 - |f|) \xrightarrow{M} \bigoplus_{i=1}^s R(b + a - \sigma_i) = \mathbb{F}_2^* \xrightarrow{d_3^*} \mathbb{F}_3^* \tag{3.14}$$

form an exact sequence.

Fix a basis y_1, \dots, y_s for $\mathbb{F}_2 = \bigoplus_{i=1}^s R(-b + a - \sigma_i)$, with the degree of y_i equal to $b + a - \sigma_i$, and let y_1^*, \dots, y_s^* be the corresponding dual basis for \mathbb{F}_2^* . Let z_1, \dots, z_s and ζ_1, \dots, ζ_s be generating sets for Z and Z^* , respectively, with $z_i = q(y_i)$ and $\zeta_i = \ell^{-1}(s^{-b}(z_i))$ for all i . (The shift function is explained in (0.5), ℓ is the isomorphism of Proposition 3.10(1), and q is defined in the paragraph before Proposition 3.10.) Form the matrix M with the element $\zeta_j(z_i)$ of R in position (i, j) . Notice that $\zeta_j(z_i)$ has degree $b + 2a - \sigma_i - \sigma_j$; and hence, the map in (3.13) is a homogeneous map. Furthermore, Proposition 3.10(2) shows that M is an alternating matrix. We have created M so that M carries the j th basis vector, v_j , of $\bigoplus R(\sigma_j + 3 - |f|)$ to $q^*(\zeta_j) \in \mathbb{F}_2^*$. Indeed, $M(v_j) = \sum \zeta_j(z_i)y_i^* \in \mathbb{F}_2^*$, and, if $y \in \mathbb{F}_2$, then $M(v_j)$ sends y to

$$\begin{aligned} \zeta_j \left(\sum y_i^*(y) \cdot z_i \right) &= \zeta_j \left(\sum y_i^*(y) \cdot q(y_i) \right) = \zeta_j \left(q \left(\sum y_i^*(y) \cdot y_i \right) \right) \\ &= \zeta_j(q(y)) = (q^*(\zeta_j))(y). \end{aligned}$$

It follows that the image of M in \mathbb{F}_2^* is equal to the image of $q^* : Z^* \rightarrow \mathbb{F}_2^*$. On the other hand, we may apply $(_)^* = \text{Hom}_R(_, R)$ to the exact sequence

$$\mathbb{F}_3 \xrightarrow{d_3} \mathbb{F}_2 \xrightarrow{q} Z \rightarrow 0$$

to obtain the exact sequence

$$0 \rightarrow Z^* \xrightarrow{q^*} \mathbb{F}_2^* \xrightarrow{d_3^*} \mathbb{F}_3^*.$$

We have now shown that (3.14) is an exact sequence.

We may apply [4, Lemma 2.1] to find a homogeneous invertible matrix

$$\varepsilon : \bigoplus_i P(\sigma_i + 3) \rightarrow \bigoplus_i P(\sigma_i + 3),$$

with entries in P , so that εD_3^T is an alternating matrix. Define Φ to be the alternating matrix $D_3 \varepsilon^T$ and Ψ to be the matrix $(\varepsilon^T)^{-1} D_4$. It is clear that (Φ, Ψ) is a matrix factorization of fI and that the complex of (3.2) is a minimal homogeneous resolution. We need only verify that Ψ is an alternating matrix. To do this, we may look in the quotient field of P , where Ψ is equal to $f\Phi^{-1} = (f/\det \Phi) \text{Adj } \Phi$. Observe that $\text{Adj } \Phi$, which is the classical adjoint of Φ , is an alternating matrix. \square

4. Examples, further comments, and questions

In this section we ask what happens in the situation of Theorem 1.1 when R has small multiplicity and the socle of R/J is pure. Also, we compare the R -resolutions of R/J and $\omega_{R/J}$.

In Proposition 4.1, R has small multiplicity and the socle of R/J lives in exactly one degree. In this case, all of the relevant information (the dimension of the socle of R/J , the degrees of the entries of the matrices d_3 and d_4 , and the degree of the socle elements of R/J) is determined by the parity of the back twist b in the P -resolution of P/I . We give an example for each parity.

Proposition 4.1. *Adopt all of the notation and hypotheses of Theorem 1.1. Assume $|f| = 3$ and the socle of R/J lives in exactly one degree.*

- (a) *If b is odd, then $\dim \text{soc } R/J = \frac{3}{2}(\mu(I) - 1)$, every entry of d_3 must have degree 2, every entry of d_4 must have degree 1, and $\sigma_i = \frac{b-1}{2}$, for all i .*
- (b) *If b is even, then $\dim \text{soc } R/J = 3(\mu(I) - 1)$, every entry of d_3 must have degree 1, every entry of d_4 must have degree 2, and $\sigma_i = \frac{b}{2} - 1$, for all i .*

Proof. Notice that, in the language of Theorem 1.1, $a = a(R) = |f| - 3 = 0$. Recall that $s = \dim \text{soc } R/J$ and that $Z = \text{im } d_2$. Let σ equal to the common value of σ_i , for $1 \leq i \leq s$, $N_2 = b - \sigma$, and $N_3 = \sigma + 3$. Theorem 1.1 establishes the exact sequence:

$$0 \rightarrow Z(-3) \rightarrow R(-N_3)^s \xrightarrow{d_3} R(-N_2)^s \rightarrow Z \rightarrow 0. \tag{4.2}$$

Every entry of the matrix d_3 has the same degree and we denote this degree by $\text{deg } d_3$. It is clear that

$$\text{deg } d_3 = N_3 - N_2 = 2\sigma + 3 - b.$$

In other words, $b + \text{deg } d_3 - 3$ must be even and

$$\sigma = \frac{b + \text{deg } d_3 - 3}{2}.$$

The matrices d_4 and d_3 can be lifted to P to give a matrix factorization of fI and $|f| = 3$. So, $\text{deg } d_3 + \text{deg } d_4 = 3$ and $\text{deg } d_3$ is equal to either 1 or 2. If $\text{deg } d_3 = 1$, then b is even and $\sigma = \frac{b}{2} - 1$. If $\text{deg } d_3 = 2$, then b is odd and $\sigma = \frac{b-1}{2}$.

Let $e(_)$ represent multiplicity. When n is large, (4.2) yields

$$\dim Z_n - \dim Z_{n-3} = s[\dim R_{n-N_2} - \dim R_{n-N_3}]. \tag{4.3}$$

The left side of (4.3) is $3e(Z)$. The right side is

$$se(R)[N_3 - N_2] = se(R) \deg d_3.$$

The R -module Z has positive rank equal to $\mu(I) - 1$; therefore,

$$e(Z) = e(R) \text{rank}(Z) = e(R)(\mu(I) - 1)$$

and $3(\mu(I) - 1) = s \deg d_3$. We have

$$s = \frac{3(\mu(I) - 1)}{\deg d_3},$$

with $\deg d_3$ equal to 1 or 2. Recall that the ideal I is a grade three Gorenstein ideal; consequently, $\mu(I)$ is automatically odd. \square

Example 4.4. Let $P = k[x, y, z]$, where k is a field of characteristic p , f be the polynomial $x^3 + y^3 + z^3$, I be the grade three Gorenstein ideal $(x^2, xz, xy + z^2, yz, y^2)$ of P (see [2, Proposition 6.2]), $R = P/(f)$ and $J = IR$. The following calculations were made using Macaulay2. We first give numerical information about the socle and minimal resolution of $R/J^{[p^e]}$, when $p = 5$. The notation $2 : 5$ under “pos 1” next to $e = 0$ means that the module in position 1 in the minimal R -resolution of R/J is $R(-2)^5$. The notation $12 : 6$ under “socle” next to $e = 1$ means that the socle of $R/J^{[p^1]}$ is minimally generated by 6 generators, each of degree 12. The hypotheses of Theorem 1.1 apply to $J^{[p^e]}$ for $1 \leq e \leq 4$. Notice that $b_e = 5(5^e)$ is odd, $s_e = \frac{3}{2}(\mu(I) - 1)$, $\deg d_{e,3} = 2$, $\deg_{e,4} = 1$, and $\sigma_{e,i} = \frac{b_e - 1}{2}$.

e	socle	pos 0	pos 1	pos 2	pos 3	pos 4
0	2 : 1	0 : 1	2 : 5	3 : 6	5 : 6	6 : 6
1	12 : 6	0 : 1	10 : 5	13 : 6	15 : 6	16 : 6
2	62 : 6	0 : 1	50 : 5	63 : 6	65 : 6	66 : 6
3	312 : 6	0 : 1	250 : 5	313 : 6	315 : 6	316 : 6
4	1562 : 6	0 : 1	1250 : 5	1563 : 6	1565 : 6	1566 : 6

Here is numerical information when $p = 2$. The hypotheses of Theorem 1.1 apply to $J^{[p^e]}$ for $2 \leq e \leq 4$. Notice that $b_e = 5(2^e)$ is even, $s_e = 3(\mu(I) - 1)$, $\deg d_{e,3} = 1$, $\deg_{e,4} = 2$, and $\sigma_{e,i} = \frac{b_e}{2} - 1$.

e	socle	pos 0	pos 1	pos 2	pos 3	pos 4
0	2 : 1	0 : 1	2 : 5	3 : 6	5 : 6	6 : 6
1	4 : 7	0 : 1	4 : 5	6 : 12	7 : 12	9 : 12
2	9 : 12	0 : 1	8 : 5	11 : 12	12 : 12	14 : 12
3	19 : 12	0 : 1	16 : 5	21 : 12	22 : 12	24 : 12
4	39 : 12	0 : 1	32 : 5	41 : 12	42 : 12	44 : 12

Return to the situation of Theorem 1.1. We notice that the infinite tails of the R -resolutions of R/J and $\omega_{R/J}$ are equal. We wonder how often this phenomenon occurs.

Proposition 4.5. Retain the notation of Theorem 1.1. If \mathbb{L}_\bullet is the minimal R -resolution of $\omega_{R/J}$, then the truncation

$$\mathbb{L}_{\geq 3} : \cdots \rightarrow \mathbb{L}_4 \rightarrow \mathbb{L}_3$$

is isomorphic, as a complex, to a shift of the truncation $\mathbb{F}_{\geq 3}$.

Proof. We know the R -resolution of Z :

$$\cdots \rightarrow \mathbb{F}_3 \rightarrow \mathbb{F}_2 \rightarrow Z \rightarrow 0. \tag{4.6}$$

The R -resolution of $\text{syz}_2^P(P/I) \otimes_P R$ is given in (3.4). Consider the short exact sequence (3.6). The map α_2 may be lifted to a comparison of complexes from (3.4) to (4.6); the mapping cone of the resulting map of complexes is a resolution of $\omega_{R/J}(-b - a)$:

$$\cdots \rightarrow \mathbb{F}_5 \rightarrow \begin{array}{c} \mathbb{T}_3 \otimes_P R \\ \oplus \\ \mathbb{F}_4 \end{array} \rightarrow \begin{array}{c} \mathbb{T}_2 \otimes_P R \\ \oplus \\ \mathbb{F}_3 \end{array} \rightarrow \mathbb{F}_2 \rightarrow \omega_{R/J}(-b - a) \rightarrow 0.$$

It follows that $\mathbb{L}_{\geq 3}$ is

$$\cdots \rightarrow \mathbb{F}_6(b + a) \rightarrow \mathbb{F}_5(b + a),$$

which is equal to $\mathbb{F}_{\geq 3}(-|f| + b + a) = \mathbb{F}_{\geq 3}(b - 3)$. \square

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