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On inexact Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems

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Abstract

We study theoretical properties of two inexact Hermitian/skew-Hermitian splitting (IHSS) iteration methods for the large sparse non-Hermitian positive definite system of linear equations. In the inner iteration processes, we employ the conjugate gradient (CG) method to solve the linear systems associated with the Hermitian part, and the Lanczos or conjugate gradient for normal equations (CGNE) method to solve the linear systems associated with the skew-Hermitian part, respectively, resulting in IHSS(CG, Lanczos) and IHSS(CG, CGNE) iteration methods, correspondingly. Theoretical analyses show that both IHSS(CG, Lanczos) and IHSS(CG, CGNE) converge unconditionally to the exact solution of the non-Hermitian positive definite linear system. Moreover, their contraction factors and asymptotic convergence rates are dominantly dependent on the spectrum of the Hermitian part, but are less dependent on the spectrum of the skew-Hermitian part, and are independent of the eigenvectors of the matrices involved. Optimal choices of the inner iteration steps in the IHSS(CG, Lanczos) and IHSS(CG, CGNE) iterations are discussed in detail by

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considering both global convergence speed and overall computation workload, and computational efficiencies of both inexact iterations are analyzed and compared deliberately.

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1. Introduction

We consider the solution of large sparse system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n} \text{ non-singular, and } x, b \in \mathbb{C}^n, \tag{1.1}$$

where A is a non-Hermitian and positive definite matrix. Because the coefficient matrix A naturally possesses a Hermitian/skew-Hermitian (HS) splitting [12,18,14,17] $A = H + S$, with

$$H = \frac{1}{2}(A + A^*) \quad \text{and} \quad S = \frac{1}{2}(A - A^*).$$

Bai et al. [5] recently presented the following *Hermitian/skew-Hermitian splitting* (HSS) method to iteratively compute a reliable and accurate approximate solution for the system of linear equations (1.1):

The HSS iteration method. Given an initial guess $x^{(0)}$. For $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{1.2}$$

where α is a given positive constant.

The HSS iteration (1.2) converges unconditionally to the exact solution of the system of linear equations (1.1), and the upper bounds of its contraction factor in a special weighted norm and its asymptotic convergence rate are only dependent on the spectrum of the Hermitian part H, but are independent of the spectrum of the skew-Hermitian part S as well as the eigenvectors of the matrices H, S and A. In addition, the optimal value of the parameter α can be determined by the lower and the upper eigenvalue bounds of the matrix H.

To invert the matrices $\alpha I + H$ and $\alpha I + S$ efficiently at each step of the HSS iteration in actual implementations, Bai et al. [5] further proposed to solve the linear systems with coefficient matrices $\alpha I + H$ and $\alpha I + S$ inexactly by iterative methods, e.g., solving the linear systems with coefficient matrix $\alpha I + H$ by the conjugate gradient (CG) method [15] and those with coefficient matrix $\alpha I + S$ by the Lanczos [12,18] or the conjugate gradient for normal equations (CGNE) method [15], to some prescribed accuracies, and obtained two special but quite practical *inexact Hermitian/skew-Hermitian splitting* (IHSS) iterations, briefly called as IHSS(CG, Lanczos) and IHSS(CG, CGNE), respectively. See [1,9,7] for a similar approach.

The main aim of this paper is to study the convergence properties of both IHSS(CG, Lanczos) and IHSS(CG, CGNE) in depth and investigate the optimal numbers of inner iteration steps in detail by considering both global convergence speed and overall computation workload. In particular, we show that the asymptotic convergence rates of IHSS(CG, Lanczos) and IHSS(CG, CGNE) are essentially the same. In addition, the convergence rates of IHSS(CG, Lanczos) and IHSS(CG, CGNE) tend to the convergence rate of HSS when the tolerances of the inner iterations tend to zero as the outer iterations increase. We also investigate their computational efficiencies of IHSS(CG, Lanczos) and IHSS(CG, CGNE). We find that they are quite comparable.

The organization of this paper is as follows. In Section 2, we establish IHSS(CG, Lanczos) and IHSS(CG, CGNE) iterations. In Section 3, we review some useful lemmas. In Section 4, we study their convergence properties. Their computational efficiencies are analyzed and compared in Section 5, and numerical results for IHSS iterations are given in Section 6. Finally, in Section 7 we draw a brief conclusion and some remarks. In Appendix, we give the proofs of the theorems presented in Section 4.

2. The IHSS iterations

The two-half steps at each HSS iterate require exact inverses of the n-by-n matrices $\alpha I + H$ and $\alpha I + S$. However, this may be very costly and impractical in actual implementations. To overcome this disadvantage and further improve efficiency of the HSS iteration, we can solve the resulting two sub-problems iteratively. More specifically, we may employ CG to solve the system of linear equations with coefficient matrix $\alpha I + H$ and some Krylov subspace method such as Lanczos or CGNE to solve the system of linear equations with coefficient matrix $\alpha I + S$ to some prescribed accuracies at each step of the HSS iteration.[3] This results in the inexact Hermitian/skew-Hermitian splitting (IHSS) iterations based on CG and Lanczos (IHSS(CG, Lanczos)), or based on CG and CGNE (IHSS(CG, CGNE)), for solving the system of linear equations (1.1).

The tolerances (or numbers of inner iteration steps) for CG and Lanczos (or CGNE) may be different and changed according to the outer iterate. Therefore, the resulted IHSS iterations are actually two non-stationary iterative methods for solving the system of linear equations (1.1). Moreover, when the tolerances of the inner iterations tend to zero as the outer iterations increase, the asymptotic convergence rates of the IHSS iterations approach that of the HSS iteration.

2.1. *IHSS(CG, Lanczos)*

For the IHSS(CG, Lanczos) iteration, employing the CG to solve linear systems with coefficient matrix $\alpha I + H$ is quite natural, because $\alpha I + H$ is Hermitian positive definite. For iteration solvers for the linear systems with the coefficient matrix $\alpha I + S$, we can choose the Lanczos method studied in [12,18]; it has a three-term recurrence form which has an unconditional convergence property, and a comparable computation workload to that of CG.

The IHSS(CG, Lanczos) iteration method. Input an initial guess $x^{(0)}$, the stopping tolerance ε for the outer iteration, the largest admissible number $k_{\max}$ of the outer iteration steps, two stopping tolerances ε_{cg} and ε_{lan} for the inner CG and the inner Lanczos iterations, and two positive integer sequences $\{\mu_k\}$ and $\{\nu_k\}$ of the largest admissible inner CG and inner Lanczos iteration steps, respectively.

1. $k := 0$.
2. $r^{(k)} = b - Ax^{(k)}$ and $\rho_k = \|r^{(k)}\|_2^2$.

[3] $\alpha I + H$ and $\alpha I + S$ are called shifted Hermitian and shifted skew-Hermitian matrices, respectively.

3. `if` $\sqrt{\rho_k} \leqslant \varepsilon\|b\|_2$ `or` $k > k_{\max}$ `then goto 10.`
4. `call cg`$(H, \alpha, r^{(k)}, \rho_k, \mu_k, \varepsilon_{cg}, y^{(\mu_k)})$.
5. $x^{(k+\frac{1}{2})} = x^{(k)} + y^{(\mu_k)}$.
6. $r^{(k+\frac{1}{2})} = \alpha y^{(\mu_k)} - Sy^{(\mu_k)}$ `and` $\rho_{k+\frac{1}{2}} = \|r^{(k+\frac{1}{2})}\|_2^2$.
7. `call Lanczos` $(S, \alpha, r^{(k+\frac{1}{2})}, \rho_{k+\frac{1}{2}}, \nu_k, \varepsilon_{lan}, z^{(\nu_k)})$.
8. $x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(\nu_k)}$.
9. `Set` $k := k + 1$ `and goto 2.`
10. `Set` $x := x^{(k)}$ `and output` x.

---

subroutine `cg`$(H, \alpha, r, \rho, \mu, \varepsilon_{cg}, y)$

1. $y := 0$, $\rho_0 := \rho$ `and` $\ell := 0$
2. `if` $\rho_\ell \leqslant \varepsilon_{cg}^2 \rho_0$ `or` $\ell > \mu$
   `then output` y
   `else`
   (a) `if` $\ell = 0$ `then` $v = r$
       `else`
       $\beta = \frac{\rho_\ell}{\rho_{\ell-1}}$ `and` $v := r + \beta v$
   (b) $w = \alpha v + Hv$
   (c) $\omega = \frac{\rho_\ell}{v^* w}$
   (d) $y := y + \omega v$
   (e) $r := r - \omega w$
   (f) $\rho_{\ell+1} = \|r\|_2^2$
   (g) $\ell := \ell + 1$

`Subroutine of CG iteration for the linear system` $(\alpha I + H)y = r$

subroutine `Lanczos`$(S, \alpha, r, \rho, \nu, \varepsilon_{lan}, y)$

1. $y = w := 0$, $\rho_0 := \rho$ `and` $\ell := 0$
2. `if` $\rho_\ell \leqslant \varepsilon_{lan}^2 \rho_0$ `or` $\ell > \nu$
   `then output` y
   `else`
   (a) `if` $\ell = 0$ `then` $\omega = 1$
       `else`
       $\beta = \frac{\rho_\ell}{\rho_{\ell-1}}$ `and` $\omega = \frac{\omega}{\omega+\beta}$
   (b) $w := \frac{\omega}{\alpha} r - (1 - \omega)w$
   (c) $u = \alpha w + Sw$
   (d) $y := y + w$
   (e) $r := r - u$
   (f) $\rho_{\ell+1} = \|r\|_2^2$
   (g) $\ell := \ell + 1$

`Subroutine of Lanczos iteration for the linear system` $(\alpha I + S)y = r$

Assume that $\chi(H)$ and $\chi(S)$ are the flops required to compute the matrix-vector products Hy and Sy, respectively, for a given vector $y \in \mathbb{C}^n$. Then straightforward computations show that each step of the CG and the Lanczos iterations requires $\chi(H) + 12n$ and $\chi(S) + 9n + 4$ flops, respectively, and each step of the IHSS(CG, Lanczos) iteration requires another $\chi(H) + 2\chi(S) + 7n$ flops besides the amount of operations of the `cg` and the `Lanczos` subroutines. Therefore, the total workload at each step of the IHSS(CG, Lanczos) iteration is

$$\mu_k(\chi(H) + 12n) + \nu_k(\chi(S) + 9n + 4) + \chi(H) + 2\chi(S) + 7n$$

flops. If we assume that each row of the matrices H and S has at most τ_h and τ_s non-zero entries, respectively, then $\chi(H) = (2\tau_h - 1)n$, $\chi(S) = (2\tau_s - 1)n$, and the total workload at each step of the IHSS(CG, Lanczos) iteration is therefore

$$W(\tau_h, \tau_s, \mu_k, \nu_k) = (2\tau_h + 11)n\mu_k + [(2\tau_s + 8)n + 4]\nu_k + (2\tau_h + 4\tau_s + 4)n.$$

Because in many applications (e.g., discretization matrices from partial differential equations) we have $\tau \equiv \tau_h = \tau_s + 1$, it immediately follows that

$$\begin{aligned} W(\tau, \mu_k, \nu_k) &\equiv W(\tau_h, \tau_s, \mu_k, \nu_k) \\ &= (2\tau + 11)n\mu_k + 2[(\tau + 3)n + 2]\nu_k + 6\tau n. \end{aligned} \tag{2.1}$$

2.2. IHSS(CG, CGNE)

Besides the Lanczos method, we can also solve the sub-systems of linear equations with coefficient matrix $\alpha I + S$ by other Krylov subspace methods such as CGNE [15] at each step of the IHSS iterate. Like Lanczos, CGNE also has a three-term recurrence form, an unconditional and monotonical convergence property, and a comparable computer storage and computation workload to that of CG.

The IHSS(CG, CGNE) iteration method. Input an initial guess $x^{(0)}$, the stopping tolerance ε for the outer iteration, the largest admissible number $k_{\max}$ of the outer iteration steps, two stopping tolerances ε_{cg} and ε_{cgne} for the inner CG and the inner CGNE iterations, and two positive integer sequences $\{\mu_k\}$ and $\{\nu_k\}$ of the largest admissible inner CG and inner CGNE iteration steps, respectively.

1. $k := 0$.
2. $r^{(k)} = b - Ax^{(k)}$ and $\rho_k = \|r^{(k)}\|_2^2$.
3. if $\sqrt{\rho_k} \leqslant \varepsilon\|b\|_2$ or $k > k_{\max}$ then goto 10.
4. call cg$(H, \alpha, r^{(k)}, \rho_k, \mu_k, \varepsilon_{cg}, y^{(\mu_k)})$.
5. $x^{(k+\frac{1}{2})} = x^{(k)} + y^{(\mu_k)}$.
6. $r^{(k+\frac{1}{2})} = \alpha y^{(\mu_k)} - Sy^{(\mu_k)}$ and $\rho_{k+\frac{1}{2}} = \|r^{(k+\frac{1}{2})}\|_2^2$.
7. call cgne$(S, \alpha, r^{(k+\frac{1}{2})}, \rho_{k+\frac{1}{2}}, \nu_k, \varepsilon_{cgne}, z^{(\nu_k)})$.
8. $x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(\nu_k)}$.
9. Set $k := k + 1$ and goto 2.
10. Set $x := x^{(k)}$ and output x.

subroutine cg$(H, \alpha, r, \rho, \mu, \varepsilon_{cg}, y)$

1. $y := 0$, $\rho_0 := \rho$ and $\ell := 0$
2. if $\rho_\ell \leqslant \varepsilon_{cg}^2\rho_0$ or $\ell > \mu$ then output y else
   (a) if $\ell = 0$ then $v = r$ else $\beta = \frac{\rho_\ell}{\rho_{\ell-1}}$ and $v := r + \beta v$
   (b) $w = \alpha v + Hv$
   (c) $\omega = \frac{\rho_\ell}{v^* w}$
   (d) $y := y + \omega v$
   (e) $r := r - \omega w$
   (f) $\rho_{\ell+1} = \|r\|_2^2$
   (g) $\ell := \ell + 1$

Subroutine of CG iteration for the linear system $(\alpha I + H)y = r$

subroutine cgne$(S, \alpha, r, \rho, \nu, \varepsilon_{cgne}, y)$

1. $y := 0$, $\rho_0 := \rho$ and $\ell := 0$
2. if $\rho_\ell \leqslant \varepsilon_{cgne}^2\rho_0$ or $\ell > \nu$ then output y else
   (a) if $\ell = 0$ then $v = r$ else $\beta = \frac{\rho_\ell}{\rho_{\ell-1}}$ and $v := r + \beta v$
   (b) $w = \alpha^2 v - S^2 v$
   (c) $\omega = \frac{\rho_\ell}{v^* w}$
   (d) $y := y + \omega v$
   (e) $r := r - \omega w$
   (f) $\rho_{\ell+1} = \|r\|_2^2$
   (g) $\ell := \ell + 1$

Subroutine of CGNE iteration for the linear system $(\alpha I + S)y = r$

After straightforward computations we know that each step of the CG and the CGNE iterations requires $\chi(H) + 12n$ and $2\chi(S) + 14n + 1$ flops, respectively, and each step of the IHSS(CG, CGNE) iteration requires another $\chi(H) + 2\chi(S) + 7n$ flops besides the amount of operations of the cg and the cgne subroutines. Therefore, the total workload at each step of the IHSS(CG, CGNE) iteration is

$$\mu_k(\chi(H) + 12n) + \nu_k(2\chi(S) + 14n + 1) + \chi(H) + 2\chi(S) + 7n$$

flops. If we assume that each row of the matrices H and S has at most τ_h and τ_s non-zero entries, respectively, then $\chi(H) = (2\tau_h - 1)n$, $\chi(S) = (2\tau_s - 1)n$, and the total workload at each step of the IHSS(CG, CGNE) iteration is thereby

$$W_o(\tau_h, \tau_s, \mu_k, \nu_k) = (2\tau_h + 11)n\mu_k + [(4\tau_s + 14)n - 1]\nu_k + (2\tau_h + 4\tau_s + 4)n.$$

When $\tau \equiv \tau_h = \tau_s + 1$, it immediately follows that

$$\begin{aligned} W_o(\tau, \mu_k, \nu_k) &\equiv W_o(\tau_h, \tau_s, \mu_k, \nu_k) \\ &= (2\tau + 11)n\mu_k + 4[(\tau + 2)n + 1]\nu_k + 6\tau n. \end{aligned} \tag{2.2}$$

In general, we could assume that the matrices H and S have the same sparsity as the matrix A so that simpler formulas about the workloads $W(\tau_h, \tau_s, \mu_k, \nu_k)$ and $W_o(\tau_h, \tau_s, \mu_k, \nu_k)$ can be obtained, but this may be not always the case. For example, for a Hessenberg-type matrix A, the numbers of the non-zero entries in H and S may be considerably different from that in A. So, a more realistic assumption on the numbers of non-zero entries for the matrices involved in the IHSS iteration methods may be the one imposed on the matrices H and S rather than on the matrix A itself, just as what we have done in the above.

It should be mentioned that if good preconditioners to the matrices $\alpha I + H$ and $\alpha I + S$ are cheaply obtainable, we can employ the preconditioned conjugate gradient method and the preconditioned Lanczos method (or the preconditioned conjugate gradient for normal equation method) instead of CG and Lanczos (or CGNE) at each of the iteration steps so that the computational efficiency of IHSS(CG, Lanczos) (or IHSS(CG, CGNE)) may be considerably improved.

3. Basic lemmas

For the positive definite matrix $A \in \mathbb{C}^{n\times n}$, let $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively; represent the lower and the upper bounds of the eigenvalues of the matrix H by $\gamma_h - \rho_h$ and $\gamma_h + \rho_h$, and the upper bound of the absolute values of the eigenvalues of the matrix S by ρ_s. We note that the lower bound of the absolute values of the eigenvalues of the matrix S is 0, and it always holds that $\gamma_h > \rho_h$. Define

$$\kappa_h = \frac{\gamma_h + \rho_h}{\gamma_h - \rho_h} \quad \text{and} \quad \kappa_{h,s} = 1 + \frac{\rho_s^2}{\gamma_h^2 - \rho_h^2}.$$

If α is a positive constant, then both matrices $\alpha I + H$ and $\alpha I + S$ are non-singular. In this situation, we can define a vector norm $|||x||| = \|(\alpha I + S)x\|_2$ $(\forall x \in \mathbb{C}^n)$, which naturally induces the matrix norm $|||X||| = \|(\alpha I + S)X(\alpha I + S)^{-1}\|_2$ $(\forall X \in \mathbb{C}^{n\times n})$.

Since S is skew-Hermitian, it holds that

$$\|\alpha I - S\|_2 = \|\alpha I + S\|_2 \quad \text{and} \quad \|(\alpha I - S)^{-1}\|_2 = \|(\alpha I + S)^{-1}\|_2.$$

Moreover, it follows from

$$\begin{cases} A(\alpha I + H)^{-1} = [(\alpha I + H) - (\alpha I - S)](\alpha I + H)^{-1} \\ \qquad\qquad\quad\;\; = I - (\alpha I - S)(\alpha I + H)^{-1}, \\ A(\alpha I + S)^{-1} = [(\alpha I + S) - (\alpha I - H)](\alpha I + S)^{-1} \\ \qquad\qquad\quad\;\; = I - (\alpha I - H)(\alpha I + S)^{-1} \end{cases}$$

that

$$\begin{cases} \|A(\alpha I + H)^{-1}\|_2 \leqslant 1 + \|\alpha I - S\|_2 \|(\alpha I + H)^{-1}\|_2, \\ \|A(\alpha I + S)^{-1}\|_2 \leqslant 1 + \|\alpha I - H\|_2 \|(\alpha I + S)^{-1}\|_2. \end{cases}$$

The following lemma summarizes the convergence property of the HSS iteration.

Lemma 3.1 [5]. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix and α be a positive constant. Then the iterative sequence $\{x^{(k)}\}$ generated by the HSS iteration can be equivalently expressed as*

$$x^{(k+1)} = M(\alpha)x^{(k)} + G(\alpha)b, \quad k = 0, 1, 2, \dots,$$

where

$$M(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)$$

and

$$G(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}.$$

Moreover, it holds that

$$|||M(\alpha)||| \leqslant \|(\alpha I + H)^{-1}(\alpha I - H)\|_2 \leqslant \max_{\gamma_h-\rho_h\leqslant\lambda\leqslant\gamma_h+\rho_h} \left|\frac{\alpha - \lambda}{\alpha + \lambda}\right| \equiv \sigma(\alpha) < 1.$$

In particular, if

$$\alpha^* \equiv \arg\min_{\alpha} \left\{ \max_{\gamma_h-\rho_h\leqslant\lambda\leqslant\gamma_h+\rho_h} \left|\frac{\alpha - \lambda}{\alpha + \lambda}\right| \right\} = \sqrt{\gamma_h^2 - \rho_h^2},$$

then

$$\sigma(\alpha^*) = \frac{\sqrt{\gamma_h + \rho_h} - \sqrt{\gamma_h - \rho_h}}{\sqrt{\gamma_h + \rho_h} + \sqrt{\gamma_h - \rho_h}} = \frac{\gamma_h}{\rho_h}\left[1 - \sqrt{1 - \left(\frac{\rho_h}{\gamma_h}\right)^2}\,\right]$$
$$\approx \frac{\rho_h}{2\gamma_h}\left(1 + \left(\frac{\rho_h}{2\gamma_h}\right)^2\right).$$

The following three lemmas describe convergence properties of CG, Lanczos and CGNE, respectively, which are essential for us to establish convergence theorems for IHSS(CG, Lanczos) and IHSS(CG, CGNE).

Lemma 3.2. *Let $H \in \mathbb{C}^{n\times n}$ be a Hermitian positive definite matrix, α be a positive constant, and $y^{(\mu_k)}$ be the μ_kth approximate solution generated by the μ_kth step of CG iteration for solving the Hermitian positive definite system of linear equations $(\alpha I + H)y = b$. Then $y^{(\mu_k)}$ is of the form*

$$y^{(\mu_k)} = y^* + p^{cg}_{\mu_k}(\alpha I + H)(y^{(0)} - y^*),$$

where $y^* = (\alpha I + H)^{-1}b$ is the exact solution, $y^{(0)}$ is an initial guess, and $p^{cg}_{\mu_k}$ is a polynomial of degree less than or equal to μ_k satisfying $p^{cg}_{\mu_k}(0) = 1$. Moreover, if

$$\sigma_h(\alpha, \mu_k) \equiv 2\left(\frac{\sqrt{\kappa(\alpha I + H)} - 1}{\sqrt{\kappa(\alpha I + H)} + 1}\right)^{\mu_k} \leqslant 2\left(\frac{\sqrt{\frac{\alpha+(\gamma_h+\rho_h)}{\alpha+(\gamma_h-\rho_h)}} - 1}{\sqrt{\frac{\alpha+(\gamma_h+\rho_h)}{\alpha+(\gamma_h-\rho_h)}} + 1}\right)^{\mu_k},$$

then it holds that

$$\|y^{(\mu_k)} - y^*\|_2 \leqslant \sigma_h(\alpha, \mu_k)\|y^{(0)} - y^*\|_2,$$

where $\kappa(\cdot)$ denotes the spectral condition number of the corresponding matrix.

Lemma 3.3 [18, 16]. *Let $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix, α be a positive constant, and $y^{(\nu_k)}$ be the ν_kth approximate solution generated by the ν_kth step of Lanczos iteration for solving the non-singular system of linear equations $(\alpha I + S)y = b$. If*

$$\begin{aligned}\sigma_s(\alpha, \nu_k) &\equiv \frac{2\sqrt{\alpha^2 + \|S\|_2^2}}{\alpha}\left(\frac{\|S\|_2}{\alpha + \sqrt{\alpha^2 + \|S\|_2^2}}\right)^{2\lfloor \frac{\nu_k}{2}\rfloor} \\ &\leqslant \frac{2\sqrt{\alpha^2 + \rho_s^2}}{\alpha}\left(\frac{\sqrt{\alpha^2 + \rho_s^2} - \alpha}{\sqrt{\alpha^2 + \rho_s^2} + \alpha}\right)^{\lfloor \frac{\nu_k}{2}\rfloor},\end{aligned}$$

then it holds that

$$\|y^{(\nu_k)} - y^*\|_2 \leqslant \sigma_s(\alpha, \nu_k)\|y^{(0)} - y^*\|_2,$$

where $y^* = (\alpha I + S)^{-1}b$ is the exact solution, $y^{(0)}$ is an initial guess, and $\lfloor\cdot\rfloor$ denotes the integer part of the corresponding positive real.

Lemma 3.4 (See Lemma 3.2). *Let $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix, α be a positive constant, and $y^{(\nu_k)}$ be the ν_kth approximate solution generated by the ν_kth step of CGNE iteration for solving the non-singular system of linear equations $(\alpha I + S)y = b$. Then $y^{(\nu_k)}$ is of the form*

$$y^{(\nu_k)} = y^* + p_{\nu_k}^{cg}(\alpha^2 I - S^2)(y^{(0)} - y^*),$$

where $y^* = (\alpha I + S)^{-1}b$ is the exact solution, $y^{(0)}$ is an initial guess, and $p_{\nu_k}^{cg}$ is a polynomial of degree less than or equal to ν_k satisfying $p_{\nu_k}^{cg}(0) = 1$. Moreover,

$$\|p_{\nu_k}^{cg}(\alpha^2 I - S^2)\|_2 \leqslant 2\left(\frac{\sqrt{\kappa(\alpha^2 I - S^2)} - 1}{\sqrt{\kappa(\alpha^2 I - S^2)} + 1}\right)^{\nu_k} \leqslant 2\left(\frac{\sqrt{\alpha^2 + \rho_s^2} - \alpha}{\sqrt{\alpha^2 + \rho_s^2} + \alpha}\right)^{\nu_k}. \tag{3.1}$$

At the end of this section, we list some useful estimates related to the matrices $\alpha I \pm H$ and $\alpha I \pm S$ in the following lemma.

Lemma 3.5. *Let $H \in \mathbb{C}^{n\times n}$ be a Hermitian positive definite matrix, $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix, and α be a positive constant. Then*

$$\begin{aligned}&\|(\alpha I + H)^{-1}\|_2 = \max\left\{\frac{1}{\alpha + (\gamma_h - \rho_h)}, \frac{1}{\alpha + (\gamma_h + \rho_h)}\right\} = \frac{1}{\alpha + (\gamma_h - \rho_h)}, \\ &\|(\alpha I + S)^{-1}\|_2 = \max\left\{\frac{1}{\alpha}, \frac{1}{\sqrt{\alpha^2 + \rho_s^2}}\right\} = \frac{1}{\alpha}, \\ &\|\alpha I - H\|_2 = \max\{|\alpha - (\gamma_h - \rho_h)|, |\alpha - (\gamma_h + \rho_h)|\}, \\ &\|\alpha I - S\|_2 = \max\left\{\alpha, \sqrt{\alpha^2 + \rho_s^2}\right\} = \sqrt{\alpha^2 + \rho_s^2}.\end{aligned}$$

Therefore,

$$c_h(\alpha) \equiv \|\alpha I - S\|_2 \|(\alpha I + H)^{-1}\|_2 \leqslant \frac{\sqrt{\alpha^2 + \rho_s^2}}{\alpha + (\gamma_h - \rho_h)},$$
$$c_s(\alpha) \equiv \|\alpha I - H\|_2 \|(\alpha I + S)^{-1}\|_2 \leqslant \frac{1}{\alpha} \max\{|\alpha - (\gamma_h - \rho_h)|, |\alpha - (\gamma_h + \rho_h)|\}.$$

In particular, when $\alpha = \alpha^* = \sqrt{\gamma_h^2 - \rho_h^2}$, *it holds that*

$$c_h(\alpha^*) \leqslant \frac{\sqrt{\kappa_h \kappa_{h,s}}}{\sqrt{\kappa_h} + 1} \quad \textit{and} \quad c_s(\alpha^*) \leqslant \sqrt{\kappa_h} - 1.$$

Proof. The equalities and inequalities follow from straightforward computations. □

4. Convergence analyses

Based on Lemmas 3.2–3.5, we can demonstrate the convergence theorems for IHSS(CG, Lanczos) and IHSS(CG, CGNE). The proofs of the theorems in this section can be found in Appendix.

4.1. Convergence of IHSS(CG, Lanczos)

The following theorem describes the convergence of IHSS(CG, Lanczos).

Theorem 4.1. *Let* $A \in \mathbb{C}^{n \times n}$ *be a positive definite matrix*, $H = \frac{1}{2}(A + A^*)$ *and* $S = \frac{1}{2}(A - A^*)$ *be its Hermitian and skew-Hermitian parts*, *respectively*, *and* α *be a positive constant*. *Let* $\{\mu_k\}$ *and* $\{\nu_k\}$ *be two sequences of positive integers*. *If the iterative sequence* $\{x^{(k)}\}$ *is generated by the IHSS*(*CG*, *Lanczos*) *iteration from an initial guess* $x^{(0)}$, *then it holds that*

$$|||x^{(k+1)} - x^*||| \leqslant (\sigma(\alpha) + \epsilon(\alpha, \mu_k, \nu_k))|||x^{(k)} - x^*|||,$$

where $x^* \in \mathbb{C}^n$ *is the exact solution of the system of linear equations* (1.1),

$$\epsilon(\alpha, \mu_k, \nu_k) = c_h(\alpha)(1 + c_s(\alpha))[(1 + c_s(\alpha))\sigma_h(\alpha, \mu_k)\sigma_s(\alpha, \nu_k) + c_s(\alpha)\sigma_h(\alpha, \mu_k) + \sigma_s(\alpha, \nu_k)]$$

with $\sigma(\alpha)$, $\sigma_h(\alpha, \mu_k)$ *and* $\sigma_s(\alpha, \nu_k)$ *being defined as in Lemmas* 3.1–3.3, *respectively. Therefore*, *if there exists a non-negative constant* $\sigma^{\text{ihss}}(\alpha) \in [0, 1)$ *such that*

$$\sigma(\alpha) + \epsilon(\alpha, \mu_k, \nu_k) \leqslant \sigma^{\text{ihss}}(\alpha), \quad k = 0, 1, 2, \ldots,$$

then the iterative sequence $\{x^{(k)}\}$ *converges to* $x^* \in \mathbb{C}^n$ *with a convergence factor being at most* $\sigma^{\text{ihss}}(\alpha)$.

Theorem 4.1 presents an estimate for the contraction factor of the IHSS(CG, Lanczos) iteration. Moreover, we can take $\alpha = \alpha^*$, the optimal parameter determined in Lemma 3.1, to further minimize the contraction factor and, consequently, accelerate the convergence speed of IHSS(CG, Lanczos). More precisely, we have the following theorem.

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and $\alpha = \alpha^* = \sqrt{\gamma_h^2 - \rho_h^2}$. Let $\{\mu_k\}$ and $\{\nu_k\}$ be two sequences of positive integers. If the iterative sequence $\{x^{(k)}\}$ is generated by the IHSS(CG, Lanczos) iteration from an initial guess $x^{(0)}$, then it holds that*

$$|||x^{(k+1)} - x^*||| \leqslant \left(\frac{\sqrt{\kappa_h} - 1}{\sqrt{\kappa_h} + 1} + \epsilon_{h,s}(\mu_k, \nu_k)\right) |||x^{(k)} - x^*|||,$$

where

$$\epsilon_{h,s}(\mu_k, \nu_k) = \frac{2\kappa_h\sqrt{\kappa_{h,s}}}{\sqrt{\kappa_h} + 1}\left[\left(\sqrt{\kappa_h} - 1\right)\left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1}\right)^{\mu_k} + \sqrt{\kappa_{h,s}}\left(\frac{\sqrt{\kappa_{h,s}} - 1}{\sqrt{\kappa_{h,s}} + 1}\right)^{\lfloor\frac{\nu_k}{2}\rfloor}\right.$$
$$\left. + 2\sqrt{\kappa_h\kappa_{h,s}}\left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1}\right)^{\mu_k}\left(\frac{\sqrt{\kappa_{h,s}} - 1}{\sqrt{\kappa_{h,s}} + 1}\right)^{\lfloor\frac{\nu_k}{2}\rfloor}\right].$$

Therefore, if $\{\mu_k\}$ and $\{\nu_k\}$ are chosen such that

$$\epsilon_{h,s}(\mu_k, \nu_k) < \frac{2}{\sqrt{\kappa_h} + 1}, \quad k = 0, 1, 2, \ldots,$$

there exists a non-negative constant $\sigma^{\text{ihss}}(\alpha^*) \in [0, 1)$ such that

$$|||x^{(k+1)} - x^*||| \leqslant \sigma^{\text{ihss}}(\alpha^*)|||x^{(k)} - x^*|||, \quad k = 0, 1, 2, \ldots,$$

and consequently, the iterative sequence $\{x^{(k)}\}$ generated by IHSS(CG, Lanczos) with the optimal parameter α^* converges to the exact solution $x^* \in \mathbb{C}^n$ of the system of linear equations (1.1).

Theorems 4.1 and 4.2 show that the contraction factor of the IHSS(CG, Lanczos) iteration is bounded by

$$\frac{\sqrt{\kappa_h} - 1}{\sqrt{\kappa_h} + 1} + \epsilon_{h,s}(\mu_k, \nu_k) = \frac{\sqrt{\kappa_h} - 1}{\sqrt{\kappa_h} + 1} + 2\kappa_h\sqrt{\kappa_{h,s}}\left(\frac{\sqrt{\kappa_h} - 1}{\sqrt{\kappa_h} + 1}\right)\left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1}\right)^{\mu_k}$$
$$+ \frac{2\kappa_h\kappa_{h,s}}{\sqrt{\kappa_h} + 1}\left(\frac{\sqrt{\kappa_{h,s}} - 1}{\sqrt{\kappa_{h,s}} + 1}\right)^{\lfloor\frac{\nu_k}{2}\rfloor}$$
$$+ \frac{4\kappa_h\sqrt{\kappa_h}\kappa_{h,s}}{\sqrt{\kappa_h} + 1}\left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1}\right)^{\mu_k}\left(\frac{\sqrt{\kappa_{h,s}} - 1}{\sqrt{\kappa_{h,s}} + 1}\right)^{\lfloor\frac{\nu_k}{2}\rfloor},$$

whose dominant term $\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}$ is approximately equal to the contraction factor of CG applied to the system of linear equations $Hy = b$. To make $\epsilon_{h,s}(\mu_k, \nu_k)$ approach to zero quickly and economically with increasing of μ_k and ν_k, we should choose the inner CG iteration step μ_k and the inner Lanczos iteration step ν_k at the kth outer iterate such that the two factors

$$2\kappa_h\sqrt{\kappa_{h,s}}\left(\frac{\sqrt{\kappa_h} - 1}{\sqrt{\kappa_h} + 1}\right)\left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1}\right)^{\mu_k} \quad \text{and} \quad \frac{2\kappa_h\kappa_{h,s}}{\sqrt{\kappa_h} + 1}\left(\frac{\sqrt{\kappa_{h,s}} - 1}{\sqrt{\kappa_{h,s}} + 1}\right)^{\lfloor\frac{\nu_k}{2}\rfloor}$$

approach to zero with comparable speeds. This could be achieved by letting

$$\left(\sqrt{\kappa_h} - 1\right)\left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1}\right)^{\mu_k} = \sqrt{\kappa_{h,s}}\left(\frac{\sqrt{\kappa_{h,s}} - 1}{\sqrt{\kappa_{h,s}} + 1}\right)^{\lfloor\frac{\nu_k}{2}\rfloor}$$

or in other words,

$$\nu_k = \begin{cases} \dfrac{2\left[\ln\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_{h,s}}}\right)+\mu_k \ln\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)\right]}{\ln\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)}, & \text{for } \nu_k \text{ even,} \\ \dfrac{2\left[\ln\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_{h,s}}}\right)+\mu_k \ln\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)\right]}{\ln\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)} + 1, & \text{for } \nu_k \text{ odd.} \end{cases} \tag{4.1}$$

In this situation, the contributions from the inner CG and the inner Lanczos processes to the kth outer IHSS(CG, Lanczos) iterate are well balanced, and it holds that

$$\epsilon_{h,s}(\mu_k, \nu_k) = 4\kappa_h\sqrt{\kappa_{h,s}}\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\right)\left[\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k} + \sqrt{\kappa_h}\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{2\mu_k}\right]$$

and

$$\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1} + \epsilon_{h,s}(\mu_k, \nu_k) \leqslant \frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\left[1 + 2\kappa_h\sqrt{\kappa_{h,s}}\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k}\right]^2. \tag{4.2}$$

If μ_k is chosen so that

$$\kappa_h\sqrt{\kappa_{h,s}}\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\inf_{k\geqslant 0}\{\mu_k\}} = c, \tag{4.3}$$

with

$$c < \frac{1}{2}\left(\sqrt{\frac{\sqrt{\kappa_h}+1}{\sqrt{\kappa_h}-1}} - 1\right), \tag{4.4}$$

or in other words,

$$\mu_k \geqslant \frac{\ln\left(\frac{c}{\kappa_h\sqrt{\kappa_{h,s}}}\right)}{\ln\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)}, \tag{4.5}$$

then we have

$$|||x^{(k+1)} - x^*||| \leqslant (2c+1)^2\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}|||x^{(k)} - x^*|||, \quad k = 0, 1, 2, \ldots.$$

The above analysis is summarized in the following theorem.

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and $\alpha = \alpha^* = \sqrt{\gamma_h^2 - \rho_h^2}$. Let $\{\mu_k\}$ and $\{\nu_k\}$ be two sequences of positive integers satisfying (4.1) and (4.5). Then the IHSS(CG, Lanczos) iteration converges to the exact solution $x^* \in \mathbb{C}^n$ of the system of linear equations (1.1), with the convergence factor being less than $(2c+1)^2\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}$, where c is defined by (4.3) and satisfies (4.4). Moreover, if $\underline{\lim}\,\mu_k = \underline{\lim}\,\nu_k = +\infty$, then the asymptotic convergence factor of the IHSS(CG, Lanczos) iteration tends to $\sigma(\alpha^*) = \frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}$ of that of the HSS iteration.*

In the HSS iteration (1.2), if the system of linear equations with coefficient matrix $\alpha I + S$ at its second-half step could be solved exactly by a direct method (see [15]), and the system of linear equations with coefficient matrix $\alpha I + H$ at its first-half step is being solved inexactly by conjugate gradient method, we can get a *partially inexact Hermitian/skew-Hermitian splitting* (PIHSS) iteration method, which has the convergence behaviour

$$|||x^{(k+1)} - x^*||| \leqslant (2\widetilde{c} + 1)\frac{\sqrt{\kappa_h} - 1}{\sqrt{\kappa_h} + 1}|||x^{(k)} - x^*|||, \quad k = 0, 1, 2, \ldots,$$

where the constant

$$\widetilde{c} = \sqrt{\kappa_h}\left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1}\right)^{\inf_{k\geqslant 0}\{\mu_k\}}$$

and satisfies $\tilde{c} < \frac{1}{\sqrt{\kappa_h}-1}$. Note that $\tilde{c} \ll c$ holds in general.

4.2. Convergence of IHSS(CG, CGNE)

The following theorem describes a tight expression for IHSS(CG, CGNE).

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and α be a positive constant. Let $\{\mu_k\}$ and $\{\nu_k\}$ be two sequences of positive integers. If the iterative sequence $\{x^{(k)}\}$ is generated by the IHSS(CG, CGNE) iteration from an initial guess $x^{(0)}$, then it is of the form*

$$x^{(k+1)} = x^* + M(\alpha, \mu_k, \nu_k)(x^{(k)} - x^*), \tag{4.6}$$

where

$$M(\alpha, \mu_k, \nu_k) = M(\alpha) + E(\alpha, \mu_k, \nu_k), \tag{4.7}$$

$$\begin{aligned} E(\alpha, \mu_k, \nu_k) = {} & (\alpha I + S)^{-1}(\alpha I - H)p^{cg}_{\mu_k}(\alpha I + H)(\alpha I + H)^{-1}A \\ & + p^{cg}_{\nu_k}(\alpha^2 I - S^2)(\alpha I + S)^{-1}A(\alpha I + H)^{-1}(\alpha I - S) \\ & + p^{cg}_{\nu_k}(\alpha^2 I - S^2)(\alpha I + S)^{-1}Ap^{cg}_{\mu_k}(\alpha I + H)(\alpha I + H)^{-1}A, \end{aligned} \tag{4.8}$$

and $M(\alpha)$ is defined by (3.1).

According to Theorem 4.4, the iteration matrix $M(\alpha, \mu_k, \nu_k)$ of IHSS(CG, CGNE) is a correction of the iteration matrix $M(\alpha)$ of HSS by the correction term $E(\alpha, \mu_k, \nu_k)$. The following result about the relationship between the iteration matrix $M(\alpha, \mu_k, \nu_k)$ and the coefficient matrix A holds.

Theorem 4.5. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and α be a positive constant. Let $\{\mu_k\}$ and $\{\nu_k\}$ be two sequences of positive integers and $M(\alpha, \mu_k, \nu_k)$ be the kth iteration matrix of IHSS (CG, CGNE) defined in Theorem 4.4. Then*

$$M(\alpha, \mu_k, \nu_k) = I - G(\alpha, \mu_k, \nu_k)A, \tag{4.9}$$

where

$$G(\alpha, \mu_k, \nu_k) = G(\alpha) - (\alpha I + S)^{-1}(\alpha I - H)p^{cg}_{\mu_k}(\alpha I + H)(\alpha I + H)^{-1}$$

$$
\begin{aligned}
&- (\alpha I + S)^{-1} p_{\nu_k}^{cg}(\alpha^2 I - S^2)(\alpha I - S)(\alpha I + H)^{-1} \\
&- (\alpha I + S)^{-1} p_{\nu_k}^{cg}(\alpha^2 I - S^2) A p_{\mu_k}^{cg}(\alpha I + H)(\alpha I + H)^{-1},
\end{aligned}
\tag{4.10}
$$

and the matrix $G(\alpha)$ is defined by (3.1). Moreover, the matrix $G(\alpha, \mu_k, \nu_k)$ is non-singular if and only if 2α is not an eigenvalue of the matrix

$$
\begin{aligned}
F(\alpha, \mu_k, \nu_k) = {} & (\alpha I - H) p_{\mu_k}^{cg}(\alpha I + H) + p_{\nu_k}^{cg}(\alpha^2 I - S^2)(\alpha I - S) \\
& + p_{\nu_k}^{cg}(\alpha^2 I - S^2) A p_{\mu_k}^{cg}(\alpha I + H).
\end{aligned}
$$

Next, we analyze the contraction factor (in the $|||\cdot|||$-norm) for the iteration matrices $M(\alpha, \mu_k, \nu_k)$ $(k = 0, 1, 2, \ldots)$ and, therefore, establish convergence theorem for the IHSS(CG, CGNE) iteration.

Theorem 4.6. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and α be a positive constant. Let $\{\mu_k\}$ and $\{\nu_k\}$ be two sequences of positive integers, and $M(\alpha, \mu_k, \nu_k)$ be the kth iteration matrix of the IHSS(CG, CGNE) iteration defined in Theorem 4.4. Then*

$$
|||M(\alpha, \mu_k, \nu_k)||| \leqslant \sigma(\alpha) + \epsilon(\alpha, \mu_k, \nu_k),
$$

where $\sigma(\alpha)$ is the contraction factor of the HSS iteration defined in Lemma 3.1, and the correction error

$$
\begin{aligned}
\epsilon(\alpha, \mu_k, \nu_k) = {} & \sigma(\alpha)(1 + c_s(\alpha))\|p_{\mu_k}^{cg}(\alpha I + H)\|_2 + (1 + c_h(\alpha))\|p_{\nu_k}^{cg}(\alpha^2 I - S^2)\|_2 \\
& + (1 + c_h(\alpha))(1 + c_s(\alpha))\|p_{\mu_k}^{cg}(\alpha I + H)\|_2 \|p_{\nu_k}^{cg}(\alpha^2 I - S^2)\|_2.
\end{aligned}
$$

Therefore, if there exists a non-negative constant $\sigma^{\mathrm{ihss}}(\alpha) \in [0, 1)$ such that

$$
\sigma(\alpha) + \epsilon(\alpha, \mu_k, \nu_k) \leqslant \sigma^{\mathrm{ihss}}(\alpha), \quad k = 0, 1, 2, \ldots,
$$

then the iterative sequence $\{x^{(k)}\}$ generated by the IHSS(CG, CGNE) iteration from an initial guess $x^{(0)}$ converges to the exact solution $x^* \in \mathbb{C}^n$ of the system of linear equations (1.1), with the convergence factor being at most $\sigma^{\mathrm{ihss}}(\alpha)$.

Theorem 4.6 presents an upper bound for the contraction factor of the IHSS(CG, CGNE) iteration. Moreover, when the eigenvalue bounds of both Hermitian part H and skew-Hermitian part S of the matrix A are available, we can use the optimal parameter α^* determined in Lemma 3.1 to improve the contraction factor and, consequently, accelerate the convergence speed of IHSS(CG, CGNE). More precisely, we have the following theorem.

Theorem 4.7. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and α be a positive constant. Then, when $\alpha = \alpha^* = \sqrt{\gamma_h^2 - \rho_h^2}$, we have*

$$
\begin{aligned}
\epsilon(\alpha^*, \mu_k, \nu_k) \leqslant {} & 2\sqrt{\kappa_h}\left(\frac{\sqrt{\kappa_h} - 1}{\sqrt{\kappa_h} + 1}\right)\left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1}\right)^{\mu_k} \\
& + 2\left(1 + \frac{\sqrt{\kappa_h \kappa_{h,s}}}{\sqrt{\kappa_h} + 1}\right)\left(\frac{\sqrt{\kappa_{h,s}} - 1}{\sqrt{\kappa_{h,s}} + 1}\right)^{\nu_k}
\end{aligned}
$$

$$
\begin{aligned}
&+4\sqrt{\kappa_h}\left(1+\frac{\sqrt{\kappa_h\kappa_{h,s}}}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k}\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)^{\nu_k}\\
&\equiv \epsilon_{h,s}(\mu_k,\nu_k)
\end{aligned}
$$

and

$$
|||M(\alpha^*,\mu_k,\nu_k)||| \leqslant \frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}+\epsilon_{h,s}(\mu_k,\nu_k),
$$

where μ_k and ν_k are the inner CG and the inner CGNE iteration steps at the kth outer iterate of the IHSS(CG, CGNE) iteration, respectively. Therefore, if $\{\mu_k\}$ and $\{\nu_k\}$ are chosen such that

$$
\epsilon_{h,s}(\mu_k,\nu_k) < \frac{2}{\sqrt{\kappa_h}+1}, \quad k=0,1,2,\ldots,
$$

there exists a non-negative constant $\sigma^{\text{ihss}}(\alpha^*)\in[0,1)$ such that

$$
|||M(\alpha^*,\mu_k,\nu_k)||| \leqslant \sigma^{\text{ihss}}(\alpha^*), \quad k=0,1,2,\ldots,
$$

and consequently, the iterative sequence $\{x^{(k)}\}$ generated by the IHSS(CG, CGNE) iteration with the optimal parameter α^* from an initial guess $x^{(0)}$ converges to the unique solution $x^*\in\mathbb{C}^n$ of the system of linear equations (1.1).

From Theorems 4.6 and 4.7, we know that the contraction factor of the IHSS(CG, CGNE) iteration is bounded by

$$
\begin{aligned}
\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}+\epsilon_{h,s}(\mu_k,\nu_k) = &\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}+2\sqrt{\kappa_h}\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k}\\
&+2\left(1+\frac{\sqrt{\kappa_h\kappa_{h,s}}}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)^{\nu_k}\\
&+4\sqrt{\kappa_h}\left(1+\frac{\sqrt{\kappa_h\kappa_{h,s}}}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k}\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)^{\nu_k},
\end{aligned}
$$

where the first term is the contraction factor of the HSS iteration, the second and the third terms are the contraction factors of the inner CG and the inner CGNE iterations, respectively, and the last term is a higher-order error due to the inexactness of the iteration, at the kth outer iterate of IHSS(CG, CGNE). Evidently, the best possible case of $|||M(\alpha^*,\mu_k,\nu_k)|||$ is $\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}$, which is approximately equal to the contraction factor of CG applied to the system of linear equations $Hy=b$. To make $\epsilon_{h,s}(\mu_k,\nu_k)$ approach to zero quickly with increasing of μ_k and ν_k, we should suitably choose the inner CG iteration step μ_k and the inner CGNE iteration step ν_k at the kth outer iterate such that the two factors

$$
\sqrt{\kappa_h}\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k} \quad \text{and} \quad \left(1+\frac{\sqrt{\kappa_h\kappa_{h,s}}}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)^{\nu_k}
$$

approach to zero with comparable speeds. Therefore, it is reasonable for us to choose μ_k and ν_k such that

$$
\sqrt{\kappa_h}\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k} = \left(1+\frac{\sqrt{\kappa_h\kappa_{h,s}}}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)^{\nu_k},
$$

or in other words,

$$\nu_k = \frac{\ln\left(\frac{\sqrt{\kappa_h}(\sqrt{\kappa_h}+1)}{\sqrt{\kappa_h}(\sqrt{\kappa_{h,s}}+1)+1}\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\right)\right) + \mu_k \ln\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)}{\ln\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)}. \tag{4.11}$$

In this situation, the contributions from the inner CG and the inner CGNE processes to the kth outer iterate of IHSS(CG, CGNE) are well balanced, and it holds that

$$\epsilon_{h,s}(\mu_k,\nu_k) = 4\sqrt{\kappa_h}\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k} + 4\kappa_h\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\right)\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{2\mu_k}$$

and

$$|||M(\alpha^*,\mu_k,\nu_k)||| \leqslant \frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\left(1+2\sqrt{\kappa_h}\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k}\right)^2. \tag{4.12}$$

If μ_k is chosen so that

$$\sqrt{\kappa_h}\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\inf_{k\geqslant 0}\{\mu_k\}} = c \tag{4.13}$$

with

$$c < \frac{1}{2}\left(\sqrt{\frac{\sqrt{\kappa_h}+1}{\sqrt{\kappa_h}-1}} - 1\right), \tag{4.14}$$

or in other words,

$$\mu_k \geqslant \frac{\ln\left(\frac{c}{\sqrt{\kappa_h}}\right)}{\ln\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)}, \tag{4.15}$$

then we have

$$|||M(\alpha^*,\mu_k,\nu_k)||| \leqslant (2c+1)^2\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1} < 1.$$

Therefore, it follows from (4.6) that

$$|||x^{(k+1)} - x^*||| \leqslant (2c+1)^2\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}|||x^{(k)} - x^*|||, \quad k = 0,1,2,\ldots.$$

The above analysis is summarized in the following theorem.

Theorem 4.8. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and $\alpha = \alpha^* = \sqrt{\gamma_h^2 - \rho_h^2}$. Let $\{\mu_k\}$ and $\{\nu_k\}$ be two sequences of positive integers satisfying (4.11) and (4.15). Then the IHSS(CG, CGNE) iteration converges to the exact solution $x^* \in \mathbb{C}^n$ of the system of linear equations (1.1), with the convergence factor being less than $(2c+1)^2\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}$, where c is defined by (4.13) and satisfies (4.14). Moreover, if $\underline{\lim}\,\mu_k = \underline{\lim}\,\nu_k = +\infty$, then the asymptotic convergence factor of the IHSS(CG, CGNE) iteration tends to $\sigma(\alpha^*) = \frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}$ of that of the HSS iteration.*

At the end of this section, we remark that the number of inner iteration steps μ_k can be optimized according to the computing efficiency of the IHSS iterations. When such a nearly optimal μ_k is obtained, the nearly optimal ν_k can be determined by (4.1) for the IHSS(CG, Lanczos) iteration and by (4.11) for the IHSS(CG, CGNE) iteration, respectively. This makes both IHSS(CG, Lanczos) and IHSS(CG, CGNE) can achieve their maximum computing efficiencies. See [1,7] and references therein for analogous analyses.

5. Efficiency analyses

A simple calculation shows that the memory required for the IHSS iterations is to store $x^{(k)}$, b, and five auxiliary vectors in the CG-type methods; see [9,1]. Moreover, it is possible for us not to store the matrices H and S, as all we need is two subroutines that perform the matrix-vector multiplications with respect to these two matrices. Therefore, the total amount of computer memory required is $\mathcal{O}(n)$, which has the same order of magnitude as the number of unknowns.

From the above convergence analyses we know that the workload of the Lanczos is one time less than that of the CGNE, but its convergence speed is one time slower than that of CGNE, when they are employed in the IHSS iteration to solve the system of linear equations with coefficient matrix $\alpha I + S$. Therefore, it is not obvious whether the IHSS(CG, Lanczos) iteration is comparable to the IHSS(CG, CGNE) iteration from only their asymptotic convergence rates or their computational workloads, although they possess almost the same contraction factor. A more reasonable and objective standard for assessing and comparing the effectiveness of these two iterations may be their computational efficiencies.

Without loss of generality, assume that each row of the matrices H and S has at most τ and $\tau - 1$ non-zero entries, respectively, and denote by

$$\begin{aligned} &\sigma_* \equiv \sigma(\alpha^*) = \frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}, \quad \sigma_o = \frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_{h,s}}}, \quad \delta_o = \frac{\sqrt{\kappa_h}(\sqrt{\kappa_h}+1)}{\sqrt{\kappa_h}(\sqrt{\kappa_{h,s}}+1)+1}, \\ &\sigma_h = \frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}, \quad \sigma_{h,s} = \frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}, \quad \text{and} \quad \theta = \frac{\ln(\sigma_h)}{\ln(\sigma_{h,s})}. \end{aligned} \tag{5.1}$$

Then we know from (2.1) and (4.2) that the computational efficiency $\mathscr{E}(\mu)$ of the IHSS(CG, Lanczos) iteration can be defined by

$$\mathscr{E}(\mu) = \frac{-\ln\left(\sigma_*[1 + 2\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu}]^2\right)}{(2\tau + 11)n\mu + 2[(\tau + 3)n + 2]\nu(\mu) + 6\tau n},$$

where

$$\nu(\mu) = \begin{cases} \frac{2(\ln(\sigma_o)+\mu\ln(\sigma_h))}{\ln(\sigma_{h,s})}, & \text{for } \nu(\mu) \text{ even}, \\ \frac{2(\ln(\sigma_o)+\mu\ln(\sigma_h))}{\ln(\sigma_{h,s})} + 1, & \text{for } \nu(\mu) \text{ odd}, \end{cases}$$

and from (2.2) and (4.12) that the computational efficiency $\mathscr{E}_o(\mu)$ of the IHSS(CG, CGNE) iteration can be defined by

$$\mathscr{E}_o(\mu) = \frac{-\ln\left(\sigma_*[1 + 2\sqrt{\kappa_h}\sigma_h^{\mu}]^2\right)}{(2\tau + 11)n\mu + 4(\tau + 2)n\nu_o(\mu) + (8\tau + 1)n},$$

where

$$\nu_o(\mu) = \frac{\ln(\delta_o\sigma_*) + \mu\ln(\sigma_h)}{\ln(\sigma_{h,s})}.$$

Here, we remark that the efficiency functions $\mathscr{E}(\mu)$ and $\mathscr{E}_o(\mu)$ are well-defined under the restrictions

$$\sigma_*[1+2\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu}]^2 < 1 \quad \text{and} \quad \sigma_*[1+2\sqrt{\kappa_h}\sigma_h^{\mu}]^2 < 1, \tag{5.2}$$

respectively.

To find positive integers μ such that $\mathscr{E}(\mu)$ and $\mathscr{E}_o(\mu)$ are maximized, we can solve the non-linear equations

$$\begin{aligned} 0 = f(\mu) = &\frac{4\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu}\ln(\sigma_h)}{(1+2\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu})\ln(\sigma_*[1+2\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu}]^2)} \\ &- \frac{(2\tau+11)n\ln(\sigma_{h,s})+4[(\tau+3)n+2]\ln(\sigma_h)}{\ln(\sigma_{h,s})[(2\tau+11)n\mu+2((\tau+3)n+2)\nu(\mu)+6\tau n]} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} 0 = f_o(\mu) = &\frac{4\sqrt{\kappa_h}\sigma_h^{\mu}\ln(\sigma_h)}{(1+2\sqrt{\kappa_h}\sigma_h^{\mu})\ln(\sigma_*[1+2\sqrt{\kappa_h}\sigma_h^{\mu}]^2)} \\ &- \frac{(2\tau+11)n\ln(\sigma_{h,s})+4(\tau+2)n\ln(\sigma_h)}{\ln(\sigma_{h,s})[(2\tau+11)n\mu+4(\tau+2)n\nu_o(\mu)+(8\tau+1)n]}, \end{aligned} \tag{5.4}$$

respectively, e.g., by the Newton's method iterating for several steps. Note that

$$f(0)f(+\infty) < 0 \quad \text{and} \quad f_o(0)f_o(+\infty) < 0,$$

we know that $f(\mu)$ and $f_o(\mu)$ have positive roots which satisfy (5.2), respectively. Let μ^* and μ_o^* be the smallest ones of such roots of the functions $f(\mu)$ and $f_o(\mu)$, correspondingly. Then we have

$$\mathscr{E}(\mu^*) = -\frac{4\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu^*}\ln(\sigma_h)\ln(\sigma_{h,s})}{(1+2\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu^*})[(2\tau+11)n\ln(\sigma_{h,s})+4((\tau+3)n+2)\ln(\sigma_h)]} \tag{5.5}$$

and

$$\mathscr{E}_o(\mu_o^*) = -\frac{4\sqrt{\kappa_h}\sigma_h^{\mu_o^*}\ln(\sigma_h)\ln(\sigma_{h,s})}{(1+2\sqrt{\kappa_h}\sigma_h^{\mu_o^*})[(2\tau+11)n\ln(\sigma_{h,s})+4(\tau+2)n\ln(\sigma_h)]}. \tag{5.6}$$

It then follows that

$$\begin{aligned} \frac{\mathscr{E}(\mu^*)}{\mathscr{E}_o(\mu_o^*)} &= \sigma_h^{\mu^*-\mu_o^*}\left(\frac{\frac{1}{2\sqrt{\kappa_h}}+\sigma_h^{\mu_o^*}}{\frac{1}{2\kappa_h\sqrt{\kappa_{h,s}}}+\sigma_h^{\mu^*}}\right)\left(\frac{2\tau+11+(4\tau+8)\theta}{2\tau+11+(4\tau+12+\frac{8}{n})\theta}\right) \\ &\approx \frac{2\tau+11+(4\tau+8)\theta}{2\tau+11+(4\tau+12)\theta}, \quad \text{for } \kappa_h \gg 1 \text{ and } n \gg 1 \\ &= 1-\frac{4\theta}{2\tau+11+4(\tau+3)\theta} \in \left(\frac{\tau+2}{\tau+3}, 1\right). \end{aligned} \tag{5.7}$$

This implies that $\mathscr{E}(\mu^*) \geqslant (\frac{\tau+2}{\tau+3})\mathscr{E}_o(\mu_o^*)$ in general, and $\mathscr{E}(\mu^*) \approx \mathscr{E}_o(\mu_o^*)$ when θ is small.

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A+A^*)$ and $S = \frac{1}{2}(A-A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and $\alpha = \alpha^* = \sqrt{\gamma_h^2-\rho_h^2}$.*

(a) *If $\{\mu_k\}$ and $\{\nu_k\}$ are, respectively, the sequences of inner CG and inner Lanczos iteration steps of the IHSS(CG, Lanczos) iteration such that $\{\mu_k\} = \{\mu^*\}$ and $\{\nu_k\} = \{\nu^*\}$, where μ^* is the* *smallest positive root of the non-linear function* $f(\mu)$ *in* (5.3) *satisfying* $\sigma_*(1+2\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu^*})^2 < 1$ *and*

$$\nu^* \equiv \nu^*(\mu^*) = \begin{cases} \frac{2(\ln(\sigma_o)+\mu^*\ln(\sigma_h))}{\ln(\sigma_{h,s})}, & \text{when } \nu^*(\mu^*) \text{ is even,} \\ \frac{2(\ln(\sigma_o)+\mu^*\ln(\sigma_h))}{\ln(\sigma_{h,s})}+1, & \text{when } \nu^*(\mu^*) \text{ is odd,} \end{cases}$$

then the IHSS(*CG*, *Lanczos*) *iteration converges to the exact solution* $x^* \in \mathbb{C}^n$ *of the system of linear equations* (1.1), *with the convergence factor being less than* $\sigma_*(1+2\kappa_h\sqrt{\kappa_{h,s}}\sigma_h^{\mu^*})^2$, *and with the computational efficiency* $\mathscr{E}(\mu^*)$ *being given by* (5.5).

(b) *If* $\{\mu_k\}$ *and* $\{\nu_k\}$ *are*, *respectively*, *the sequences of inner CG and inner CGNE iteration steps of the IHSS*(*CG*, *CGNE*) *iteration such that* $\{\mu_k\}=\{\mu_o^*\}$ *and* $\{\nu_k\}=\{\nu_o^*\}$, *where* μ_o^* *is the smallest positive root of the non-linear function* $f_o(\mu)$ *in* (5.4) *satisfying* $\sigma_*(1+2\sqrt{\kappa_h}\sigma_h^{\mu_o^*})^2 < 1$ *and*

$$\nu_o^* \equiv \nu_o^*(\mu_o^*) = \frac{\ln\left(\left(\frac{\sqrt{\kappa_h}(\sqrt{\kappa_h}+1)}{\sqrt{\kappa_h}(\sqrt{\kappa_{h,s}}+1)+1}\right)\left(\frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}\right)\right)+\mu_o^*\ln\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)}{\ln\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)},$$

then the IHSS(*CG*, *CGNE*) *iteration converges to the exact solution* $x^* \in \mathbb{C}^n$ *of the system of linear equations* (1.1), *with the convergence factor being less than* $\sigma_*(1+2\sqrt{\kappa_h}\sigma_h^{\mu_o^*})^2$, *and with the computational efficiency* $\mathscr{E}_o(\mu_o^*)$ *being given by* (5.6).

Furthermore, *the efficiency* $\mathscr{E}(\mu^*)$ *of IHSS*(*CG*, *Lanczos*) *is about* $1-\frac{4\theta}{2\tau+11+4(\tau+3)\theta}$ *times as much as the efficiency* $\mathscr{E}_o(\mu_o^*)$ *of IHSS*(*CG*, *CGNE*) *when* $n \gg 1$ *and* $\rho_h \gg 0$. *Here*, *the constants* $\sigma_*, \sigma_o, \sigma_h, \sigma_{h,s}, \delta_o$ *and* θ *are defined by* (5.1).

In actual computations, we easily know the stopping tolerance ε and the corresponding CPU time $\mathscr{T}$ (or $\mathscr{T}_o$) required by the IHSS(CG, Lanczos) (or the IHSS(CG, CGNE)) iteration. Therefore, we can define the average computational efficiencies $\overline{\mathscr{E}}$ and $\overline{\mathscr{E}}_o$ of IHSS(CG, Lanczos) and IHSS(CG, CGNE) by

$$\overline{\mathscr{E}} = -\frac{\ln(\varepsilon)}{\mathscr{T}} \quad \text{and} \quad \overline{\mathscr{E}}_o = -\frac{\ln(\varepsilon)}{\mathscr{T}_o},$$

respectively, which are computable approximations to the asymptotic computational efficiencies $\mathscr{E}(\mu)$ (or $\mathscr{E}(\mu^*)$) and $\mathscr{E}_o(\mu_o)$ (or $\mathscr{E}_o(\mu_o^*)$). In this sense, when $\mathscr{T}$ and $\mathscr{T}_o$ are available, we can easily make comparison between the computational efficiencies of IHSS(CG, Lanczos) and IHSS(CG, CGNE) iterations as $\frac{\overline{\mathscr{E}}}{\overline{\mathscr{E}}_o} = \frac{\mathscr{T}_o}{\mathscr{T}}$. See [2] for more discussions about the average and the asymptotic computational efficiencies.

We remark that for specific linear systems arising from applications the quantities κ_h and $\kappa_{h,s}$ may be simply expressed in the orders of magnitudes with respect to certain problem parameters, e.g., the discretization stepsize h and the mesh Reynolds number R_e for a finite-difference matrix of a partial differential equation. It then follows that the constant factors appearing in Theorems 4.2, 4.3, 4.7, 4.8 and 5.1 can be intuitively expressed in the orders of magnitudes with respect to those problem parameters. See [5].

6. A numerical example

In this section, we test the IHSS iterations by numerical experiments. All tests are started from the zero vector, performed in MATLAB with machine precision 10^{-16}, and terminated when the current iterate satisfies $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-6}$, where $r^{(k)}$ is the residual of the kth IHSS iteration.

We solve the two-dimensional convection–diffusion equations

$$-(u_{xx} + u_{yy}) + q\exp(x + y)(xu_x + yu_y) = f(x, y)$$

on the unit square $\Omega = [0, 1] \times [0, 1]$ with the homogeneous Dirichlet boundary conditions. The numbers N of grid points in the two directions are the same, and the linear systems with respect to the N^2-by-N^2 coefficient matrices $\alpha I + H$ and $\alpha I + S$ are solved by the preconditioned conjugate gradient (PCG) method, and the preconditioned Lanczos (PLanczos) or preconditioned CGNE (PCGNE) method, respectively, with transform-based preconditioners [5]. The use of the preconditioning techniques can speed up the convergence of the inner iteration solvers for the shifted Hermitian and the shifted skew-Hermitian linear sub-systems. Here we remark that $n = N^2$. In our computations, the inner PCG and the inner PLanczos/PCGNE iterates are terminated if the current residuals of the inner iterations satisfy

$$\frac{\|p^{(j)}\|_2}{\|r^{(k)}\|_2} \leqslant 1 \times 10^{-\delta_H} \quad \text{and} \quad \frac{\|q^{(j)}\|_2}{\|r^{(k)}\|_2} \leqslant 1 \times 10^{-\delta_S},$$

where $p^{(j)}$ and $q^{(j)}$ are, respectively, the residuals of the jth inner PCG and the jth inner PLanczos/PCGNE iterates at the kth outer IHSS iterate. Here δ_H and δ_S are the control tolerances for iterations about the shifted Hermitian and the shifted skew-Hermitian linear sub-problems, respectively. In our tests, we take δ_H and δ_S to be 1, 2, 3 and 4.

In Tables 6.1–6.6, we list numerical results for the centered difference scheme for $N = 64$ when $q = 10$, 100 and 1000. The optimal parameters α are set to be the values given in Table 5.1 of [5]. In the tables, "$**$" denotes that the number of IHSS iterations is larger than 1000. We remark that the numbers of HSS iterations are 127, 39 and 53 for $q = 10$, 100 and 1000, respectively, where in each HSS iteration, we solve the linear systems with the coefficient matrices $\alpha I + H$ and $\alpha I + S$ exactly by using direct solvers.

Table 6.1
Number of outer (average inner PCG, average inner PLanczos) iterations for $q = 10$

δ_H	δ_S			
	1	2	3	4
1	$**$	372 (1.73, 3.96)	281 (1.72, 7.04)	296 (1.72, 11.18)
2	$**$	353 (3.37, 4.03)	348 (3.17, 7.46)	240 (3.34, 13.37)
3	$**$	327 (5.01, 4.24)	242 (5.14, 7.63)	239 (5.02, 13.03)
4	$**$	332 (7.28, 3.92)	242 (7.48, 7.12)	239 (7.23, 12.83)

Table 6.2
Number of outer (average inner PCG, average inner PLanczos) iterations for $q = 100$

δ_H	δ_S			
	1	2	3	4
1	45 (1.82, 1.56)	40 (2.00, 4.55)	40 (1.88, 8.30)	40 (1.75, 15.53)
2	46 (3.33, 1.50)	39 (3.79, 4.51)	39 (3.72, 7.97)	39 (3.54, 15.21)
3	47 (5.19, 1.51)	40 (5.53, 4.73)	39 (5.56, 7.87)	39 (5.15, 14.97)
4	46 (7.15, 1.52)	40 (7.92, 4.63)	39 (7.67, 7.85)	39 (7.26, 15.02)

Table 6.3
Number of outer (average inner PCG, average inner PLanczos) iterations for $q = 1000$

δ_H	δ_S			
	1	2	3	4
1	54 (1.13, 3.63)	55 (1.13, 7.49)	56 (1.14, 11.71)	56 (1.14, 14.48)
2	64 (2.09, 3.02)	65 (2.08, 6.29)	65 (2.08, 10.06)	65 (2.09, 12.49)
3	65 (3.00, 2.91)	65 (2.94, 6.20)	66 (2.94, 9.86)	66 (2.94, 12.26)
4	65 (4.26, 2.85)	65 (4.14, 6.32)	65 (4.18, 9.98)	66 (4.21, 12.27)

Table 6.4
Number of outer (average inner PCG, average inner PCGNE) iterations for $q = 10$

δ_H	δ_S			
	1	2	3	4
1	623 (1.59, 1.15)	290 (1.73, 1.65)	305 (1.71, 4.34)	293 (1.71, 7.18)
2	224 (2.74, 1.78)	227 (3.37, 1.92)	238 (3.16, 3.95)	241 (3.33, 7.13)
3	214 (4.72, 1.89)	218 (5.08, 1.94)	239 (5.14, 3.18)	241 (5.01, 6.36)
4	214 (7.00, 1.79)	218 (7.27, 2.02)	240 (7.48, 3.15)	241 (7.24, 6.24)

Table 6.5
Number of outer (average inner PCG, average inner PCGNE) iterations for $q = 100$

δ_H	δ_S			
	1	2	3	4
1	53 (1.26, 1.23)	42 (1.93, 4.19)	41 (1.90, 7.10)	41 (1.76, 8.68)
2	45 (3.40, 1.42)	39 (3.77, 4.23)	40 (3.60, 7.18)	40 (3.45, 8.83)
3	46 (5.33, 1.35)	40 (5.45, 4.23)	40 (5.45, 6.68)	40 (5.13, 8.73)
4	46 (7.11, 1.28)	40 (8.00, 4.08)	40 (7.48, 6.88)	40 (7.13, 8.85)

Table 6.6
Number of outer (average inner PCG, average inner PCGNE) iterations for $q = 1000$

δ_H	δ_S			
	1	2	3	4
1	65 (1.70, 3.09)	58 (1.12, 5.72)	57 (1.12, 8.18)	57 (1.12, 12.54)
2	81 (1.69, 2.36)	69 (2.00, 4.57)	66 (2.06, 6.91)	66 (2.09, 11.32)
3	81 (2.40, 2.28)	69 (2.75, 4.43)	67 (2.97, 6.69)	67 (2.88, 11.18)
4	82 (3.36, 2.22)	69 (3.93, 4.36)	67 (4.07, 6.71)	67 (4.18, 11.01)

In Figs. 6.1–6.3, we further show the number of IHSS iterations, the total number of inner PCG iterations, and the total numbers of inner PLanczos and inner PCGNE iterations for different values of α, δ_H and δ_S, when $q = 100$ and $N = 64$.

In the following, we summarize the observations from Tables 6.1–6.6 and Figs. 6.1–6.3:

- When q is large, the numbers of IHSS(PCG, PLanczos) and IHSS(PCG, PCGNE) iterations are about the same as those of HSS iterations required for convergence. However, when q is small, the numbers of IHSS(PCG, PLanczos) and IHSS(PCG, PCGNE) iterations are larger

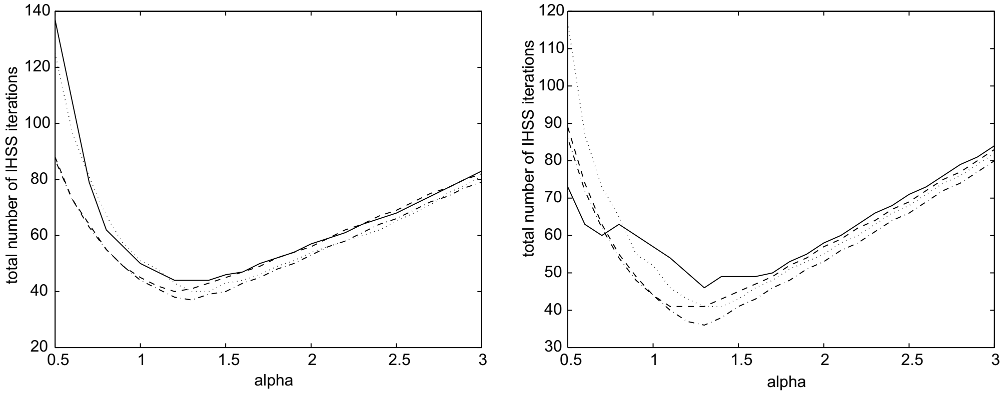

Fig. 6.1. Number of IHSS(PCG, PLanczos) (left) and IHSS(PCG, PCGNE) (right) iterations for different α, δ_H and δ_S.

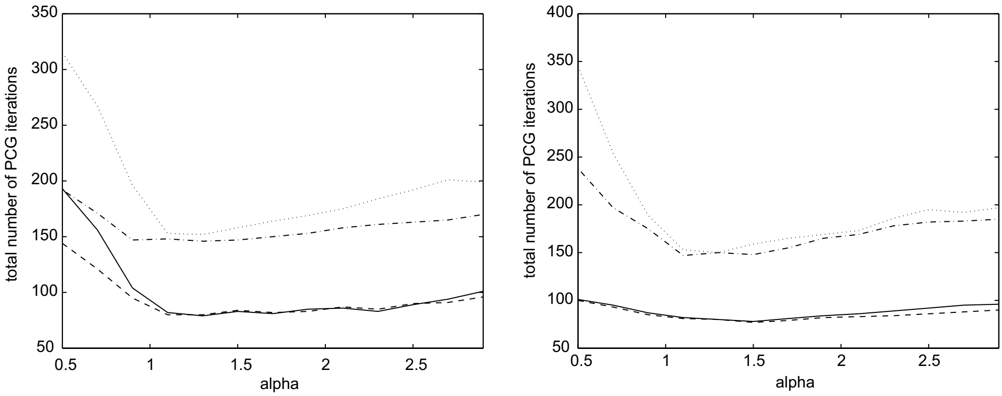

Fig. 6.2. Total numbers of inner PCG iterations in IHSS(PCG, PLanczos) (left) and IHSS(PCG, PCGNE) (right) for different α, δ_H and δ_S.

than those of HSS iterations required for convergence. These results suggest the use of inexact iterations when the skew-Hermitian part is dominant.

- We also record the number of operations required by each iteration. We find that the total computational cost of IHSS(PCG, PCGNE) iteration is smaller than that of IHSS(PCG, PLanczos) iteration. These results imply that the computational efficiency of IHSS(PCG, PCGNE) iteration is higher than that of IHSS(PCG, PLanczos) iteration. See (5.7).
- When δ_H or δ_S increases linearly, the number of inner PCG, inner PLanczos or inner PCGNE iteration also increases linearly.
- For different values of δ_H and δ_S, the optimal parameters α_{ihss} for both IHSS(PCG, PLanczos) and IHSS(PCG, PCGNE) iterations are about the same as the optimal parameters for the HSS iterations.
- In most cases, we find that the total computation cost is the least when $\delta_H = 1$ and $\delta_S = 1$ are used in IHSS(PCG, PLanczos) or IHSS(PCG, PCGNE). These results suggest that we can use the inexact iterations with large tolerances.

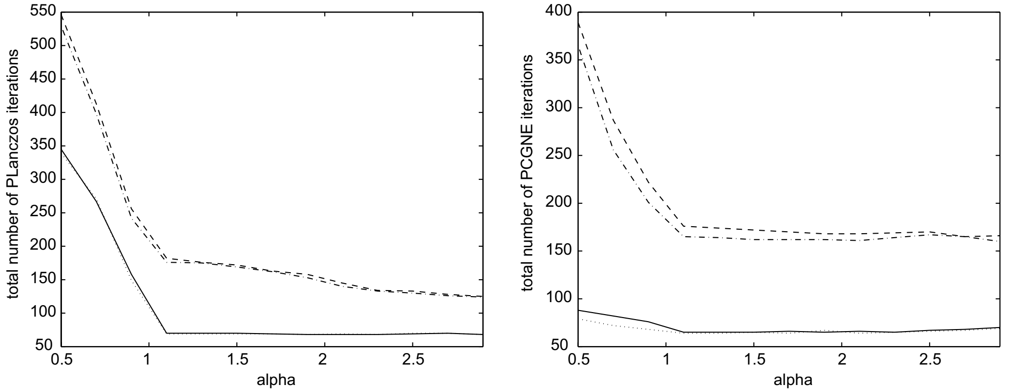

Fig. 6.3. Total numbers of inner PLanczos iterations in IHSS(PCG, PLanczos) (left) and inner PCGNE iterations in IHSS(PCG, PCGNE) (right) for different α, δ_H and δ_S.

- For $\alpha < \alpha_{\text{ihss}}$ or $\alpha > \alpha_{\text{ihss}}$, the number of IHSS iterations is significantly larger than that of IHSS iterations when $\alpha = \alpha_{\text{ihss}}$, but the total number of inner PCG iterations is slightly larger than that for the optimal case. It is interesting to note that the total number of inner PLanczos or inner PCGNE iterations is almost the same for $\alpha > \alpha_{\text{ihss}}$. This phenomenon appears for different values of δ_H and δ_S. Again, these results show that the inexact iterations can be applied especially when the skew-Hermitian part is dominant.

7. Conclusion and remarks

For the non-Hermitian positive definite system of linear equations, we study two specific but very practical inexact Hermitian/skew-Hermitian splitting methods based on some Krylov subspace iterations such as CG, Lanczos and CGNE, and demonstrate that they, like the Hermitian/skew-Hermitian splitting method, converge unconditionally to the exact solution of the linear system.

Moreover, instead of Lanczos and CGNE, we can employ other efficient CG-type methods to solve the system of linear equations with coefficient matrix $\alpha I + S$ involved at each step of the HSS iteration. In particular, when GMRES is applied to the linear system with coefficient matrix $\alpha I + S$, it automatically reduces to a three-term recurrence process, and its convergence property is only dependent on the eigenvalues, but independent of the eigenvectors, of the shifted skew-Hermitian matrix $\alpha I + S$. The corresponding convergence theory of the resulted inexact iteration can be demonstrated in an analogous way to IHSS(CG, CGNE).

Recently, the preconditioned HSS iterations and the extension of the HSS method to positive definite and positive semidefinite linear systems have been studied in [6,10,4,11,13,8,3]. In these papers. the authors have studied how to precondition HSS iteration to speed up the convergence rate of the method, and extended the HSS method to a larger class of linear systems. However, the inexact HSS iterations were not investigated in these works. We remark that the results and techniques in this paper can be equally employed to the preconditioned HSS iterations and the extensions of the HSS iteration as well.

8. Appendix

The proofs of the convergence theorems presented in Section 4 are listed in this section.

Proof of Theorem 4.1. For a fixed iterate index k, define $x^{(k+\frac{1}{2},*)}$ and $x^{(k+1,*)}$ by

$$\begin{cases} x^{(k+\frac{1}{2},*)} = (\alpha I + H)^{-1}[(\alpha I - S)x^{(k)} + b], \\ x^{(k+1,*)} = (\alpha I + S)^{-1}[(\alpha I - H)x^{(k+\frac{1}{2})} + b], \end{cases} \tag{8.1}$$

respectively. Then from Lemmas 3.2 and 3.3 we have

$$\|x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}\|_2 \leqslant \sigma_h(\alpha, \mu_k)\|x^{(k)} - x^{(k+\frac{1}{2},*)}\|_2 \tag{8.2}$$

and

$$\|x^{(k+1)} - x^{(k+1,*)}\|_2 \leqslant \sigma_s(\alpha, \nu_k)\|x^{(k+\frac{1}{2})} - x^{(k+1,*)}\|_2. \tag{8.3}$$

Because x^* is the exact solution of the system of linear equations (1.1), it satisfies the sub-systems of linear equations

$$\begin{cases} (\alpha I + H)x^* = (\alpha I - S)x^* + b, \\ (\alpha I + S)x^* = (\alpha I - H)x^* + b. \end{cases} \tag{8.4}$$

After subtracting x^* from (8.1) and making use of (8.4), accordingly, we obtain

$$\begin{cases} (\alpha I + H)(x^{(k+\frac{1}{2},*)} - x^*) = (\alpha I - S)(x^{(k)} - x^*), \\ (\alpha I + S)(x^{(k+1,*)} - x^*) = (\alpha I - H)(x^{(k+\frac{1}{2})} - x^*). \end{cases} \tag{8.5}$$

The equalities in (8.5) straightforwardly yield

$$\begin{aligned} x^{(k+1,*)} - x^* &= (\alpha I + S)^{-1}(\alpha I - H)(x^{(k+\frac{1}{2})} - x^*) \\ &= (\alpha I + S)^{-1}(\alpha I - H)(x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}) \\ &\quad + (\alpha I + S)^{-1}(\alpha I - H)(x^{(k+\frac{1}{2},*)} - x^*) \\ &= (\alpha I + S)^{-1}(\alpha I - H)(x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}) \\ &\quad + (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)(x^{(k)} - x^*) \end{aligned} \tag{8.6}$$

and

$$\begin{aligned} x^{(k+1,*)} - x^{(k+\frac{1}{2},*)} &= (x^{(k+1,*)} - x^*) - (x^{(k+\frac{1}{2},*)} - x^*) \\ &= (\alpha I + S)^{-1}(\alpha I - H)(x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}) \\ &\quad + (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)(x^{(k)} - x^*) \\ &\quad - (\alpha I + H)^{-1}(\alpha I - S)(x^{(k)} - x^*) \\ &= (\alpha I + S)^{-1}(\alpha I - H)(x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}) \\ &\quad + [(\alpha I + S)^{-1}(\alpha I - H) - I](\alpha I + H)^{-1}(\alpha I - S)(x^{(k)} - x^*). \end{aligned}$$

Therefore,

$$\begin{aligned} x^{(k+\frac{1}{2})} - x^{(k+1,*)} &= (x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}) - (x^{(k+1,*)} - x^{(k+\frac{1}{2},*)}) \\ &= [I - (\alpha I + S)^{-1}(\alpha I - H)](x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}) \\ &\quad + [I - (\alpha I + S)^{-1}(\alpha I - H)](\alpha I + H)^{-1}(\alpha I - S)(x^{(k)} - x^*). \end{aligned} \tag{8.7}$$

Besides, from (8.1) and the equations $b = Ax^*$ we can get

$$\begin{aligned} x^{(k)} - x^{(k+\frac{1}{2},*)} &= x^{(k)} - (\alpha I + H)^{-1}[(\alpha I - S)x^{(k)} + b] \\ &= (\alpha I + H)^{-1}[(\alpha I + H)x^{(k)} - (\alpha I - S)x^{(k)} - b] \\ &= (\alpha I + H)^{-1}A(x^{(k)} - x^*), \end{aligned} \tag{8.8}$$

and from (8.6) we can get

$$\begin{aligned} x^{(k+1)} - x^* &= (x^{(k+1)} - x^{(k+1,*)}) + (x^{(k+1,*)} - x^*) \\ &= (x^{(k+1)} - x^{(k+1,*)}) + (\alpha I + S)^{-1}(\alpha I - H)(x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}) \\ &\quad + (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)(x^{(k)} - x^*). \end{aligned}$$

It then follows that

$$\begin{aligned} |||x^{(k+1)} - x^*||| \leqslant\, & |||x^{(k+1)} - x^{(k+1,*)}||| \\ & + |||(\alpha I + S)^{-1}(\alpha I - H)||| \cdot |||x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}||| \\ & + |||(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)||| \cdot |||x^{(k)} - x^*|||. \end{aligned} \tag{8.9}$$

From (8.2) and (8.8) it holds that

$$\begin{aligned} \|x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}\|_2 &\leqslant \sigma_h(\alpha, \mu_k)\|x^{(k)} - x^{(k+\frac{1}{2},*)}\|_2 \\ &= \sigma_h(\alpha, \mu_k)\|(\alpha I + H)^{-1}A(x^{(k)} - x^*)\|_2 \\ &\leqslant \sigma_h(\alpha, \mu_k)\|(\alpha I + H)^{-1}A(\alpha I + S)^{-1}\|_2|||x^{(k)} - x^*|||, \end{aligned}$$

and from (8.3) and (8.7) it holds that

$$\begin{aligned} \|x^{(k+1)} - x^{(k+1,*)}\|_2 &\leqslant \sigma_s(\alpha, \nu_k)\|x^{(k+\frac{1}{2})} - x^{(k+1,*)}\|_2 \\ &\leqslant \sigma_s(\alpha, \nu_k)\{\|I - (\alpha I + S)^{-1}(\alpha I - H)\|_2\|x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}\|_2 \\ &\quad + \|[I - (\alpha I + S)^{-1}(\alpha I - H)](\alpha I + H)^{-1} \\ &\quad \times (\alpha I - S)(\alpha I + S)^{-1}\|_2|||x^{(k)} - x^*|||\} \\ &\leqslant \sigma_s(\alpha, \nu_k)\{\|I - (\alpha I + S)^{-1}(\alpha I - H)\|_2 \\ &\quad \times \sigma_h(\alpha, \mu_k)\|(\alpha I + H)^{-1}A(\alpha I + S)^{-1}\|_2 \\ &\quad + \|[I - (\alpha I + S)^{-1}(\alpha I - H)](\alpha I + H)^{-1} \\ &\quad \times (\alpha I - S)(\alpha I + S)^{-1}\|_2\}|||x^{(k)} - x^*|||. \end{aligned}$$

Through substituting these two estimates into (8.9), applying Lemma 3.5, and considering the fact that

$$Q(\alpha) \equiv (\alpha I - S)(\alpha I + S)^{-1}$$

is a Cayley transform and thus unitary, we obtain

$$\begin{aligned} |||x^{(k+1)} - x^*||| \leqslant\, & \|\alpha I + S\|_2\|x^{(k+1)} - x^{(k+1,*)}\|_2 \\ & + \|(\alpha I - H)(\alpha I + S)^{-1}\|_2\|\alpha I + S\|_2\|x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}\|_2 \\ & + \|(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)(\alpha I + S)^{-1}\|_2|||x^{(k)} - x^*||| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \{\|\alpha I + S\|_2 \sigma_s(\alpha, \nu_k) \|I - (\alpha I + S)^{-1}(\alpha I - H)\|_2 \\
&\quad \times \sigma_h(\alpha, \mu_k) \|(\alpha I + H)^{-1} A (\alpha I + S)^{-1}\|_2 \\
&\quad + \|\alpha I + S\|_2 \sigma_s(\alpha, \nu_k) \|[I - (\alpha I + S)^{-1}(\alpha I - H)](\alpha I + H)^{-1}\|_2 \\
&\quad + \|\alpha I + S\|_2 \|(\alpha I - H)(\alpha I + S)^{-1}\|_2 \\
&\quad \times \sigma_h(\alpha, \mu_k) \|(\alpha I + H)^{-1} A (\alpha I + S)^{-1}\|_2 \\
&\quad + \|(\alpha I - H)(\alpha I + H)^{-1}\|_2\} |||x^{(k)} - x^*||| \\
&\leqslant \{\|\alpha I + S\|_2 \|[I - (\alpha I + S)^{-1}(\alpha I - H)](\alpha I + H)^{-1}\|_2 \\
&\quad \times [\|I - (\alpha I + S)^{-1}(\alpha I - H)\|_2 \sigma_s(\alpha, \nu_k) \sigma_h(\alpha, \mu_k) \\
&\quad + \sigma_s(\alpha, \nu_k) + \|(\alpha I - H)(\alpha I + S)^{-1}\|_2 \sigma_h(\alpha, \mu_k)] \\
&\quad + \|(\alpha I - H)(\alpha I + H)^{-1}\|_2\} |||x^{(k)} - x^*||| \\
&\leqslant \{\|\alpha I + S\|_2 \|(\alpha I + H)^{-1}\|_2 (1 + \|(\alpha I + S)^{-1}\|_2 \|\alpha I - H\|_2) \\
&\quad \times [(1 + \|(\alpha I + S)^{-1}\|_2 \|\alpha I - H\|_2) \sigma_h(\alpha, \mu_k) \sigma_s(\alpha, \nu_k) \\
&\quad + \sigma_s(\alpha, \nu_k) + \|\alpha I - H\|_2 \|(\alpha I + S)^{-1}\|_2 \sigma_h(\alpha, \mu_k)] \\
&\quad + \|(\alpha I - H)(\alpha I + H)^{-1}\|_2\} |||x^{(k)} - x^*||| \\
&= \{c_h(\alpha)(1 + c_s(\alpha))[(1 + c_s(\alpha)) \sigma_h(\alpha, \mu_k) \sigma_s(\alpha, \nu_k) + \sigma_s(\alpha, \nu_k) \\
&\quad + c_s(\alpha) \sigma_h(\alpha, \mu_k)] + \sigma(\alpha)\} |||x^{(k)} - x^*||| \\
&= (\sigma(\alpha) + \epsilon(\alpha, \mu_k, \nu_k)) |||x^{(k)} - x^*|||.
\end{aligned}
$$

Therefore, the conclusion what we were proving follows. □

Proof of Theorem 4.2. By substituting $\alpha = \alpha^* = \sqrt{\gamma_h^2 - \rho_h^2}$ into the quantities $\sigma_h(\alpha, \mu_k)$ in Lemma 3.2, $\sigma_s(\alpha, \nu_k)$ in Lemma 3.3, as well as $\epsilon(\alpha, \mu_k, \nu_k)$ in Theorem 4.1, respectively, and noticing that $\sigma(\alpha^*) = \frac{\sqrt{\kappa_h}-1}{\sqrt{\kappa_h}+1}$ and that $c_h(\alpha^*) \leqslant \frac{\sqrt{\kappa_h \kappa_{h,s}}}{\sqrt{\kappa_h}+1}$ and $c_s(\alpha^*) \leqslant \sqrt{\kappa_h} - 1$ from Lemma 3.5, we can obtain the estimates

$$
\sigma_h(\alpha^*, \mu_k) \leqslant 2 \left(\frac{\sqrt[4]{\kappa_h} - 1}{\sqrt[4]{\kappa_h} + 1} \right)^{\mu_k}, \quad \sigma_s(\alpha^*, \nu_k) \leqslant 2\sqrt{\kappa_{h,s}} \left(\frac{\sqrt{\kappa_{h,s}} - 1}{\sqrt{\kappa_{h,s}} + 1} \right)^{\lfloor \frac{\nu_k}{2} \rfloor},
$$

and

$$
\epsilon(\alpha^*, \mu_k, \nu_k) \leqslant \epsilon_{h,s}(\mu_k, \nu_k).
$$

Therefore, the conclusion what we were proving follows immediately from Theorem 4.1. □

Proof of Theorem 4.4. For a fixed iterate index k, define $x^{(k+\frac{1}{2},*)}$ and $x^{(k+1,*)}$ be the exact solutions of the systems of linear equations

$$
\begin{cases}
(\alpha I + H) x^{(k+\frac{1}{2},*)} = (\alpha I - S) x^{(k)} + b, \\
(\alpha I + S) x^{(k+1,*)} = (\alpha I - H) x^{(k+\frac{1}{2})} + b,
\end{cases} \tag{8.10}
$$

respectively. Then by Lemmas 3.2 and 3.4, we have

$$
x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)} = p_{\mu_k}^{cg}(\alpha I + H)(x^{(k)} - x^{(k+\frac{1}{2},*)}) \tag{8.11}
$$

and

$$
x^{(k+1)} - x^{(k+1,*)} = p_{\nu_k}^{cg}(\alpha^2 I - S^2)(x^{(k+\frac{1}{2})} - x^{(k+1,*)}). \tag{8.12}
$$

By multiplying the matrix $\alpha I + H$ on both sides of (8.11) and the matrix $\alpha I + S$ on both sides of (8.12), we get

$$\begin{aligned} &b + (\alpha I - S)x^{(k)} - (\alpha I + H)x^{(k+\frac{1}{2})} \\ &\quad = p^{cg}_{\mu_k}(\alpha I + H)[b + (\alpha I - S)x^{(k)} - (\alpha I + H)x^{(k)}] \end{aligned}$$

and

$$\begin{aligned} &b + (\alpha I - H)x^{(k+\frac{1}{2})} - (\alpha I + S)x^{(k+1)} \\ &\quad = p^{cg}_{\nu_k}(\alpha^2 I - S^2)[b + (\alpha I - H)x^{(k+\frac{1}{2})} - (\alpha I + S)x^{(k+\frac{1}{2})}]. \end{aligned}$$

Let x^* be the exact solution of the system of linear equations (1.1). Then x^* satisfies (8.4). A combination of (8.10) and (8.4) results in the identity

$$x^{(k+\frac{1}{2},*)} - x^* = (\alpha I + H)^{-1}(\alpha I - S)(x^{(k)} - x^*).$$

By applying this identity to (8.11) we obtain

$$\begin{aligned} x^{(k+\frac{1}{2})} - x^* &= (x^{(k+\frac{1}{2})} - x^{(k+\frac{1}{2},*)}) + (x^{(k+\frac{1}{2},*)} - x^*) \\ &= p^{cg}_{\mu_k}(\alpha I + H)(x^{(k)} - x^{(k+\frac{1}{2},*)}) + (x^{(k+\frac{1}{2},*)} - x^*) \\ &= p^{cg}_{\mu_k}(\alpha I + H)(x^{(k)} - x^*) + [I - p^{cg}_{\mu_k}(\alpha I + H)](x^{(k+\frac{1}{2},*)} - x^*) \\ &= \{p^{cg}_{\mu_k}(\alpha I + H) + [I - p^{cg}_{\mu_k}(\alpha I + H)](\alpha I + H)^{-1}(\alpha I - S)\} \\ &\quad \times (x^{(k)} - x^*). \end{aligned} \tag{8.13}$$

On the other hand, by using (8.10), (8.12) and (8.4) and following a similar argument to (8.13), we have

$$\begin{aligned} x^{(k+1)} - x^* &= \{p^{cg}_{\nu_k}(\alpha^2 I - S^2) + [I - p^{cg}_{\nu_k}(\alpha^2 I - S^2)](\alpha I + S)^{-1}(\alpha I - H)\} \\ &\quad \times (x^{(k+\frac{1}{2})} - x^*). \end{aligned} \tag{8.14}$$

Substituting (8.13) into (8.14), we then immediately obtain (4.6), where

$$\begin{aligned} M(\alpha, \mu_k, \nu_k) = &\{p^{cg}_{\nu_k}(\alpha^2 I - S^2) + [I - p^{cg}_{\nu_k}(\alpha^2 I - S^2)](\alpha I + S)^{-1}(\alpha I - H)\} \\ &\times \{p^{cg}_{\mu_k}(\alpha I + H) + [I - p^{cg}_{\mu_k}(\alpha I + H)](\alpha I + H)^{-1}(\alpha I - S)\}. \end{aligned}$$

Finally, it straightforwardly follows from $A = H + S$ that the matrix $M(\alpha, \mu_k, \nu_k)$ can be expressed as the sum of the matrices $M(\alpha)$ and $E(\alpha, \mu_k, \nu_k)$. □

Proof of Theorem 4.5. According to Lemma 3.1 we know that $M(\alpha) = I - G(\alpha)A$. Because

$$\begin{aligned} A(\alpha I + H)^{-1}(\alpha I - S)A^{-1} &= A(\alpha I + H)^{-1}[(\alpha I + H) - A]A^{-1} \\ &= I - A(\alpha I + H)^{-1} \\ &= [(\alpha I + H) - A](\alpha I + H)^{-1} \\ &= (\alpha I - S)(\alpha I + H)^{-1}, \end{aligned}$$

(4.9) and (4.10) follow straightforwardly from (4.7) and (4.8), respectively. Moreover, Noticing that

$$\begin{aligned}(\alpha I+S)G(\alpha,\mu_k,\nu_k)(\alpha I+H) &= 2\alpha I-(\alpha I-H)p^{cg}_{\mu_k}(\alpha I+H)\\ &\quad -p^{cg}_{\nu_k}(\alpha^2 I-S^2)(\alpha I-S)\\ &\quad -p^{cg}_{\nu_k}(\alpha^2 I-S^2)Ap^{cg}_{\mu_k}(\alpha I+H)\\ &= 2\alpha I-F(\alpha,\mu_k,\nu_k),\end{aligned}$$

we immediately see that $G(\alpha,\mu_k,\nu_k)$ is non-singular if and only if 2α is not an eigenvalue of the matrix $F(\alpha,\mu_k,\nu_k)$. $\square$

Proof of Theorem 4.6. We note from Theorem 4.4 that

$$|||M(\alpha,\mu_k,\nu_k)||| = |||M(\alpha)+E(\alpha,\mu_k,\nu_k)||| \leqslant |||M(\alpha)|||+|||E(\alpha,\mu_k,\nu_k)|||,$$

and from Lemma 3.1 that

$$|||M(\alpha)||| \leqslant \sigma(\alpha).$$

In addition, from (4.8) we can obtain

$$\begin{aligned}|||E(\alpha,\mu_k,\nu_k)||| &= \|(\alpha I+S)E(\alpha,\mu_k,\nu_k)(\alpha I+S)^{-1}\|_2\\ &\leqslant \|p^{cg}_{\mu_k}(\alpha I+H)(\alpha I+H)^{-1}(\alpha I-H)A(\alpha I+S)^{-1}\|_2\\ &\quad +\|p^{cg}_{\nu_k}(\alpha^2 I-S^2)A(\alpha I+H)^{-1}(\alpha I-S)(\alpha I+S)^{-1}\|_2\\ &\quad +\|p^{cg}_{\nu_k}(\alpha^2 I-S^2)Ap^{cg}_{\mu_k}(\alpha I+H)(\alpha I+H)^{-1}A(\alpha I+S)^{-1}\|_2\\ &\leqslant \|p^{cg}_{\mu_k}(\alpha I+H)\|_2\|(\alpha I+H)^{-1}(\alpha I-H)\|_2\|A(\alpha I+S)^{-1}\|_2\\ &\quad +\|p^{cg}_{\nu_k}(\alpha^2 I-S^2)\|_2\|A(\alpha I+H)^{-1}\|_2\\ &\quad +\|p^{cg}_{\nu_k}(\alpha^2 I-S^2)\|_2\|A(\alpha I+H)^{-1}\|_2\\ &\quad \times\|p^{cg}_{\mu_k}(\alpha I+H)\|_2\|A(\alpha I+S)^{-1}\|_2\\ &\leqslant \epsilon(\alpha,\mu_k,\nu_k).\end{aligned}$$

Therefore, the conclusion what we were proving follows immediately. $\square$

Proof of Theorem 4.7. By substituting $\alpha=\alpha^*=\sqrt{\gamma_h^2-\rho_h^2}$ into the quantities $\sigma_h(\alpha,\mu_k)$, the upper bound of (3.1), $c_h(\alpha)$ and $c_s(\alpha)$ in Lemma 3.5, as well as $\epsilon(\alpha,\mu_k,\nu_k)$ and $|||M(\alpha,\mu_k,\nu_k)|||$ in Theorem 4.6, respectively, we then obtain the estimates about $c_h(\alpha^*)$ and $c_s(\alpha^*)$ in Lemma 3.5, as well as

$$\|p^{cg}_{\mu_k}(\alpha^* I+H)\|_2 \leqslant 2\left(\frac{\sqrt[4]{\kappa_h}-1}{\sqrt[4]{\kappa_h}+1}\right)^{\mu_k}$$

and

$$\|p^{cg}_{\nu_k}((\alpha^*)^2 I-S^2)\|_2 \leqslant 2\left(\frac{\sqrt{\kappa_{h,s}}-1}{\sqrt{\kappa_{h,s}}+1}\right)^{\nu_k},$$

and thereby, $\epsilon(\alpha^*,\mu_k,\nu_k)$ and $|||M(\alpha^*,\mu_k,\nu_k)|||$, correspondingly. $\square$

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