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Riordan matrices in the reciprocation of quadratic polynomials

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We iterate contractive one-degree polynomials with coefficients in the ring $\mathbb{K}[[x]]$ of formal power series to calculate the reciprocal in $\mathbb{K}[[x]]$ of a quadratic polynomial. Doing this we meet the structure of Riordan array. We interpret certain changes of variable as a Riordan array. We finish the paper by using our techniques to find new ways to get known formulas for the sum of powers of natural numbers involving Stirling and Eulerian numbers.

1. Introduction

In this paper, we deal with two types of arithmetical triangles. Those of the first type are remainders that appear in an iterative process, those of the second one can be interpreted as changes of variables. All elements in both types are invertible Riordan arrays (or Riordan matrices) and then they are elements

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of the Riordan group. This group was first introduced in [26] but we are referring to a larger group treated for example in [30] and [14] and also known under the name of Riordan group. Recently, Sprugnoli in [32] has written a list of bibliography on this topic.

The usual approach to Riordan arrays is through the concept of a formal power series. This series have been exposed, for example, by Henrici [12] or in the classical texts of Combinatorial Analysis as Comtet [4] or Riordan [22] or more recently by Graham et al. in [11]. Formal power series correspond to generating functions, and this approach, developed in the previous texts, originated the so-called Method of Coefficients introduced by Egorychev [9], see also Merlini et al. [18] for a recent perspective.

The main idea on the structure of Riordan arrays is to generalize, in a suitable way, the structure of the Pascal triangle, see [28,24], for some earlier work in this direction.

In [16], we constructed the elements in the Riordan group from an iterative process to calculate $\frac{f}{g}$ using a mild generalization of the Banach Fixed Point Theorem.

There, we showed that the structure of Riordan arrays, the reciprocation operation in the ring $\mathbb{K}[[x]]$ and some fixed point problems are intrinsically related. Recall:

Banach's Fixed Point Theorem (BFPT). *Let (X, d) be a complete metric space and $f : X \to X$ contractive. Then f has a unique fixed point x_0 and $f^n(x) \to x_0$ for every $x \in X$.*

In the above statement $f^n = f \circ f \circ \cdots \circ f$. Recall that a map is contractive, concretely c-contractive, if there is a real number $c \in [0, 1)$ such that $d(f(x), f(y)) \leqslant cd(x, y)$. We recommend, for example, [8] for the description of some of the applications of this result.

To reach our goal we used, in [16], an ultrametric d in the ring of formal power series. The idea of considering the ring of formal power series as a topological, or even metric, space goes back to the later nineteen century or the earlier twenty century. To put more recent examples let us say that it is implicitly or explicitly used by Rota and collaborators in their program of re-foundation of combinatorics. In particular it is used by Roman and Rota [25] in their formulation of Umbral Calculus which is also a suitable framework to approach Riordan arrays.

There are, at least, two usual different notations for Riordan arrays, see different authors: [2,10,14,17,27,30,34].

A Riordan array D is traditionally represented by a pair of formal power series $d(t)$, $h(t)$

$$D = (d(t), h(t)) \quad \text{or} \quad D = \mathscr{R}(d(t), h(t)),$$

where $d(0), h(0) \neq 0$. With this notation the action of D on a formal power series $f(t)$ is given by the formula

$$\mathscr{R}(d(t), h(t)) * f(t) = d(t) f(th(t)).$$

The other usual notation is just as above but supposing directly that $d(0) \neq 0$, $h(0) = 0$ and $h'(0) \neq 0$, where h' is the derivative. In this case the action is given by

$$\mathscr{R}(d(t), h(t)) * f(t) = d(t) f(h(t)).$$

Our notation, introduced in [16], and used again herein, is different from both of them. For us a Riordan array is represented by the symbol $T(f|g)$ where f and g are formal power series with $f(0)$, $g(0) \neq 0$. The main difference is that the action of $T(f|g)$ on a power series m is given by

$$T(f|g)(m) = \frac{f}{g} m\left(\frac{t}{g}\right),$$

if t is the indeterminate. We usually do not use the indeterminate to reinforce the idea that a power series here is just a point in a metric space. So the t in the previous formula is no more than the generating function of the sequence $(0, 1, 0, \ldots)$. As we showed in [16] if f, g, d, h are power series with $f(0), g(0), d(0), h(0) \neq 0$, then the conversion formula to pass from our notation to the first traditional one, and viceversa, is given by:

$$T(f|g) = \mathscr{R}(d(t), h(t))$$

where

$$d(t) = \frac{f(t)}{g(t)} \quad \text{and} \quad h(t) = \frac{1}{g(t)} \quad \text{or} \quad f = \frac{d}{h} \quad \text{and} \quad g = \frac{1}{h}$$

With this formula one can construct tables of conversion from one to the other notation. For example the following equality, basic for this work: $T(f|g) = T(f|1)T(1|g)$ which in other notations becomes $\mathscr{R}\left(\frac{f(x)}{g(x)}, \frac{1}{g(x)}\right) = \mathscr{R}(f(x), 1) * \mathscr{R}\left(\frac{1}{g(x)}, \frac{1}{g(x)}\right)$ or $\mathscr{R}\left(\frac{f(x)}{g(x)}, \frac{x}{g(x)}\right) = \mathscr{R}(f(x), x) * \mathscr{R}\left(\frac{1}{g(x)}, \frac{x}{g(x)}\right)$. Also the identity matrix: $T(1|1)$ or $\mathscr{R}(1, 1)$ or $\mathscr{R}(1, x)$. Another example is the Pascal triangle P: $T(1|1-x)$ or $\mathscr{R}\left(\frac{1}{1-x}, \frac{1}{1-x}\right)$ or $\mathscr{R}\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$.

The main reason for our notation is the way we approached the Pascal triangle in [16] using an iterative process related to some fixed point problems. The reason to maintain this notation herein is because it reflects better the kind of problems we are treating. At a first look our notation could appear as cumbersome and complex, but just the fact to get mixed the roles of both parameters in the action of $T(f|g)$ allowed us to obtain an algorithm to construct $T(f|g)$ depending only on the algorithm to obtain the coefficients of the power series $\frac{f}{g}$.

In this paper we are not going to study some prefixed kinds of Riordan arrays, we just *run into* this structure. It *appears* naturally as a reminder associated to an iterative process to calculate the reciprocal of certain power series. Actually, we show that these remainders are generalizations of the Pascal triangle, $T(1|1-x)$ according to our notation. In fact we want to state that the arithmetical pattern used to construct the Pascal triangle is, intrinsically, in the reciprocation of any quadratic polynomial. So, this paper describes a natural framework where Riordan arrays appear as a consequence and not as the main objective. The other main idea is that the successive approximation method gives rise to such pattern of behavior. To carry out the iterations we will use the Banach Fixed Point Theorem.

In Section 2, we develop, in a significant example, the theory that we will describe in the following sections.

In Section 3, we consider the polynomial function induced by a certain polynomial $P(S) \in \mathbb{K}[[x]][S]$ of degree one which is contractive for a suitable complete metric in $\mathbb{K}[[x]]$ and whose unique fixed point is just the reciprocal $\frac{1}{Q}$ of a quadratic polynomial $Q(x) = a + bx + cx^2$, with $a \neq 0$. Later we compare the iterations at $S = 0$ of P with the Taylor polynomials of $\frac{1}{Q}$. In this way we define the remainder. We identify this remainder with an element of the Riordan group. We define the family of polynomials associated to a Riordan array (which, in some sense, is to consider those arrays by rows not by columns as usual) and studying those we find that doing linear changes of variables in the polynomials we arrive to what we will call the Pascal triangle associated to the series $\frac{1}{Q}$. Finally we get the Pascal triangle as product of certain matrices. There are similar results in the literature. See [19,20,21]. We show a factorization of the Pascal Triangle in terms of the remainder $T\left(\frac{1}{Q} \middle| \frac{a+bx}{-c}\right)$ and the corresponding two changes of variables.

The remainders, $T\left(\frac{1}{Q} \middle| \frac{a+bx}{-c}\right)$ in our notation, have been used in the literature. When $Q(1) = 0$ and $ac < 0, ab < 0$ the corresponding remainder is related to the description of a probabilistic modelling of a certain movement of a particle in the plane, see [5]. Actually all our remainders and linear changes of variables are 7-matrices as called in [5]. Moreover the used iteration process generates all 7-matrices if we allow to start at any series as initial condition. We will not treat it here. On the contrary we always start to iterate at $S = 0$.

In Section 4, we focus on the study of the change of variables obtained in Section 3. Actually, we show that, in general, if we multiply any element of Riordan group, $T(f|g)$ by $T(1|a + bx)$ we are doing a change of variables in the associated family of polynomials of $T(f|g)$. We note that the set of treated changes of variables has a representation as a subgroup of the Riordan group. Inside this subgroup we find different elements of order 2 and some conjugation relations.

In Section 5, we go back to our motivating example. Using the family of polynomials associated in this case, we get some known formulas of the sums of powers of natural numbers.

From now on, we consider $\mathbb{N} = \{0, 1, 2, \ldots\}$ the natural numbers in the field $\mathbb{K}$ of characteristic 0. Recall that a field $\mathbb{K}$ is of *characteristic 0* if the minimal subfield inside $\mathbb{K}$ is isomorphic to the rational numbers. Equivalently if the n-fold sum $1+1+\cdots+1$ is never null. The notation $f^{(k)}$ represents the kth derivative of the power series f.

2. Motivation: BFPT and the arithmetic–geometric series

There is an obvious way to sum the geometric series $\sum_{k=0}^{\infty} x^k$ using the BFPT. See [1] for a proof without words. It is natural to wonder if we can sum the arithmetic–geometric series $\sum_{k=1}^{\infty} kx^{k-1}$ using the BFPT. It is easy to see that there are not any one-degree polynomial $f(t)=g(x)t+h(x)$ and any point x_0 such that the partial sum $\sum_{k=0}^{n}(k+1)x^k=f^{n+1}(x_0)$. Since $f(x_0)=g(x)x_0+h(x)=1, f^2(x_0)=f(f(x_0))=f(1)=g(x)+h(x)=1+2x$ and $f^3(x_0)=f(f(f(x_0)))=f(1+2x)=g(x)(1+2x)+h(x)$ we obtain that $x_0=-\frac{1}{3}$ and $f(x)=\frac{3}{2}xt+1+\frac{1}{2}x$ and from here $f^4\left(-\frac{1}{3}\right)\neq 1+2x+3x^2+4x^3$. In view of this, we are going to iterate a polynomial whose fixed point is the sum of the arithmetic–geometric series, that is $\sum_{k=1}^{\infty} kx^{k-1}=\frac{1}{(1-x)^2}$. Since the equality $t=\frac{1}{(1-x)^2}$ can be converted to $t=1+(2x-x^2)t$, we consider the polynomial $f(t)=1+(2x-x^2)t$ and we do the first iterations in $t=0$:

$$\begin{aligned}
&f(0)=1,\\
&f^2(0)=1+2x-x^2,\\
&f^3(0)=1+2x+3x^2-4x^3+x^4,\\
&f^4(0)=1+2x+3x^2+4x^3-11x^4+6x^5-x^6,\\
&f^5(0)=1+2x+3x^2+4x^3+5x^4-26x^5+23x^6-8x^7+x^8,\\
&f^6(0)=1+2x+3x^2+4x^3+5x^4+6x^5-57x^6+72x^7-39x^8+10x^9-x^{10}.
\end{aligned}$$

We can observe that in each iteration the partial sum appears plus a remainder. We want to control the difference with the partial sum. For it, we began to write the coefficients of the remainder, that is:

$$A_1=\begin{pmatrix}
-1 &&&&&&\\
-4 & 1 &&&&&\\
-11 & 6 & -1 &&&&\\
-26 & 23 & -8 & 1 &&&\\
-57 & 72 & -39 & 10 & -1 &&\\
-120 & 201 & -150 & 59 & -12 & 1 &\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{1}$$

Observing the above triangle we can see some resemblances with the Pascal Triangle as we will prove in the last section. For example: The rule of construction is similar to that of the Pascal triangle, each element is twice the above element minus the element above to the left side. The elements in the first column are Eulerian numbers except for the sign. The sum of the elements in any row are triangular numbers with negative sign. For every element, the sum of all elements in its row to the right and all elements above in its column is zero.

Returning to the iterations, we can write the $(n+2)$-iteration as

$$f^{n+2}(0)=\mathsf{T}_{n+1}\left(\frac{1}{(1-x)^2}\right)+x^{n+2}p_n(x),$$

where $\mathsf{T}_n(S)$ is the n-degree Taylor polynomial of $S(x)$ at $x=0$ and we call $(p_n(x))_{n\in\mathbb{N}}$ *the family of polynomials associated to this triangle*. In this case:

$$\begin{aligned}
&p_0(x)=-1,\\
&p_1(x)=-4+x,\\
&p_2(x)=-11+6x-x^2,\\
&p_3(x)=-26+23x-8x^2+x^3,\\
&p_4(x)=-57+72x-39x^2+10x^3-x^4.
\end{aligned}$$

In general and using the recurrence of the elements of the above triangle we get

$$p_{n+1}(x) = (2-x)p_n(x) - (n+2) \quad \text{and} \quad p_n(x) = -\sum_{k=0}^{n}(k+1)(2-x)^{n-k}.$$

In view of the last expression we do the changes $t = 2-x$ and $q_n(t) = -p_n(2-t)$. If we consider the coefficients as above, for the new family of polynomials, we obtain:

$$A_2 = \begin{pmatrix} 1 & & & & & & \\ 2 & 1 & & & & & \\ 3 & 2 & 1 & & & & \\ 4 & 3 & 2 & 1 & & & \\ 5 & 4 & 3 & 2 & 1 & & \\ 6 & 5 & 4 & 3 & 2 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Working with this family and its derivatives we obtain the next expressions:

$$q_n(t) = \sum_{k=0}^{n}(n+1-k)t^k \quad \text{and} \quad q_n^{(k)}(1) = k!\binom{n+2}{k+2}.$$

Once again, in view of the last expression it is reasonable to expand $q_n(t)$ at $t = 1$,

$$q_n(t) = \sum_{k=0}^{n}\binom{n+2}{k+2}(t-1)^k.$$

Now we can do the change $s = t-1$ and consider the family $r_n(s) = q_n(s+1)$. If we put the coefficients as above we get the Pascal triangle with the two first columns deleted:

$$A_3 = \begin{pmatrix} 1 & & & & & & \\ 3 & 1 & & & & & \\ 6 & 4 & 1 & & & & \\ 10 & 10 & 5 & 1 & & & \\ 15 & 20 & 15 & 6 & 1 & & \\ 21 & 35 & 35 & 21 & 7 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As we will point out, in a more general setting in the next section, the three above matrices are Riordan arrays. In fact $A_1 = T\left(\frac{1}{(1-x)^2}\middle| 2x-1\right)$. After the first change of variable it is transformed into $A_2 = T\left(\frac{1}{(1-x)^2}\middle| 1\right)$. After the second change of variable it becomes $A_3 = T\left(\frac{1}{(1-x)^2}\middle| 1-x\right)$. We also have the following equalities relating them:

$$T\left(\frac{1}{(1-x)^2}\middle| 2x-1\right) = T\left(\frac{1}{(1-x)^2}\middle| 1\right)T(1|2x-1)$$

and

$$T\left(\frac{1}{(1-x)^2}\middle| 1\right) = T\left(\frac{1}{(1-x)^2}\middle| 1-x\right)T(1|1+x).$$

Note that this means that the matrices $T(1|2x-1)$ and $T(1|1+x)$ can be interpreted as the changes of variables in the family of polynomials made before.

This example motivates the next section where we will do the analogous development for any quadratic polynomial.

3. An iterative process to calculate the reciprocal of quadratic polynomials

In this section, we want to show that the development described in the previous one is no more than a particular example of a general phenomenon about quadratic polynomials.

For any quadratic polynomial Q we find a contractive first degree polynomial function P_Q in $\mathbb{K}[[x]]$ whose unique fixed point is just $\frac{1}{Q}$. We iterate this function. So, in each iteration the partial sum of $\frac{1}{Q}$ appears plus a remainder. The study of these remainders and the identification of them as Riordan arrays is the first main aim of this section. The second main aim is the proof, by means of changes of variables, that the structure of Pascal triangle is in each remainder. This is the reason why we define the Pascal triangle associated to a power series.

Along this section we use [16] for notation and basic results. In particular, as supposed there, $\mathbb{K}$ is always a field with characteristic zero and $\mathbb{K}[[x]]$ is the ring of formal power series over $\mathbb{K}$. Denote by $\omega(f)$ the *order* of $f=\sum_{n\geqslant 0} f_n x^n$. Recall that $\omega(f)$ is the smallest nonnegative integer number p such that $f_p \neq 0$ if any exist. Otherwise, that is if $f=0$, we write $\omega(f)=\infty$. See [23] for details and for the main properties of ultrametrics. We recall here some facts in [16] that we need:

Proposition 1. *The map $d:\mathbb{K}[[x]]\times\mathbb{K}[[x]]\to\mathbb{R}_+$ defined by $d(f,g)=\frac{1}{2^{\omega(f-g)}}$ is a complete ultrametric on $\mathbb{K}[[x]]$. Moreover $d(f,g)\leqslant\frac{1}{2^{k+1}}$ if and only if $\mathsf{T}_k(f)=\mathsf{T}_k(g)$. Finally the sum and product of series are continuous if we consider the corresponding product topology in $\mathbb{K}[[x]]\times\mathbb{K}[[x]]$.*

Proposition 2. *Let $f,h\in\mathbb{K}[[x]]$ with $f(0)=0$. Then the first degree polynomial map $P:\mathbb{K}[[x]]\to\mathbb{K}[[x]]$ defined by $P(S)=fS+h$ is $\frac{1}{2}$-contractive independently on f and h. In fact $d(P(S_1),P(S_2))=\frac{1}{2^{\omega(f)}}d(S_1,S_2)$. Moreover the unique fixed point of P is just $\frac{h}{1-f}$ and consequently*

$$\frac{h}{1-f}=\left(\sum_{n\geqslant 0} f^n\right)h.$$

Corollary 3. *Let $f,g\in\mathbb{K}[[x]]$ with $g(0)\neq 0$. If $f=\sum_{n\geqslant 0} f_n x^n$ and $g=\sum_{n\geqslant 0} g_n x^n$ and $\frac{f}{g}=\sum_{n\geqslant 0} d_n x^n$, then $d_n=-\frac{g_1}{g_0}d_{n-1}-\frac{g_2}{g_0}d_{n-2}-\cdots-\frac{g_n}{g_0}d_0+\frac{f_n}{g_0}$, for $n\geqslant 1$, $d_0=\frac{f_0}{g_0}$.*

Note that in the above corollary we iterate the function $P:\mathbb{K}[[x]]\to\mathbb{K}[[x]]$ defined by $P(S)=\left(\frac{g_0-g}{g_0}\right)S+\frac{f}{g_0}$ which is at least $\frac{1}{2}$-contractive and whose unique fixed point is $\frac{f}{g}$.

Another of the results in [16] is the following:

Algorithm for $T(f|g)$

$f=\sum_{n\geqslant 0} f_n x^n$, $g=\sum_{n\geqslant 0} g_n x^n$ with $g_0\neq 0$, $T(f|g)=(d_{i,j})$ with $i,j\geqslant 0$.

$$\begin{array}{l|llllll}
f_0 & & & & & & \\
f_1 & d_{0,0} & d_{0,1} & d_{0,2} & d_{0,3} & d_{0,4} & \cdots \\
f_2 & d_{1,0} & d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & \cdots \\
f_3 & d_{2,0} & d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
f_{n+1} & d_{n,0} & d_{n,1} & d_{n,2} & d_{n,3} & d_{n,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

with $d_{i,j}=0$ if $j>i$ and the following rules for $i\geqslant j$:

If $j>0$

$$d_{i,j}=-\frac{g_1}{g_0}d_{i-1,j}-\frac{g_2}{g_0}d_{i-2,j}-\cdots-\frac{g_i}{g_0}d_{0,j}+\frac{d_{i-1,j-1}}{g_0}=\frac{1}{g_0}\left(d_{i-1,j-1}-\sum_{k=1}^{i} g_k d_{i-k,j}\right)$$

and if $j=0$

$$d_{i,0}=-\frac{g_1}{g_0}d_{i-1,0}-\frac{g_2}{g_0}d_{i-2,0}-\cdots-\frac{g_i}{g_0}d_{0,0}+\frac{f_i}{g_0}=\frac{1}{g_0}\left(f_i-\sum_{k=1}^{i} g_k d_{i-k,0}\right).$$

Since the empty sum evaluates to 0 we have that $d_{0,0} = \frac{f_0}{g_0}$. Then, in the 0-column are just the coefficients of $\frac{f}{g}$, i.e. $d_{i,0} = d_i$.

If $Q(x) = a + bx + cx^2$ with $a \neq 0$ and from Proposition 2, we have that the first degree polynomial function in $\mathbb{K}[[x]]$, $P_Q : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ defined by $P_Q(S) = ((\frac{-b}{a})x + (\frac{-c}{a})x^2)S + \frac{1}{a}$ where $S \in \mathbb{K}[[x]]$, is contractive and its unique fixed point is $\frac{1}{Q}$. So we iterate this function and we compare these iterations with the corresponding Taylor polynomial of $\frac{1}{Q}$, $\mathsf{T}_n(\frac{1}{Q})$. In this process a lower triangle matrix $(d_{n,k})$, depending on Q, appears. This matrix describes the remainder as we did in the motivating example. The following theorem shows, in particular, an algorithm to get the entries of the matrix $(d_{n,k})$. Studying this matrix, and using our **Algorithm for** $T(f|g)$ above we realize that $(d_{n,k})$ is a Riordan array.

Theorem 4. *Let $Q(x) = a + bx + cx^2$ be a polynomial with $a \neq 0$. Consider the one degree polynomial function in the ring $K[[x]]$, $P_Q(S) = \left(\left(\frac{-b}{a}\right)x + \left(\frac{-c}{a}\right)x^2\right)S + \frac{1}{a}$, where $S \in \mathbb{K}[[x]]$. If $\frac{1}{Q} = \sum_{k\geqslant 0} d_k x^k$ then $P_Q^{n+1}(0) = \mathsf{T}_n\left(\frac{1}{Q}\right) + x^{n+1}\left(\sum_{k=1}^{\infty} d_{n,k}x^{k-1}\right)$ with*

(1) *$d_0 = 1/a$, $d_1 = (-b/a)d_0$ and $d_n = (-b/a)d_{n-1} + (-c/a)d_{n-2}, n \geqslant 2$.*
(2) *$d_{n,k} = 0$, $\forall k > n$. If we call $d_{n,0} = d_n$ for $n \in \mathbb{N}$, we have $d_{n,k} = (-b/a)d_{n-1,k} + (-c/a)d_{n-1,k-1}$ for $n, k \geqslant 1$.*

Proof. First, since P_Q is at least $\frac{1}{2}$-contractive then $\mathsf{T}_n\left(\frac{1}{Q}\right)$ is just the n-degree polynomial in the expression $P_Q^{n+1}(0)$, that is $\mathsf{T}_n(\frac{1}{Q})$ is obtained from $P_Q^{n+1}(0)$ eliminating all powers, in the unknown, greater than n.

Part (1) is well-known and it was proved again in the Corollary 3. We are going to use induction in order to prove (2). Note first that $P_Q(0) = \frac{1}{a} = T_0\left(\frac{1}{Q}\right)$. So, $d_{0,k} = 0$ for $k > 0$, now

$$P_Q^2(0) = ((-b/a)x + (-c/a)x^2)\frac{1}{a} + \frac{1}{a} = \frac{1}{a} + (-b/a)\frac{1}{a}x + x^2(-c/a^2)$$

consequently $d_{1,1} = -c/a^2 = (-b/a)d_{0,1} + (-c/a)d_{0,0}$, because $d_{0,1} = 0$ and $d_{0,0} = d_0 = \frac{1}{a}$ and $d_{1,k} = 0$ for $k > 1$. Suppose that the result is true for $m - 1 \in \mathbb{N}$ then

$$\begin{aligned}
P_Q^{m+1}(0) &= ((-b/a)x + (-c/a)x^2)\left(\sum_{k=0}^{m-1} d_k x^k + x^m\left(\sum_{k=1}^{\infty} d_{m-1,k}x^{k-1}\right)\right) + \frac{1}{a} \\
&= 1/a + \sum_{k=0}^{m-1}(-b/a)d_k x^{k+1} + \sum_{k=0}^{m-2}(-c/a)d_k x^{k+2} + (-c/a)d_{m-1}x^{m+1} + x^{m+1} \\
&\quad \times \left(\sum_{k=1}^{\infty}((-b/a) + (-c/a)x)d_{m-1,k}x^{k-1}\right) \\
&= 1/a + (-b/a)d_0 x + \sum_{k=2}^{m}((-b/a)d_{k-1} + (-c/a)d_{k-2})x^k \\
&\quad + x^{m+1}\left((-b/a)d_{m-1,1} + (-c/a)d_{m-1} + \sum_{k=2}^{\infty}(-b/a)d_{m-1,k}x^{k-1} + \sum_{k=1}^{\infty}(-c/a)d_{m-1,k}x^k\right) \\
&= T_m(1/Q) + x^{m+1} \times \left(\sum_{k=1}^{\infty}((-b/a)d_{m-1,k} + (-c/a)d_{m-1,k-1})x^{k-1}\right).
\end{aligned}$$

Consequently, $d_{m,k} = (-b/a)d_{m-1,k} + (-c/a)d_{m-1,k-1}$. Now if $k > m$ then $d_{m,k} = (-b/a)d_{m-1,k} + (-c/a)d_{m-1,k-1}$. By induction hypothesis we have $d_{m,k} = 0$. $\square$

Let $C_Q = (d_{n,k})_{n,k\in\mathbb{N}}$. Note that it is a lower triangular matrix. Let us expand a few terms of C_Q avoiding, a priori, the null entries:

$$\left.\begin{array}{l}
\frac{1}{a}\\
\frac{-b}{a^2}\\
\frac{b^2}{a^3}-\frac{c}{a^2}\\
\frac{-b^3}{a^4}+\frac{2bc}{a^3}\\
\vdots\\
\frac{1}{Q}
\end{array}\right|
\begin{array}{llll}
 & & & \\
\frac{-c}{a^2} & & & \\
\frac{2bc}{a^3} & \frac{c^2}{a^3} & & \\
\frac{-3cb^2}{a^4}+\frac{c^2}{a^3} & \frac{-3bc^2}{a^4} & \frac{-c^3}{a^4} & \\
\vdots & \vdots & \vdots & \ddots\\
\frac{1}{Q}\frac{-xc}{a+bx} & \frac{1}{Q}\frac{x^2c^2}{(a+bx)^2} & \frac{1}{Q}\frac{-x^3c^3}{(a+bx)^3} & \cdots
\end{array}$$

Let $R_Q = (r_{n,k})_{n,k\in\mathbb{N}}$ with $r_{n,k} = d_{n+1,k+1}$, that is, C_Q without the first column and the first row.

As an easy and direct application of our algorithm for $T(f|g)$ quoted above, we obtain the structure of these matrices in the following:

Corollary 5. *Let $Q = a + bx + cx^2$ with $a, c \neq 0$. The matrices R_Q and C_Q are the following Riordan arrays*:

$$R_Q = T\left(\frac{1}{Q}\,\middle|\,\frac{a+bx}{-c}\right) \quad \text{and} \quad C_Q = T\left(\frac{1}{Q}\frac{a+bx}{-c}\,\middle|\,\frac{a+bx}{-c}\right).$$

Now, we are going to find a formula for the general term of $T\left(\frac{1}{Q}\,\middle|\,\frac{a+bx}{-c}\right)$, that is $r_{n,k}$, in terms of the numbers $\{d_k\}_{k\in\mathbb{N}}$. Recall that $\frac{1}{Q} = \sum_{k=0}^{\infty} d_k x^k$:

Proposition 6. *Let $Q(x) = a + bx + cx^2$ with $a, c \neq 0$. Then*

$$r_{n,k} = \sum_{l=0}^{n-k} d_l \binom{n-l}{k} (-c/a)^{k+1}(-b/a)^{n-k-l}.$$

Proof. It is easy to verify the equality for $r_{0,0}, r_{1,0}, r_{1,1}$. Now by induction on n fixing k we have by part (2) in Theorem 4:

$$\begin{aligned}
r_{n,k} &= (-b/a)r_{n-1,k} + (-c/a)r_{n-1,k-1}\\
&= \sum_{l=0}^{n-k-1} d_l \binom{n-1-l}{k}(-c/a)^{k+1}(-b/a)^{n-k-l} + \sum_{l=0}^{n-k} d_l \binom{n-1-l}{k-1}(-c/a)^{k+1}(-b/a)^{n-k-l}\\
&= \sum_{l=0}^{n-k-1} d_l \left(\binom{n-1-l}{k} + \binom{n-1-l}{k-1}\right)(-c/a)^{k+1}(-b/a)^{n-k-l} + d_{n-k}(-c/a)^{k+1}\\
&= \sum_{l=0}^{n-k} d_l \binom{n-l}{k}(-c/a)^{k+1}(-b/a)^{n-k-l}. \qquad \square
\end{aligned}$$

One of the principal tools in this paper is:

Definition 7. Consider $T(f|g)$ and suppose that $(d_{i,j})_{i,j\in\mathbb{N}}$ is the associated matrix to $T(f|g)$. Then *the family of polynomials associated to $T(f|g)$*, which we denote by $(p_n)_{n\in\mathbb{N}}$, is

$$p_n(x) = \sum_{j=0}^{n} d_{n,j} x^j \quad \text{with } n \in \mathbb{N}.$$

Let us consider the sequence of polynomials associated to R_Q defined by $p_0(x) = r_{0,0}$, $p_n(x) = \sum_{k=0}^{n} r_{n,k}x^k$, $n \in \mathbb{N}$ where $r_{n,k} = d_{n+1,k+1}$ are those described in Theorem 4 for the quadratic polynomial Q. With this notation we can rewrite the main formula in Theorem 4 as:

$$P_Q^{n+2}(0) = \mathsf{T}_{n+1}(1/Q) + x^{n+2}p_n(x) \quad \text{or} \quad p_n(x) = \frac{P_Q^{n+2}(0) - \mathsf{T}_{n+1}\left(\frac{1}{Q}\right)}{x^{n+2}}.$$

So using Theorem 4 again, we obtain easily

Proposition 8

$$p_{n+1}(x) = \left(\frac{b+cx}{-a}\right)p_n(x) + (-c/a)d_{n+1}, \quad \text{for } n \geqslant 0.$$

Consequently

$$p_n(x) = (-c/a)\left(\sum_{k=0}^{n} d_{n-k}\left(\frac{b+cx}{-a}\right)^k\right).$$

In view of the above proposition it is natural to consider the change of variable, supposing $c \neq 0$, $t = \frac{b+cx}{-a}$. Define $q_n(t) = (-a/c)p_n\left(\frac{at+b}{-c}\right)$. Consequently

$$q_n(t) = \sum_{k=0}^{n} d_{n-k}t^k. \tag{2}$$

If we consider the matrix $M = (m_{n,k})_{n,k\in\mathbb{N}}$ where the entries in the row n are the coefficients of the polynomial q_n in increasing power order we have:

Proposition 9. $M = T\left(\frac{1}{Q}\middle| 1\right)$.

Proof. We know that $T\left(\frac{1}{Q}\middle| 1\right)$ is a lower triangular Toeplitz matrix whose columns are, beginning at the main diagonal, the coefficients of $\frac{1}{Q}$, that is, d_n. If we now read $T\left(\frac{1}{Q}\middle| 1\right)$ by rows, avoiding the a priori null entries, we get for the first row d_0, the second: d_1, d_0, the third d_2, d_1, d_0 and so on, that is, the matrix M. □

Related to the polynomials q_n defined in (2) we obtain the next expression for the number $q_n^{(k)}(1)$, where $q_n^{(k)}$ means the kth derivative of q_n:

$$q_n^{(k)}(1) = \sum_{j=k}^{n} \frac{j!}{(j-k)!}d_{n-j}.$$

Now, expanding $q_n(t)$ at $t = 1$,

$$q_n(t) = \sum_{k=0}^{n}\sum_{j=k}^{n}\binom{j}{k}d_{n-j}(t-1)^k.$$

Once more, in view of the above expression, it is natural to consider the change of variable $s = t - 1$. We define the next family of polynomials: $r_n(s) = q_n(s+1)$. Consequently

$$r_n(s) = \sum_{k=0}^{n}\sum_{j=k}^{n}\binom{j}{k}d_{n-k}s^k.$$

If we define the matrix $L = (l_{n,k})_{n,k\in\mathbb{N}}$ where the nth row consists of the coefficients of r_n, as before, we get:

Proposition 10. $L = T\left(\frac{1}{Q}\middle| 1-x\right)$.

Proof. We write the matrix L

$$
\begin{array}{cccccc}
d_0 & & & & & \\
d_0+d_1 & d_0 & & & & \\
d_0+d_1+d_2 & 2d_0+d_1 & d_0 & & & \\
d_0+d_1+d_2+d_3 & 3d_0+2d_1+d_2 & 3d_0+d_1 & \ddots & & \\
\downarrow & \downarrow & \downarrow & & \searrow & \\
\frac{1}{Q}\frac{1}{1-x} & \frac{1}{Q}\frac{x}{(1-x)^2} & \frac{1}{Q}\frac{x^2}{(1-x)^3} & \cdots & \frac{1}{Q}\frac{x^{n-1}}{(1-x)^n} & \cdots
\end{array}
$$

that is $T\left(\frac{1}{Q}\middle| 1-x\right)$. $\square$

Remark 11. The above proposition reflects the known ubiquity of the Pascal triangle. More concretely the rule of construction of the Pascal triangle is in any of the remainders described before, up to some *linear* changes of variables.

Given a power series $f = \sum_{n\geqslant 0} f_n x^n$, if we call $T(f|1-x)$ *the Pascal triangle associated to power series f*, then the Pascal triangle associated to the power series $f \equiv 1$ is just the classical Pascal triangle. Moreover the rule of construction of $T(f|1-x)$ is just the same as that of Pascal triangle. Note that $T(f|1-x) = T(f|1)T(1|1-x)$. So, the last equality says that, up the changes of variables, to construct these remainders one only has to know the coefficients of $\frac{1}{Q}$ and the rule of construction of Pascal triangle.

If we look again at Proposition 8 and to the first change of variable, we can say that to make this change of variable is just the same thing as to multiply the matrix $T\left(\frac{1}{Q}\middle|\frac{a+bx}{-c}\right)$ by $T\left(1\middle|\frac{c+bx}{-a}\right)$ because

$$T\left(\frac{1}{Q}\middle| 1\right) = T\left(\frac{1}{Q}\middle|\frac{a+bx}{-c}\right)T\left(1\middle|\frac{c+bx}{-a}\right)$$

note that $T^{-1}\left(1\middle|\frac{a+bx}{-c}\right) = T\left(1\middle|\frac{c+bx}{-a}\right)$.

In a similar way the last change of variable, after Proposition 9, is the same as to multiply the matrix $T\left(\frac{1}{Q}\middle| 1\right)$ by the Pascal triangle $T(1|1-x)$ because

$$T\left(\frac{1}{Q}\middle| 1-x\right) = T\left(\frac{1}{Q}\middle| 1\right)T(1|1-x).$$

We can summarize all above in the next factorization theorem:

Theorem 12.

$$T\left(\frac{1}{Q}\middle|\frac{a+bx}{-c}\right) = T\left(\frac{1}{Q}\middle| 1-x\right)T^{-1}(1|1-x)T\left(1\middle|\frac{a+bx}{-c}\right).$$

Remark 13. Note that the global change of variable in the sequence of associated polynomials given by

$$s = \frac{a+b+cx}{-a},$$
$$r_n(s) = \left(\frac{-a}{c}\right)p_n\left(\frac{as+a+b}{-c}\right)$$

corresponds to multiply the remainder by the matrix $T\left(1\left|\frac{a+(a+b)x}{-c}\right.\right)$.

4. The group of generalized linear change of variables

In this section, we are going to relate the change of variables represented by $T(1|a+bx)$, and some mild generalizations, with some results and problems in the related literature. First, to reinforce the idea of change of variable, we can generalize the procedure described in the previous section to a general arithmetical triangle $T(f|g)$:

Proposition 14. *Let $p_n(x)=\sum_{k=0}^{n} d_{n,k}x^k$ be the associated polynomials to $T(f|g)$ with general term $(d_{n,k})_{n,k\in\mathbb{N}}$. Then the associated polynomials to $T(f|ag+bx)$, with $a\neq 0$, are $q_n(t)=\frac{1}{a}\sum_{k=0}^{n} d_{n,k}t^k$ with $t=\frac{x-b}{a}$.*

Proof. First, we observe that $T(f|ag+bx)=T(f|g)T(1|a+bx)$. On the other hand, it is easy to find the general term, $(l_{k,m})_{n,k\in\mathbb{N}}$, of $T(1|a+bx)$, in fact: $l_{k,m}=\frac{1}{a^{m+1}}\binom{k}{m}\left(-\frac{b}{a}\right)^{k-m}$. We define the associated polynomials to $T(f|ag+bx)$ as $\hat{p}_n(x)=\sum_{m=0}^{n}\alpha_{n,m}x^m$ where $\alpha_{n,m}=\sum_{k=m}^{n} d_{n,k}l_{k,m}$. Then

$$\begin{aligned}
\hat{p}_n(x) &= \sum_{m=0}^{n}\alpha_{n,m}x^m=\sum_{m=0}^{n}\sum_{k=m}^{n} d_{n,k}l_{k,m}x^m=\sum_{k=0}^{n}\sum_{m=0}^{k} d_{n,k}l_{k,m}x^m\\
&=\sum_{k=0}^{n}\sum_{m=0}^{k} d_{n,k}\left(\frac{1}{a^{m+1}}\binom{k}{m}\left(-\frac{b}{a}\right)^{k-m}\right)x^m=\sum_{k=0}^{n} d_{n,k}\frac{1}{a^{k+1}}\sum_{m=0}^{k}\binom{k}{m}(-b)^{k-m}x^m\\
&=\sum_{k=0}^{n} d_{n,k}\frac{1}{a^{k+1}}(x-b)^k=\frac{1}{a}\sum_{k=0}^{n} d_{n,k}\left(\frac{x-b}{a}\right)^k. \quad \square
\end{aligned}$$

Consider now the sets LCV= $\{T(1|\alpha+\beta x)$ with $\alpha,\beta\in\mathbb{K}$ and $\alpha\neq 0\}$, we call it the set of linear changes of variable, and GLCV= $\{T(\lambda|\alpha+\beta x)$ with $\lambda,\alpha,\beta\in\mathbb{K}$ and $\lambda,\alpha\neq 0\}$, we call it the set of generalized linear changes of variables. For this set we have

Proposition 15. *GLCV is a subgroup of the Riordan group and, of course, LCV is a subgroup of GLCV.*

Proof. $T(1|1)\in GLCV$ and $T(\lambda_1|\alpha_1+\beta_1x)T^{-1}(\lambda_2|\alpha_2+\beta_2x)=T\left(\frac{\lambda_1}{\lambda_2}\left|\frac{\alpha_1+(\beta_1-\beta_2)x}{\alpha_2}\right.\right)$. $\square$

In order to compare with some results in the literature we have to say that the group LCV contains all the Pascal-like triangle denoted by P_b in [2]. In our notation $P_b=T(1|1-bx)$.

It is clear that if the field $\mathbb{K}$ is algebraically closed (remember that we supposed always of characteristic zero) then GLCV contains elements of any finite order. In fact if ω_n is a primitive nth root of unity and λ is a nth root of unity then $T(\lambda|\omega_n+\alpha x)$ has order n for any $\alpha\in\mathbb{K}$. For reasons explained by Shapiro in [29] and Cameron and Nkwanta in [2], the researchers in combinatorics focus on Riordan matrices of order 2.

We realize that, for combinatorial interest it is common to concentrate on elements with nonnegative entries. Following [2] we consider $M=T(-1|-1)$ so $R=T(f|g)$ has pseudo order 2 if and only if $RM=T(-f|-g)$ has order 2. Note that M has order 2. Consider also the order 2 matrix $\widetilde{M}=T(1|-1)=-M$. Obviously $\widetilde{M}$ is not conjugated to M in the Riordan group. That is, there is not a Riordan matrix R with $\widetilde{M}=RMR^{-1}$ but it is not a negative answer to question Q8 of Shapiro in [29] because there all elements of order 2 considered had 1 as the first entry in the main diagonal (because of the definition of Riordan matrix considered). Our result, related to the group GLCV, is the following:

Proposition 16. *Let* $\alpha \in \mathbb{K}$, *with* $\alpha \neq 0$. *Then*:

(i) *The elements* $T(1|\alpha x - 1)$ *have order 2 and all of them are conjugated to* $\widetilde{M}$ *in the group LCV and hence in the Riordan group*.

(ii) *The elements* $T(-1|\alpha x - 1)$ *have order* 2 *and all of them are conjugated to* M *in the group GLCV and hence in the Riordan group*.

Proof. The proof of the whole proposition follows from the equality, easy to check,

$$T(1|\alpha x - 1) = T(1|\alpha x - 2)T(1|-1)T^{-1}(1|\alpha x - 2).$$

We found this by using elementary operations in certain related matrices. In fact this equality proves all in (i) now $T(-1|\alpha x - 1) = T(-1|1)T(1|\alpha x - 1)$ but $T(-1|1) = -T(1|1)$ and then commutes with any other. Then (ii) follows multiplying by the left in (i) by $T(-1|1)$. Note also that $T(-1|1)\widetilde{M} = M$. $\square$

For elements of order 2 in the Riordan group we have some results in the next proposition but *caution*: Abusing the language, order 2 in the next proposition means that the square is the identity, that is we allow the identity to have order 2. We do it to avoid some unnecessary restrictions.

Proposition 17. (i) $T(f|g)$ *is of order 2 if and only if* $T(-f|g)$ *is of order 2*.

(ii) $T(f|g)$ *is of order 2 if and only if* $T(f^n|g)$ *is of order 2* $\forall\, n \in Z$.

(iii) $T(f|g)$ *is of order 2 if and only if* $T(fg^n|g)$ *is of order 2* $\forall\, n \in Z$.

Proof. In our notation an element $T(f|g)$ has order 2 if and only if $T^2(f|g) = T(1|1)$ but $T^2(f|g) = T\left(ff\left(\frac{x}{g}\right)|gg\left(\frac{x}{g}\right)\right)$. So $T(f|g)$ has order 2 if and only if $gg\left(\frac{x}{g}\right) = 1$ and $ff\left(\frac{x}{g}\right) = 1$. Let us prove only the part (iii). Since $gg\left(\frac{x}{g}\right) = 1$ and $ff\left(\frac{x}{g}\right) = 1$, then $fg^nf\left(\frac{x}{g}\right)g^n(\frac{x}{g}) = 1$ because of the commutativity of the product of the series. $\square$

The above and the below propositions allow us to get more elements of order 2 in the associated subgroup. Moreover these new elements of order 2 are also conjugated in this subgroup of the Riordan group, to the matrix M as asked by Shapiro in [29].

Proposition 18. *Let* $\alpha \in \mathbb{K}$, *with* $\alpha \neq 0$. *Then the elements, of the associated subgroup,* $T(\alpha x - 1|\alpha x - 1)$ *have order* 2 *and all of them are conjugated, inside the associated subgroup, to* $M = T(-1|-1)$.

Proof. Recall that

$$T(1|\alpha x - 1) = T(1|\alpha x - 2)T(1|-1)T^{-1}(1|\alpha x - 2)$$

then

$$T(\alpha x - 1|\alpha x - 1) = T(\alpha x - 2|\alpha x - 2)T(-1|-1)T^{-1}(\alpha x - 2|\alpha x - 2). \qquad \square$$

Remark 19. Recently some positive answers to some problems posed by Shapiro in [29] have been given in [6,7]. They are related to involutions in the Riordan group and to the problem of conjugation with the Matrix M. After some communications with the authors of [6] we realize that a good choice of an invertible matrix B in their Theorem 2.5 for Riordan involutions $D = (g(x), f(x))$ conjugated to the matrix M is, using their notation, $B = \left(\exp\left(\frac{\Phi(x,xf(x))}{2}\right), \pm(1 - f(x))\right)$, because the original one proposed in [6] is not always invertible. We choose this corresponding B in the special case in Proposition 18.

5. A special remainder: sums of powers of natural numbers

As we mentioned in the introduction one of the applications of Riordan arrays is to use them to find numerical or combinatorial identities, see for example [26,28,30,31,34], among many others. The usual way to do that is by means of the action of a Riordan array to a particular power series or using the inverses of elements in the Riordan group. We hope that our way to use the change of variables could help, in the future, to get some other identities. But now, we want to show how the use of the family of polynomials associated to a $T(f\,|\,g)$ can be used to get, in a different way, some equalities. To do that we choose our arithmetic–geometric series motivating example.

So, when we try to sum $\sum_{k=0}^{\infty}(k+1)x^k$ using the corresponding iteration process, we consider $Q_1(x)=(1-x)^2$, and the remainder, in this case, is $R_{Q_1}=T\left(\frac{1}{(1-x)^2}\,\Big|\,2x-1\right)$. In this case

$$R_{Q_1}=(r_{n,k})=A_1,$$

where A_1 is the matrix displayed in (1).

Recall that our matrix R_{Q_1} is specially embedded into the matrix

$$C_{Q_1}=(d_{n,k})=\begin{pmatrix}1&&&&&&\\2&-1&&&&&\\3&-4&1&&&&\\4&-11&6&-1&&&\\5&-26&23&-8&1&&\\6&-57&72&-39&10&-1&\\7&-120&201&-150&59&-12&1\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix},$$

we have the relation $r_{n,k}=d_{n+1,k+1}$ for $n,k\in\mathbb{N}$.

Some of the main properties of the above arithmetical triangle are the following:

Theorem 20. (i) *The rule of contruction is*: $d_{n,k}=2d_{n-1,k}-d_{n-1,k-1}$ *for any* $n,k\geqslant 1, d_{0,k}=0$ *if* $k\geqslant 1$ *and* $d_{n,0}=n+1$ *for* $n\in\mathbb{N}$.

(ii) *For every* $d_{i,j}$, *the sum of all elements to the right in its row and all elements above in its column is zero. That is* $\sum_{k=0}^{i-1}d_{k,j}+\sum_{k=j+1}^{i}d_{i,k}=0$.

(iii) *The sum of the elements in any row in* R_{Q_1} *are triangular numbers with negative sign*.

(iv) *The general term is* $d_{n,j}=n+j+1+\sum_{k=1}^{j}(-1)^k\binom{n+j+1-k}{+j+2-2k}2^{n+j+2-2k}$.

(v) *The entries in the first column of* R_{Q_1} *are Eulerian numbers except for the sign*.

(vi) $n=\sum_{k=1}^{n}(-1)^{k-1}\binom{2n-k}{k-1}4^{n-k}$.

Proof. (i) It is a direct consequence of Theorem 4 for $a=1,\ b=-2,\ c=1$.

(ii) Suppose first that $j=0$, then using the sequence of polynomials (p_n) associated to R_{Q_1} and the corresponding sequence (q_n) obtained after the first change of variables we have that $\sum_{k=0}^{i-1}d_{k,0}=1+2+3+\cdots+i=q_{i-1}(1)$. Moreover $\sum_{k=1}^{i}d_{i,k}=p_{i-1}(1)$ but in this case the relation is just $q_n(t)=-p_n(2-t)$ because $a=1,\ b=-2,\ c=1$. So we have proved this particular case.

For the rest of the cases, i.e. $j\geqslant 1$, let us proceed by induction on $i-j$. If $i-j=0$ it is clear because it is a lower triangular matrix.

Suppose now that the proposition is true for $i-j=n$. This means that

$$\sum_{k=1}^{n}a_{i-k,j}+\sum_{k=1}^{n}a_{i,j+k}=0,$$

because it is a lower triangular matrix. From (i) above we obtain

$$\begin{aligned}\sum_{k=1}^{n+1} a_{i-k,j} + \sum_{k=1}^{n+1} a_{i,j+k} &= a_{i-1,j} + \sum_{k=2}^{n+1} a_{i-k,j} + 2a_{i-1,j+1} - a_{i-1,j} + \sum_{k=2}^{n+1}(2a_{i-1,j+k} - a_{i-1,j+k-1}) \\ &= \sum_{k=2}^{n+1} a_{i-k,j} + a_{i-1,j+1} + a_{i-1,j+1} + \sum_{k=2}^{n+1} a_{i-1,j+k} + \sum_{k=2}^{n+1} a_{i-1,j+k} - \sum_{k=2}^{n+1} a_{i-1,j+k-1}.\end{aligned}$$

Since $a_{i-1,i} = 0$ we have that

$$\sum_{k=1}^{n+1} a_{i-k,j} + \sum_{k=1}^{n+1} a_{i,j+k} = \sum_{k=1}^{i-j-1} a_{i-1-k,j} + \sum_{k=1}^{i-j-1} a_{i-1,j+k},$$

which is null by the induction hypothesis.

(iii) Note that this is obtained taking $j = 0$ in (ii).

(iv) It can be obtained from Proposition 6 after some computations.

(v) Using the formula in (iv) we see that the entries in the first column in R_{Q_1} are the numbers

$$d_{n+1,1} = r_{n,0} = n + 3 - 2^{n+2} = -\left\langle {n+2 \atop 1} \right\rangle$$

see the table in page 268 in [11].

(vi) Since $d_{n,n} + d_{n+1,n+1} = 0$ and using the formula in (iv) we get the result after some minor computations. $\square$

We are going to obtain two different formulas for the sums of powers of natural numbers. Although both of them are known, what is new is our way to obtain them from the family of polynomials associated to $T\left(\frac{1}{(1-x)^2} \,\middle|\, 2x-1\right)$.

After our first change of the variable in the associated sequence of polynomials $t = \frac{b+cx}{-a}$, we obtain the sequence of polynomials $q_n(t) = (-a/c)p_n\left(\frac{at+b}{-c}\right)$ that was described as

$$q_n(t) = \sum_{k=0}^{n} d_{n-k} t^k.$$

For the particular case of Q_1 this sequence is

$$q_n(t) = (n+1) + nt + (n-1)t^2 + \cdots + 2t^{n-1} + t^n$$

with the recurrence

$$q_{n+1}(t) = tq_n(t) + (n+2).$$

In this case, $q_n^{(k)}(1) = k!\binom{n+2}{k+2}$ for $n, k \in \mathbb{N}$. Where $f^{(k)}$ represents the kth derivative of f.

For this sequence of polynomials we can obtain the following formula for the sequence of the derivatives, we leave the proof to the reader.

Proposition 21

$$q'_{n+1}(t) = (n+1)q_n(t) - 2\sum_{k=0}^{n-1} q_k(t).$$

Let us denote by $(Q_p)_{p\in\mathbb{N}}$ the sequence of the upper factorial polynomials. That is, $Q_p(x) = x(x+1)\cdots(x+p-1)$, $p \in \mathbb{N}$, $Q_0(x) \equiv 1$. We have our own proof of the following fact:

Proposition 22

$$\sum_{k=1}^{n} Q_p(k) = \frac{Q_{p+1}(n)}{p+1}.$$

Proof. Let $p \in \mathbb{N}$. By differentiating $p-2$ times in the expression in the previous proposition and valuating at $t=1$ we have

$$q_{n+p-1}^{(p-1)}(1) = (n+p-1)q_{n+p-2}^{(p-2)}(1) - 2\sum_{k=0}^{n+p-3} q_k^{(p-2)}(1),$$

so

$$(p-1)!\binom{n+p+1}{p+1} = (n+p-1)(p-2)!\binom{n+p}{p} - 2\sum_{k=0}^{n+p-3}(p-2)!\binom{k+2}{p}.$$

Avoiding the null terms and changing the variable, $l = k-p+3$, in the sum we have $2\sum_{l=1}^{n}\binom{l+p-1}{p} = 2\frac{(n+p)!n}{n!(p+1)!}$ that is no more than another way to write the stated result. □

We are now going to find a formula for the sum of powers of natural numbers, using the Stirling numbers, that is equivalent to that in [11, p. 275] (see also [15]).

Following [33, Lemma 1.3.3, p. 18], the numbers $s(n,k) = (-1)^{n-k}c(n,k)$ are known as the Stirling numbers of the first kind and $c(n,k)$ are called the signless Stirling numbers of the first kind. Where the numbers $c(n,k)$ satisfy the recurrence $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$ $n,k \geqslant 1$ with the initial conditions $c(n,k) = 0$ if $n \leqslant 0$ or $k \leqslant 0$ except for $c(0,0) = 1$. On the other hand, the Stirling numbers of the second kind satisfy the following basic recurrence: $S(n,k) = kS(n-1,k) + S(n-1,k-1)$, $n,k \geqslant 1$ with the initial condition $S(0,0) = 1$. Thus the matrix $s = (s(n,k))_{n,k\in\mathbb{N}}$ and the matrix $S = (S(n,k))_{n,k\in\mathbb{N}}$ are mutually inverses (see [11] or [33]). Let I be the identity matrix then,

$$((-1)^{n-k}c(n,k))_{n,k\in\mathbb{N}} \cdot (S(n,k))_{n,k\in\mathbb{N}} = I.$$

Consequently the inverse of the matrix $C = (c(n,k))_{n,k\in\mathbb{N}}$ is $\overline{S} = ((-1)^{n+k}S(n,k))_{n,k\in\mathbb{N}}$.

Corollary 23. $\sum_{k=1}^{n} k^p = \sum_{j=1}^{p}(-1)^{p+j}S(p,j)j!\binom{n+j}{n-1}$ *where* $S(p,j)$ *are the Stirling numbers of the second kind.*

Proof. In [33], we can find that $\sum_{k=0}^{p} c(p,k)x^k = Q_p(x)$.

Hence

$$\sum_{k=1}^{n} Q_p(k) = \sum_{j=0}^{p} c(p,j)\sum_{k=1}^{n} k^j.$$

From the previous proposition, we have

$$\sum_{j=0}^{p} c(p,j)\sum_{k=1}^{n} k^j = \frac{Q_{p+1}(n)}{p+1}.$$

So consider the $p+1$ equalities

$$\sum_{j=0}^{m} c(m,j)\sum_{k=1}^{n} k^j = \frac{Q_{m+1}(n)}{m+1}, \quad m = 0,1,\ldots,p$$

Then using the inverse matrix $\overline{S}$ of C we obtain the announced formula. □

We are going to finish by describing what we think a very singular way to obtain a formula for the sum of powers using L'Hopital rule where Eulerian numbers appear *without* invoking them. Apart from the L'Hopital rule the main tools to obtain this formula are some equalities described in [13].

For each $p \geqslant 1$, consider the sequence of polynomials

$$q_{n,p}(t) = (n+1)^p + n^p t + (n-1)^p t^2 + \cdots + 2^p t^{n-1} + 1^p t^n.$$

Note that for $p = 1$ we have the above sequence $q_n(t)$.

Proposition 24. *For $t \neq 1$,*

$$q_{n-1,p}(t) = \frac{g(t) + (-1)^{p+1} \sum_{k=1}^{p} \left\langle {p \atop p-k} \right\rangle t^{n+k}}{(1-t)^{p+1}},$$

where $g(t)$ is a p-degree polynomial and $\left\langle {p \atop p-k} \right\rangle$ are just the Eulerian numbers.

Proof. Since $q_{n,p}(t) = (n+1)^p + t q_{n-1,p}(t)$ and

$$q_{n,p}(t) - q_{n-1,p}(t) = \sum_{k=1}^{n+1} (k^p - (k-1)^p) t^{n+1-k}.$$

Then, if $t \neq 1$

$$q_{n-1,p}(t) = \frac{n^p - \sum_{k=1}^{n} (k^p - (k-1)^p) t^{n+1-k}}{1-t}.$$

So if $t \neq 1$,

$$q_{n-1,p}(t) = \frac{(1-t)^p n^p - \sum_{k=1}^{n} ((k-1)^p - k^p) t^{n+1-k} (1-t)^p}{(1-t)^{p+1}}.$$

Expanding the numerator in powers of t, for our purpose we are only interested in the coefficients from t^{p+1} to t^{n+p}, and after some computations we find that the numerator is:

$$g(t) + \sum_{k=1}^{n-p} \sum_{j=0}^{p+1} (-1)^{p+1-j} \binom{p+1}{j} (p+k-j)^p t^{n+1-k}$$
$$+ \sum_{k=1}^{p} \sum_{j=0}^{p-k} (-1)^{p+1-j} \binom{p+1}{j} (p+1-k-j)^p t^{n+k},$$

where $g(t)$ is a p-degree polynomial. All the coefficients in the middle term are zero (see [3] or [13, p. 244]). Finally, as one can see in [11, p. 255], the coefficients in the third term of the expression are, up the factor $(-1)^{p+1}$, the Eulerian numbers $\left\langle {p \atop p-k} \right\rangle$. □

Corollary 25. $\sum_{k=1}^{n} k^p = \sum_{k=1}^{p} \sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (p+1-k-j)^p \binom{n+k}{p+1}$.

Proof. Since $q_{n-1,p}(1) = \sum_{k=1}^{n} k^p$, we only need to apply the L'Hopital rule $p+1$-times to the formula in the previous proposition. □

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