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Mapping properties of weakly arcwise open dendroids

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Abstract

A dendroid X is said to be *weakly arcwise open* if for each point p of X each arc component of $X \setminus \{p\}$ either is open or has empty interior. We study various mapping properties of these dendroids. The leading problem is what classes of mappings between dendroids preserve the property of being weakly arcwise open.

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A dendroid means an arcwise connected and hereditarily unicoherent metric continuum. Investigation of shore points (special kind of noncut points) in dendroids, see [11,13], leads in a natural way to establish a class of dendroids called weakly arcwise open (WAO), being a generalization of dendroids having the property of Kelley, see [12]. The class of WAO dendroids was studied in [12], [13, Section 3], [14,3]. The aim of this paper is to continue this study, especially with respect to mapping properties of WAO dendroids.

0. Preliminaries

A *continuum* means a compact connected metric space. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two of its subcontinua is connected. A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. Recall that each subcontinuum of a dendroid is a dendroid, and that for every two points a and b of a dendroid X there is a unique arc $ab \subset X$ joining a and b.

Given a subset S of a dendroid X, we denote by $\mathfrak{A}(S)$ the family of all arc components of S in X. A point p of a dendroid X is called a *ramification point* (an *end point*) of X provided that $\operatorname{card}(\mathfrak{A}(X \setminus \{p\})) \geqslant 3$ (if $\operatorname{card}(\mathfrak{A}(X \setminus \{p\})) = 1$, respectively). If a dendroid X has exactly one ramification point, then it is called a *fan*, and the ramification point is named the *top* of the fan.

A dendroid X is said to be *arcwise open* (*closed, respectively*) *at a point* $p \in X$ provided that each element of $\mathfrak{A}(X \setminus \{p\})$ is open (closed, respectively) with respect to the subspace $X \setminus \{p\}$ of X. Since $X \setminus \{p\}$ is an open subset

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of X, it follows that a dendroid X is arcwise open at p if and only if each arc component of $X \setminus \{p\}$ is open. A dendroid X is said to be *weakly arcwise open at a point* $p \in X$ provided that each element of $\mathfrak{A}(X \setminus \{p\})$ either is open or has empty interior. If the mentioned condition holds at each point $p \in X$, then X is said to be *arcwise open, weakly arcwise open* (shortly WAO), respectively.

More generally, if K is a nonempty closed subset of a dendroid X, we say that X *is WAO at* K provided that each element of $\mathfrak{A}(X \setminus K)$ either is open or it has empty interior. Connections between being WAO at K and being WAO at each point of K were studied in [14].

A dendroid X is said to be *smooth at a point* $p \in X$ provided that for each point $a \in X$, and for each sequence of points $\{a_n\}$ converging to a in X the sequence of the arcs pa_n converges to the arc pa (with respect to the Hausdorff metric); it is said to be *smooth* provided that there exists a point $p \in X$ at which X is smooth (see e.g. [6, p. 194]).

The following notation will be used in the paper. $\mathbb{N}$ stands for the set of all positive integers. $\mathcal{C}$ means the Cantor ternary set in the closed unit interval $[0, 1]$. In the Euclidean plane we denote by $\overline{uv}$ the straight line segment with end points u and v.

A surjective mapping $f : X \to Y$ between continua is said to be:

- *open* provided that the images of open sets under f are open;
- *light* provided that for each point $y \in Y$ the set $f^{-1}(y)$ has one-point components (note that this condition is equivalent to the property that $f^{-1}(y)$ is zero-dimensional);
- a *retraction* provided that $Y \subset X$ and $f|Y$ is the identity;
- *monotone* provided that for each point $y \in Y$ the set $f^{-1}(y)$ is connected; equivalently, if for each subcontinuum Q of Y, the inverse image $f^{-1}(Q)$ is connected;
- *almost monotone* provided that for each subcontinuum Q of Y with nonempty interior, the inverse image $f^{-1}(Q)$ is connected;
- *quasi-monotone* provided that for each subcontinuum Q of Y with nonempty interior, the inverse image $f^{-1}(Q)$ has finitely many components, each of which is mapped onto Q under f;
- *weakly monotone* provided that for each subcontinuum Q of Y with nonempty interior, each component of the inverse image $f^{-1}(Q)$ is mapped onto Q under f;
- *feebly monotone* provided that if Y_1 and Y_2 are proper subcontinua of Y such that $Y = Y_1 \cup Y_2$, then their inverse images $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$ are connected;
- *locally monotone* provided that for each point $x \in X$ there exists a closed neighborhood U of x such that $f(x) \in \operatorname{int}(f(U))$ and $f|U : U \to f(U)$ is monotone;
- *monotone relative to a point* $x \in X$ provided that for each subcontinuum Q of Y with $f(x) \in Q$, the inverse image $f^{-1}(Q)$ is connected;
- *confluent* provided that for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto Q under f;
- *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;
- *weakly semi-confluent* provided that for each subcontinuum Q of Y with nonempty interior, and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;
- *weakly confluent* provided that for each subcontinuum Q of Y some component of $f^{-1}(Q)$ is mapped onto Q under f.

Given a class $\mathfrak{M}$ of mappings between continua, a mapping $f : X \to Y$ is said to be *hereditarily* $\mathfrak{M}$ provided that for each subcontinuum $K \subset X$ the partial mapping $f|K : K \to f(K) \subset Y$ is in $\mathfrak{M}$. The reader is referred to [10, Table II, p. 28] to see interrelations between most of the classes of mappings mentioned above. Some properties of feebly monotone mappings are studied in [1].

1. Auxiliary results

The following property of a dendroid Y is considered in [3]:

$$\text{There exists a point } p \in Y \text{ such that for every point } y \in Y \setminus \{p\} \text{ the arc component of } Y \setminus \{y\} \text{ which contains } p \text{ is open.} \tag{1.1}$$

Below we collect some facts and assertions needed in the sequel. They are all proved in [3]. Recall that $\mathcal{C}$ means the Cantor set.

Proposition 1.2. *If a dendroid Y satisfies condition* (1.1)*, then the continuum X defined by*

$$X = (Y \times \mathcal{C})/\big(\{p\} \times \mathcal{C}\big) \tag{1.3}$$

is a WAO dendroid.

Theorem 1.4. *Each dendroid Y satisfying condition* (1.1) *can be embedded into a WAO dendroid.*

Since for each fan Y with the top v if $y \in Y \setminus \{v\}$ then the set $Y \setminus \{y\}$ has exactly two arc components, and the one not containing v is an arc without an end point, we infer that

$$\text{any fan } Y \text{ satisfies (1.1) with the top } v \text{ as } p. \tag{1.5}$$

Therefore we have a corollary to Theorem 1.4.

Corollary 1.6. *Each fan can be embedded into a WAO fan.*

We will need the following example of a fan that is not WAO. In the Cartesian coordinates in the plane put $s = (0,0)$, $a = (1,0)$, $p = (2,0)$ and $a_n = (1, \frac{1}{n})$ for each $n \in \mathbb{N}$. Define

$$X = \overline{sp} \cup \bigcup \{\overline{sa_n}\colon\ n \in \mathbb{N}\}. \tag{1.7}$$

A dendroid Y is said to be (see [5, pp. 192–193]):

- *of type* 1 provided that there are in Y a point p and a sequence of points $\{a_n\}$ converging to a point a such that $Y = \mathrm{cl}(\bigcup\{pa_n\colon\ n \in \mathbb{N}\})$, the sequence of arcs $\{pa_n\}$ converges to a subcontinuum L of Y, and there are an end point s of L and an open neighborhood U of s in Y such that $s \neq a$ and, if C is the component of $L \cap U$ containing s, then $C \cap (\bigcup\{pa_n\colon\ n \in \mathbb{N}\}) = \emptyset$;
- *of type* 2 provided that there are in Y two points s and t, two sequences of points $\{a_n\}$ and $\{b_n\}$ which converge to points a and b, respectively, such that $Y = (\bigcup\{sa_n\colon\ n \in \mathbb{N}\}) \cup sabt \cup (\bigcup\{tb_n\colon\ n \in \mathbb{N}\})$, where $sabt$ stands for the arc from s to t passing through a and b, the sequences of arcs $\{sa_n\}$ and $\{tb_n\}$ converge to the arcs sa and tb, respectively, and $\mathrm{diam}(sa_n \cap st)$ as well as $\mathrm{diam}(tb_n \cap st)$ tend to 0.

The above concepts were used to characterize smooth dendroids in the next quoted theorem, see [5, Theorem 1, p. 194].

Theorem 1.8. *A dendroid is not smooth if and only if it contains a subdendroid of either type* 1 *or type* 2.

Dendroids of type 1 were used as a tool in study relations between WAO property at closed subsets and at singletons, in the following context.

Let K be a closed subset of a dendroid X. Consider the implication

$$X \text{ is WAO at } p \text{ for each } p \in K \quad \Rightarrow \quad X \text{ is WAO at } K. \tag{1.9}$$

The following proposition is shown in [14, Proposition 1.7, p. 117].

Proposition 1.10. *If the dendroid X does not contain a dendroid of type* 1*, then implication* (1.9) *holds for each closed subset K of X.*

An example is presented in [14, p. 118] showing that containing of no subdendroid of type 1 is an essential assumption in this result. The same example shows that "type 2" cannot be substituted for "type 1" in that proposition: the example in [14, Fig. 1, p. 118] does not contain any subdendroid of type 2, and implication (1.9) does not hold. We will also use the following result (see [14, Proposition 1.6, p. 117]).

Proposition 1.11. *Implication* (1.9) *holds for any dendroid X provided that the set K has a finite number of components.*

Still the same example (i.e., [14, Fig. 1, p. 118]) shows that the finiteness of the number of components is an indispensable assumption in the result.

2. Mapping properties—negative results

We start with some consequences of previously quoted results. Let a dendroid Y be smooth at a point $p \in Y$, let X be defined by (1.3) and define $Y_0 = (Y \times \{0\})/(p, 0) \subset X$. Since the natural projection $\pi : X \to Y_0$ defined by $\pi((y, c)) = y$ for each $y \in Y_0$ and $c \in \mathcal{C}$ is an open and light retraction, we have the following result.

Proposition 2.1. *Each dendroid Y satisfying condition* (1.1) *is an open light retract of some WAO dendroid. In particular, by* (1.5)*, each fan is an open light retract of some WAO fan.*

Since there are fans which are not WAO (e.g., the fan X defined by (1.7)), we have the following consequence of Proposition 2.1.

Observation 2.2. *Open light retractions do not preserve WAO property, even for fans.*

Recall that in [14, Example 2.8, p. 121] an example of a WAO dendroid X is presented such that an open light retract of X is a dendroid which is not WAO.

Since each open mapping of a compact space is confluent, [15, Theorem 7.5, p. 148], each confluent mapping is weakly monotone and each weakly monotone mapping is weakly semi-confluent (just by the definitions), Observation 2.2 implies a corollary.

Corollary 2.3. *Weakly monotone mappings, and thus weakly semi-confluent ones, do not preserve WAO property, even for fans.*

Given spaces X and Y, a cover $\{A_j\colon\ j \in J\}$ of X (where J is an arbitrary set of indices), and a family of mappings $f_j : A_j \to Y$, we say that the mappings f_j are *compatible* if for each pair j_1, j_2 of elements of J we have $f_{j_1}|(A_{j_1} \cap A_{j_2}) = f_{j_2}|(A_{j_1} \cap A_{j_2})$. Letting $f(x) = f_j(x)$ for each $x \in A_j$ we define a mapping $f : X \to Y$ called the *combination of the mappings f_j* (see [4, p. 71]).

A surjective mapping $f : X \to Y$ is called a *local homeomorphism in the large sense* provided that for each point $x \in X$ there exists an open neighborhood $U(x)$ such that the partial mapping $f|U(x) : U(x) \to f(U(x))$ is a homeomorphism. Further, if $f(U(x))$ is an open subset of Y for each $x \in X$, then f is called a *local homeomorphism* (see [8, p. 51] and [15, p. 199]).

Let us consider the following example.

Example 2.4. There are fans X and Y and a surjective mapping $f : X \to Y$ such that

(2.4.1) X is WAO;
(2.4.2) Y is not WAO;
(2.4.3) f is the combination of homeomorphisms;
(2.4.4) f is a local homeomorphism in the large sense.

Proof. Indeed, in the Cartesian coordinates in the plane put $s = (0, 0)$, $a = (1, 0)$, $p = (2, 0)$ and $a_n = (1, \frac{1}{n})$ for each $n \in \mathbb{N}$. Define $X = \overline{sa} \cup \bigcup\{\overline{sa_n}\colon\ n \in \mathbb{N}\}$. Thus X is a WAO fan (in fact, the harmonic fan) with the top s. Define $Y = \overline{sp} \cup \bigcup\{\overline{sa_n}\colon\ n \in \mathbb{N} \setminus \{1\}\}$. Thus Y is a fan that is not WAO (it is homeomorphic to the fan X defined by (1.7)). Finally let $f_1 : \overline{sa_1} \to \overline{sp}$ be the linear homeomorphism with $f_1(s) = s$, take $f|\overline{sa}$ and $f|\overline{sa_n}$ for $n \geqslant 2$ as the identities, and define $f : X \to Y$ as the combination of the above homeomorphisms. Note that f is a local homeomorphism in the large sense. So, the argument is complete. □

The above example justifies the next fact.

Observation 2.5. *Neither the combination of homeomorphisms nor local homeomorphisms in the large sense preserve WAO property, even for fans.*

Recall that a mapping is a local homeomorphism if and only if it is open and a local homeomorphism in the large sense (see [8, Théorème 1, p. 54]). In the light of Observation 2.5 it is natural to ask if local homeomorphisms preserve WAO property. The (trivial) answer is yes, because each local homeomorphism between dendroids is a homeomorphism (see [10, (6.1), p. 50] for a more general result).

Looking for mappings that preserve WAO property, recall the following three results, the third of which being the main result of Section 2 of [14] and a consequence of the two others, see [14, Propositions 1.6, 2.1 and Theorem 2.2, pp. 117, 119 and 120, respectively].

Proposition 2.6. *Let X be a dendroid and K a closed subset of X that has a finite number of components. Then implication* (1.9) *is true.*

Proposition 2.7. *Let a monotone mapping $f : X \to Y$ be defined on a dendroid X. Then Y is a dendroid, and the following implication holds for each point $q \in Y$.*

$$X \text{ is WAO at } f^{-1}(q) \quad \Rightarrow \quad Y \text{ is WAO at } q. \tag{2.8}$$

Theorem 2.9. *Monotone mappings preserve WAO property for dendroids, i.e., if X is a WAO dendroid and $f : X \to Y$ is a monotone surjection, then Y is a WAO dendroid.*

The above invariance of WAO property (under monotone mappings) cannot be strengthened to the invariance of the property at a point. More precisely, if a dendroid X is WAO at a point $p \in X$ and a mapping $f : X \to Y$ is a monotone surjection, then the dendroid Y need not be WAO at $f(p)$. An appropriate example, with X and Y being smooth fans, is presented in [14, p. 119].

A class of mappings that is essentially larger than that of monotone mappings is the class of mappings which are monotone relative to a given point of the domain. So, a natural question arises if Theorem 2.9 can be extended from monotone mappings to this wider class. To answer this question recall that [2, Theorem 4.1, p. 14] says that any confluent mapping between fans is monotone relative to the top of the domain. Since each open mapping between compact spaces is confluent, Observation 2.2 implies the next one.

Observation 2.10. *WAO property is not preserved under mappings which are monotone relative to a point, even for fans.*

Notice that the mapping $f : X \to Y$ in Example 2.4 is monotone relative to s. Thus that example gives another argument for Observation 2.10.

3. Mapping properties—positive results

Below we give a proposition with a diagram that illustrates relations between other known classes of mappings that are larger than the class of monotone mappings.

Proposition 3.1. *The following implications, illustrated in the diagram below, are consequences of the definitions*:

$$\begin{array}{ccccc}
\text{monotone} & \Longrightarrow & \text{locally monotone} & & \\
\Downarrow & & \Downarrow & & \\
\text{almost monotone} & \Longrightarrow & \text{quasi-monotone} & \Longrightarrow & \text{weakly monotone} \\
\Downarrow & & & & \Downarrow \\
\text{feebly monotone} & & \Longrightarrow & & \text{weakly semi-confluent}
\end{array} \tag{3.2}$$

Recall that weakly semi-confluent mappings (thus all its subclasses, in particular weakly monotone mappings) preserve the property of being a dendroid (see [9, Remark, p. 180, and Theorem 3, p. 179]; see also [10, Table IV, pp. 69–70]). Thus we have the following:

Statement 3.3. *If X is a dendroid and $f : X \to Y$ is a surjection belonging to any of the classes of diagram* (3.2)*, then Y is a dendroid.*

In the light of Statement 3.3 and Theorem 2.9 the following question is natural.

Question 3.4. What classes of mappings of diagram (3.2) (larger than that of monotone mappings) preserve WAO property?

Note that the preserving of WAO property can be split into two implications. Namely we have the following (obvious) proposition.

Proposition 3.5. *Let a surjective mapping $f : X \to Y$ between dendroids X and Y satisfy, for each point $q \in Y$, the following two implications*:

(3.5.1) *X is WAO at p for each $p \in f^{-1}(q) \Rightarrow X$ is WAO at $f^{-1}(q)$*;
(3.5.2) *X is WAO at $f^{-1}(q) \Rightarrow Y$ is WAO at q.*

Then f preserves WAO property, i.e.,

(3.5.3) *X is WAO $\Rightarrow Y$ is WAO.*

Proposition 3.5 allows to specify Question 3.4 as follows.

Question 3.6. For what classes of surjective mappings $f : X \to Y$ (where X is a dendroid) of diagram (3.2) (larger than that of monotone mappings):

(a) does implication (3.5.1) hold for each $q \in Y$?
(b) does implication (3.5.2) hold for each $q \in Y$?

The following partial answer to Question 3.6(a) is obtained for locally monotone mappings.

Proposition 3.7. *Let a locally monotone surjection $f : X \to Y$ be defined on a dendroid X. Then Y is a dendroid and implication* (3.5.1) *holds for each point $q \in Y$.*

Proof. Indeed, Y is a dendroid according to Statement 3.3 (compare also [10, Table IV, p. 69]). Since f is locally monotone, for each $q \in Y$ its preimage $f^{-1}(q)$ has finitely many components, see [10, Theorem 4.32, p. 22]. Thus $f^{-1}(q)$ can be taking as K in Proposition 1.11, so (1.9) holds, and therefore (3.5.1) is satisfied. □

In the light of Propositions 3.5 and 3.7 the question whether locally monotone mappings between dendroids preserve WAO property reduces to the following one.

Question 3.8. Let a locally monotone surjection $f : X \to Y$ be defined on a dendroid X, and let $q \in Y$. Does implication (3.5.2) hold?

Partial answers to the above question are obtained under additional assumptions concerning the mapping f, see below Theorems 3.23 and 3.24.

Implication (3.5.1) holds under additional assumptions on X, even without any conditions concerning mappings. The next result shows this.

Proposition 3.9. *Let a dendroid X do not contain any subdendroid of type* 1*. Then for an arbitrary mapping $f : X \to Y$ onto a dendroid Y implication* (3.5.1) *is satisfied for each $q \in Y$.*

Proof. In fact, it is enough to apply Proposition 1.10 with $K = f^{-1}(q)$ for any $q \in Y$. □

Recall that implication (3.5.2) holds for quasi-monotone mappings $f : X \to Y$ between dendroids under additional assumptions on Y. Namely we have the following result (see [14, Lemma 2.4, p. 120]).

Proposition 3.10. *Let a quasi-monotone surjective mapping $f : X \to Y$ be defined on a dendroid X. Then Y is a dendroid. Let $q \in Y$. If Y satisfies the condition*

(3.10.1) *for each arc component $B \in \mathfrak{A}(Y \setminus \{q\})$ with* $\operatorname{int}(B) \neq \emptyset$ *there exists a continuum K such that $\emptyset \neq \operatorname{int}(K) \subset K \subset B$,*

then implication (3.5.2) *holds.*

The following two results, that concern mappings of fans, are consequences of Proposition 3.10. They are shown as Proposition 2.5 and Theorem 2.6 of [14, pp. 120 and 121].

Proposition 3.11. *Let $f : X \to Y$ be a quasi-monotone mapping of a WAO fan X onto a fan Y that maps the top of X to the top of Y. Then Y is WAO.*

Proposition 3.12. *Let $f : X \to Y$ be a locally monotone mapping of a WAO fan X onto a continuum Y. Then Y is a WAO fan* (*or an arc*)*.*

The next result is also a consequence of Proposition 3.10.

Proposition 3.13. *Let dendroids X and Y be such that X does not contain any subdendroid of type* 1*, and Y satisfies condition* (3.10.1) *for each $q \in Y$. Let $f : X \to Y$ be a quasi-monotone surjective mapping. If X is WAO, Then Y is WAO, too.*

Proof. Proposition 3.9 gives (3.5.1), and Proposition 3.10 gives (3.5.2). Then the conclusion follows from Theorem 3.5. □

Consequently, by Theorem 1.8, we obtain a corollary, which has been shown as Corollary 2.7 in [14, p. 121].

Proposition 3.14. *Let $f : X \to Y$ be a quasi-monotone mapping of a smooth WAO dendroid X onto a dendroid Y such that Y satisfies condition* (3.10.1) *for each point $q \in Y$. Then Y is WAO.*

In our investigation of Questions 3.6 for various classes of mappings related to monotone ones, we have seen that some extensions of the classes of almost monotone, quasi-monotone and weakly monotone mappings are very natural in this study. So, we consider the following definitions:

Definition 3.15. A surjective mapping $f : X \to Y$ between continua is said to be:

- *arcwise almost monotone* (abbreviated AAM) provided that for each subarc Q of Y with nonempty interior, the inverse image $f^{-1}(Q)$ is connected;
- *arcwise quasi-monotone* (abbreviated AQM) provided that for each subarc Q of Y with nonempty interior, the inverse image $f^{-1}(Q)$ has finitely many components, each of which is mapped onto Q under f;
- *arcwise weakly monotone* (abbreviated AWM) provided that for each subarc Q of Y with nonempty interior, each component of the inverse image $f^{-1}(Q)$ is mapped onto Q under f.

Proposition 3.16. *The following implications, illustrated in the diagram below, are consequences of the definitions*:

$$\begin{array}{ccccc} \text{almost monotone} & \Longrightarrow & \text{quasi-monotone} & \Longrightarrow & \text{weakly monotone} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{AAM} & \Longrightarrow & \text{AQM} & \Longrightarrow & \text{AWM} \end{array} \tag{3.17}$$

Remark 3.18. Note that none of the three classes of mappings introduced in Definition 3.15 preserves the property of being a dendroid (in contrast to Statement 3.3). It is enough to show this for AAM mappings. To present the needed example we apply the cone operation.

Recall that for a space X we define $\mathrm{Cone}(X) = (X \times [0,1])/(X \times \{1\})$. Points of $\mathrm{Cone}(X)$ are denoted by (x,t) for $x \in X$ and $t \in [0,1]$. In particular, $(x_1, 1) = (x_2, 1)$ for every $x_1, x_2 \in X$. For a mapping $f: X \to Y$ the induced mapping $\mathrm{Cone}(f): \mathrm{Cone}(X) \to \mathrm{Cone}(Y)$ is defined by $\mathrm{Cone}(f)(x,t) = (f(x),t)$. Indeed, consider the well known Cantor–Lebesgue step function $\varphi: \mathcal{C} \to [0,1]$ (see e.g. [7, §16, II, (8), p. 150]; compare [15, Chapter II, §4, p. 35]). Then

$$f = \mathrm{Cone}(\varphi): X = \mathrm{Cone}(\mathcal{C}) \to Y = \mathrm{Cone}\big([0,1]\big)$$

maps the Cantor fan X onto the two-cell Y and it is AAM, since the condition of the definition of an AAM mapping is satisfied vacuously: there is no arc with nonempty interior in Y.

Remark 3.19. Since weakly semi-confluent mappings preserve the property of being a dendroid (see [9, Remark, p. 180, and Theorem 3, p. 179]; compare Statement 3.3), it follows from Remark 3.18 that the classes of AAM, AQM and AWM mappings are not contained in the classes of feebly monotone or weakly semi-confluent mappings. In particular, the second row of diagram (3.17) cannot be inserted between the second and the third rows of diagram (3.2).

The next lemma was proved in [14, Lemma 1.5(b), p. 117] for monotone mappings. We extend it to AAM and to feebly monotone mappings.

Lemma 3.20. *Let $f: X \to Y$ be either an* AAM *or a feebly monotone mapping between dendroids X and Y. For a fixed closed subset W of Y, let $B \in \mathfrak{A}(Y \setminus W)$ and $A \in \mathfrak{A}(X \setminus f^{-1}(W))$. Then the following implication holds*:

(3.20.1) $A \cap f^{-1}(\mathrm{int}(B)) \neq \emptyset \Rightarrow f^{-1}(\mathrm{int}(B)) \subset \mathrm{int}(A)$.

Proof. Take a point $u \in A \cap f^{-1}(\mathrm{int}(B))$. Then $f(u) \in f(A) \cap \mathrm{int}(B) \subset f(A) \cap B$, whence it follows from the definition of B that $f(A) \subset B$.

Consider two cases.

Case 1. $f^{-1}(\mathrm{int}(B)) \not\subset A$.

Then there exists a point $s \in f^{-1}(\mathrm{int}(B)) \setminus A$. Therefore $f(s) \in \mathrm{int}(B) \subset B$, whence it follows that the arc $K = f(s)f(u) \subset B$ has nonempty interior.

Subcase (a). The mapping f is AAM.

Since f is AAM, the set $f^{-1}(K)$ is connected (and compact, thus it is a subdendroid of X). Note that $s, u \in f^{-1}(K)$, and that $u \in A$, while $s \notin A$. So, u and s are in different arc components of $X \setminus f^{-1}(W)$, whence the arc $su \subset f^{-1}(K)$ intersects $f^{-1}(W)$ by the definition of A. Thus $f^{-1}(K) \cap f^{-1}(W) = f^{-1}(K \cap W) \neq \emptyset$, and therefore $K \cap W \neq \emptyset$, a contradiction with $K \subset B \subset Y \setminus W$. This contradiction proves that subcase (a) of Case 1 is not possible.

Subcase (b). The mapping f is feebly monotone.

Let $L_1, L_2 \subset Y$ be defined by

$$L_1 \in \mathfrak{A}\big(Y \setminus \{f(u)\}\big) \quad \text{with } f(s) \in L_1 \quad \text{and} \quad L_2 \in \mathfrak{A}\big(Y \setminus \{f(s)\}\big) \quad \text{with } f(u) \in L_2.$$

Putting $Y_i = \mathrm{cl}(L_i)$ for $i \in \{1,2\}$ we see that Y_i are proper subcontinua of Y such that $Y = Y_1 \cup Y_2$ and $K \subset Y_1 \cap Y_2$. This implies that $\emptyset \neq \mathrm{int}(K) \subset \mathrm{int}(Y_1 \cap Y_2)$. Since f is feebly monotone, it follows that $f^{-1}(Y_i)$ are continua.

Let $L = Y_1 \cap Y_2$. Thus $f^{-1}(L) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$ is a subcontinuum of X as the intersection of two subcontinua in a hereditarily unicoherent continuum X. Note that $s, u \in f^{-1}(L)$, and that (as previously) $u \in A$, while $s \notin A$. So, u and s are in different arc components of $X \setminus f^{-1}(W)$, whence the arc $su \subset f^{-1}(L)$ intersects $f^{-1}(W)$ by the definition of A. Thus $f^{-1}(L) \cap f^{-1}(W) = f^{-1}(K \cap W) \neq \emptyset$, and therefore $L \cap W \neq \emptyset$, a contradiction with $L \subset B \subset Y \setminus W$. This contradiction proves that subcase (b) of Case 1 is not possible.

Therefore it follows that Case 1 cannot hold.

Case 2. $f^{-1}(\mathrm{int}(B)) \subset A$.

Then $\mathrm{int}(f^{-1}(\mathrm{int}(B))) \subset \mathrm{int}(A)$. Since $\mathrm{int}(f^{-1}(\mathrm{int}(B))) = f^{-1}(\mathrm{int}(B))$ by openness of $f^{-1}(\mathrm{int}(B))$, the conclusion follows. □

To prove the next result we will use the following lemma (see [14, Lemma 1.5(a), p. 116]).

Lemma 3.21. *Let $f : X \to Y$ be a mapping between dendroids X and Y, and W be a closed subset of Y. If an arc component $B \in \mathfrak{A}(Y \setminus W)$ is such that $B \setminus \mathrm{int}(B) \neq \emptyset$, then there exists an arc component $A \in \mathfrak{A}(X \setminus f^{-1}(W))$ such that*

$$\big(A \setminus \mathrm{int}(A)\big) \cap f^{-1}\big(B \setminus \mathrm{int}(B)\big) \neq \emptyset.$$

Moreover, for each point $v \in B \setminus \mathrm{int}(B)$ there is a point $u \in f^{-1}(v)$ such that $u \in A(u)$, where $A(u) \in \mathfrak{A}(X \setminus f^{-1}(W))$.

The proof of the next result is modeled on the ideas of the one of [14, Proposition 2.1, p. 119].

Proposition 3.22. *If a surjection $f : X \to Y$ between dendroids is either* AAM *or feebly monotone, then f satisfies implication* (3.5.2) *for each $q \in Y$.*

Proof. Let W, A and B be as above, and let $q \in Y$. To show that implication (3.5.2) is true, assume that X is WAO at $f^{-1}(q)$. We have to prove that Y is WAO at q.

To this aim suppose on the contrary that there is an arc component $B \in \mathfrak{A}(Y \setminus \{q\})$ such that

(3.22.1) $\mathrm{int}(B) \neq \emptyset \neq B \setminus \mathrm{int}(B)$.

Then by the first inequality of (3.22.1) it follows that $A \cap f^{-1}(\mathrm{int}(B)) \neq \emptyset$ for some arc component $A \in (X \setminus f^{-1}(q))$. Applying Lemma 3.20 we infer from implication (3.20.1) that $f^{-1}(\mathrm{int}(B)) \subset \mathrm{int}(A)$. Thus $\mathrm{int}(A) \neq \emptyset$. Consequently, since X is WAO at $f^{-1}(q)$, it follows that A is open.

The second inequality of (3.22.1) and Lemma 3.21 imply that

(3.22.2) $(S \setminus \mathrm{int}(S)) \cap f^{-1}(B \setminus \mathrm{int}(B)) \neq \emptyset$

for some arc component $S \in \mathfrak{A}(X \setminus f^{-1}(q))$. In particular, $S \setminus \mathrm{int}(S) \neq \emptyset$, i.e., S is not open. Since X is WAO at $f^{-1}(q)$, it follows that $\mathrm{int}(S) = \emptyset$. This proves that $S \neq A$.

Take now two points: $v_1 \in \mathrm{int}(B)$ and $v_2 \in B \setminus \mathrm{int}(B)$. Taking the arc $K = v_1v_2 \subset B$ we see that $\mathrm{int}(K) \neq \emptyset$.

If f is AAM, it follows that the set $f^{-1}(K)$ is connected (and compact, thus it is a subdendroid) of $X \setminus f^{-1}(q)$). Note that $K \subset B$ implies $\emptyset \neq \mathrm{int}(K) \subset \mathrm{int}(B)$, whence it follows that $\emptyset \neq f^{-1}(\mathrm{int}(K)) \subset f^{-1}(\mathrm{int}(B)) \subset \mathrm{int}(A)$. Therefore $f^{-1}(K) \cap A \neq \emptyset$. Further, the second part of the conclusion of Lemma 3.21 implies that there is a point $u \in f^{-1}(v_2)$ such that $u \in A(u) \setminus \mathrm{int}(A(u))$, where $A(u) \in \mathfrak{A}(X \setminus f^{-1}(q))$. Therefore, as previously in (3.22.2), we have $(A(u) \setminus \mathrm{int}(A(u))) \cap f^{-1}(B \setminus \mathrm{int}(B)) \neq \emptyset$, which implies that $A \neq A(u)$. Thus $u \in f^{-1}(K) \cap A(u)$, so $f^{-1}(K) \cap A(u) \neq \emptyset$. Therefore the dendroid $f^{-1}(K)$ intersects two distinct arc components of $X \setminus f^{-1}(q)$, whence $f^{-1}(K) \cap f^{-1}(q) \neq \emptyset$, which implies $q \in K$, a contradiction with $K \subset B$.

If f is feebly monotone, we define

$$L_1 \in \mathfrak{A}\big(Y \setminus \{v_1\}\big) \quad \text{with } v_2 \in L_1 \quad \text{and} \quad L_2 \in \mathfrak{A}\big(Y \setminus \{v_2\}\big) \quad \text{with } v_1 \in L_2.$$

Putting $Y_i = \mathrm{cl}(L_i)$ for $i \in \{1, 2\}$ we see that Y_i are proper subcontinua of Y such that $Y = Y_1 \cup Y_2$ and $K \subset Y_1 \cap Y_2$. This implies that $\emptyset \neq \mathrm{int}(K) \subset \mathrm{int}(Y_1 \cap Y_2)$. Since f is feebly monotone, it follows that $f^{-1}(Y_i)$ are continua.

Let $L = Y_1 \cap Y_2$. Thus $f^{-1}(L) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$ is a subcontinuum of X as the intersection of two subcontinua in a hereditarily unicoherent continuum X. Note that $K \subset L \subset B$ implies $\emptyset \neq \mathrm{int}(K) \subset \mathrm{int}(L) \subset \mathrm{int}(B)$, whence it follows that $\emptyset \neq f^{-1}(\mathrm{int}(L)) \subset f^{-1}(\mathrm{int}(B)) \subset \mathrm{int}(A)$. Therefore $f^{-1}(L) \cap A \neq \emptyset$. The rest of the proof runs in the same way as for the AAM mapping with L in place of K.

The proof is complete. □

Theorem 3.23. *Let a dendroid X be WAO. If a surjective mapping $f : X \to Y$ onto a dendroid Y is both locally monotone and* AAM*, then Y is WAO.*

Proof. The mapping f satisfies, for each $q \in Y$, implication (3.5.1) by Proposition 3.7, and implication (3.5.2) by Proposition 3.22. Thus all the assumptions of Theorem 3.5 are satisfied, whence the conclusion follows. □

Using Statement 3.3 and arguing as in the proof of Theorem 3.23 we get the next result.

Theorem 3.24. *Let a dendroid X be WAO. If a surjective mapping $f : X \to Y$ onto a continuum Y is both locally monotone and feebly monotone, then Y is a WAO dendroid.*

Theorem 3.23 (or Theorem 3.24), Statement 3.3 and Proposition 3.16 imply the following corollary.

Corollary 3.25. *Let a dendroid X be WAO. If a surjective mapping $f : X \to Y$ onto a continuum Y is both locally monotone and almost monotone, then Y is a WAO dendroid.*

Theorem 3.26. *Let a WAO dendroid X do not contain any subdendroid of type* 1*. If a surjective mapping $f : X \to Y$ onto a dendroid Y is either* AAM *or feebly monotone, then Y is WAO.*

Proof. Implication (3.5.1) is satisfied by Proposition 3.9, and (3.5.2) holds according to Proposition 3.22. Thus the result follows by Theorem 3.5. □

Since a smooth dendroid does not contain any subdendroid of type 1 (compare Theorem 1.8), we get a corollary.

Corollary 3.27. *If a surjective mapping $f : X \to Y$ from a smooth dendroid X onto a dendroid Y is either* AAM *or feebly monotone, then Y is WAO.*

In connection with the above results the following questions are natural.

Question 3.28. Are the assumptions on the mapping f of being (a) locally monotone, (b) AAM essential in Theorem 16? (c) The same for "feebly monotone" in Theorem 3.24.

Question 3.29.

(a) Is noncontaining of any subdendroid of type 1 an indispensable assumption in Theorem 3.26?
(b) Is smoothness of X a necessary assumption in Corollary 3.27?

Since each weakly monotone mapping is AWM, Corollary 2.3 implies the following.

Corollary 3.30. AWM *mappings between dendroids do not preserve WAO property, even for fans.*

Summarizing, the present status of our knowledge in the considered area is the following (look at diagrams (3.2) and (3.17)):

(1) WAO property is preserved under mappings which are either (a) monotone, or (b) both locally monotone and AAM, or (c) both locally monotone and feebly monotone. In particular mappings which are both locally monotone and almost monotone preserve WAO.

(2) WAO property is not preserved under mappings which are either weakly semi-confluent or AWM. In particular, weakly monotone mappings are such.
(3) For quasi-monotone and AQM mappings we have open questions (in general); partial results are obtained, among other results, for fans and smooth dendroids.

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