ELSEVIER

CrossMark

Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 879 [FS] (2014) 143–161

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Consistent non-minimal couplings of massive higher-spin particles

Ignacio Cortese [a], Rakibur Rahman [a,*], M. Sivakumar [b]

[a] *Physique Théorique et Mathématique & International Solvay Institutes, Université Libre de Bruxelles, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium*

[b] *School of Physics, University of Hyderabad, Hyderabad 500046, India*

Received 30 September 2013; received in revised form 6 December 2013; accepted 10 December 2013

Available online 14 December 2013

Abstract

The mutual compatibility of the dynamical equations and constraints describing a massive particle of arbitrary spin, though essential for consistency, is generically lost in the presence of interactions. The conventional Lagrangian approach avoids this difficulty, but fails to ensure light-cone propagation and becomes very cumbersome. In this paper, we take an alternative route—the involutive form of the equations and constraints—to guarantee their algebraic consistency. This approach enormously simplifies the search for consistent interactions, now seen as deformations of the involutive system, by keeping manifest the causal propagation of the correct number of degrees of freedom. We consider massive particles of arbitrary integer spin in electromagnetic and gravitational backgrounds to find their possible non-minimal local couplings. Apart from easily reproducing some well-known results, we find restrictions on the backgrounds for consistent propagation of such a particle in isolation. The results can be altered by non-local interactions that may arise from additional massive states in the interacting theory.

1. Introduction

Any fundamental particle described in Quantum Field Theory carries an irreducible unitary representation of the Poincaré group. Massive particles of arbitrary spin, which belong to the first

* Corresponding author.
E-mail addresses: icortese@ulb.ac.be (I. Cortese), rakibur.rahman@ulb.ac.be (R. Rahman), mssp@uohyd.ernet.in (M. Sivakumar).

http://dx.doi.org/10.1016/j.nuclphysb.2013.12.005

Wigner class, are customarily represented by symmetric traceless tensors (bosons) or symmetric γ-traceless tensor-spinors (fermions).[1] A spin-s bosonic field of mass m, which we denote by $\varphi_{\mu_1\ldots\mu_s}$, is required to satisfy the Klein–Gordon equation,

$$\left(\partial^2 - m^2\right)\varphi_{\mu_1\ldots\mu_s} = 0, \tag{1.1}$$

and is subject to the divergence condition,

$$\partial\cdot\varphi_{\mu_1\ldots\mu_{s-1}} \equiv \partial^{\mu_s}\varphi_{\mu_1\ldots\mu_s} = 0. \tag{1.2}$$

Of course, the field $\varphi_{\mu_1\ldots\mu_s}$ is traceless to begin with:

$$\varphi'_{\mu_1\ldots\mu_{s-2}} \equiv \varphi_{\mu_1\ldots\mu_{s-1}}{}^{\mu_{s-1}} = 0. \tag{1.3}$$

The dynamical equation (1.1) and the constraints (1.2) and (1.3) comprise a set of Fierz–Pauli conditions, from which one finds that in d dimensions the total number of propagating degrees of freedom (DoF) is given by

$$\mathfrak{D} = \binom{d-4+s}{s} + 2\binom{d-4+s}{s-1}. \tag{1.4}$$

In particular when $d = 4$, this number reduces to $2s + 1$ as expected.

As first noted by Fierz and Pauli [1], turning on interactions for these higher-spin (HS) fields at the level of equations of motion (EoM) and constraints, by replacing ordinary derivatives with covariant ones in Eqs. (1.1)–(1.3), results in inconsistencies. Consider, for example, a massive spin-s field, $\varphi_{\mu_1\ldots\mu_s}$, minimally coupled to electromagnetism (EM). The naïve covariantization, $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$, of Eqs. (1.1)–(1.3) gives

$$\left(D^2 - m^2\right)\varphi_{\mu_1\ldots\mu_s} = 0, \qquad D\cdot\varphi_{\mu_1\ldots\mu_{s-1}} = 0, \qquad \varphi'_{\mu_1\ldots\mu_{s-2}} = 0. \tag{1.5}$$

The Klein–Gordon equation and the transversality condition, however, yield

$$\left[D^{\mu_1}, D^2 - m^2\right]\varphi_{\mu_1\ldots\mu_s} = 0, \tag{1.6}$$

which results in unwarranted constraints because covariant derivatives do not commute. For a constant EM field strength $F_{\mu\nu}$, for example, one gets

$$ieF^{\mu_1\rho}D_\rho\varphi_{\mu_1\ldots\mu_s} = 0. \tag{1.7}$$

This constraint disappears when the interaction is turned off, and so the system (1.5) does not describe the same number of DoFs as the free theory. To avoid such difficulties, Fierz and Pauli suggested [1] that one take recourse to the Lagrangian formulation, which would automatically render the resulting EoMs and constraints algebraically consistent.

However, a Lagrangian formulation guarantees neither that no unphysical DoFs start propagating nor that the physical ones propagate only within the light cone. Indeed, superluminal propagation can occur in non-trivial external EM backgrounds even for infinitesimally small values of the EM field invariants. This is the notorious Velo–Zwanziger problem [2]. This pathology manifests itself in general for all charged massive HS particles with $s \geqslant 3/2$. Field theoretically it is quite challenging to construct consistent interactions of massive HS particles since this problem persists for a wide class of non-minimal generalizations of the theory and also for other interactions [3–5].

[1] While spin is not a good quantum number in $d > 4$, the rank of the symmetric (γ-)traceless Lorentz tensor(-spinor) continues to define "spin" in arbitrary dimensions.

Addition of non-minimal terms and/or new dynamical DoFs may rescue causality. For a massive charged spin-$\frac{3}{2}$ field, the problem is elegantly solved by $\mathcal{N}=2$ (broken) supergravity [6–8] or by judiciously constructed non-minimal models [9]. For $s \geqslant 2$, the only explicit solution known to date comes from open string field theory [10,11], which spells out highly non-minimal terms so that any field belonging to the first Regge trajectory propagates causally in a constant EM background. Explicit string-theoretic Lagrangians are known for $s=2$ [10] and $s=3$ [12], and they are guaranteed to exist for any HS field [11]. These horribly complicated Lagrangians give rise in the critical dimension a very simple but consistent set of Fierz–Pauli conditions [10,11]:

$$\begin{aligned}
&\left(D^2-m^2\right)\varphi_{\mu_1\ldots\mu_s}-2ies F^{\alpha}_{\ (\mu_1}\varphi_{\mu_2\ldots\mu_s)\alpha}=0,\\
&D\cdot\varphi_{\mu_1\ldots\mu_{s-1}}=0, \qquad \varphi'_{\mu_1\ldots\mu_{s-2}}=0.
\end{aligned} \tag{1.8}$$

The enormous simplicity at the level of EoMs and constraints makes one wonder whether a Lagrangian formulation, originally proposed in [1], is really the best way of understanding HS interactions. After all, in the context of massless HS fields consistent interacting theories appear in AdS space at the level of EoMs [13], which have resisted so far any embedding into a Lagrangian framework, if it exists at all.

Given this, one can step back to revisit the issue of introducing interactions at the level of the EoMs and constraints. Notice that the free system (1.1)–(1.3) and its consistent deformation (1.8) are strikingly similar: they both not only set the divergence and trace exactly to zero, but also have in common some not-so-apparent features that are important for consistency.[2] While the naïve covariantization (1.5) fails, the consistent set (1.8) is not a major modification either. Is it possible to find a systematic procedure to deform the free equations in the presence of interactions without hurting their algebraic consistency? The answer is yes. Indeed, the authors of Ref. [14] have addressed precisely this issue and proposed a universal covariant method for constructing consistent interactions at the level of field equations. Unlike the Lagrangian framework, their method may not need any auxiliary fields and relies on the involution and preservation of gauge symmetries and identities (to be explained in Section 2) of the EoMs and constraints, which guarantee algebraic consistency. This approach may simplify the search for consistent interactions to a great extent by keeping manifest the causal propagation of the correct number of DoFs. In this paper, we employ this method to find non-minimal local couplings of massive particles of arbitrary integer spin exposed to external EM and gravitational backgrounds.

The organization of the paper is as follows. In the remaining of this section we clarify our conventions and notations and present our main results. In Section 2 we get familiar with the formalism proposed in Ref. [14] by explaining some key ideas and working out warm-up examples of a massive spin-1 particle in EM and gravitational backgrounds. Section 3 is devoted to both EM and gravitational interactions of massive particles of arbitrary integer spin. A general methodology is developed throughout this section, which we apply in either case to find the possible non-minimal local couplings and identify the backgrounds that may consistently propagate such a particle in isolation. The resulting couplings are the magnetic dipole and the gravitational quadrupole moments, quantified respectively by the g- and the h-factors. Their values are examined in Section 4, where we also see how they may get modified by the presence of non-locality and/or additional dynamical fields. We conclude in Section 5 with some remarks.

[2] We will investigate and exploit these features, à la Ref. [14], to construct consistent interactions.

1.1. Conventions and notations

We work with a mostly positive metric. The notation $(i_1 \dots i_n)$ means totally symmetric expression in all the indices $i_1, \dots, i_n$ with the normalization factor $\frac{1}{n!}$. The tensor $\eta^{\alpha_1 \cdots \alpha_n, \mu_1 \cdots \mu_n} \equiv \eta^{\alpha_1 \beta_1} \dots \eta^{\alpha_n \beta_n} \delta^{\mu_1 \dots \mu_n}_{\beta_1 \dots \beta_n}$ will appear in many places. We denote EM and gravitational covariant derivatives respectively as D_μ and ∇_μ, whose commutators obey

$$[D_\mu, D_\nu] = ieF_{\mu\nu},$$
$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma.$$

1.2. Results

- In isolation a massive charged HS particle with local EM interactions may consistently propagate only as a probe in an EM background. For $s = 1$, the background is required to satisfy the source-free Maxwell equations: $\partial_\mu F^{\mu\nu} = 0$, whereas for $s \geqslant 2$, the symmetrized gradient of the field strength must vanish: $\partial_{(\mu} F_{\nu)\rho} = 0$.
- Consistent local gravitational interactions of a solitary massive HS particle may exist only in an external gravitational background. The Ricci tensor of this manifold must be covariantly constant (Ricci symmetric space), and for $s \geqslant 3$, the gradient of the Weyl tensor must also satisfy: $\nabla_{(\mu} W_\nu{}^\alpha{}_{\rho)}{}^\beta = 0$. No such restrictions exist for $s = 1$.
- The covariant transversality condition in both cases requires no modification.
- The above results—derived for irreducible representations: symmetric Lorentz tensors with vanishing trace—hold whether or not the system comes from a Lagrangian, provided that interactions are local and no other DoFs are present. Therefore, consistent propagation of massive HS particles in arbitrary EM and gravitational backgrounds calls for non-locality and/or a (possibly infinite) tower of massive states.
- The gyromagnetic ratio or g-factor, that quantifies the magnetic dipole moment of the particle, must be $g = 2$. Other values are possible only when non-local interactions and/or additional massive states are present.
- The gravimagnetic ratio or h-factor of the particle, that quantifies its gravitational quadrupole moment, must be $h = 1$. This value may get altered again by non-local interactions and/or the presence of other massive particles.

2. Formalism and warm-up examples

In this section, we explain some basic notions to summarize the formalism—the deformation of involutive equations—proposed in Ref. [14] for covariant construction of consistent interactions. We will skip some technical details; readers may take a look at Ref. [14] and references therein. While our ultimate goal is to study the EM and gravitational couplings of massive particles of any integer spin, we will consider all along the example of spin 1 for the sake of simplicity. The methodology for arbitrary spin will be developed in Section 3, which will then be employed to find consistent non-minimal couplings.

2.1. Involution

Let us consider a system of partial differential equations

$$T^a\big[\Phi^i, \partial_\mu \Phi^i, \dots, \partial_{\mu_1} \cdots \partial_{\mu_q} \Phi^i\big] = 0, \quad a = 1, 2, \dots, t, \tag{2.1}$$

that governs the dynamics of some fields Φ^i, $i = 1, 2, \ldots, f$. The maximal order of these equations defines the order of this system, which is q. The system (2.1) is said to be involutive if it contains all the differential consequences of order $\leqslant p$ derivable from any order-p subsystem: $T^b[\Phi^i, \partial_\mu \Phi^i, \ldots, \partial_{\mu_1} \cdots \partial_{\mu_p} \Phi^i] = 0$, $b \subset a$, $p \leqslant q$.

To illustrate the difference between involutive and non-involutive systems, let us consider the second order Lagrangian EoMs of the Proca field φ^μ in flat space-time

$$\partial^2 \varphi_\mu - \partial_\mu \partial \cdot \varphi - m^2 \varphi_\mu = 0, \quad m^2 \neq 0. \tag{2.2}$$

Its divergence however gives rise to the first order transversality condition:

$$\partial \cdot \varphi = 0. \tag{2.3}$$

The Proca system (2.2) does not include this lower order differential consequence, and is therefore non-involutive. On the other hand, when the EoMs (2.2) are supplemented by Eq. (2.3), they leave us with a second-order involutive system of equations:

$$T^\mu = \left(\partial^2 - m^2\right)\varphi^\mu = 0, \qquad T = \partial \cdot \varphi = 0, \tag{2.4}$$

which is of course equivalent to the original non-involutive one (2.2). The involutive system (2.4) is non-Lagrangian as it consists of $d + 1$ equations—too many to result directly from the variation of a Lagrangian functional of the d-component field φ_μ.

As a matter of fact, any field theory can be brought to an involutive form [14], which is equivalent to the original system in that they both have the same solution space. Generically, an involutive system may or may not be a Lagrangian one. Free massive HS fields, in particular, have non-involutive Lagrangian equations, but they can also be described by an involutive non-Lagrangian system, namely Eqs. (1.1)–(1.3). The involutive form retains all the symmetries of the original system, and can be very useful in the study of covariant field equations, as we will see.

2.2. *Gauge symmetries and gauge identities*

Let the system (2.1) be involutive. In general, it may enjoy local gauge symmetries

$$\delta_\varepsilon \Phi^i = \varepsilon^\alpha R^i_\alpha, \qquad \delta_\varepsilon T_a|_{T=0} = 0, \quad \alpha = 1, 2, \ldots, r, \tag{2.5}$$

where ε^α are the gauge parameters, while R^i_α are the gauge symmetry generators, which are differential operators of finite order for local symmetries.

More importantly, the involutive system may possess non-trivial gauge identities, which may or may not be related to gauge symmetries. Their schematic form is

$$L^A \triangleright T \equiv L^A_a T^a = 0, \quad A = 1, 2, \ldots, l, \tag{2.6}$$

with the gauge identity generators L^A_a being local differential operators. Note that for Lagrangian systems there is an isomorphism between symmetries and Noether identities. For generic non-Lagrangian involutive systems, no such correspondence exists; still one can have non-trivial gauge identities. In other words, gauge identities are more generic than Noether ones, and may exist even in the absence of any gauge symmetry. However, the two coincide for a set of Lagrangian equations if they are involutive from the outset [14].

Our spin-1 example do not have any gauge symmetry, but it is easy to see that the involutive form (2.4) possesses the gauge identity

$$L \triangleright T \equiv L^\mu T_\mu + LT = \partial^\mu T_\mu - \left(\partial^2 - m^2\right)T = 0, \tag{2.7}$$

where the gauge identity generators are given by

$$L^{\mu} = \partial^{\mu}, \qquad L = -\left(\partial^2 - m^2\right). \tag{2.8}$$

Eq. (2.7) is a third order gauge identity. In general, the order of the gauge identity (2.6) is defined as the maximal order of individual terms appearing in the summation $L_a^A T^a$.

The gauge identities of an involutive system are important in that they reflect algebraic consistency and play a crucial role in the DoF count, to which we now turn.

2.3. Compatibility and DoF count

The compatibility coefficient of an involutive system, say Eq. (2.1), is defined as

$$\Delta = f - \sum_k (t_k - l_k + r_k) = f - t + l - r, \tag{2.9}$$

where t_k, l_k and r_k are respectively the number of equations, independent gauge identities and gauge symmetries of order k. If $\Delta = 0$, the system is said to be absolutely compatible.

Again, the Proca system (2.4) in d dimension consists of d second order equations T^{μ}, and another first order one T. As we have seen, this system has a gauge identity (2.7), but no gauge symmetries. Because the field φ_{μ} contains $f = d$ components to begin with, one finds from definition (2.9) that $\Delta = d - (d + 1) + 1 = 0$, so that the system (2.4) is absolutely compatible. The same is true for arbitrary spin, as we will show in Section 3.

In fact, all known physical systems are absolutely compatible. It is plausible that any reasonable field theory has $\Delta = 0$. If an involutive system of equations is absolutely compatible, the number of DoFs it describes is given by [14]:

$$\mathfrak{D} = \frac{1}{2} \sum_k k(t_k - l_k - r_k), \tag{2.10}$$

provided both the gauge symmetry and gauge identity generators are irreducible. This simple formula enables one to covariantly control the number of physical DoFs.

Accordingly, the involutive Proca system (2.4) has the DoF count

$$\mathfrak{D} = \frac{1}{2}\left[1 \times (1 - 0 - 0) + 2 \times (d - 0 - 0) + 3 \times (0 - 1 - 0)\right] = d - 1,$$

which is indeed the correct number of physical polarizations of a massive spin-1 field. For massive particles of arbitrary spin, we will see in Section 3 that the counting (2.10) matches with the formula (1.4), as expected.

2.4. Consistent deformations

As suggested in Ref. [14], one can control the consistency of interactions by exploiting the involutive form and the gauge symmetries and identities of the free field equations. Given an original system of free fields, the following procedure enables one to introduce consistent couplings at the level of EoMs.

1. The free system of equations are written down in an involutive form.
2. All the gauge symmetries and identities of the free involutive system are identified.

3. Interactions are realized through deformations of the equations, gauge symmetries and identities. Perturbatively in some coupling constant λ, the deformations are:

$$T^a = T_0^a + \lambda T_1^a + \lambda^2 T_2^a + \cdots,$$
$$L_a^A = L_{0a}^A + \lambda L_{1a}^A + \lambda^2 L_{2a}^A + \cdots,$$
$$R_\alpha^i = R_{0\alpha}^i + \lambda R_{1\alpha}^i + \lambda^2 R_{2\alpha}^i + \cdots. \tag{2.11}$$

4. The deformations (2.11) are chosen such that at every order in λ three requirements are fulfilled: (a) the system remains involutive and absolutely compatible; (b) the deformed system has the same number of gauge symmetry and identities, i.e., the quantities $l = \sum l_k$ and $r = \sum r_k$ remain the same; (c) the number of physical polarizations given by Eq. (2.10) remains the same as in the free theory.

The requirements (a) and (b) guarantee that the system remains algebraically consistent with perturbatively included interactions, while condition (c) ensures the deformed system has the same number of physical DoFs as the original one.[3]

2.5. *Warm-up example: Spin 1*

The free involutive Proca system (2.4) and its gauge identity generators (2.8) can be minimally coupled to EM by the substitution $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$, so that we have

$$T_0^\mu = (D^2 - m^2)\varphi^\mu = 0, \qquad T_0 = D \cdot \varphi = 0, \tag{2.12}$$
$$L_0^\mu = D^\mu, \qquad L_0 = -(D^2 - m^2). \tag{2.13}$$

This is indeed the zeroth order deformation in the coupling constant e, because the associated gauge identity fails or become anomalous only at $\mathcal{O}(e)$:

$$L_0 \rhd T_0 \equiv L_0^\mu T_{0\mu} + L_0 T_0 = [D_\mu, D^2]\varphi^\mu = \mathcal{O}(e), \tag{2.14}$$

due to the non-commutativity of the covariant derivatives. This is precisely the anomaly noticed by Fierz and Pauli [1]. But this failure can be rectified at $\mathcal{O}(e)$ by the inclusion of appropriate first order deformations, such that

$$(L_0 + L_1) \rhd (T_0 + T_1) = L_0 \rhd T_0 + L_0 \rhd T_1 + L_1 \rhd T_0 + L_1 \rhd T_1 = \mathcal{O}(e^2). \tag{2.15}$$

That is, the first order deformations must obey

$$L_0 \rhd T_1 + L_1 \rhd T_0 = -L_0 \rhd T_0 + \mathcal{O}(e^2) = [D^2, D_\mu]\varphi^\mu + \mathcal{O}(e^2). \tag{2.16}$$

One can explicitly compute the commutator on the right hand side; it is given by

$$[D^2, D_\mu]\varphi^\mu = D_\mu(2ieF^{\mu\nu}\varphi_\nu) - ie\partial_\mu F^{\mu\nu}\varphi_\nu. \tag{2.17}$$

The left hand side of consistency condition (2.16) can also be made more explicit:

$$L_0 \rhd T_1 + L_1 \rhd T_0 = D_\mu T_1^\mu - (D^2 - m^2)T_1 + L_1^\mu(D^2 - m^2)\varphi_\mu + L_1(D \cdot \varphi). \tag{2.18}$$

[3] Notice that the orders of the equations, gauge identities and symmetries may increase (but never decrease) at any order in λ. One should be careful about possible inclusion of higher derivative kinetic terms, which may signal propagating ghosts.

One can now compare Eqs. (2.17) and (2.18) to identify $T_1^\mu = 2ieF^{\mu\nu}\varphi_\nu$. The term $\partial_\mu F^{\mu\nu}\varphi_\nu$, however, cannot be identified with anything else if one wants to avoid non-local deformations containing the operator $(D^2 - m^2)^{-1}$. Local deformations are still possible if the photon is a background that obeys the source-free Maxwell equations:

$$\partial_\mu F^{\mu\nu} = 0. \tag{2.19}$$

Then, a set of consistent deformations up to $\mathcal{O}(e)$ is given by

$$T^\mu = \left(D^2 - m^2\right)\varphi^\mu + 2ieF^{\mu\nu}\varphi_\nu = 0, \qquad T = D\cdot\varphi = 0, \tag{2.20}$$

$$L^\mu = D^\mu, \qquad L = -\left(D^2 - m^2\right). \tag{2.21}$$

Actually, these deformations are correct up to all orders since $L \triangleright T$ vanishes. It is easy to see that the deformed system (2.20) is involutive. Actually, the individual values of t_k and l_k do not change $\forall k$. Therefore, this system is algebraically consistent and describes the same number of DoFs as the free theory, namely $d-1$. The propagation of these DoFs is manifestly causal since the deformed equations contain no higher derivative kinetic terms.

Similarly, for gravitational coupling the zeroth order deformations[4] are obtained from Eqs. (2.4) and (2.8) by the minimal substitution $\partial_\mu \to \nabla_\mu$. That is,

$$T_0^\mu = \left(\nabla^2 - m^2\right)\varphi^\mu = 0, \qquad T_0 = \nabla\cdot\varphi = 0, \tag{2.22}$$

$$L_0^\mu = \nabla^\mu, \qquad L_0 = -\left(\nabla^2 - m^2\right), \tag{2.23}$$

where $[\nabla_\mu, \nabla_\nu]\varphi^\rho = R^\rho{}_{\sigma\mu\nu}\varphi^\sigma$. The gauge identity anomaly in this case is given by

$$L_0 \triangleright T_0 = -\left[\nabla^2, \nabla_\mu\right]\varphi^\mu = \nabla_\mu\left(R^{\mu\nu}\varphi_\nu\right). \tag{2.24}$$

This anomaly is cured up to all orders, without any restrictions on the gravitational field, by first order deformations with only $T_1^\mu = -R^{\mu\nu}\varphi_\nu$ non-vanishing:

$$T^\mu = \left(\nabla^2 - m^2\right)\varphi^\mu - R^{\mu\nu}\varphi_\nu = 0, \qquad T = \nabla\cdot\varphi = 0, \tag{2.25}$$

$$L^\mu = \nabla^\mu, \qquad L = -\left(\nabla^2 - m^2\right). \tag{2.26}$$

The Proca field therefore interacts consistently with an arbitrary gravitational field because indeed the above deformations identically satisfy the gauge identity to all orders:

$$L \triangleright T = -\left[\nabla^2, \nabla_\mu\right]\varphi^\mu - \nabla_\mu\left(R^{\mu\nu}\varphi_\nu\right) = 0. \tag{2.27}$$

Having worked out the simple but instructive examples of spin 1, we are now ready to consider the EM and gravitational couplings of arbitrary-spin particles.

3. Arbitrary spin: Non-minimal couplings

In this section we are going to deform the free massive HS system to construct consistent couplings to EM and gravitational backgrounds, using the formalism previously explained. As we go along, we will develop a methodology meant for deformations of this particular system. The same methodology works for EM as well as for gravitational couplings.

[4] In this case, one may think that the Riemann curvature incorporates the deformation parameter.

3.1. Free involutive system

The staring point is the free involutive system for a massive spin-s particle, which is

$$T_{\mu_1\ldots\mu_s} = \left(\partial^2 - m^2\right)\varphi_{\mu_1\ldots\mu_s} = 0, \tag{3.1}$$

$$T_{\mu_1\ldots\mu_{s-1}} = \partial \cdot \varphi_{\mu_1\ldots\mu_{s-1}} = 0. \tag{3.2}$$

Note that, unlike the Fierz–Pauli conditions (1.1)–(1.3), this system does not incorporate the trace constraint as a zeroth order differential equation. In fact, the field $\varphi_{\mu_1\ldots\mu_s}$ appearing in the involutive system (3.1)–(3.2) is an irreducible representation of the Lorentz group: a symmetric *traceless* rank-s tensor. Because of this reason, the number of second order equations in the system (3.1)–(3.2) is the same as the number of independent components of a rank-s symmetric traceless tensor,[5] which is

$$t_2 = \binom{d-1+s}{s} - \binom{d-3+s}{s-2}. \tag{3.3}$$

Similarly, the transversality condition amounts to

$$t_1 = \binom{d-2+s}{s-1} - \binom{d-4+s}{s-3} \tag{3.4}$$

first order equations. On the other hand, the system possesses third order gauge identities:

$$\partial^{\mu_s} T_{\mu_1\ldots\mu_s} - \left(\partial^2 - m^2\right)T_{\mu_1\ldots\mu_{s-1}} = \left[\partial^{\mu_s}, \partial^2 - m^2\right]\varphi_{\mu_1\ldots\mu_s} = 0. \tag{3.5}$$

In the compact form (2.6), they read

$$L^{\alpha_1\ldots\alpha_{s-1}} \triangleright T = \left[\partial_\mu, \partial^2\right]\varphi^{\mu\alpha_1\ldots\alpha_{s-1}} = 0, \tag{3.6}$$

where the gauge identity generators are given by

$$L^{\alpha_1\ldots\alpha_{s-1},\mu_1\ldots\mu_s} = \eta^{\alpha_1\ldots\alpha_{s-1},(\mu_1\ldots\mu_{s-1}}\partial^{\mu_s)}, \tag{3.7}$$

$$L^{\alpha_1\ldots\alpha_{s-1},\mu_1\ldots\mu_{s-1}} = -\eta^{\alpha_1\ldots\alpha_{s-1},\mu_1\ldots\mu_{s-1}}\left(\partial^2 - m^2\right). \tag{3.8}$$

Notice that the trace of the identity (3.6) is vanishing on account of the tracelessness of the field itself. The number of independent gauge identities is again given by that of the independent components of a of a rank-$(s-1)$ symmetric traceless tensor, namely

$$l_3 = \binom{d-2+s}{s-1} - \binom{d-4+s}{s-3}. \tag{3.9}$$

Finally, the system (3.1)–(3.2) enjoys no gauge symmetries whatsoever. To summarize, at k-th order in derivatives, the number of equations t_k, independent gauge identities l_k, and gauge symmetries r_k are respectively given by

$$t_k = t_1\delta_k^1 + t_2\delta_k^2, \qquad l_k = l_3\delta_k^3, \qquad r_k = 0. \tag{3.10}$$

Clearly, the compatibility coefficient (2.9) vanishes,

$$\Delta = f - t_1 - t_2 + l_3 = 0, \tag{3.11}$$

[5] This is also the number of fields f described by the system.

on account of the equalities $f = t_2$ and $t_1 = l_3$. That is, the system is absolutely compatible. Then the number of physical DoFs is given by formula (2.10); it is

$$\mathfrak{D} = \frac{1}{2}\sum_k k(t_k - l_k) = \frac{1}{2}(t_1 + 2t_2 - 3l_3) = t_2 - t_1. \tag{3.12}$$

In view of the expressions (3.3), (3.4) and (3.9), this coincides with the formula (1.4) for the propagating DoFs of a massive spin-s particle in d dimensions.

3.2. Deformation in EM background

As we have identified the gauge identities of the free system, we would like to exploit its involutive form to introduce consistent interactions. First, we consider EM coupling. With this end in view, we consider the deformations of the equations,

$$T_{\text{free}}^{\mu_1\ldots\mu_s} \rightarrow T^{\mu_1\ldots\mu_s} = \sum_{n=0}^{\infty} T_n^{\mu_1\ldots\mu_s} = 0, \tag{3.13}$$

$$T_{\text{free}}^{\mu_1\ldots\mu_{s-1}} \rightarrow T^{\mu_1\ldots\mu_{s-1}} = \sum_{n=0}^{\infty} T_n^{\mu_1\ldots\mu_{s-1}} = 0, \tag{3.14}$$

and also of the gauge identity generators,

$$L_{\text{free}}^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_s} \rightarrow L^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_s} = \sum_{n=0}^{\infty} L_n^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_s}, \tag{3.15}$$

$$L_{\text{free}}^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}} \rightarrow L^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}} = \sum_{n=0}^{\infty} L_n^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}}, \tag{3.16}$$

where n denotes the order of the perturbative expansion in the EM charge e. We require that the deformed system satisfy the gauge identities

$$L^{\alpha_1\ldots\alpha_{s-1}} \triangleright T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(L_m^{\alpha_1\ldots\alpha_{s-1}} \triangleright T_n\right) = 0. \tag{3.17}$$

If the system (3.13)–(3.14) remains involutive, then at each order of the deformation the equations (i.e., $\forall n$ the quantities $T_n^{\mu_1\ldots\mu_s}$ and $T_n^{\mu_1\ldots\mu_{s-1}}$) must be symmetric and traceless.[6] Similarly, the gauge identities (3.17) must also remain symmetric and traceless since otherwise it would mean an unwarranted change in their total number.

The deformed gauge identities (3.17) break down into a cascade, order by order in e. The zeroth order gauge identities may become anomalous at $\mathcal{O}(e)$, and they read

$$\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}} \equiv -L_0^{\alpha_1\ldots\alpha_{s-1}} \triangleright T_0 = \mathcal{O}(e), \tag{3.18}$$

where $\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}}$ is called the (symmetric traceless) anomaly tensor. At first order this anomaly must be rectified so as to push the failure of the gauge identities to $\mathcal{O}(e^2)$. The first order gauge identities can be rewritten as

$$\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}} = L_0^{\alpha_1\ldots\alpha_{s-1}} \triangleright T_1 + L_1^{\alpha_1\ldots\alpha_{s-1}} \triangleright T_0 + \cdots, \tag{3.19}$$

[6] Otherwise, at any given order one would find unwarranted constraints on the field, which vanish when the interaction is turned off. The involutive form eliminates this unacceptable possibility.

where the ellipses stand for $\mathcal{O}(e^2)$ terms. It is important to note that if there are no such $\mathcal{O}(e^2)$ terms, then the system requires no higher order deformations provided that

$$\mathcal{C}^{\alpha_1\ldots\alpha_{s-1}} \equiv L_1^{\alpha_1\ldots\alpha_{s-1}} \rhd T_1 = 0. \tag{3.20}$$

Eqs. (3.18)–(3.20) constitute the core of our analysis. The program is to compute the anomaly tensor (3.18) and rewrite it in a suitable form, so that one can read off the first order deformations from Eq. (3.19). If these identifications leave us with no second order terms and if Eq. (3.20) is satisfied, then the deformations consistently stop at first order.

Note that the zeroth order equations are given not by the free system (3.1)–(3.2) itself, but by the minimally coupled version resulting from $\partial_\mu \to D_\mu$ for EM interactions:

$$T_0^{\mu_1\ldots\mu_s} = \left(D^2 - m^2\right)\varphi^{\mu_1\ldots\mu_s} = 0, \tag{3.21}$$

$$T_0^{\mu_1\ldots\mu_{s-1}} = D \cdot \varphi^{\mu_1\ldots\mu_{s-1}} = 0, \tag{3.22}$$

while the zeroth order gauge identity generators follow similarly from Eqs. (3.7)–(3.8):

$$L_0^{\alpha_1\ldots\alpha_{s-1},\mu_1\ldots\mu_s} = \eta^{\alpha_1\ldots\alpha_{s-1},(\mu_1\ldots\mu_{s-1}} D^{\mu_s)}, \tag{3.23}$$

$$L_0^{\alpha_1\ldots\alpha_{s-1},\mu_1\ldots\mu_{s-1}} = -\eta^{\alpha_1\ldots\alpha_{s-1},\mu_1\ldots\mu_{s-1}}\left(D^2 - m^2\right). \tag{3.24}$$

These indeed qualify as the correct zeroth order deformations since the anomaly tensor is

$$\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}} = \left[D^2, D_\mu\right]\varphi^{\mu\alpha_1\ldots\alpha_{s-1}} = \mathcal{O}(e). \tag{3.25}$$

Next, one has to rewrite this expression for the anomaly tensor in a form that facilitates the comparison with the first order deformations through Eq. (3.19). We are particularly interested in finding the non-minimal couplings that show up as corrections to the Klein–Gordon equation (3.21). In Eq. (3.19) they appear in the following form

$$L_0^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_s} T_1^{\mu_1\ldots\mu_s} = D_{\alpha_s} T_1^{\alpha_1\ldots\alpha_s}. \tag{3.26}$$

Therefore, we should extract from the anomaly tensor total derivatives of symmetric traceless objects. Divergence of the field may also appear in Eq. (3.19) through the term

$$L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}} T_0^{\mu_1\ldots\mu_{s-1}} = L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}} D \cdot \varphi^{\mu_1\ldots\mu_{s-1}}. \tag{3.27}$$

The anomaly tensor may also give rise to terms which are neither of the above two forms. The sum $\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}}$ of all such terms must be identified through Eq. (3.19) as

$$\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}} = L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_s}\left(D^2 - m^2\right)\varphi^{\mu_1\ldots\mu_s} - \left(D^2 - m^2\right) T_1^{\alpha_1\ldots\alpha_{s-1}}. \tag{3.28}$$

This identification, however, can only give non-local solutions for the relevant first order deformations.[7] Problematic for local deformations, such terms must vanish if locality has to be preserved in the absence of additional DoFs. This may impose some restrictions on the external background. Constraints on the field, however, must be avoided lest the very involutive form of the system should be ruined.

In order to rewrite the anomaly tensor (3.25), we use properties of the commutator and the product rule for covariant derivatives. Thus we obtain

$$\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}} = -2ieD_\mu\left(F^{\rho\mu}\varphi_\rho{}^{\alpha_1\ldots\alpha_{s-1}}\right) - ie\partial_\mu F^{\mu\nu}\varphi_\nu{}^{\alpha_1\ldots\alpha_{s-1}}. \tag{3.29}$$

[7] We will comment on non-locality in the next section.

The total derivative term can be cast into the form (3.26) if the rank-s tensor inside the parentheses is made symmetric and traceless. While this is done at the cost of adding and subtracting some total derivatives, the left-over terms give rise to divergence pieces of the form (3.27) on account of the product rule. The final result is

$$\begin{aligned}\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}} &= D_\mu\big[-2iesF^{\rho(\mu}\varphi_\rho{}^{\alpha_1\ldots\alpha_{s-1})}\big]+2ie(s-1)F^{\rho(\alpha_1}D\cdot\varphi_\rho{}^{\alpha_2\ldots\alpha_{s-1})}\\ &\quad+2ie(s-1)\partial_\mu F_\nu{}^{(\alpha_1}\varphi^{\alpha_2\ldots\alpha_{s-1})\mu\nu}-ie\partial_\mu F^{\mu\nu}\varphi_\nu{}^{\alpha_1\ldots\alpha_{s-1}}.\end{aligned}\tag{3.30}$$

In view of Eq. (3.19), the first line of the above expression gives us the identifications

$$T_1^{\mu_1\ldots\mu_s}=-2iesF^{\rho(\mu_1}\varphi_\rho{}^{\mu_2\ldots\mu_s)},\tag{3.31}$$

$$L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}}=-2ie(s-1)\,\delta^{\alpha_1\ldots\alpha_{s-1}}_{\rho(\mu_1\ldots\mu_{s-2}}F^\rho{}_{\mu_{s-1})},\tag{3.32}$$

thanks to Eqs. (3.26) and (3.27). On the other hand, the second line of Eq. (3.30) is identified as $\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}}$, which should vanish, as we already pointed out from Eq. (3.28). Splitting the gradient of the field strength into irreducible Lorentz tensors, one obtains

$$\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}}=2ie(s-1)Q_{\mu\nu}{}^{(\alpha_1}\varphi^{\alpha_2\ldots\alpha_{s-1})\mu\nu}-ie\left(\frac{2s+d-3}{d-1}\right)\partial_\mu F^{\mu\nu}\varphi_\nu{}^{\alpha_1\ldots\alpha_{s-1}},\tag{3.33}$$

where $Q_{\mu\nu}{}^\alpha$ is the symmetric traceless gradient of the field strength in d dimensions:

$$Q_{\mu\nu}{}^\alpha\equiv\partial_{(\mu}F_{\nu)}{}^\alpha-\left(\frac{1}{d-1}\right)\big[\eta_{\mu\nu}\partial_\rho F^{\rho\alpha}+\delta^\alpha_{(\mu}\partial^\rho F_{\nu)\rho}\big],\tag{3.34}$$

and the (anti)symmetric tensors $\partial_{(\mu}F_{\nu\alpha)}$ and $\partial_{[\mu}F_{\nu\alpha]}$ are zero identically. For $s>1$, it is clear from Eq. (3.33) that $\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}}$ can be zero, without incurring unwarranted constraints on the HS field, if and only if both the quantities $Q_{\mu\nu}{}^\alpha$ and $\partial_\mu F^{\mu\nu}$ vanish. The above conditions are tantamount to requiring that the EM background satisfy:

$$\partial_{(\mu}F_{\nu)\rho}=0.\tag{3.35}$$

This admits, in particular, $F_{\mu\nu}=$ constant as a consistent background.[8] For $s=1$, however, it suffices to require that the background obey the source-free Maxwell equations: $\partial_\mu F^{\mu\nu}=0$. Finally, the deformations can be made consistent up to all orders by the choice:

$$T_1^{\mu_1\ldots\mu_{s-1}}=0,\qquad L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_s}=0.\tag{3.36}$$

Indeed, this choice renders the tensor $\mathcal{C}^{\alpha_1\ldots\alpha_{s-1}}$ appearing in Eq. (3.20) vanishing.

To summarize, we have found the following consistently deformed involutive system:

$$T^{\mu_1\ldots\mu_s}=\big(D^2-m^2\big)\varphi^{\mu_1\ldots\mu_s}-2iesF^{\rho(\mu_1}\varphi_\rho{}^{\mu_2\ldots\mu_s)}=0,\tag{3.37}$$

$$T^{\mu_1\ldots\mu_{s-1}}=D\cdot\varphi^{\mu_1\ldots\mu_{s-1}}=0,\tag{3.38}$$

for a class of EM backgrounds. Augmented by the implicit trace condition, $\varphi'_{\mu_1\ldots\mu_{s-2}}=0$, the same equations show up, quite curiously, in string theory as well [10,11]. This system is algebraically consistent by construction. The DoF count is also correct for an obvious reason: this system and the free one shares the same set of t_k and l_k given in Eq. (3.10). The Laplacian kinetic operator in Eq. (3.37) also makes causal propagation manifest.

8 Non-constant EM backgrounds may also qualify. In $d=4$, for example, the generic solution of Eq. (3.35) is: $F^{0i}=\epsilon^i+\varepsilon^{ijk}\alpha_j x_k$ and $F^{ij}=\varepsilon^{ijk}(\beta_k+\alpha_k t)$, with $\epsilon_i,\alpha_i,\beta_i$ constants and $i=1,2,3$.

3.3. Deformation in gravitational background

Now we turn to gravitational coupling. In this case, the Riemann curvature is assumed to have incorporated the deformation parameter, and the covariant derivatives are denoted by ∇_μ. Modulo these, the steps and analyses of the previous subsection hold verbatim in this case until one writes down an explicit expression like (3.29) for the anomaly tensor. While

$$\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}} = \left[\nabla^2, \nabla_\mu\right]\varphi^{\mu\alpha_1\ldots\alpha_{s-1}}, \tag{3.39}$$

similar steps lead to the gravitational counterpart of Eq. (3.29), which reads

$$\begin{aligned}\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}} = {} & \nabla_\mu\left(2\sum_{i=1}^{s-1} R^{\mu}{}_{\nu}{}^{\alpha_i}{}_{\rho}\,\varphi^{\alpha_1\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_{s-1}\nu\rho} - R^{\rho\mu}\varphi_{\rho}{}^{\alpha_1\ldots\alpha_{s-1}}\right) \\ & - \sum_{i=1}^{s-1}\nabla_\mu R^{\mu}{}_{\nu}{}^{\alpha_i}{}_{\rho}\,\varphi^{\alpha_1\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_{s-1}\nu\rho}.\end{aligned} \tag{3.40}$$

In deriving the above we have used the symmetry properties of the Riemann tensor, which imply in particular $R^{\mu}{}_{(\rho}{}^{\nu}{}_{\sigma)} = R^{(\mu}{}_{\rho}{}^{\nu)}{}_{\sigma}$. Again, from the total derivative term in the first line of Eq. (3.40) one can extract the gravitational counterpart of Eq. (3.26), namely

$$L_0^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_s} T_1^{\mu_1\ldots\mu_s} = \nabla_{\alpha_s} T_1^{\alpha_1\ldots\alpha_s}. \tag{3.41}$$

The feat is achieved by rendering the rank-s tensor inside the parentheses symmetric and traceless. The left-over terms from the last step again produce covariant divergences—the gravitational counterpart of Eq. (3.27)—of the form

$$L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}} T_0^{\mu_1\ldots\mu_{s-1}} = L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}} \nabla\cdot\varphi^{\mu_1\ldots\mu_{s-1}}, \tag{3.42}$$

thanks to the product rule. On the other hand, one can massage the second line of Eq. (3.40) by using the contracted Bianchi identity: $\nabla_\mu R^{\mu}{}_{\nu}{}^{\alpha}{}_{\rho} = \nabla^\alpha R_{\nu\rho} - \nabla_\rho R_\nu{}^\alpha$. All these steps lead us to the following expression for the anomaly tensor:

$$\begin{aligned}\mathcal{A}^{\alpha_1\ldots\alpha_{s-1}} = {} & \nabla_\mu\left[s(s-1)R^{(\mu}{}_{\nu}{}^{\alpha_1}{}_{\rho}\varphi^{\alpha_2\ldots\alpha_{s-1})\nu\rho} - sR^{\rho(\mu}\varphi_{\rho}{}^{\alpha_1\ldots\alpha_{s-1})}\right] \\ & - (s-1)\left[(s-2)R^{(\alpha_1}{}_{\mu}{}^{\alpha_2}{}_{\nu}\nabla\cdot\varphi^{\alpha_3\ldots\alpha_{s-1})\mu\nu} - R^{\mu(\alpha_1}\nabla\cdot\varphi_{\mu}{}^{\alpha_2\ldots\alpha_{s-1})}\right] \\ & + (s-1)\left[2\nabla_\mu R_\nu{}^{(\alpha_1}\varphi^{\alpha_2\ldots\alpha_{s-1})\mu\nu} - \nabla^{(\alpha_1}R_{\mu\nu}\varphi^{\alpha_2\ldots\alpha_{s-1})\mu\nu}\right] \\ & - (s-1)(s-2)\nabla_\mu R^{(\alpha_1}{}_{\nu}{}^{\alpha_2}{}_{\rho}\,\varphi^{\alpha_3\ldots\alpha_{s-1})\mu\nu\rho}.\end{aligned} \tag{3.43}$$

When plugged into the first order gauge identity (3.19), the first line of this expression gives us, in view of Eq. (3.41), the following identification for the deformation of equations:

$$T_1^{\mu_1\ldots\mu_s} = s(s-1)R^{(\mu_1}{}_{\nu}{}^{\mu_2}{}_{\rho}\varphi^{\mu_3\ldots\mu_s)\nu\rho} - sR^{\rho(\mu_1}\varphi_{\rho}{}^{\mu_2\ldots\mu_s)}. \tag{3.44}$$

Similarly, from the second line one identifies, on account of Eq. (3.42),

$$\begin{aligned}& L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_{s-1}} \\ & \quad = -(s-1)\left[(s-2)\,\delta^{\alpha_1\ldots\alpha_{s-1}}_{\rho\sigma(\mu_1\ldots\mu_{s-3}}R^{\rho}{}_{\mu_{s-2}}{}^{\sigma}{}_{\mu_{s-1})} - \delta^{\alpha_1\ldots\alpha_{s-1}}_{\rho(\mu_1\ldots\mu_{s-2}}R^{\rho}{}_{\mu_{s-1})}\right].\end{aligned} \tag{3.45}$$

The remaining third and fourth lines in the expression (3.43) for the anomaly tensor are identified as $\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}}$, which must be set to zero in order to avoid non-locality. Now $\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}}$ contains gradients of the Riemann and Ricci tensors, and the latter quantities can be split into irreducible Lorentz tensors, so that one obtains the expression

$$\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}} = -\frac{(s-1)(s-2)}{d-2}\big[(d-2)X_{\mu\nu\rho}{}^{(\alpha_1\alpha_2}\varphi^{\alpha_3\ldots\alpha_{s-1})\mu\nu\rho} + Y_{\mu\nu\rho}\, g^{(\alpha_1\alpha_2}\varphi^{\alpha_3\ldots\alpha_{s-1})\mu\nu\rho}\big]$$
$$+\left(\frac{s-1}{d-2}\right)\bigg[(2s+d-6)\,Y_{\mu\nu}{}^{(\alpha_1}\varphi^{\alpha_2\ldots\alpha_{s-1})\mu\nu}$$
$$-\left(\frac{s+2d-6}{3}\right)Z_{\mu\nu}{}^{(\alpha_1}\varphi^{\alpha_2\ldots\alpha_{s-1})\mu\nu}\bigg]$$
$$+\frac{2(s-1)(s+d-2)}{(d-1)(d+2)}(\nabla_\mu R)\varphi^{\alpha_1\ldots\alpha_{s-1}\,\mu}, \tag{3.46}$$

where $X_{\mu\nu\rho}{}^{\alpha\beta}$, $Y_{\mu\nu}{}^{\alpha}$ and $Z_{\mu\nu}{}^{\alpha}$ are the following irreducible Lorentz tensors:

$$X_{\mu\nu\rho}{}^{\alpha\beta} = \nabla_{(\mu}W_{\nu}{}^{\alpha}{}_{\rho)}{}^{\beta} - \left(\frac{2}{d+2}\right)g_{(\mu\nu}\nabla^{\sigma}W_{\rho)}{}^{(\alpha}{}_{\sigma}{}^{\beta)}, \tag{3.47}$$

$$Y_{\mu\nu\rho} = \nabla_{(\mu}R_{\nu\rho)} - \left(\frac{2}{d+2}\right)g_{(\mu\nu}\nabla_{\rho)}R, \tag{3.48}$$

$$Z_{\mu\nu\rho} = 2\nabla_{[\rho}R_{\mu]\nu} + \left(\frac{1}{d-1}\right)g_{\nu[\rho}\nabla_{\mu]}R + (\mu \leftrightarrow \nu), \tag{3.49}$$

with $W_{\mu\alpha\nu\beta}$ denoting the Weyl tensor.[9] Now that $\mathcal{B}^{\alpha_1\ldots\alpha_{s-1}}$ must be set to zero without imposing any further constraints on the HS field, we have three different cases:

- $s=1$: Because all the dangerous terms in Eq. (3.46) are proportional to $s-1$, they vanish automatically for the Proca field and pose no restrictions on the background.
- $s=2$: In this case, the first line in Eq. (3.46) vanishes. In order to kill the other terms, one must set to zero all the quantities $Y_{\mu\nu\rho}$, $Z_{\mu\nu\rho}$ and $\nabla_\mu R$, which is tantamount to having a covariantly constant Ricci tensor:

$$\nabla_\mu R_{\nu\rho} = 0. \tag{3.50}$$

  Thus, consistency requires that the gravitational background be Ricci symmetric.
- $s \geqslant 3$: For higher spins, on top of having a Ricci symmetric space, one needs additional conditions on the Weyl tensor, namely $X_{\mu\nu\rho}{}^{\alpha\beta}=0$. Because the divergence of the Weyl tensor can be expressed in terms of $Z_{\mu\nu\rho}$ and $\nabla_\mu R$ as a consequence of the Bianchi identities, we have the equivalent set of conditions:

$$\nabla_\mu R_{\nu\rho} = 0, \qquad \nabla_{(\mu}W_{\nu}{}^{\alpha}{}_{\rho)}{}^{\beta} = 0, \tag{3.51}$$

  which a gravitational background must satisfy in order to propagate consistently an arbitrary-spin particle in isolation, under the assumption of locality. Note, in particular, that symmetric spaces do qualify as consistent backgrounds, since they have covariantly constant Riemann tensors: $\nabla_\mu R_{\nu\alpha\rho\beta} = 0$.

Given one of the appropriate restrictions, one can render the deformations (3.44) and (3.45) consistent up to all orders by simply choosing

$$T_1^{\mu_1\ldots\mu_{s-1}} = 0, \qquad L_1^{\alpha_1\ldots\alpha_{s-1},}{}_{\mu_1\ldots\mu_s} = 0. \tag{3.52}$$

[9] The quantities (3.47)–(3.49) are all traceless, thanks to the identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$.

Surely, the tensor $\mathcal{C}^{\alpha_1\dots\alpha_{s-1}}$ appearing in Eq. (3.20) vanishes with this choice. Thus, we end up having the following consistently deformed involutive system:

$$\begin{aligned} T^{\mu_1\dots\mu_s} &= (\nabla^2 - m^2)\varphi^{\mu_1\dots\mu_s} + \left[s(s-1)R^{(\mu_1}{}_\nu{}^{\mu_2}{}_\rho\varphi^{\mu_3\dots\mu_s)\nu\rho} - sR^{\rho(\mu_1}\varphi_\rho{}^{\mu_2\dots\mu_s)}\right] \\ &= 0, \end{aligned} \tag{3.53}$$

$$T^{\mu_1\dots\mu_{s-1}} = \nabla\cdot\varphi^{\mu_1\dots\mu_{s-1}} = 0, \tag{3.54}$$

with the aforementioned restrictions on the gravitational background. The algebraic consistency, preservation of the correct number of DoFs and causal propagation are guaranteed precisely the same way as they are in an EM background and in the free theory.

4. On g- and h-factors and non-locality

The EM and gravitational non-minimal couplings we found in Section 3 are respectively called the magnetic dipole and the gravitational quadrupole terms. The magnetic dipole moment is quantified by the so-called gyromagnetic ratio or the g-factor. For a spin-s boson of mass m and charge e, the g-factor appears at the level of EoM as follows [5]:

$$(D^2 - m^2)\varphi^{\mu_1\dots\mu_s} - iegsF^{\rho(\mu_1}\varphi_\rho{}^{\mu_2\dots\mu_s)} = 0. \tag{4.1}$$

Direct comparison of this with our result (3.37) reveals that, for all spins,

$$g = 2. \tag{4.2}$$

This may not come as a surprise, since $g = 2$ turns out to be the "preferred" tree-level value for any spin [15,16]. Moreover, open string theory predicts the same universal value for g [10, 11,16]. On the other hand, it has been observed that Kaluza–Klein (KK) reductions of consistent higher dimensional models give $g = 1$ for all spins [17–19]. How does this fact go along with our results?

The answer lies in non-locality—a possibility we did not explore. In fact, KK theories describe a tower of massive particles, not a single one. If one is interested in the dynamics of a particular state, one may integrate out the other ones, some of which are of comparable mass. This results in a non-local theory. The conclusion is that additional dynamical states and/or non-local terms may change the value of g. Indeed, sum rules from low-energy Compton scattering can show that in the presence of other massive state $g - 2$ may become $\mathcal{O}(1)$ [20]. Let us see how this could be understood within our framework.

For simplicity, let us consider $s = 2$. Under the condition (3.35), the anomaly tensor (3.30) can be rewritten, for an arbitrary gyromagnetic ratio g, as

$$\mathcal{A}^\alpha = D_\mu\left[-2iegF^{\rho(\mu}\varphi_\rho{}^{\alpha)}\right] + iegF^{\rho\alpha}D\cdot\varphi_\rho - ie(g-2)F^{\mu\nu}D_\mu\varphi_\nu^\alpha. \tag{4.3}$$

The first two terms on the right hand side can again be incorporated into local first order deformations. Note, in particular, that the first term gives rise to the value g for the gyromagnetic ratio. Even when the background satisfies the condition (3.35), the last term with $g \neq 2$ signals breakdown of locality. More explicitly, the consistency of the first order gauge identity (3.19) requires the identification

$$\begin{aligned} ie(g-2)F^{\mu\nu}D_\mu\varphi_\nu^\alpha &= -L_0^{\alpha,}{}_\mu T_1^\mu - L_1^{\alpha,}{}_{\mu\nu}T_0^{\mu\nu} \\ &= (D^2 - m^2)T_1^\alpha - L_1^{\alpha,}{}_{\mu\nu}(D^2 - m^2)\varphi^{\mu\nu}. \end{aligned} \tag{4.4}$$

This may admit only non-local solutions for the relevant first order deformations, like

$$L_1^{\alpha,\mu\nu} = 0, \qquad T_1^{\mu} = ie(g-2)\left(\frac{1}{D^2 - m^2}\right)\left(F^{\rho\sigma} D_\rho \varphi_\sigma^\mu\right), \tag{4.5}$$

that modify the transversality condition. The presence of the operator $(D^2 - m^2)^{-1}$ is tantamount to non-locality, which might have arisen from integrating out other massive states of the theory. Thus non-locality and/or additional DoFs of comparable mass may give $g \neq 2$. Non-local interactions, however, are beyond the scope of our present analysis.

On the other hand, the gravitational quadrupole moment is quantified analogously by the gravimagnetic ratio or the h-factor. A careful definition of the h-factor was given in Ref. [21]. At the level of EoM it shows up as

$$\left(\nabla^2 - m^2\right)\varphi^{\alpha(s)} + h\left[R_{\mu\nu\rho\sigma}\frac{1}{2}\left(\Sigma^{\mu\nu}\right)^{\alpha(s)}{}_{\beta(s)}\frac{1}{2}\left(\Sigma^{\mu\nu}\right)^{\beta(s)}{}_{\gamma(s)}\right]\varphi^{\gamma(s)} + \cdots = 0, \tag{4.6}$$

where the ellipses denote possible on-shell vanishing terms, and

$$\left(\Sigma^{\mu\nu}\right)_{\alpha(s)}{}^{\beta(s)} \equiv 2s\delta^{[\mu}_{(\alpha_1}\eta^{\nu](\beta_1}\delta^{\beta_2\ldots\beta_s)}_{\alpha_2\ldots\alpha_s)} = -\left(\Sigma^{\mu\nu}\right)^{\beta(s)}{}_{\alpha(s)}, \tag{4.7}$$

are the components of the Lorentz generators in the spin-s bosonic representation.

These are antisymmetric in $\mu\nu$, and symmetric in the indices of the individual sets $\alpha(s)$ and $\beta(s)$. These symmetry properties result in the tracelessness of $\Sigma^{\mu\nu}$ in the $\alpha(s)$ indices and hence in the $\beta(s)$ indices as well.[10] A straightforward computation gives

$$\left(\nabla^2 - m^2\right)\varphi^{\alpha_1\ldots\alpha_s} + h\left[s(s-1)R_\mu{}^{(\alpha_1}{}_\nu{}^{\alpha_2}\varphi^{\alpha_3\ldots\alpha_s)\mu\nu} - sR^{\mu(\alpha_1}\varphi_\mu{}^{\alpha_2\ldots\alpha_s)}\right] + \cdots = 0. \tag{4.8}$$

Upon comparing our results (3.53) with this definition, one immediately concludes

$$h = 1. \tag{4.9}$$

This also happens to be the preferred field theory value from considerations of tree-level unitarity and supersymmetry [22]. However, the study of three-point functions in superstring theory suggests $h \neq 1$ in general [21], and this happens because of the existence of a whole tower of states of arbitrarily large masses and spins. Quite similarly to the g-factor analysis, one can show for gravitational couplings that $h \neq 1$ necessarily results in non-locality, thanks to the consistency of the first order gauge identity (3.19). For a spin-2 particle in a Ricci symmetric space, the gravitational counterpart of the identification (4.4) admits a non-local solution

$$L_1^{\alpha,\mu\nu} = 0, \qquad T_1^{\sigma} = -(h-1)\left(\frac{1}{\nabla^2 - m^2}\right)\left(2R^{\mu\nu\rho\sigma} + R^{\mu\nu}g^{\rho\sigma}\right)\nabla_\mu\varphi_{\nu\rho}. \tag{4.10}$$

Here the presence of the operator $(\nabla^2 - m^2)^{-1}$ is a telltale sign of non-locality. This might be taken as a hint for the existence of other interacting massive DoFs, which when taken into account, may restore locality in the effective theory.

[10] The definition of the Lorentz generators given in Ref. [21], though, is plagued with typographic errors. We thank M. Porrati for pointing this out and clarifying the properties of $\Sigma^{\mu\nu}$. Note also that the antisymmetry (4.7) between the $\alpha(s)$ and $\beta(s)$ sets renders any cross contraction of indices vanishing.

5. Concluding remarks

In this paper, we have exploited properties of involutive differential equations to consistently couple an arbitrary-spin massive particle to EM and gravitational backgrounds. Originally developed in [14], the method works at the level of field equations with a consistent perturbative scheme to introduce interactions. The virtues of this framework are manifold—manifest covariance, built-in algebraic consistency, stability of the number of physical DoFs and their manifest causal propagation, and striking simplicity.

It is surprising to see how easily this approach may produce non-trivial results and shed light on some intricacies of interacting massive HS particles. Among others, we could reproduce the preferred field theory values of the g- and h-factors, and see how these values may/do get altered in the presence of other massive states. For EM coupling, we have seen that a solitary Proca field requires a background obeying source-free Maxwell equations, whereas an arbitrary HS field requires that the EM field strength satisfy the condition $\partial_{(\mu} F_{\nu)\rho} = 0$. We also find that an isolated massive HS particle may have consistent non-minimal local interactions with a gravitational background, which (a) has no restrictions for $s = 1$, (b) must be a Ricci symmetric space for $s = 2$, and (c) must be a Ricci symmetric space with additional conditions for $s \geqslant 3$ on the Weyl tensor: $\nabla_{(\mu} W_{\nu}{}^{\alpha}{}_{\rho)}{}^{\beta} = 0$. Curiously, the consistency of the Lagrangian dynamics of spinning particles in various dimensions imposes similar restrictions on the backgrounds [23,24]. Also, the study of gravitating partially massless spin-2 fields leads naturally to Ricci symmetric spaces [25], which however do not suffice for consistency in this case.

Let us emphasize that our results for the non-minimal couplings show up as consistent deformations on top of minimal coupling, whose existence has been implicitly assumed. For EM interactions this assumption may not hold; indeed, massive HS fields may not have an EM charge but still possess higher multipoles. In this case our conclusions will not be valid. For gravitational interactions, however, our assumption is well justified since minimal coupling is required in order for the principle of equivalence to hold [26].

A couple of comments on our consistent local deformations are due. First, the choices (3.36) and (3.52) for the first order corrections respectively for EM and gravity are unique in that they not only keep locality intact, but also make the inclusion of higher order deformations unnecessary. In particular, they leave the covariant transversality conditions undeformed. Any other choices for these first order deformations will call for either non-locality or higher order corrections or both. Last but not the least, the restrictions on the EM and gravitational backgrounds are required only for the sake of locality. This does not mean that a massive HS field cannot propagate consistently in an arbitrary background, but that it can do so only if non-local interactions are allowed and/or other interacting DoFs are present. It is possible that if one begins with an enlarged (possibly infinite) set of HS fields, one may find consistent local deformations of the free theory for arbitrary backgrounds. While from bottom-up it is a priori not possible to know which set of fields, if any, may achieve this feat, top-down approaches like KK incarnations of string theory or supergravity provide some concrete examples of this kind.

Our results cover a wider range of possibilities since the approach does not assume any Lagrangian embedding. Requiring that the system of equations results from a local Lagrangian may therefore result in further restrictions. Indeed, Lagrangians for a charged massive HS field are known to exist for a constant EM background only in $d = 26$ [10,11]; it is not clear whether they exist in arbitrary dimensions, which may require additional dynamical fields [27]. Another example is a massive spin-2 field in a gravitational background; a local Lagrangian approach seem to call for Einstein manifolds [28]—a subset of Ricci symmetric manifolds we require.

In short, the requirements we find are necessary but may not be sufficient for the existence of local Lagrangians.

The possible non-existence of a local Lagrangian is perhaps the most uncomfortable feature of this formalism. How does one quantize such non-Lagrangian involutive systems? The authors in Refs. [29,30] have tried to address this issue through a formalism based on a generalized Lagrangian structure, called the Lagrange anchor. This structure contains information beyond the solutions to the EoMs, much like the Lagrangian does, that can help one perform a path integral for the non-Lagrangian system.

The approach of deformation of involutive equations is a powerful one that can be applied to many different systems. The immediate things to consider are fermions and mixed symmetry tensors in EM and gravitational backgrounds [31]. One may also try partially massless fields to see if the results of [25] could be reproduced. Note that the Lagrangian formulation requires auxiliary fields, which can be incorporated into the trace of some otherwise traceless tensor fields. In order to make more contact with the Lagrangian framework, it is desirable to start with traceful fields. In this case, as one casts the original system into an involutive form, the free system enjoys a much richer gauge structure with many more gauge identities [31]. The correct DoF count crucially depends on taking all the independent identities into account. Unlike in the traceless set-up, there exist relations among the gauge identities themselves; one needs to be careful while deforming the free system. We leave this as future work.

Acknowledgements

R.R. and M.S. would like to thank S.L. Lyakhovich and A. Waldron for stimulating discussions and useful comments, the Galileo Galilei Institute for Theoretical Physics (where this work was initiated) for hospitality and the INFN for partial support. I.C. acknowledges partial support by CONACyT's program "Estancias Sabáticas y Posdoctorales al Extranjero" (México) 2011–2012, grant number 172676. R.R. is a Postdoctoral Fellow of the Fonds de la Recherche Scientifique – FNRS. The work of I.C. and R.R. is partially supported by IISN-Belgium (conventions 4.4511.06 and 4.4514.08) and by the "Communauté Française de Belgique" through the ARC program and by the ERC through the "SyDuGraM" Advanced Grant. M.S. thanks M. Henneaux and H. Nicolai for hospitality respectively at Université Libre de Bruxelles and at AEI for Gravitational Physics (Potsdam), where part of his work was done. He acknowledges support from DST (India) in the form of a project.

References

[1] M. Fierz, W. Pauli, Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. 173 (1939) 211.
[2] G. Velo, D. Zwanziger, Phys. Rev. 186 (1969) 1337;
G. Velo, D. Zwanziger, Phys. Rev. 188 (1969) 2218;
G. Velo, Nucl. Phys. B 43 (1972) 389.
[3] A. Shamaly, A.Z. Capri, Ann. Phys. 74 (1972) 503;
M. Hortacsu, Phys. Rev. D 9 (1974) 928.
[4] M. Kobayashi, A. Shamaly, Phys. Rev. D 17 (1978) 2179;
M. Kobayashi, A. Shamaly, Prog. Theor. Phys. 61 (1979) 656.
[5] S. Deser, V. Pascalutsa, A. Waldron, Phys. Rev. D 62 (2000) 105031, arXiv:hep-th/0003011;
S. Deser, A. Waldron, Nucl. Phys. B 631 (2002) 369, arXiv:hep-th/0112182.
[6] S. Ferrara, P. van Nieuwenhuizen, Phys. Rev. Lett. 37 (1976) 1669;
D.Z. Freedman, A.K. Das, Nucl. Phys. B 120 (1977) 221.
[7] J. Scherk, J.H. Schwarz, Phys. Lett. B 82 (1979) 60;

J. Scherk, J.H. Schwarz, Nucl. Phys. B 153 (1979) 61;
B. de Wit, P.G. Lauwers, A. Van Proeyen, Nucl. Phys. B 255 (1985) 569.
[8] S. Deser, B. Zumino, Phys. Rev. Lett. 38 (1977) 1433.
[9] M. Porrati, R. Rahman, Phys. Rev. D 80 (2009) 025009, arXiv:0906.1432 [hep-th].
[10] P.C. Argyres, C.R. Nappi, Phys. Lett. B 224 (1989) 89.
[11] M. Porrati, R. Rahman, A. Sagnotti, Nucl. Phys. B 846 (2011) 250, arXiv:1011.6411 [hep-th].
[12] S.M. Klishevich, Int. J. Mod. Phys. A 15 (2000) 395, arXiv:hep-th/9805174.
[13] E.S. Fradkin, M.A. Vasiliev, Phys. Lett. B 189 (1987) 89;
E.S. Fradkin, M.A. Vasiliev, Nucl. Phys. B 291 (1987) 141;
M.A. Vasiliev, Phys. Lett. B 243 (1990) 378;
M.A. Vasiliev, Class. Quantum Gravity 8 (1991) 1387;
M.A. Vasiliev, Phys. Lett. B 285 (1992) 225;
M.A. Vasiliev, Phys. Lett. B 567 (2003) 139, arXiv:hep-th/0304049.
[14] D.S. Kaparulin, S.L. Lyakhovich, A.A. Sharapov, J. High Energy Phys. 1301 (2013) 097, arXiv:1210.6821 [hep-th].
[15] S. Weinberg, Lectures on Elementary Particles and Quantum Field Theory, Proceedings of the Summer Institute, Brandeis University, 1970, vol. I, MIT Press, Cambridge, MA, 1970.
[16] S. Ferrara, M. Porrati, V.L. Telegdi, Phys. Rev. D 46 (1992) 3529;
S. Ferrara, M. Porrati, Phys. Lett. B 288 (1992) 85.
[17] A. Hosoya, K. Ishikawa, Y. Ohkuwa, K. Yamagishi, Phys. Lett. B 134 (1984) 44.
[18] S.D. Rindani, M. Sivakumar, J. Phys. G 12 (1986) 1335;
S.D. Rindani, M. Sivakumar, Z. Phys. C 49 (1991) 601.
[19] K.-Y. Kim, M. Taylor, arXiv:1304.6729 [hep-th].
[20] H.R. Grigoryan, M. Porrati, J. High Energy Phys. 1207 (2012) 048, arXiv:1204.1064 [hep-th].
[21] I. Giannakis, J.T. Liu, M. Porrati, Phys. Rev. D 59 (1999) 104013, arXiv:hep-th/9809142.
[22] A. Cucchieri, M. Porrati, S. Deser, Phys. Rev. D 51 (1995) 4543, arXiv:hep-th/9408073;
I. Giannakis, J.T. Liu, M. Porrati, Phys. Lett. B 469 (1999) 129, arXiv:hep-th/9909012.
[23] S.L. Lyakhovich, A.Y. Segal, A.A. Sharapov, Phys. Rev. D 54 (1996) 5223, arXiv:hep-th/9603174;
S.L. Lyakhovich, A.A. Sharapov, K.M. Shekhter, J. Math. Phys. 38 (1997) 4086, arXiv:hep-th/9609133;
S.L. Lyakhovich, A.A. Sharapov, K.M. Shekhter, Int. J. Mod. Phys. A 15 (2000) 4287, arXiv:hep-th/0002247;
K.B. Alkalaev, S.L. Lyakhovich, Mod. Phys. Lett. A 14 (1999) 2727.
[24] K. Hallowell, A. Waldron, SIGMA 3 (2007) 089, arXiv:0707.3164 [math.DG];
K. Hallowell, A. Waldron, Commun. Math. Phys. 278 (2008) 775, arXiv:hep-th/0702033.
[25] S. Deser, E. Joung, A. Waldron, J. Phys. A 46 (2013) 214019, arXiv:1208.1307 [hep-th].
[26] S. Weinberg, Phys. Rev. 135 (1964) B1049;
M. Porrati, Phys. Rev. D 78 (2008) 065016, arXiv:0804.4672 [hep-th].
[27] M. Porrati, R. Rahman, Phys. Rev. D 84 (2011) 045013, arXiv:1103.6027 [hep-th].
[28] I.L. Buchbinder, D.M. Gitman, V.D. Pershin, Phys. Lett. B 492 (2000) 161, arXiv:hep-th/0006144;
I.L. Buchbinder, V.A. Krykhtin, P.M. Lavrov, Phys. Lett. B 685 (2010) 208, arXiv:0912.0611 [hep-th].
[29] P.O. Kazinski, S.L. Lyakhovich, A.A. Sharapov, J. High Energy Phys. 0507 (2005) 076, arXiv:hep-th/0506093;
S.L. Lyakhovich, A.A. Sharapov, J. High Energy Phys. 0503 (2005) 011, arXiv:hep-th/0411247;
S.L. Lyakhovich, A.A. Sharapov, J. High Energy Phys. 0602 (2006) 007, arXiv:hep-th/0512119.
[30] D.S. Kaparulin, S.L. Lyakhovich, A.A. Sharapov, J. Math. Phys. 51 (2010) 082902, arXiv:1001.0091 [math-ph];
D.S. Kaparulin, S.L. Lyakhovich, A.A. Sharapov, J. High Energy Phys. 1109 (2011) 006, arXiv:1106.4252 [hep-th].
[31] I. Cortese, R. Rahman, M. Sivakumar, work in preparation.