A trace inequality arising from quantum information theory

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Abstract

We introduce a skew information of Lieb’s type

\[ S_{f,g}(A, X) = \text{Tr} f(A)Xg(A)X - \text{Tr} f(A)g(A)X^2 \]

for selfadjoint matrices \( A, X \). We give conditions for \( f \) and \( g \) so that \( S_{f,g} \) is positive or negative. As another important application, we settle the problem posed by Yanagi, Furuichi and Kuriyama from quantum information theory.

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1. Introduction

In [7], Holevo discussed the reliability function of a classical-quantum channel and used the trace function
\[ \mu(s, \pi) = -\log \text{Tr} \left[ \left( \sum_k \pi_k S_k^{1+s} \right)^{1+s} \right], \]

where \( \{\pi_k\}^n_1 \) is a probability distribution and \( \{S_k\}^n_1 \) is a family of density matrices. He posed a question if \( \mu(s) \) is a concave function. When all density matrices commute, this is the case. In [6], Furuichi et al. showed that it is concave if the inequality

\[
\text{Tr} \left[ \left( \sum_k \pi_k S_k^{1+s} \right) \sum_k \pi_k S_k^{1+s} \left( \log S_k^{1+s} \right)^2 \right] \geq \text{Tr} \left[ \left( \sum_k \pi_k S_k^{1+s} \right)^{s-1} \left( \sum_k \pi_k S_k^{1+s} \log S_k^{1+s} \right)^2 \right]
\]

holds. They discussed the special case \( n = 2 \) with \( \pi_1 = \pi_2 = \frac{1}{2} \). In this special case, with \( A = S_1^{1+s} \) and \( B = S_2^{1+s} \), the inequality under consideration is equivalent to the following.

**Problem 1.** *If \( A \) and \( B \) are positive-definite contractive matrices, then

\[
\text{Tr} \left( A + B \right)^s \{ A (\log A)^2 + B (\log B)^2 \} \geq \text{Tr} \left[ (A + B)^{s-1} (A \log A + B \log B)^2 \right]
\]

holds for any real number \( s \in [0, 1] \).*

We consider this problem by introducing another informational quantity for real-valued (continuous) functions \( f \) and \( g \):

\[
S_{f,g}(A, X) = \text{Tr} f(A)Xg(A)X - \text{Tr} f(A)g(A)X^2
\]

for selfadjoint matrices \( A \) and \( X \). Here we call \( S_{f,g}(A, X) \) a *generalized skew information* since it is an extension of Lieb’s skew information in [8]:

\[
S_s(\rho, X) = \text{Tr} \rho^{1-s}X\rho^sX - \text{Tr} \rho X^2
\]

for positive numbers \( s \in (0, 1) \), a density matrix \( \rho \geq 0 \) and an observable \( X = X^* \). In this note, we show that the above problem is solved by the positivity of \( S_{f,g}(A, X) \) for \( f(x) = x^s \) and \( g(x) = 1/x \).

### 2. Skew information

Lieb’s skew information \( S_s(\rho, X) \) is non-positive by the following classical Hilbert–Schmidt norm inequality:

\[
\|\rho^{\frac{1}{2}} X \rho^{\frac{1}{2}}\|_2 \leq \|\rho^{\frac{1}{2}} X\|_2.
\]
We try to find conditions for $f$ and $g$ so that the generalized skew information should be negative or positive, which will enables us in Section 3 to solve Problem 1. According to Bourin [3,4], $(f,g)$ is called a monotone pair (resp. antimonotone pair) of functions on the domain $D \subset \mathbb{R}$ if

$$(f(a) - f(b))(g(a) - g(b)) \geq 0 \quad \text{(resp. } (f(a) - f(b))(g(b) - g(a)) \geq 0)$$

for any $a, b \in D$. For example, for a positive function $f$ on $D$ and real numbers $r, s, t$ with $t > 0$, put $g(x) = tf(x)^t + s$. Then $(f,g)$ is a monotone (resp. antimonotone) pair if $r \geq 0$ (resp. $r \leq 0$). If $f$ and $g$ are positive, the following theorem was shown in [3,4] (see also [1]). Here we give a direct proof:

**Theorem 2** (Bourin). If $(f,g)$ is a monotone (resp. antimonotone) pair, then the generalized skew information is non-positive, $S_{f,g}(A, X) \leq 0$ (resp. non-negative, $S_{f,g}(A, X) \geq 0$).

**Proof.** Since $(f,g)$ is a monotone pair if and only if $(f,-g)$ is an antimonotone one, it suffices to consider the monotone case, that is,

$$f(a)g(b) + f(b)g(a) \leq f(a)g(a) + f(b)g(b).$$

We may assume that $A$ is diagonal; $A = \text{diag}(t_1, \ldots, t_n)$. Since $x_{ij}x_{ji} = |x_{ij}|^2$ for selfadjoint $X = (x_{ij})$, we have

$$\text{Tr } f(A)Xg(A)X = \sum_{k=1}^{n} f(t_k)g(t_k)|x_{kk}|^2 + \sum_{k<j} (f(t_k)g(t_j) + f(t_j)g(t_k))|x_{kj}|^2 \leq \sum_{k=1}^{n} f(t_k)g(t_k)|x_{kk}|^2 + \sum_{k<j} (f(t_k)g(t_k) + f(t_j)g(t_j))|x_{kj}|^2 = \sum_{k,j=1}^{n} f(t_k)g(t_k)|x_{kj}|^2 = \text{Tr } f(A)g(A)X^2.$$

Thus $S_{f,g}(A, X) \leq 0$. □

The functions $x^{1-s}$ and $x^s$ for $0 < s < 1$ form a monotone pair, which also implies that the original Lieb skew information $S_s(\rho, X)$ is non-positive.

3. Yanagi–Furuichi–Kuriyama inequality

Yanagi et al. [9] solved Problem 1 when the matrices are $2 \times 2$ ones. For general matrices, they solved it only for $s = 0$ and $s = 1$ in [9]. Here we solve it completely as an application of Theorem 2:
Theorem 3. If $A$ and $B$ are $n \times n$ positive-definite matrices with $m \leq A, B \leq M$ for positive numbers $m$ and $M$, then the following inequalities hold for any $s \geq 0$:

$$S_{x^s,1/s}(A + B, A \log A + B \log B) + \frac{(\log \frac{M}{m})^2}{4} \text{Tr}(A + B)^s + 1 \geq \text{Tr}[(A + B)^s(A \log A + B \log B)]$$

$$- \text{Tr}[(A + B)^{s-1}(A \log A + B \log B)^2] \geq S_{x^s,1/s}(A + B, A \log A + B \log B) \geq 0.$$

The inside of the trace in the expression $\text{Tr}[(A + B)^s(A \log A + B \log B)^2]$ may be understood as the following positive-definite matrices;

$$(A + B)^{s/2}(A \log A + B \log B)^{s/2}$$

or

$$\frac{(A \log A)^2 + B \log B}{2} (A + B)^{s/2}(A \log A + B \log B)^{s/2}.$$

Then, note that Problem 1 cannot be extended to operator inequalities even for $2 \times 2$ matrices with $s = 1$ (see [5]).

$$(A + B)^{1/2}(A \log A + B \log B)^{1/2} \neq (A \log A + B \log B),$$

$$(A \log A)^2 + B \log B \geq (A \log A + B \log B)^2.$$

In the original problem of Yanagi, Furuichi and Kuriyama, $A$ and $B$ are assumed to be contractive since they use the Jensen’s operator inequality on the positive interval $[0, \infty)$. To prove Theorem 3, recall the following Jensen’s inequality (e.g., [2]; Theorem V.2.3): If $C^*C + D^*D \leq 1$, then

$$C^*X^2C + D^*Y^2D \geq (C^*XC + D^*YD)^2$$

holds for any Hermitian operators $X$, $Y$ since $f(x) = x^2$ is operator convex on any interval. To estimate the upper bound in Theorem 3, we need the following lemma:

Lemma 4. If $X$ and $Y$ are Hermitian with $\ell \leq X, Y \leq L$ for real numbers $\ell$ and $L$ and if $C^*C + D^*D = 1$, then

$$C^*X^2C + D^*Y^2D \leq (C^*XC + D^*YD)^2 + \frac{(L - \ell)^2}{4}.$$

Proof. For any selfadjoint $Z$ such that $\ell \leq Z \leq L$, we have obviously

$$Z^2 \leq (L + \ell)Z - \ell L \leq Z^2 + \frac{(L - \ell)^2}{4}.$$

Apply the first inequality to $X$ and $Y$ in place of $Z$ to get
Since $\ell \leq C^*XC + D^*YD \leq L$, apply the second inequality to $C^*XC + D^*YD$ in place of $Z$ to get

$$(L + \ell)(C^*XC + D^*YD) - \ell L \leq (C^*XC + D^*YD)^2 + \frac{(L - \ell)^2}{4}.$$ 

Combining those two inequalities we arrive at the inequality in the assertion. $\square$

**Proof of Theorem 3.** Putting $C = A^{1/2}(A + B)^{-1/2}$ and $D = B^{1/2}(A + B)^{-1/2}$, we can apply the above Jensen’s inequality:

$$C^*(\log A)^2 C + D^*(\log B)^2 D \geq [C^*(\log A)C + D^*(\log B)D]^2,$$

and hence

$$(A + B)^{s/2}[A(\log A)^2 + B(\log B)^2](A + B)^{s/2} \geq (A + B)^{(s+1)/2}[C^*(\log A)C + D^*(\log B)D]^2(A + B)^{(s+1)/2}.$$ 

Put $E = A \log A + B \log B$. Then it follows that

$$\text{Tr}[(A + B)^t[A(\log A)^2 + B(\log B)^2]] = \text{Tr}[(A + B)^{t+1/2}[C^*(\log A)C + D^*(\log B)D]^2(A + B)^{(s+1)/2}].$$

Since $(x^t, 1/x)$ is an antimonotone pair of functions, we have by Theorem 2

$$\text{Tr}[(A + B)^t[A(\log A)^2 + B(\log B)^2]] \leq \text{Tr}[(A + B)^{t-1/2}[C^*(\log A)C + D^*(\log B)D] + (\log M/m)^2/4](A + B)^{(s+1)/2}.$$ 

This proves the second inequality of Theorem 3. Applying Lemma 4 for $\ell = \log m$ and $L = \log M$, we also have

$$\text{Tr}[(A + B)^t[A(\log A)^2 + B(\log B)^2]] = \text{Tr}[(A + B)^{t+1/2}[C^*(\log A)^2 C + D^*(\log B)^2 D](A + B)^{t+1/2}] \leq \text{Tr}[(A + B)^{t+1/2}[C^*(\log A)C + D^*(\log B)D]^2 + (\log M/m)^2/4](A + B)^{t+1/2}.$$ 

$$= \text{Tr}[(A + B)^tE(A + B)^{-1}E] + \frac{(\log M/m)^2}{4}\text{Tr}(A + B)^{t+1}.$$
and hence

\[ S_{x,1/x}(A + B, E) + \frac{(\log \frac{M}{m})^2}{4} \text{Tr} (A + B)^{x+1} \]
\[ = \text{Tr} (A + B)^x E (A + B)^{-1} E + \frac{(\log \frac{M}{m})^2}{4} \text{Tr} (A + B)^{x+1} \]
\[ - \text{Tr} [(A + B)^{x-1} E^2] \]
\[ \geq \text{Tr} [(A + B)^x (A \log A)^2 + B (\log B)^2] \]
\[ - \text{Tr} [(A + B)^{x-1} (A \log A + B \log B)^2] \]

This proves the first inequality of Theorem 3. □

The inequality in Problem 1 follows from the second inequality of Theorem 3.

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References