On non abelian tensor analogues of 3-Engel groups

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Abstract

Tensor analogues of n-Engel groups were introduced by Moravec. He described the structure of tensor analogues of 2-Engel groups. In this paper, some new results for the tensor analogues of 3-Engel groups shall be presented.

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1. Introduction

For any group $G$, the nonabelian tensor square $G \otimes G$ is a group generated by the symbols $g \otimes h$, subject to the relations,

$gg' \otimes h = (g^g' \otimes h^{g'}) (g' \otimes h)$ and $g \otimes hh' = (g \otimes h') (g^h' \otimes h^{h'})$,

where $g, g', h, h' \in G$ and $g^h = h^{-1}gh$.

The more general concept of nonabelian tensor product of groups acting on each other in certain compatible way was introduced by Brown and Loday [3]. The set of right $n$-Engel elements of a group $G$ is defined by $R_n(G) = \{a \in G : [a, x] = 1 \text{ for all } x \in G\}$. (Here $[a, x]$ denotes $a^{-1}x^{-1}ax$, and $[a, x, \ldots]$ denotes the left-normed commutator $[[\ldots [a, x], \ldots], x]$.) It is evident that $R_2(G)$ is a subgroup of $G$. In contrast, it has been shown that for $n \geq 3$, the set

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$R_n(G)$ is not necessarily a subgroup [6]. The set of right $n_{\otimes}$-Engel elements of a group can be defined as

$$R_n^{\otimes}(G) = \{a \in G : [a, n_{-1} x] \otimes x = 1_{\otimes} \forall x \in G\}.$$

Biddle and Kappe [1] proved that $R_2^{\otimes}(G)$ is always a characteristic subgroup of $G$ which is contained in $R_2(G)$. A group $G$ is an $n$-Engel group if, $[x, n y] = 1$ for all $x, y \in G$. The group $G$ is said to be $n_{\otimes}$-Engel, when $[x, n_{-1} y] \otimes y = 1_{\otimes}$ for any $x, y \in G$. It is obvious that every $n_{\otimes}$-Engel group is also $n$-Engel. Kappe and Kappe [5] proved that a group $G$ is a 3-Engel group if and only if the normal closure of every element in $G$ is a 2-Engel group. Here, a tensor analogue of this result is obtained. Also, it will be demonstrated that a group $G$ is a 3$\otimes$-Engel group if and only if $(x^G) R_2^{\otimes}(G)$ is a 2$\otimes$-Engel group for all $x \in G$. For $x \in G$, $x^G$ denotes the conjugacy class of $x$ in $G$.

Let $\beta$ be the group-theoretic property. A group $G$ is said to have a finite covering by $\beta$-subgroups, if $G$ is a union of finite family of $\beta$-subgroups. The finite coverings of groups by their 2-Engel subgroups have been studied by Kappe [4]. She has shown that a group $G$ has a finite covering by 2-Engel subgroups if and only if $|G : R_2(G)| < \infty$. There is a similar approach in the context of 2$\otimes$-Engel groups. Moravec [7] proved that a group $G$ can be covered by a finite family of 2$\otimes$-Engel subgroups if and only if $|G : R_2^{\otimes}(G)| < \infty$. Kappe also showed that $G$ has a finite covering by 2-Engel normal subgroups if and only if $G$ is a 3-Engel and $|G : R_2(G)| < \infty$. We show that a tensor analogue of this result also exists by proving that a group $G$ has a finite covering by 2$\otimes$-Engel normal subgroups if and only if $G$ is a 3$\otimes$-Engel and $|G : R_2^{\otimes}(G)| < \infty$.

2. Preliminary results

This section contains basic results on nonabelian tensor squares and the results concerning 2$\otimes$-Engel groups which will be used throughout the paper.

**Lemma 2.1** ([2]). Let $g, g', h, h' \in G$. The following relations hold in $G \otimes G$:

(a) $(g^{-1} \otimes h)^{g} = (g \otimes h)^{-1} = (g \otimes h^{-1})^h$.

(b) $(g' \otimes h')^{g \otimes h} = (g' \otimes h')^{[g, h]}$.

(c) $[g, h] \otimes g' = (g \otimes h)^{-1} (g \otimes h)^{g'}$.

(d) $g' \otimes [g, h] = (g \otimes h)^{-g'} (g \otimes h)$.

(e) $[g, h] \otimes [g', h'] = [g \otimes h, g' \otimes h']$.

For a group $G$ and a nonempty subset $X$, we define the tensor annihilator of $X$ in $G$ to be the set $C_G^{\otimes}(X) = \{a \in G : a \otimes x = 1_{\otimes} \forall x \in X\}$.

**Proposition 2.2** ([7]). Let $G$ be a group, $x, y, z \in G$ and $a \in R_2^{\otimes}(G)$.

(a) $[a, x] \otimes y = ([a, y] \otimes x)^{-1}$.

(b) $[a, x] \in C_G^{\otimes}(x^G)$.

(c) $[a, x] \otimes [y, z] = 1_{\otimes}$.

(d) $[x, a] \otimes a = 1_{\otimes}$.

(e) $[x, y] \otimes a = ([x, a] \otimes y)^2$.

**Corollary 2.3.** Let $G$ be a group, $x, y, z, t \in G$ and $a \in R_2^{\otimes}(G)$.

(a) $[x, a, y] \otimes [z, t] = 1_{\otimes}$.

(b) $[x, y, a] \otimes z = 1_{\otimes}$. 
Proof. We have \([x, a] \in R_2^\otimes(G)\), since \(R_2^\otimes(G)\) is always a characteristic subgroup of \(G\). Hence, (a) follows from part (c) of Proposition 2.2. To prove (b) from part (a) of Lemma 2.1, we have \([x, y, a] \otimes z = ([a, [x, y]] \otimes z)^{-[x, y, a]}\). Now, (b) follows by parts (a) and (c) of Proposition 2.2. \(\square\)

Theorem 2.4 ([7]). A group \(G\) has a finite covering by \(2_\otimes\)-Engel subgroups if and only if \(|G : R_2^\otimes(G)| < \infty\).

3. Main results

In this section, we will show tensor analogues of results regarding 3-Engel groups [4,5].

Theorem 3.1. A group \(G\) is a \(3_\otimes\)-Engel group if and only if the normal closure of every element in \(G\) is a \(2_\otimes\)-Engel group.

Proof. Let \(x\) and \(y\) be arbitrary elements of \(G\). Then we have, \([y, x, x] \otimes x = 1_\otimes \iff [y, x^2, x] \otimes x = 1_\otimes \iff [y, x^{-1}, y] \otimes x = 1_\otimes \iff [x^{-y}, x] \otimes x = 1_\otimes \iff [x^y, x^{-1}] \otimes x^{-1} = 1_\otimes \iff [x^y, x] \otimes x = 1_\otimes\), as required. \(\square\)

Corollary 3.2. Let \(G\) be a \(3_\otimes\)-Engel group and \(a, b, g, h \in G\) then, \(\langle (a \otimes b), (a \otimes b)^{g \otimes h} \rangle\) is nilpotent of class at most 2.

Proof. From parts (b) and (e) in Lemma 2.1 and using Theorem 3.1, we have \(\langle (a \otimes b)^{g \otimes h}, a \otimes b, a \otimes b \rangle = \langle [a, b]^{[g, h]}, [a, b] \rangle \otimes [a, b] = 1_\otimes\). Again, using parts (b) and (e) in Lemma 2.1 and considering Theorem 3.1, we have,

\[
[(a \otimes b)^{g \otimes h}, a \otimes b, (a \otimes b)^{g \otimes h}] = [[a, b]^{[g, h]}, [a, b]] \otimes [a, b]^{[g, h]}
\]

\[
= [[a, b], [a, b]^{[g, h]-1}] \otimes [a, b]^{[g, h]}
\]

\[
= ([a, b], [a, b]^{[g, h]-1}) \otimes [a, b]^{[g, h]-1}
\]

\[
= 1_\otimes.
\]

Hence \(\langle (a \otimes b)^{g \otimes h}, a \otimes b \rangle \in Z((a \otimes b, (a \otimes b)^{g \otimes h}))\), as required. \(\square\)

Theorem 3.3. A group \(G\) is a \(3_\otimes\)-Engel group if and only if \(\langle x^G \rangle R_2^\otimes(G)\) is a \(2_\otimes\)-Engel group for all \(x \in G\).

Proof. It is sufficient to show that for any \(3_\otimes\)-Engel group \(G\), the normal subgroup of \(\langle x^G \rangle R_2^\otimes(G)\) is a \(2_\otimes\)-Engel group of \(G\) for all \(x \in G\), i.e. \([ua, vb] \otimes vb = 1_\otimes\) for all \(u, v \in \langle x^G \rangle\) and all \(a, b \in R_2^\otimes(G)\). By Theorem 3.1, the reverse direction is obvious. Using Proposition 2.2, we conclude that \(R_2^\otimes(G) = \{a \in G; [ax, y] \otimes y = [x, y] \otimes y \text{ for all } x, y \in G\}\). Since \(a \in R_2^\otimes(G), [ua, vb] \otimes vb = [u, vb] \otimes vb\), as \(ua^{-1}u^{-1} \in R_2^\otimes(G)\). Expanding \([u, vb] \otimes vb\), we obtain

\[
[u, vb] \otimes vb = [u, b][u, v]^b \otimes vb
\]

\[
= ([u, b] \otimes vb)[u, v]^b ([u, v]^b \otimes vb)
\]

\[
= ([u, b] \otimes b)([u, v]^b \otimes v)^b ([u, v]^b \otimes b)([u, v]^b \otimes v)^b
\]

\[
= ([u, v]^b \otimes b)([u, v]^b \otimes v)^b \text{ (by (d) and (b) Proposition 2.2)}
\]
\[ = ([u, v] \otimes b)^b([u, v][u, v, b] \otimes v)^b \]
\[ = ([u, v][u, v, b] \otimes v)^b \quad \text{(by (e) and (b) Proposition 2.2)} \]
\[ = ([u, v] \otimes v)^b[u, v, b]^b([u, v, b] \otimes v)^b \]
\[ = ([u, v, b] \otimes v)^b \quad \text{(by Theorem 3.1)} \]
\[ = 1_\otimes \quad \text{(by (b) Corollary 2.3).} \]

Kappe [4] proved that a group has a finite covering by 2-Engel subgroup if and only if \(|G : R_2(G)| < \infty\). Our proof of the tensor analogue follows the lines of Kappe’s proof.

**Theorem 3.4.** A group \(G\) has a finite covering by \(2_\otimes\)-Engel normal subgroups if and only if \(G\) is a \(3_\otimes\)-Engel and \(|G : R_2^\otimes(G)| < \infty\).

**Proof.** Let \(G = \bigcup_{i=1}^n H_i\), where \(H_i\) are normal \(2_\otimes\)-Engel subgroups of \(G\). It follows immediately by Theorem 2.4 that \(G/R_2^\otimes(G)\) is finite. For every \(x \in G\), there exists an \(H_i\) in the normal covering of \(G\) with \(\langle x^G \rangle \subseteq H_i\). Hence, every normal closure \(\langle x^G \rangle\) is \(2_\otimes\)-Engel, as it is a subgroup of a \(2_\otimes\)-Engel group \(H_i\). Now by Theorem 3.1, \(G\) is a \(3_\otimes\)-Engel group. Conversely, assume that \(G\) is a \(3_\otimes\)-Engel group and that \(G/R_2^\otimes(G)\) is finite. Let \(T = \{g_1, \ldots, g_n\}\) be a transversal of \(R_2^\otimes(G)\) in \(G\). Since each \(g \in G\) can be written as \(g_iw\) for some \(w \in R_2^\otimes(G)\) and some \(g_i \in T\), we obtain that \(G = \bigcup_{i=1}^n \langle g_i^G \rangle R_2^\otimes(G)\). By Theorem 3.3, \(\langle x^G \rangle R_2^\otimes(G)\) is a normal \(2_\otimes\)-Engel subgroup for every \(x \in G\). Hence, \(G\) has a covering by finitely many normal \(2_\otimes\)-Engel subgroups. \(\square\)

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**References**