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Eigentime identity for transient Markov chains ☆

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Abstract

An eigentime identity is proved for transient symmetrizable Markov chains. For general Markov chains, if the trace of Green matrix is finite, then the expectation of first leap time is uniformly bounded, both of which are proved to be equivalent for single birth processes. For birth–death processes, the explicit formulas are presented. As an application, we give the bounds of exponential convergence rates of (sub-) Markov semigroup P_t from l_∞ to l_∞.

Keywords: Markov chain; Eigenvalue; Essential spectrum; Eigentime identity; Green matrix

1. Introduction and main results

Let Q-matrix $Q = (q_{ij}\colon\ i, j \in E)$ be conservative, totally stable and irreducible on a countably infinite state spaces E. Let X_t, $t \geqslant 0$, be the corresponding continuous-time Markov chain with the minimal Q-function $P(t) = (p_{ij}(t)\colon\ i, j \in E)$ for its transition function. See [2,3] for more details.

Let ξ_n be the successive jumps, that is,

$$\xi_0 = 0, \qquad \xi_n = \inf\{t\colon\ t > \xi_{n-1},\ X_t \neq X_{\xi_{n-1}}\}, \quad n \geqslant 1,$$

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and the first leap time $\xi = \lim_{n\to\infty} \xi_n$. From [2, Chapter 2], we know that

$$p_{ij}(t) = \mathbb{P}_i[X_t = j,\ t < \xi]. \tag{1.1}$$

The process is assumed to be transient, that is, the Green matrix $G = (g_{ij}\colon\ i, j \in E)$ satisfies

$$g_{ij} := \int_0^\infty p_{ij}(t)\,dt < \infty, \quad \text{for } i, j \in E,$$

so that $\lim_{t\to\infty} p_{ij}(t) = 0$.

In this paper we will study the relationship among the trace of Green matrix G, the expectation of the first leap time and eigenvalues of the corresponding Markov generator. These results extend the so-called eigentime identity for ergodic Markov chains to the present setting. The explicit formulas are given for single birth processes (or upward skip-free process) and especially for birth–death processes.

We remark that the term of "eigentime identity" comes from Aldous and Fill [1], which is initially proved for finite Markov chains in a form different from (1.3). For an irreducible finite Markov chain, discrete or continuous time, it will always be ergodic. In [7], this identity was extended to continuous-time ergodic Markov chains on infinitely countable state space.

First of all, we have

Theorem 1.1. *Let* $\mathrm{tr}(G) := \sum_{j\in E} g_{jj}$, *then* $\sup_{i\in E} \mathbb{E}_i\xi \leqslant \mathrm{tr}(G)$. *Therefore the process is explosive if* $\mathrm{tr}(G) < \infty$.

In general the converse assertion of Theorem 1.1 is not true. See the paragraph just following Proposition 1.7. However for the single birth process (or upward skip-free process), this converse assertion is still true. For the precise definition of a single birth process, see Section 3.

Theorem 1.2. *For a single birth process, we have* $\mathrm{tr}(G) = \sup_{i\in E} \mathbb{E}_i\xi$.

In what follows we will focus on the symmetrizable processes. Suppose that Q be symmetric with respect to a measure $\mu = (\mu_i > 0,\ i \in E)$, that is, $\mu_i q_{ij} = \mu_j q_{ij}$ for any $i, j \in E$, whose total mass is infinite ($\sum_{i\in E} \mu_i = \infty$). Let L be the self-adjoint operator in $L^2(\mu)$ associated with $Q = (q_{ij})$ and $(P_t,\ t \geqslant 0)$ be the Markov semi-group with the Markov generator L. Denote by $\sigma(L)$ and $\sigma_{\mathrm{ess}}(L)$ the spectrum and essential spectrum of $-L$ in $L^2(\mu)$, respectively. The essential spectrum consists of continuous spectrum and eigenvalues with infinite multiplicity. Since E is infinite, $\sigma_{\mathrm{ess}}(L)$ may be non-empty. When $\sigma_{\mathrm{ess}}(L) = \emptyset$, denote by $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ all the eigenvalues of $-L$, counting multiplicity. Actually, we will prove

Theorem 1.3. *Assume that the process is symmetrizable. If* $\mathrm{tr}(G) < \infty$, *then* P_t *is a Hilbert–Schmidt operator for any* $t > 0$. *Therefore* $\sigma_{\mathrm{ess}}(L) = \emptyset$ *and* $\sum_{n\geqslant 0} e^{-\lambda_n t} < \infty$ *for any* $t > 0$.

Since $\mu(E)=\infty$, we have $\lambda_1>0$ when $\sigma_{\text{ess}}(L)=\emptyset$. Now let $f^{(n)}$ be an eigenfunction corresponding to λ_n such that $\{f^{(n)}\colon n\geqslant 1\}$ is an orthonormal sequence in $L^2(\mu)$. By [3, Theorem 6.7], P_t is (weakly) symmetric with respect to μ, then by Kendall's representation (cf. [2, Section 1.6]), we have

$$p_{ij}(t)=\mu_j\sum_{n\geqslant 1}e^{-\lambda_n t}f_i^{(n)}f_j^{(n)}. \tag{1.2}$$

The proof of (1.2) is given in Section 2.

Now we will investigate the probability meaning of $\operatorname{tr}(G)$ for Q-process X_t. Let $\tau_j^+=\inf\{t\colon \xi_1<t<\xi,\ X_t=j\}$ be the (first) return time to j after finite times jumps, with the convention $\inf\emptyset=\infty$. We have the following eigentime identity.

Theorem 1.4. *Assume that the process is symmetrizable and $\sigma_{\text{ess}}(L)=\emptyset$, then*

$$\sum_{n\geqslant 1}\lambda_n^{-1}=\operatorname{tr}(G)=\sum_{j\in E}\frac{1}{q_j\mathbb{P}_j[\tau_j^+=\infty]}. \tag{1.3}$$

In particular, if $\operatorname{tr}(G)<\infty$, *then $\lambda_n^{-1}=o(n^{-1})$ as $n\to\infty$.*

Next, we give the explicit eigentime formula for birth–death processes, the unique type of single birth processes which is symmetrizable. The birth–death q-matrix $Q=(q_{ij})$ on $\mathbb{Z}_+$ is defined by: $q_{i,i+1}=b_i$ $(i\geqslant 0)$, $q_{i,i-1}=a_i$ $(i\geqslant 1)$ and $q_{ij}=0$ for all $|i-j|\geqslant 2$. Define the potential coefficients to be $\mu_0=1$, $\mu_i=b_0b_1\cdots b_{i-1}/a_1a_2\cdots a_i$ $(i\geqslant 1)$. Set

$$R=\sum_{i=0}^{\infty}\frac{1}{\mu_i b_i}\sum_{j=0}^{i}\mu_j=\sum_{i=0}^{\infty}\mu_i\sum_{j=i}^{\infty}\frac{1}{\mu_j b_j}.$$

Let $(X_t,\ t\geqslant 0)$ be the minimal birth–death process for Q, according to [2, Chapter 8], R is the expectation of first passage time of X_t from 0 to ∞ and also $R=\sup_i\mathbb{E}_i\xi=\mathbb{E}_0\xi$. When $R<\infty$, the corresponding q-process is not unique, for details see [2, Chapter 8] or [3, Chapter 4]. This is the reason why we consider here only the minimal birth–death process. Actually, R is just the trace of Green matrix and as a consequence of Theorems 1.2 and 1.4, we have

Corollary 1.5. *Let $P(t)=(p_{ij}(t))$ be the transition probability matrix of minimal birth–death process $(X_t,t\geqslant 0)$ for Q. Assume that $\sum_{i\in E}\mu_i=\infty$ and $R<\infty$, then the assertions in Theorem* 1.4 *hold and the eigentime formula reads*:

$$\sum_{n\geqslant 1}\lambda_n^{-1}=R. \tag{1.4}$$

Example 1.6. Let $a_i=i^\gamma$ for $i\geqslant 1$ and $b_i=(i+1)^\gamma$ for $i\geqslant 0$, then $\mu_i=1$ and so that $\mu=\infty$ and for $\gamma>2$,

$$R=\sum_{i=0}^{\infty}(i+1)^{1-\gamma}<\infty.$$

Thus it holds that

$$\sum_{n\geqslant 1}\lambda_n^{-1}=\sum_{i=0}^{\infty}(i+1)^{1-\gamma}.$$

Theorem 1.4 enables us to provide a counterexample to the converse of Theorem 1.1 based on Example 1.6.

Proposition 1.7. *Assume $\gamma>2$. Let X_t be the associated minimal process with the Markov generator L as in Example 1.6. Then there exists a constant $C<\infty$ such that*

$$\lambda_n\leqslant Cn^{\gamma},\quad n\geqslant 1. \tag{1.5}$$

Assume $2<\gamma\leqslant 3$. Now let X_t^i, $i=1,2,3$, be three independent copies of X_t, then $\tilde{X}_t=(X_t^1,X_t^2,X_t^3)$ has the Markov generator $\tilde{L}=L\otimes I\otimes I+I\otimes L\otimes I+I\otimes I\otimes L$ with $\sigma(\tilde{L})=\{\lambda_n+\lambda_m+\lambda_l\colon n,m,l\geqslant 1\}$, so by Theorem 1.4 and (1.5) we have

$$\begin{aligned}\operatorname{tr}(\tilde{G})&=\sum_{n\geqslant 1}\sum_{m\geqslant 1}\sum_{l\geqslant 1}\frac{1}{\lambda_n+\lambda_m+\lambda_l}\geqslant C_1\sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{1}{(n+m)^{\gamma-1}}\\&\geqslant C_2\sum_{n\geqslant 1}\frac{1}{(n+1)^{\gamma-2}}=\infty.\end{aligned}$$

Let $\tilde{\xi}_n,\xi_n^1$ be the successive jumps for the corresponding processes $\tilde{X}_t$ and X_t^1, respectively, then obviously $\tilde{\xi}_n\leqslant\xi_n^1$, so that $\tilde{\xi}:=\lim_{n\to\infty}\tilde{\xi}_n\leqslant\xi^1$. Thus it follows from Theorems 1.1 and 1.4 that $\sup_{i_1,i_2}\mathbb{E}_{i_1,i_2}\tilde{\xi}\leqslant\sup_{i_1,i_2}\mathbb{E}_{i_1,i_2}\xi^1=\sup_{i_1}\mathbb{E}_{i_1}\xi^1\leqslant\operatorname{tr}(G^1)=\sum_{n\geqslant 1}1/n^{\gamma-1}<\infty$.

Before moving to the detailed proofs of the results, we would like to give an application to the uniform decay of $p_{ij}(t)$.

Let l_∞ be Banach space of bounded functions on E with the sup-norm $\|f\|_\infty=\sup_{i\in E}|f_i|$. The transition matrix $P_t=(p_{ij}(t))$ gives rise to a bounded linear operator from l_∞ to l_∞, with operator norm $\|P_t\|_{\infty\to\infty}=\sup_{i\in E}\sum_{j\in E}p_{ij}(t)$. Cf. [2, Chapter 1]. We will study the convergence of $\|P_t\|_{\infty\to\infty}$ as $t\to\infty$. Since $\|P_{t+s}\|_{\infty\to\infty}\leqslant\|P_t\|_{\infty\to\infty}\|P_s\|_{\infty\to\infty}$, we have $\lim_{t\to\infty}\|P_t\|_{\infty\to\infty}=0$ if and only if there exist $\epsilon>0$ and $C<\infty$ such that $\|P_t\|_{\infty\to\infty}\leqslant Ce^{-\epsilon t}$. Thus we define

$$\beta=\sup\bigl\{\epsilon>0\colon \exists C<\infty \text{ such that } \forall t\geqslant 0,\ \|P_t\|_{\infty\to\infty}\leqslant Ce^{-\epsilon t}\bigr\} \tag{1.6}$$

to be the (exponential) uniform decay rate. We have

Theorem 1.8.

(1) *In general, we have $\beta\geqslant(\sup_{i\in E}\mathbb{E}_i\xi)^{-1}$.*
(2) *If in addition the process is symmetrizable and $\sigma_{\mathrm{ess}}(L)=\emptyset$, then $(\sum_{n\geqslant 1}\lambda_n^{-1})^{-1}\leqslant\beta\leqslant\lambda_1$.*

Corollary 1.9. *For birth–death processes, there exists $C < \infty$ such that $R^{-1} \leqslant \beta \leqslant CR^{-1}$. Furthermore, if $R < \infty$, then $(\sum_{n\geqslant 1}\lambda_n^{-1})^{-1} \leqslant \beta \leqslant C(\sum_{n\geqslant 1}\lambda_n^{-1})^{-1}$.*

2. Proofs of Theorems 1.1, 1.3 and 1.4

Proof of Theorem 1.1. Recall that $\tau_j^+ = \inf\{t\colon \xi_1 < t < \xi,\ X_t = j\}$ is the return time of j after finite times jumps, so that $[\tau_j^+ = \infty] = [\tau_j^+ \geqslant \xi]$. Let $F_{ij}(t) = \mathbb{P}_i[\tau_j^+ \leqslant t]$ be the distribution function of τ_j^+, and define the Laplace transforms

$$P_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)\,dt, \qquad \mathscr{F}_{ij}(\lambda) = \int_0^\infty e^{-\lambda t}\,dF_{ij}(t), \quad \lambda > 0.$$

It follows from Kolmogorov's backward equation that

$$p_{ij}(t) = \delta_{ij}e^{-q_i t} + \int_0^t p_{jj}(t-s)\,dF_{ij}(s). \tag{2.1}$$

Taking Laplace transforms in (2.1), we have

$$P_{jj}(\lambda) = \frac{1}{(\lambda + q_j)(1 - \mathscr{F}_{jj}(\lambda))} \quad \text{and} \quad \text{for } i \neq j, \quad P_{ij}(\lambda) = \mathscr{F}_{ij}(\lambda)P_{jj}(\lambda).$$

By letting $\lambda \to 0$ and noting that $1 - \mathscr{F}_{ij}(0) = \mathbb{P}_i[\tau_j^+ = \infty] = \mathbb{P}_i[\tau_j^+ \geqslant \xi]$, we get

$$g_{jj} = \frac{1}{q_j \mathbb{P}_j[\tau_j^+ = \infty]} \quad \text{and} \quad \text{for } i \neq j, \quad g_{ij} = g_{jj}\mathbb{P}_i[\tau_j^+ < \infty]. \tag{2.2}$$

Since $\mathbb{P}_i[\tau_j^+ < \infty] \leqslant 1$, we have for any $i \in E$,

$$\sum_{j\in E} g_{ij} \leqslant \sum_{j\in E} g_{jj} = \operatorname{tr}(G),$$

hence $\sup_i \sum_j g_{ij} \leqslant \operatorname{tr}(G) < \infty$ by assumption.

On the other hand, by (1.1) we have

$$\sum_{j\in E} p_{ij}(t) = \sum_{j\in E} \mathbb{P}_i[X_t = j,\ \xi > t] = \mathbb{P}_i[\xi > t].$$

Thus

$$\sum_{j\in E} g_{ij} = \int_0^\infty \mathbb{P}_i[\xi > t]\,dt = \mathbb{E}_i\xi.$$

Therefore, $\sup_{i\in E} \mathbb{E}_i\xi \leqslant \operatorname{tr}(G) < \infty$. $\square$

To prove Theorem 1.3, we need the following lemma.

Lemma 2.1. *If* $\operatorname{tr}(G) < \infty$, *then for any* $t > 0$, $\phi(t) := \sum_{j \in E} p_{jj}(t) < \infty$.

Proof. Note that

$$\operatorname{tr}(G) = \int_0^\infty \phi(t)\,dt < \infty,$$

then $\phi(t) < \infty$ for a.e. $t \in (0, \infty)$. Since for any $j \in E$, $p_{jj}(t)$ is decreasing in $t \in (0, \infty)$ by Kendall's representation (cf. [2]), so is $\phi(t)$, then it follows that $\phi(t) < \infty$ for any $t \in (0, \infty)$. □

Proof of Theorem 1.3. Fix $t > 0$, set $k_{ij} = p_{ij}(t)/\mu_j$. To prove that P_t is a Hilbert–Schmidt operator, by [6, Proposition 1.b.15] we need only to show that $(k_{ij}) \in L^2(\mu \times \mu)$. In fact,

$$\begin{aligned}\sum_{i,j \in E} k_{ij}^2 \mu_i \mu_j &= \sum_{i,j \in E} \mu_i p_{ij}(t)^2/\mu_j = \sum_{ij} \mu_j p_{ji}(t) p_{ij}(t)/\mu_j \\ &= \sum_{ij} p_{ji}(t) p_{ij}(t) \leqslant \sum_j p_{jj}(2t) < \infty\end{aligned}$$

by Lemma 2.1. The other assertions will follow easily (cf. [5, Chapter 4]). □

Before going to prove Theorem 1.4, we prove (1.2) first. Let $(\cdot,\cdot)$ be the inner product in $L^2(\mu)$. Since P_t is self-adjoint in $L^2(\mu)$ for any $t \geqslant 0$, then by spectral theorem (cf. [10]), there exists a family of projections $E_n = (\cdot\,, f^{(n)}) f^{(n)}$ such that

$$P_t = \sum_{n \geqslant 1} e^{-\lambda_n t} E_n.$$

Let e^j be defined by $e_i^j = \delta_{ij}$, we have

$$\mu_i p_{ij}(t) = \left(P_t e^j, e^i\right) = \sum_k \sum_n \mu_k e^{-\lambda_n t} \left(E_n e^j\right)_k e_k^i = \mu_i \sum_n e^{-\lambda_n t} \left(E_n e^j\right)_i. \tag{2.3}$$

Since $E_n e^j = (e^j, f^{(n)}) f^{(n)} = \sum_k \mu_k e_k^j f_k^{(n)} f^{(n)} = \mu_j f_j^{(n)} f^{(n)}$, then (1.2) follows from (2.3).

Proof of Theorem 1.4. Since $\sigma_{\mathrm{ess}}(L) = \emptyset$, it follows from (1.2) that

$$g_{jj} = \int_0^\infty \sum_{n \geqslant 1} \mu_j e^{-\lambda_n t} \left[f_j^{(n)}\right]^2 dt = \mu_j \sum_{n \geqslant 1} \lambda_n^{-1} \left[f_j^{(n)}\right]^2,$$

thus

$$\operatorname{tr}(G) = \sum_j g_{jj} = \sum_j \mu_j \left[f_j^{(n)}\right]^2 \sum_{n \geqslant 1} \lambda_n^{-1} = \sum_{n \geqslant 1} \lambda_n^{-1}.$$

The second equality in (1.3) follows from (2.2) directly. □

3. Single birth processes

In this section, we will prove Theorems 1.2 and 1.5 concerning about single birth and birth–death processes. For these purposes, we shall compute out the trace of Green matrix $\mathrm{tr}(G)$ for these processes. By Theorem 1.4, we need compute out $P_j[\tau_j^+ = \infty]$ for any $j \geqslant 0$.

Recall that the Q-matrix $Q = (q_{ij}\colon\ i, j \in \mathbb{Z}_+)$ of a single birth process is defined by: $q_{i,i+1} > 0$, $q_{i,i+j} = 0$ for all $i \in \mathbb{Z}_+$ and $j \geqslant 2$. Assume that Q is totally stable and conservative: $q_i = -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$ for all $i \in \mathbb{Z}_+$. Define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ for $0 \leqslant k < n$ $(k, n \in \mathbb{Z}_+)$ and

$$m_n = \sum_{k=0}^{n} F_n^{(k)}/q_{k,k+1}, \quad n \geqslant 0,$$

$$F_n^{(n)} = 1, \qquad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leqslant i < n.$$

Then

$$R := \sum_{n=0}^{\infty} m_n = \sum_{k \geqslant 0} \frac{1}{q_{k,k+1}} \sum_{n \geqslant k} F_n^{(k)}. \tag{3.1}$$

Especially, if in addition $q_{ij} = 0$ for $j \leqslant i - 2$, then the corresponding single birth process is called a birth–death process. For the birth–death process, we have

$$R = \sum_{i=0}^{\infty} \frac{1}{\mu_i b_i} \sum_{j=0}^{i} \mu_j = \sum_{i=0}^{\infty} \mu_i \sum_{j=i}^{\infty} \frac{1}{\mu_j b_j}.$$

Proof of Theorem 1.2. Let $\tau = \lim_{i \to \infty} \tau_i^+$ be the first time arriving ∞, then $\tau = \xi$, a.e. and by the property of single birth, $\sup_{i \geqslant 0} \mathbb{E}_i \xi = \sup_{i \geqslant 0} \mathbb{E}_i \tau = \mathbb{E}_0 \tau = R$. See [11, Theorem 1.1] and [9, §6.3].

From [11, Theorem 1.2], we have

$$\mathbb{P}_j\big[\tau_j^+ = \infty\big] = 1 - \mathbb{P}_j\big[\tau_j^+ < \xi\big] = \frac{q_{j,j+1}}{q_j \sum_{k \geqslant j} F_k^{(j)}}.$$

It follows from (2.2) and (3.1) that

$$\mathrm{tr}(G) = \sum_{j \geqslant 0} \frac{1}{q_j \mathbb{P}_j[\tau_j = \infty]} = \sum_{j \geqslant 0} \frac{1}{q_{j,j+1}} \sum_{k \geqslant j} F_k^{(j)} = R. \qquad \square$$

Proof of Proposition 1.7. To prove (1.5), we need the following classical result due to the max–min principle (cf. [8, Proposition 5.1]). Let $(D, \mathcal{D}(D))$ be the Dirichlet form associated with the Markov generator L. Suppose $\sigma_{\mathrm{ess}}(L) = \emptyset$. For any $n \geqslant 1$, if there is $\{g^{(k)}\}_{k=0}^{n} \subset \mathcal{D}(D)$ with $\mu(|g^{(k)}|^2) = 1$ and $\mu(g^{(k)} g^{(l)}) = D(g^{(k)}, g^{(l)}) = 0$ for $k \neq l$, then

$$\lambda_n \leqslant \max\big\{D\big(g^{(k)}, g^{(k)}\big)\colon\ 1 \leqslant k \leqslant n\big\}. \tag{3.2}$$

Since $\gamma > 2$, we have $\sigma_{\mathrm{ess}}(L) = \emptyset$ by Theorem 1.3. Let e^j be defined by $e_i^j = \delta_{ij}$ and set $g^{(k)} = e^{2k}$. Noting that $\mu_i = 1$, it is easy to check that $\mu(|g^{(k)}|^2) = 1$ and $\mu(g^{(k)} g^{(l)}) = D(g^{(k)}, g^{(l)}) = 0$ for $k \neq l$. For $k \leqslant n$, we have

$$D\big(g^{(k)}, g^{(k)}\big) = \sum_{i \geqslant 1} \mu_i a_i \big(g_i^{(k)} - g_{i-1}^{(k)}\big)^2 = a_{2k} + a_{2k+1} \leqslant 2(2k+1)^\gamma.$$

Hence $\lambda_n \leqslant C n^\gamma$ for some $C < \infty$. $\square$

4. Uniform decay

To prove Theorem 1.8, we need the following lemma.

Lemma 4.1. *Let $M = \sup_{i \in E} \mathbb{E}_i \xi$, then for any $n \geqslant 0$, $\sup_{i \in E} \mathbb{E}_i \xi^n \leqslant n! M^n$. Therefore $\sup_{i \in E} \mathbb{E}_i e^{\lambda \xi} \leqslant (1 - \lambda M)^{-1}$ for $\lambda < 1/M$.*

Proof. Since $\mathbb{P}_i[\xi > t] = \sum_{j \in E} p_{ij}(t)$ and $p_{ij}(t)$ is the minimal non-negative solution for backward integral equation

$$p_{ij}(t) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i s} \sum_{k \neq i} q_{ik} p_{kj}(t-s)\, ds,$$

by (1.1) we have

$$\mathbb{P}_i[\xi > t] = e^{-q_i t} + \int_0^t e^{-q_i s} \sum_{k \neq i} q_{ik} \mathbb{P}_k[\xi > t-s]\, ds.$$

Then

$$\begin{aligned}
\frac{\mathbb{E}_i \xi^{n+1}}{n+1} &= \int_0^\infty t^n \mathbb{P}_i[\xi > t]\, dt = \frac{n!}{q_i^{n+1}} + \int_0^\infty t^n\, dt \sum_{k \neq i} \int_0^t \mathbb{P}_k[\xi > t-s] e^{-q_i s} q_{ik}\, ds \\
&= \frac{n!}{q_i^{n+1}} + \sum_{k \neq i} \int_0^\infty e^{-q_i s}\, ds \int_0^\infty (s+u)^n \mathbb{P}_k[\xi > u] q_{ik}\, du \\
&= \frac{n!}{q_i^{n+1}} + \sum_{k \neq i} q_{ik} \sum_{m=0}^n \binom{n}{m} \frac{(n-m)!}{q_i^{n-m+1}} \frac{\mathbb{E}_k \xi^{m+1}}{m+1} \\
&= \frac{n!}{q_i^{n+1}} + \sum_{k \neq i} q_{ik} \sum_{m=0}^n \frac{n!}{(m+1)!} \frac{\mathbb{E}_k \xi^{m+1}}{q_i^{n-m+1}},
\end{aligned}$$

that is,

$$\frac{q_i^{n+1}\mathbb{E}_i\xi^{n+1}}{(n+1)!} = 1 + \sum_{m=0}^{n-1}\sum_{k\neq i} q_{ik}\frac{q_i^m\mathbb{E}_i\xi^{m+1}}{(m+1)!} + \sum_{k\neq i} q_{ik}\frac{q_i^n\mathbb{E}_i\xi^{n+1}}{(n+1)!}$$
$$= \frac{q_i^n\mathbb{E}_i\xi^n}{n!} + \sum_{k\neq i} q_{ik}\frac{q_i^n\mathbb{E}_i\xi^{n+1}}{(n+1)!}.$$

Since $p_{ij}(t)$ is minimal, it follows that $(\mathbb{E}_i\xi^n, i \in E)$ is the minimal non-negative solution of

$$x_i = \frac{1}{q_i}\sum_{k\neq i} q_{ik}x_k + \frac{n}{q_i}\mathbb{E}_i\xi^{n-1}, \quad i \in E,\ n \geqslant 1. \tag{4.1}$$

When $n = 1$, set $\mathbb{E}_i\xi^0 = 1$. Since $\mathbb{E}_i\xi \leqslant M$, we assume by induction that $\mathbb{E}_i\xi^{n-1} \leqslant (n-1)!M^{n-1}$. Let $(x_i^*,\ i \in E)$ be the minimal non-negative solution of

$$x_i = \frac{1}{q_i}\sum_{k\neq i} q_{ik}x_k + \frac{n!M^{n-1}}{q_i}.$$

When comparing with (4.1) for $n = 1$, it follows from [4, Corollary 3.3.3] that $x_i^* = n!M^{n-1}\mathbb{E}_i\xi \leqslant n!M^n$. But by the comparison theorem (cf. [3, Chapter 2]), we get $\mathbb{E}_i\xi^n \leqslant x_i^* \leqslant n!M^n$. $\square$

Proof of Theorem 1.8. (1) From the proof of Theorem 1.1, we know that $\|P_t\|_{\infty\to\infty} = \sup_{i\in E}\mathbb{P}_i[\xi > t]$. Then it follows from Lemma 4.1 that for any $\lambda < 1/M$,

$$\|P_t\|_{\infty\to\infty} \leqslant \frac{e^{-\lambda t}}{1-\lambda M},$$

hence $\beta \geqslant \lambda$ for any $\lambda < 1/M$, so that $\beta \geqslant 1/M$.

(2) From (1) and Theorems 1.1 and 1.4, it follows that $\beta \geqslant (\sum_{n\geqslant 1}\lambda_n^{-1})^{-1}$. We will prove $\beta \leqslant \lambda_1$. First note that by symmetry $(\mu_i p_{ij}(t) = \mu_j p_{ji}(t))$, we have

$$\begin{aligned}
\|P_t\|_{L^1(\mu)\to L^1(\mu)} &= \sup_{\|f\|_{L^1(\mu)}\leqslant 1}\sum_{i\in E}\mu_i\Big|\sum_{j\in E}p_{ij}(t)f_j\Big| \\
&\leqslant \sup_{\|f\|_{L^1(\mu)}\leqslant 1}\sum_{i\in E}\mu_i\sum_{j\in E}p_{ij}(t)|f_j| \\
&= \sup_{\|f\|_{L^1(\mu)}\leqslant 1}\sum_{i\in E}\sum_{j\in E}\mu_j p_{ji}(t)|f_j| \\
&\leqslant \sup_{j\in E}\sum_{i\in E}p_{ji}(t) \\
&= \|P_t\|_{\infty\to\infty}.
\end{aligned}$$

Applying the interpolation theorem, we have $\|P_t\|_{L^2(\mu)\to L^2(\mu)} \leqslant \|P_t\|_{\infty\to\infty}$, so that

$$\lambda_1 = \lim_{t\to\infty} -\frac{1}{t}\log\|P_t\|_{L^2(\mu)\to L^2(\mu)} \geqslant \beta. \qquad \square$$

Proof of Corollary 1.9. By Theorem 1.8, we need to prove that there exists $C < \infty$ such that $\beta \leqslant C(\sum_{n\geqslant 1} \lambda_n^{-1})^{-1}$. Indeed, by the definition of β, there exists $C < \infty$ such that $\sup_{i\in E} \mathbb{P}_i[\xi > t] = \|P_t\|_{\infty\to\infty} \leqslant Ce^{-\beta t/2}/2$, thus

$$\sup_{i\in E} \mathbb{E}_i \xi \leqslant \int_0^\infty \sup_{i\in E} \mathbb{P}_i[\xi > t]\, dt \leqslant C/\beta.$$

The assertion follows from Corollary 1.5. □

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