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Biharmonic submanifolds with parallel mean curvature vector field in spheres

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ABSTRACT

We present some results on the boundedness of the mean curvature of proper biharmonic submanifolds in spheres. A partial classification result for proper biharmonic submanifolds with parallel mean curvature vector field in spheres is obtained. Then, we completely classify the proper biharmonic submanifolds in spheres with parallel mean curvature vector field and parallel Weingarten operator associated to the mean curvature vector field.
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1. Introduction

Biharmonic maps between two Riemannian manifolds (M,g) and (N,h), M compact, generalize harmonic maps (see [13]) and represent the critical points of the *bienergy functional*

$$E_2 : C^\infty(M,N) \to \mathbb{R}, \qquad E_2(\phi) = \frac{1}{2}\int_M \left|\tau(\phi)\right|^2 v_g,$$

where $\tau(\phi) = \operatorname{trace} \nabla d\phi$ denotes the *tension field* associated to the map ϕ. We recall that harmonic maps are characterized by the vanishing of the tension field (see, for example, [12]).

The first variation of E_2, obtained by G.Y. Jiang in [16], shows that ϕ is a biharmonic map if and only if its *bitension field* vanishes

$$\begin{aligned} \tau_2(\phi) &= -J\big(\tau(\phi)\big) = -\Delta\tau(\phi) - \operatorname{trace} R^N\big(d\phi\cdot, \tau(\phi)\big)\, d\phi\cdot \\ &= 0, \end{aligned} \tag{1.1}$$

i.e. $\tau(\phi) \in \operatorname{Ker} J$, where J is, formally, the Jacobi operator associated to ϕ. Here Δ denotes the rough Laplacian on sections of the pull-back bundle $\phi^{-1}(TN)$ and R^N denotes the curvature operator on (N,h), and we use the following sign conventions

$$\Delta V = -\operatorname{trace} \nabla^2 V, \quad \forall V \in C\big(\phi^{-1}(TN)\big),$$
$$R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \forall X, Y \in C(TN).$$

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When M is not compact a map $\phi : (M, g) \to (N, h)$ is said to be biharmonic if it is a solution of Eq. (1.1). As J is a linear operator, any harmonic map is biharmonic. We call *proper biharmonic* the non-harmonic biharmonic maps, and the submanifolds with non-harmonic (non-minimal) biharmonic inclusion map are called *proper biharmonic submanifolds*.

One can easily construct proper biharmonic maps between Euclidean spaces, for example by choosing third order polynomial maps or by using the Almansi property (see [3]). Regarding proper biharmonic Riemannian immersions into the Euclidean space, they are characterized by the equation $\Delta H = 0$, where H denotes the mean curvature vector field, i.e. they are also biharmonic in the sense of Chen (see [9]).

A nonexistence result for proper biharmonic maps was obtained by requesting a compact domain and a non-positively curved codomain [16]. Moreover, the nonexistence of proper biharmonic Riemannian immersions with constant mean curvature in non-positively curved spaces was proved (see [22]). Other nonexistence results, mainly regarding proper biharmonic Riemannian immersions into non-positively curved manifolds can be found in [4–6,10,11,14,19]. Surprisingly, in [23] the author constructed examples of proper biharmonic Riemannian immersions (of non-constant mean curvature) in conformally flat negatively curved spaces.

On the other hand there are many examples of proper biharmonic submanifolds in positively curved spaces.

In this paper we study proper biharmonic submanifolds in Euclidean spheres with additional extrinsic properties: parallel mean curvature vector field or parallel Weingarten operator associated to the mean curvature vector field, obtaining some rigidity results.

The paper is organized as follows. In the preliminary section we gather some known results on proper biharmonic submanifolds in the unit Euclidean sphere $\mathbb{S}^n$. This section also recalls the Moore decomposition lemma.

In the main section we first prove, for compact proper biharmonic submanifolds of $\mathbb{S}^n$, a boundedness condition involving the mean curvature $|H|$ and the norm $|A_H|$ of the Weingarten operator associated to the mean curvature vector field (Theorem 3.2).

Then, the proper biharmonic submanifolds with parallel mean curvature vector field in unit Euclidean spheres are studied. It is known that a constant mean curvature proper biharmonic submanifold in $\mathbb{S}^n$ satisfies $|H| \in (0, 1]$, and $|H| = 1$ if and only if it is minimal in a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2})$ (see [21]). We prove here that the mean curvature of the proper biharmonic submanifolds M^m with parallel mean curvature vector field in $\mathbb{S}^n$ takes values in $(0, \frac{m-2}{m}] \cup \{1\}$, and we determine the proper biharmonic submanifolds with parallel mean curvature vector field and $|H| = \frac{m-2}{m}$ (Theorem 3.11).

Finally, we investigate proper biharmonic submanifolds in spheres with parallel mean curvature vector field, parallel Weingarten operator associated to the mean curvature vector field, and $|H| \in (0, \frac{m-2}{m})$. We first prove that such submanifolds have exactly two distinct principal curvatures in the direction of H (Corollary 3.15) and then, using the Moore Lemma, we determine all of them (Theorem 3.16).

We shall work in the C^∞ category, i.e. all manifolds, metrics, connections, maps, sections are assumed to be smooth. All manifolds are assumed to be connected.

2. Preliminaries

The biharmonic equation (1.1) for the inclusion map $\imath : M^m \to \mathbb{S}^n$ of a submanifold M in $\mathbb{S}^n$ writes

$$\Delta H = mH,$$

where H denotes the mean curvature vector field of M in $\mathbb{S}^n$. Although simple, this equation is not used in order to obtain examples and classification results. The following characterization, obtained by splitting the bitension field in its normal and tangent components, proved to be more useful.

Theorem 2.1. *(See [22].)*

(i) *The canonical inclusion $\imath : M^m \to \mathbb{S}^n$ of a submanifold M in an n-dimensional unit Euclidean sphere is biharmonic if and only if*

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B\big(A_H(\cdot), \cdot\big) - mH = 0, \\ 4 \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) + m \operatorname{grad}\big(|H|^2\big) = 0, \end{cases} \tag{2.1}$$

where A denotes the Weingarten operator, B the second fundamental form, H the mean curvature vector field, $\nabla^\perp$ and $\Delta^\perp$ the connection and the Laplacian in the normal bundle of M in $\mathbb{S}^n$, and grad *denotes the gradient on M.*

(ii) *If M is a submanifold with parallel mean curvature vector field, i.e. $\nabla^\perp H = 0$, in $\mathbb{S}^n$, then M is biharmonic if and only if*

$$\operatorname{trace} B\big(A_H(\cdot), \cdot\big) = mH, \tag{2.2}$$

or equivalently,

$$\begin{cases} |A_H|^2 = m|H|^2, \\ \langle A_H, A_\eta \rangle = 0, \quad \forall \eta \in C(NM),\ \eta \perp H, \end{cases} \tag{2.3}$$

where NM denotes the normal bundle of M in $\mathbb{S}^n$.

We recall that the small hypersphere

$$\mathbb{S}^{n-1}(1/\sqrt{2}) = \{(x, 1/\sqrt{2}) \in \mathbb{R}^{n+1}\colon\ x \in \mathbb{R}^n,\ |x|^2 = 1/2\} \subset \mathbb{S}^n \tag{2.4}$$

and the standard products of spheres $\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2})$, given by

$$\{(x, y) \in \mathbb{R}^{n+1}\colon\ x \in \mathbb{R}^{n_1+1},\ y \in \mathbb{R}^{n_2+1},\ |x|^2 = |y|^2 = 1/2\} \subset \mathbb{S}^n, \tag{2.5}$$

$n_1 + n_2 = n - 1$ and $n_1 \neq n_2$, are the main examples of proper biharmonic submanifolds in $\mathbb{S}^n$ (see [7,16]). Inspired by these examples, by using their minimal submanifolds, two methods of construction for proper biharmonic submanifolds in spheres were given.

Theorem 2.2. *(See [6].) Let M be a submanifold in a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$. Then M is proper biharmonic in $\mathbb{S}^n$ if and only if M is minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$.*

We note that the proper biharmonic submanifolds of $\mathbb{S}^n$ obtained from minimal submanifolds of the proper biharmonic hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2})$ are pseudo-umbilical, i.e. $A_H = |H|^2\,\mathrm{Id}$, have parallel mean curvature vector field and mean curvature $|H| = 1$. Clearly, $\nabla A_H = 0$.

Theorem 2.3. *(See [6].) Let n_1, n_2 be two positive integers such that $n_1 + n_2 = n - 1$, and let M_1 be a submanifold in $\mathbb{S}^{n_1}(1/\sqrt{2})$ of dimension m_1, with $0 \leqslant m_1 \leqslant n_1$, and let M_2 be a submanifold in $\mathbb{S}^{n_2}(1/\sqrt{2})$ of dimension m_2, with $0 \leqslant m_2 \leqslant n_2$. Then $M_1 \times M_2$ is proper biharmonic in $\mathbb{S}^n$ if and only if*

$$\begin{cases} m_1 \neq m_2, \\ \tau_2(\imath_1) + 2(m_2 - m_1)\tau(\imath_1) = 0, \\ \tau_2(\imath_2) - 2(m_2 - m_1)\tau(\imath_2) = 0, \\ |\tau(\imath_1)| = |\tau(\imath_2)|, \end{cases}$$

where $\imath_1 : M_1 \to \mathbb{S}^{n_1}(1/\sqrt{2})$ and $\imath_2 : M_2 \to \mathbb{S}^{n_2}(1/\sqrt{2})$ are the canonical inclusions.

Obviously, if M_2 is minimal in $\mathbb{S}^{n_2}(1/\sqrt{2})$, then $M_1 \times M_2$ is biharmonic in $\mathbb{S}^n$ if and only if M_1 is minimal in $\mathbb{S}^{n_1}(1/\sqrt{2})$. The proper biharmonic submanifolds obtained in this way are no longer pseudo-umbilical, but still have parallel mean curvature vector field and their mean curvature is $|H| = \frac{|m_1 - m_2|}{m} \in (0, 1)$, where $m = m_1 + m_2$. Moreover, $\nabla A_H = 0$ and the principal curvatures in the direction of H, i.e. the eigenvalues of A_H, are constant on M and given by $\lambda_1 = \cdots = \lambda_{m_1} = \frac{m_1 - m_2}{m}$, $\lambda_{m_1+1} = \cdots = \lambda_{m_1+m_2} = -\frac{m_1 - m_2}{m}$.

In the proof of the main results of this paper we shall also use the following lemma.

Lemma 2.4 *(Moore Lemma). (See [18].) Suppose that M_1 and M_2 are connected Riemannian manifolds and that*

$$\varphi : M_1 \times M_2 \to \mathbb{R}^r$$

is an isometric immersion of the Riemannian product. If the second fundamental form $\tilde{B}$ of φ has the property

$$\tilde{B}(X, Y) = 0,$$

for all X tangent to M_1, Y tangent to M_2, then φ is a product immersion $\varphi = \varphi_0 \times \varphi_1 \times \varphi_2$, where $\varphi_0 : M_1 \times M_2 \to \mathbb{R}^{n_0}$ is constant, $\varphi_i : M_i \to \mathbb{R}^{n_i}$, $i = 1, 2$, and $\mathbb{R}^r = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ is an orthogonal decomposition. Moreover, $\mathbb{R}^{n_1}$ is the subspace of $\mathbb{R}^r$ generated by all vectors tangent to $M_1 \times \{p_2\}$, for all $p_2 \in M_2$, and $\mathbb{R}^{n_2}$ is the subspace generated by all vectors tangent to $\{p_1\} \times M_2$, for all $p_1 \in M_1$.

3. Main results

3.1. Compact proper biharmonic submanifolds in spheres

The following result for proper biharmonic constant mean curvature submanifolds in spheres was obtained.

Theorem 3.1. *(See [21].) Let M be a proper biharmonic submanifold with constant mean curvature in $\mathbb{S}^n$. Then $|H| \in (0, 1]$. Moreover, if $|H| = 1$, then M is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.*

If the condition on the mean curvature to be constant is replaced by the condition on the submanifold to be compact, we obtain the following.

Theorem 3.2. *Let M be a compact proper biharmonic submanifold of $\mathbb{S}^n$. Then either*

(i) *there exists a point $p \in M$ such that $|A_H(p)|^2 < m|H(p)|^2$,*

or

(ii) $|A_H|^2 = m|H|^2$. *In this case, M has parallel mean curvature vector field and $|H| \in (0,1]$.*

Proof. Let M be a proper biharmonic submanifold of $\mathbb{S}^n$. The first equation of (2.1) implies that

$$\langle \Delta^\perp H, H\rangle = m|H|^2 - |A_H|^2,$$

and by using the Weitzenböck formula,

$$\frac{1}{2}\Delta|H|^2 = \langle \Delta^\perp H, H\rangle - \left|\nabla^\perp H\right|^2,$$

we obtain

$$\frac{1}{2}\Delta|H|^2 = m|H|^2 - |A_H|^2 - \left|\nabla^\perp H\right|^2. \tag{3.1}$$

As M is compact, by integrating Eq. (3.1) on M we get

$$\int_M \left(m|H|^2 - |A_H|^2\right) v_g \geqslant 0,$$

and (i) and the first part of (ii) follow. Then, it is easy to see that

$$m|H|^4 \leqslant |A_H|^2,$$

for any submanifold of a given Riemannian manifold, so when $|A_H|^2 = m|H|^2$ we get $|H| \in (0,1]$. Moreover, by integrating (3.1), we obtain $\nabla^\perp H = 0$ and we conclude the proof. □

Regarding the mean curvature, from Theorem 3.2, we get the following result.

Corollary 3.3. *Let M be a compact proper biharmonic submanifold of $\mathbb{S}^n$. Then either*

(i) *there exists a point $p \in M$ such that $|H(p)| < 1$,*

or

(ii) $|H| = 1$. *In this case M is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.*

3.2. Biharmonic submanifolds with $\nabla^\perp H = 0$ in spheres

The following result concerning proper biharmonic surfaces with parallel mean curvature vector field was proved.

Theorem 3.4. *(See [5].) Let M^2 be a proper biharmonic surface with parallel mean curvature vector field in $\mathbb{S}^n$. Then M is minimal in a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2})$ in $\mathbb{S}^n$.*

We shall further see that, when $m > 2$, the situation is more complex and, apart from 1, the mean curvature can assume other lower values, as expected in view of Theorem 2.3.

First, let us prove an auxiliary result, concerning non-full proper biharmonic submanifolds of $\mathbb{S}^n$, which generalizes Theorem 5.4 in [5].

Proposition 3.5. *Let M^m be a submanifold of a small hypersphere $\mathbb{S}^{n-1}(a)$ in $\mathbb{S}^n$, $a \in (0,1)$. Then M is proper biharmonic in $\mathbb{S}^n$ if and only if either $a = 1/\sqrt{2}$ and M is minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$, or $a > 1/\sqrt{2}$ and M is minimal in a small hypersphere $\mathbb{S}^{n-2}(1/\sqrt{2})$ in $\mathbb{S}^{n-1}(a)$. In both cases, $|H| = 1$.*

Proof. The converse follows immediately by using Theorem 2.2.

In order to prove the other implication, denote by $\mathbf{j}$ and $\mathbf{i}$ the inclusion maps of M in $\mathbb{S}^{n-1}(a)$ and of $\mathbb{S}^{n-1}(a)$ in $\mathbb{S}^n$, respectively.

Up to an isometry of $\mathbb{S}^n$, we can consider

$$\mathbb{S}^{n-1}(a)=\left\{\left(x^1,\dots,x^n,\sqrt{1-a^2}\right)\in\mathbb{R}^{n+1}\colon \sum_{i=1}^n\left(x^i\right)^2=a^2\right\}\subset\mathbb{S}^n.$$

Then

$$C\left(T\mathbb{S}^{n-1}(a)\right)=\left\{\left(X^1,\dots,X^n,0\right)\in C\left(T\mathbb{R}^{n+1}\right)\colon \sum_{i=1}^n x^iX^i=0\right\},$$

while $\eta=\frac{1}{c}(x^1,\dots,x^n,-\frac{a^2}{\sqrt{1-a^2}})$ is a unit section in the normal bundle of $\mathbb{S}^{n-1}(a)$ in $\mathbb{S}^n$, where $c^2=\frac{a^2}{1-a^2}$, $c>0$. The tension and bitension fields of the inclusion $\imath=\mathbf{i}\circ\mathbf{j}:M\to\mathbb{S}^n$, are given by

$$\tau(\imath)=\tau(\mathbf{j})-\frac{m}{c}\eta,\qquad \tau_2(\imath)=\tau_2(\mathbf{j})-\frac{2m}{c^2}\tau(\mathbf{j})+\frac{1}{c}\left\{\left|\tau(\mathbf{j})\right|^2-\frac{m^2}{c^2}\left(c^2-1\right)\right\}\eta.$$

Since M is biharmonic in $\mathbb{S}^n$, we obtain

$$\tau_2(\mathbf{j})=\frac{2m}{c^2}\tau(\mathbf{j}) \tag{3.2}$$

and

$$\left|\tau(\mathbf{j})\right|^2=\frac{m^2}{c^2}\left(c^2-1\right)=\frac{m^2}{a^2}\left(2a^2-1\right).$$

From here $a\geqslant 1/\sqrt{2}$. Also,

$$\left|\tau(\imath)\right|^2=\left|\tau(\mathbf{j})\right|^2+\frac{m^2}{c^2}=m^2.$$

This implies that the mean curvature of M in $\mathbb{S}^n$ is 1.

The case $a=1/\sqrt{2}$ is solved by Theorem 2.2.

Consider $a>1/\sqrt{2}$, thus $\tau(\mathbf{j})\neq 0$. As $|H|=1$, by applying Theorem 3.1, M is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2})\subset\mathbb{S}^n$, so it is pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{S}^n$ [8]. From here it can be proved that M is also pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{S}^{n-1}(a)$. As M is not minimal in $\mathbb{S}^{n-1}(a)$, it follows that M is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-2}(b)$ in $\mathbb{S}^{n-1}(a)$. By a straightforward computation, Eq. (3.2) implies $b=1/\sqrt{2}$ and the proof is completed. □

Since every small sphere $\mathbb{S}^{n'}(a)$ in $\mathbb{S}^n$, $a\in(0,1)$, is contained into a great sphere $\mathbb{S}^{n'+1}$ of $\mathbb{S}^n$, from Proposition 3.5 we have the following.

Corollary 3.6. *Let M^m be a submanifold of a small sphere $\mathbb{S}^{n'}(a)$ in $\mathbb{S}^n$, $a\in(0,1)$. Then M is proper biharmonic in $\mathbb{S}^n$ if and only if either $a=1/\sqrt{2}$ and M is minimal in $\mathbb{S}^{n'}(1/\sqrt{2})$, or $a>1/\sqrt{2}$ and M is minimal in a small hypersphere $\mathbb{S}^{n'-1}(1/\sqrt{2})$ in $\mathbb{S}^{n'}(a)$. In both cases, $|H|=1$.*

Let M^m be a submanifold in $\mathbb{S}^n$. For our purpose it is convenient to define, following [1] and [2], the $(1,1)$-tensor field $\Phi=A_H-|H|^2I$, where I is the identity on $C(TM)$. We notice that Φ is symmetric, $\operatorname{trace}\Phi=0$ and

$$|\Phi|^2=|A_H|^2-m|H|^4. \tag{3.3}$$

Moreover, $\Phi=0$ if and only if M is pseudo-umbilical.

By using the Gauss equation of M in $\mathbb{S}^n$, one gets the curvature tensor field of M in terms of Φ as follows.

Lemma 3.7. *Let M^m be a submanifold in $\mathbb{S}^n$ with nowhere zero mean curvature vector field. Then the curvature tensor field of M is given by*

$$\begin{aligned}R(X,Y)Z={}&\left(1+|H|^2\right)\left(\langle Z,Y\rangle X-\langle Z,X\rangle Y\right)\\&+\frac{1}{|H|^2}\left(\left\langle Z,\Phi(Y)\right\rangle\Phi(X)-\left\langle Z,\Phi(X)\right\rangle\Phi(Y)\right)\\&+\left\langle Z,\Phi(Y)\right\rangle X-\left\langle Z,\Phi(X)\right\rangle Y+\langle Z,Y\rangle\Phi(X)-\langle Z,X\rangle\Phi(Y)\\&+\sum_{a=1}^{k-1}\left\{\left\langle Z,A_{\eta_a}(Y)\right\rangle A_{\eta_a}(X)-\left\langle Z,A_{\eta_a}(X)\right\rangle A_{\eta_a}(Y)\right\},\end{aligned} \tag{3.4}$$

for all $X, Y, Z \in C(TM)$, where $\{H/|H|, \eta_a\}_{a=1}^{k-1}$, $k = n - m$, denotes a local orthonormal frame field in the normal bundle of M in $\mathbb{S}^n$.

In the case of hypersurfaces, i.e. $k = 1$, the previous result holds by making the convention that $\sum_{a=1}^{k-1}\{\ldots\} = 0$.
For what concerns the expression of trace $\nabla^2\Phi$, which will be needed further, the following result holds.

Lemma 3.8. *Let M^m be a submanifold in $\mathbb{S}^n$ with nowhere zero mean curvature vector field. If $\nabla^\perp H = 0$, then $\nabla\Phi$ is symmetric and*

$$\big(\operatorname{trace}\nabla^2\Phi\big)(X) = -|\Phi|^2 X + \left(m + m|H|^2 - \frac{|\Phi|^2}{|H|^2}\right)\Phi(X) + m\Phi^2(X) - \sum_{a=1}^{k-1}\langle\Phi, A_{\eta_a}\rangle A_{\eta_a}(X). \tag{3.5}$$

Proof. From the Codazzi equation, as $\nabla^\perp H = 0$, we get $(\nabla A_H)(X, Y) = (\nabla A_H)(Y, X)$, for all $X, Y \in C(TM)$, where

$$(\nabla A_H)(X, Y) = (\nabla_X A_H)(Y) = \nabla_X A_H(Y) - A_H(\nabla_X Y).$$

As the mean curvature of M is constant we have $\nabla\Phi = \nabla A_H$, thus $\nabla\Phi$ is symmetric.
We recall the Ricci commutation formula

$$\big(\nabla^2\Phi\big)(X, Y, Z) - \big(\nabla^2\Phi\big)(Y, X, Z) = R(X, Y)\Phi(Z) - \Phi\big(R(X, Y)Z\big), \tag{3.6}$$

for all $X, Y, Z \in C(TM)$, where

$$\begin{aligned}\big(\nabla^2\Phi\big)(X, Y, Z) &= (\nabla_X\nabla\Phi)(Y, Z)\\ &= \nabla_X\big((\nabla\Phi)(Y, Z)\big) - (\nabla\Phi)(\nabla_X Y, Z) - (\nabla\Phi)(Y, \nabla_X Z).\end{aligned}$$

Consider $\{X_i\}_{i=1}^m$ to be a local orthonormal frame field on M and $\{H/|H|, \eta_a\}_{a=1}^{k-1}$, $k = n - m$, a local orthonormal frame field in the normal bundle of M in $\mathbb{S}^n$. As η_a is orthogonal to H, we get trace $A_{\eta_a} = 0$, for all $a = 1, \ldots, k-1$. Using also the symmetry of Φ and $\nabla\Phi$, (3.6) and (3.4), we have

$$\begin{aligned}\big(\operatorname{trace}\nabla^2\Phi\big)(X) &= \sum_{i=1}^m\big(\nabla^2\Phi\big)(X_i, X_i, X) = \sum_{i=1}^m\big(\nabla^2\Phi\big)(X_i, X, X_i)\\ &= \sum_{i=1}^m\big\{\big(\nabla^2\Phi\big)(X, X_i, X_i) + R(X_i, X)\Phi(X_i) - \Phi\big(R(X_i, X)X_i\big)\big\}\\ &= \sum_{i=1}^m\big(\nabla^2\Phi\big)(X, X_i, X_i) - |\Phi|^2 X + \left(m + m|H|^2 - \frac{|\Phi|^2}{|H|^2}\right)\Phi(X) + m\Phi^2(X)\\ &\quad + \sum_{a=1}^{k-1}\big\{(A_{\eta_a}\circ\Phi - \Phi\circ A_{\eta_a})\big(A_{\eta_a}(X)\big) - \langle\Phi, A_{\eta_a}\rangle A_{\eta_a}(X)\big\}.\end{aligned}$$

By a straightforward computation,

$$\sum_{i=1}^m\big(\nabla^2\Phi\big)(X, X_i, X_i) = \nabla_X(\operatorname{trace}\nabla\Phi) = \nabla_X\operatorname{grad}(\operatorname{trace}\Phi) = 0.$$

Moreover, from the Ricci equation, since $\nabla^\perp H = 0$, we obtain $A_{\eta_a}\circ A_H = A_H\circ A_{\eta_a}$, thus $A_{\eta_a}\circ\Phi = \Phi\circ A_{\eta_a}$, and we end the proof of this lemma. □

We shall also use the following lemma.

Lemma 3.9. *Let M^m be a submanifold in $\mathbb{S}^n$ with nowhere zero mean curvature vector field. If $\nabla^\perp H = 0$ and A_H is orthogonal to A_{η_a}, for all $a = 1, \ldots, k-1$, then*

$$-\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + \left(m + m|H|^2 - \frac{|\Phi|^2}{|H|^2}\right)|\Phi|^2 + m\operatorname{trace}\Phi^3. \tag{3.7}$$

Proof. Since A_H is orthogonal to A_{η_a} and $\operatorname{trace} A_{\eta_a} = 0$, we get $\langle \Phi, A_{\eta_a}\rangle = 0$, for all $a = 1, \dots, k-1$, and (3.5) becomes

$$\big(\operatorname{trace} \nabla^2 \Phi\big)(X) = -|\Phi|^2 X + \left(m + m|H|^2 - \frac{|\Phi|^2}{|H|^2}\right)\Phi(X) + m\Phi^2(X). \tag{3.8}$$

Now, the Weitzenböck formula,

$$-\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + \big\langle \Phi, \operatorname{trace} \nabla^2\Phi\big\rangle,$$

together with the symmetry of Φ and (3.8), leads to the conclusion. $\square$

We also recall here the Okumura Lemma.

Lemma 3.10 *(Okumura Lemma). (See [20].) Let $b_1, \dots, b_m$ be real numbers such that $\sum_{i=1}^m b_i = 0$. Then*

$$-\frac{m-2}{\sqrt{m(m-1)}}\left(\sum_{i=1}^m b_i^2\right)^{3/2} \leqslant \sum_{i=1}^m b_i^3 \leqslant \frac{m-2}{\sqrt{m(m-1)}}\left(\sum_{i=1}^m b_i^2\right)^{3/2}.$$

Moreover, equality holds in the right-hand (respectively, left-hand) side if and only if $(m-1)$ of the b_i's are non-positive (respectively, non-negative) and equal.

By using the above lemmas we obtain the following result on the boundedness of the mean curvature of proper biharmonic submanifolds with parallel mean curvature vector field in spheres, as well as a partial classification result. We shall see that $|H|$ does not fill out all the interval $(0, 1]$.

Theorem 3.11. *Let M^m, $m > 2$, be a proper biharmonic submanifold with parallel mean curvature vector field in $\mathbb{S}^n$ and $|H| \in (0, 1)$. Then $|H| \in (0, \frac{m-2}{m}]$. Moreover, $|H| = \frac{m-2}{m}$ if and only if M is an open part of a standard product*

$$M_1^{m-1} \times \mathbb{S}^1(1/\sqrt{2}\,) \subset \mathbb{S}^n,$$

where M_1 is a minimal submanifold in $\mathbb{S}^{n-2}(1/\sqrt{2}\,)$.

Proof. Consider the tensor field Φ associated to M. Since it is traceless, Lemma 3.10 implies that

$$\operatorname{trace} \Phi^3 \geqslant -\frac{m-2}{\sqrt{m(m-1)}}|\Phi|^3. \tag{3.9}$$

By (2.3), as M is proper biharmonic with parallel mean curvature vector field, $|A_H|^2 = m|H|^2$ and $\langle A_H, A_\eta\rangle = 0$, for all $\eta \in C(NM)$, η orthogonal to H. From (3.3) we obtain

$$|\Phi|^2 = m|H|^2\big(1 - |H|^2\big), \tag{3.10}$$

thus $|\Phi|$ is constant. We can apply Lemma 3.9 and, using (3.9) and (3.10), Eq. (3.7) leads to

$$0 \geqslant m^2|H|^3\big(1 - |H|^2\big)\left(2|H| - \frac{m-2}{\sqrt{m-1}}\sqrt{1 - |H|^2}\,\right),$$

thus $|H| \in (0, \frac{m-2}{m}]$.

The condition $|H| = \frac{m-2}{m}$ holds if and only if $\nabla\Phi = 0$ and we have equality in (3.9). This is equivalent to the fact that $\nabla A_H = 0$ and, by the Okumura Lemma, the principal curvatures in the direction of H are constant functions on M and given by

$$\begin{aligned}
&\lambda_1 = \dots = \lambda_{m-1} = \lambda = \frac{m-2}{m},\\
&\lambda_m = \mu = -\frac{m-2}{m}.
\end{aligned} \tag{3.11}$$

Further, we consider the distributions

$$\begin{aligned}
&T_\lambda = \big\{X \in TM\colon\ A_H(X) = \lambda X\big\}, \qquad \dim T_\lambda = m-1,\\
&T_\mu = \big\{X \in TM\colon\ A_H(X) = \mu X\big\}, \qquad \dim T_\mu = 1.
\end{aligned}$$

One can easily verify that, as A_H is parallel, T_λ and T_μ are mutually orthogonal, smooth, involutive and parallel, and the de Rham decomposition theorem (see [17]) can be applied.

Thus, for every $p_0 \in M$ there exists a neighborhood $U \subset M$ which is isometric to a product $\tilde{M}_1^{m-1} \times I$, $I = (-\varepsilon, \varepsilon)$, where $\tilde{M}_1$ is an integral submanifold for T_λ through p_0 and I corresponds to the integral curves of the unit vector field $Y_1 \in T_\mu$ on U. Moreover $\tilde{M}_1$ is a totally geodesic submanifold in U and the integral curves of Y_1 are geodesics in U. We note that Y_1 is a parallel vector field on U.

In the following, we shall prove that the integral curves of Y_1, thought of as curves in $\mathbb{R}^{n+1}$, are circles of radius $1/\sqrt{2}$, all lying in parallel 2-planes. In order to prove this, consider $\{H/|H|, \eta_a\}_{a=1}^{k-1}$ to be an orthonormal frame field in the normal bundle and $\{X_\alpha\}_{\alpha=1}^{m-1}$ an orthonormal frame field in T_λ, on U. We have

$$\begin{aligned} \operatorname{trace} B\big(A_H(\cdot),\cdot\big) &= \sum_{\alpha=1}^{m-1} B\big(A_H(X_\alpha), X_\alpha\big) + B\big(A_H(Y_1), Y_1\big) \\ &= \lambda m H - 2\lambda B(Y_1, Y_1). \end{aligned}$$

This, together with (2.2) and (3.11), leads to

$$B(Y_1, Y_1) = -\frac{1}{\lambda} H, \tag{3.12}$$

so $|B(Y_1, Y_1)| = 1$. From here, since A_{η_a} and A_H commute, we obtain

$$A_{\eta_a}(Y_1) = 0, \quad \forall a = 1, \ldots, k-1. \tag{3.13}$$

We also note that

$$\nabla^{\mathbb{S}^n}_{Y_1} B(Y_1, Y_1) = -\frac{1}{\lambda}\big(\nabla^\perp_{Y_1} H - A_H(Y_1)\big) = -Y_1. \tag{3.14}$$

Consider $c : I \to U$ to be an integral curve for Y_1 and denote by $\gamma : I \to \mathbb{S}^n$, $\gamma = \imath \circ c$, where $\imath : M \to \mathbb{S}^n$ is the inclusion map. Denote $E_1 = \dot{\gamma} = Y_1 \circ \gamma$. Since Y_1 is parallel, c is a geodesic on M and, using Eqs. (3.12) and (3.14), we obtain the following Frenet equations for the curve γ in $\mathbb{S}^n$,

$$\begin{aligned} &\nabla^{\mathbb{S}^n}_{\dot{\gamma}} E_1 = B(Y_1, Y_1) = -\frac{1}{\lambda} H = E_2, \\ &\nabla^{\mathbb{S}^n}_{\dot{\gamma}} E_2 = -E_1. \end{aligned} \tag{3.15}$$

Let now $\tilde{\gamma} = \jmath \circ \gamma : I \to \mathbb{R}^{n+1}$, where $\jmath : \mathbb{S}^n \to \mathbb{R}^{n+1}$ denotes the inclusion map. Denote $\tilde{E}_1 = \dot{\tilde{\gamma}} = Y_1 \circ \tilde{\gamma}$. From (3.15) we obtain the Frenet equations for $\tilde{\gamma}$ in $\mathbb{R}^{n+1}$,

$$\begin{aligned} &\nabla^{\mathbb{R}^{n+1}}_{\dot{\tilde{\gamma}}} \tilde{E}_1 = -\frac{1}{\lambda} H - \tilde{\gamma} = \sqrt{2}\,\tilde{E}_2, \\ &\nabla^{\mathbb{R}^{n+1}}_{\dot{\tilde{\gamma}}} \tilde{E}_2 = -\sqrt{2}\,\tilde{E}_1, \end{aligned}$$

thus $\tilde{\gamma}$ is a circle of radius $1/\sqrt{2}$ in $\mathbb{R}^{n+1}$ and it lies in a 2-plane with corresponding vector space generated by $\tilde{E}_1(0)$ and $\tilde{E}_2(0)$.

Since Y_1 and $-\frac{1}{\lambda}H - \mathrm{x}$, with x the position vector field, are parallel in $\mathbb{R}^{n+1}$ along any curve of $\tilde{M}_1$, we conclude that the 2-planes determined by the integral curves of Y_1 have the same corresponding vector space, thus are parallel.

Consider the immersions

$$\phi : \tilde{M}_1 \times I \to \mathbb{S}^n,$$

and

$$\tilde{\phi} = \jmath \circ \phi : \tilde{M}_1 \times I \to \mathbb{R}^{n+1}.$$

Using the fact that $\tilde{M}_1$ is an integral submanifold of T_λ and (3.13), it is not difficult to verify that $\tilde{B}(X, Y) = 0$, for all $X \in C(T\tilde{M}_1)$ and $Y \in C(TI)$, thus we can apply Lemma 2.4. As the 2-planes determined by the integral curves of Y_1 have the same corresponding vector space and by Corollary 3.6, we obtain the orthogonal decomposition

$$\mathbb{R}^{n+1} = \mathbb{R}^{n-1} \oplus \mathbb{R}^2 \tag{3.16}$$

and $U = M_1 \times M_2$, where $M_1^{m-1} \subset \mathbb{R}^{n-1}$ and $M_2 \subset \mathbb{R}^2$ is a circle of radius $1/\sqrt{2}$. We can see that the center of this circle is the origin of $\mathbb{R}^2$. Thus $M_1 \subset \mathbb{S}^{n-2}(1/\sqrt{2}) \subset \mathbb{R}^{n-1}$ and from Theorem 2.3, since U is biharmonic in $\mathbb{S}^n$, we conclude that M_1 is a minimal submanifold in $\mathbb{S}^{n-2}(1/\sqrt{2}) \subset \mathbb{R}^{n-1}$. Consequently, the announced result holds locally.

In order to prove the global result we use the connectedness of M. Let $p \in M$ and let V be an open neighborhood of p given by the de Rham Theorem, as above, such that $U \cap V \neq \emptyset$. Consider c_U and c_V two integral curves for T_μ, such that c_U lies in U and c_V lies in V and $c_U \cap c_V \neq \emptyset$. It is clear that the 2-plane in $\mathbb{R}^{n+1}$ where c_U lies coincides with the 2-plane where c_V lies. Therefore, the decomposition (3.16) does not depend on the choice of p_0.

We can thus conclude that M is an open part of a standard product

$$M_1 \times \mathbb{S}^1(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_1 is a minimal submanifold in $\mathbb{S}^{n-2}(1/\sqrt{2})$. □

By a standard argument, using the universal covering, we also obtain the following result.

Corollary 3.12. *Let M^m, $m > 2$, be a proper biharmonic submanifold with parallel mean curvature vector field in $\mathbb{S}^n$ and $|H| \in (0,1)$. Assume that M is complete. Then $|H| \in (0, \frac{m-2}{m}]$ and $|H| = \frac{m-2}{m}$ if and only if*

$$M = M_1^{m-1} \times \mathbb{S}^1(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_1 is a complete minimal submanifold of $\mathbb{S}^{n-2}(1/\sqrt{2})$.

If we consider the case of hypersurfaces, the condition on the mean curvature vector field to be parallel is equivalent to the condition on the mean curvature to be constant and Theorem 3.11 leads to the following result.

Corollary 3.13. *Let M^m, $m > 2$, be a proper biharmonic constant mean curvature hypersurface with $|H| \in (0,1)$ in $\mathbb{S}^{m+1}$. Then $|H| \in (0, \frac{m-2}{m}]$. Moreover, $|H| = \frac{m-2}{m}$ if and only if M is an open part of $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

Proof. We recall that $|H| = \frac{m-2}{m}$ if and only if $\nabla A_H = 0$ and the principal curvatures of M in the direction of H are constant, one of multiplicity 1 and one of multiplicity $m-1$. This implies that M is an isoparametric hypersurface and, using a result in [5,15], we conclude. □

3.3. Biharmonic submanifolds with $\nabla^\perp H = 0$ and $\nabla A_H = 0$ in spheres

Inspired by the case $|H| = \frac{m-2}{m}$ of Theorem 3.11, in the following we shall study proper biharmonic submanifolds in $\mathbb{S}^n$ with parallel mean curvature vector field and parallel Weingarten operator associated to the mean curvature vector field.

We shall also need the following general result.

Proposition 3.14. *Let M^m be a submanifold in $\mathbb{S}^n$ with nowhere zero mean curvature vector field. If $\nabla^\perp H = 0$, $\nabla A_H = 0$ and A_H is orthogonal to A_η, for all $\eta \in C(NM)$, $\eta \perp H$, then M is either pseudo-umbilical, or it has two distinct principal curvatures in the direction of H. Moreover, the principal curvatures in the direction of H are solutions of the equation*

$$mt^2 + \left(m - \frac{|A_H|^2}{|H|^2}\right)t - m|H|^2 = 0. \tag{3.17}$$

Proof. As $\nabla A_H = 0$, the principal curvatures in the direction of H are constant on M. Denote by $\{X_i\}_{i=1}^m$ a local orthonormal frame field on M such that $A_H(X_i) = \lambda_i X_i$, $i = 1, \dots, m$. Clearly, $\sum_{i=1}^m \lambda_i = m|H|^2$.

Since A_H is parallel, $\nabla_X A_H(Y) = A_H(\nabla_X Y)$, thus $R(X,Y)$ and A_H commute for all $X, Y \in C(TM)$. In particular,

$$R(X_i, X_j)A_H(X_j) = A_H\big(R(X_i, X_j)X_j\big),$$

and by considering the scalar product with X_j and using the symmetry of A_H, we get

$$(\lambda_i - \lambda_j)\big\langle R(X_i, X_j)X_j, X_i\big\rangle = 0, \quad \forall i, j = 1, \dots, m. \tag{3.18}$$

Consider $\{H/|H|, \eta_a\}_{a=1}^{k-1}$, $k = n - m$, a local orthonormal frame field in the normal bundle of M in $\mathbb{S}^n$. We have

$$B(X_i, X_i) = \frac{\lambda_i}{|H|^2}H + \sum_{a=1}^{k-1}\big\langle A_{\eta_a}(X_i), X_i\big\rangle \eta_a, \tag{3.19}$$

and for $\lambda_i \neq \lambda_j$, as X_i is orthogonal to X_j and $A_H \circ A_{\eta_a} = A_{\eta_a} \circ A_H$, for all $a = 1, \dots, k-1$, we obtain

$$B(X_i, X_j) = \frac{1}{|H|^2}\big\langle A_H(X_i), X_j\big\rangle H + \sum_{a=1}^{k-1}\big\langle A_{\eta_a}(X_i), X_j\big\rangle \eta_a = 0. \tag{3.20}$$

By using (3.19) and (3.20) in the Gauss equation for M in $\mathbb{S}^n$, one gets

$$\big\langle R(X_i,X_j)X_j,X_i\big\rangle = 1+\frac{\lambda_i\lambda_j}{|H|^2}+\sum_{a=1}^{k-1}\big\langle A_{\eta_a}(X_i),X_i\big\rangle\big\langle A_{\eta_a}(X_j),X_j\big\rangle. \tag{3.21}$$

In fact, (3.18), together with (3.21), implies

$$(\lambda_i-\lambda_j)\left(1+\frac{\lambda_i\lambda_j}{|H|^2}+\sum_{a=1}^{k-1}\big\langle A_{\eta_a}(X_i),X_i\big\rangle\big\langle A_{\eta_a}(X_j),X_j\big\rangle\right)=0,\quad \forall i,j=1,\dots,m. \tag{3.22}$$

Summing on i in (3.22) we obtain

$$0=m|H|^2-\left(m-\frac{|A_H|^2}{|H|^2}\right)\lambda_j-m\lambda_j^2+\sum_{a=1}^{k-1}\langle A_{\eta_a},A_H\rangle\big\langle A_{\eta_a}(X_j),X_j\big\rangle-\sum_{a=1}^{k-1}\operatorname{trace}A_{\eta_a}\big\langle A_{\eta_a}(X_j),A_H(X_j)\big\rangle.$$

Since $\operatorname{trace}A_{\eta_a}=0$ and $\langle A_H,A_{\eta_a}\rangle=0$, for all $a=1,\dots,k-1$, we conclude the proof. $\square$

Corollary 3.15. *Let M^m, $m>2$, be a proper biharmonic submanifold in $\mathbb{S}^n$. If $\nabla^\perp H=0$, $\nabla A_H=0$ and $|H|\in(0,\frac{m-2}{m}]$, then M has two distinct principal curvatures λ and μ in the direction of H, of different multiplicities m_1 and m_2, respectively, and*

$$\begin{cases}\lambda=\dfrac{m_1-m_2}{m},\\ \mu=-\dfrac{m_1-m_2}{m},\\ |H|=\dfrac{|m_1-m_2|}{m}.\end{cases} \tag{3.23}$$

Proof. Since M is proper biharmonic, all the hypotheses of Proposition 3.14 are satisfied. Taking into account (2.3), from (3.17) follows that the principal curvatures of M in the direction of H satisfy the equation $t^2=|H|^2$. As $|H|\in(0,\frac{m-2}{m}]$, M cannot be pseudo-umbilical, thus it has two distinct principal curvatures $\lambda=-\mu\neq0$ in the direction of H. If m_1 denotes the multiplicity of λ and m_2 the multiplicity of μ, from $\operatorname{trace}A_H=m|H|^2$ we have $(m_1-m_2)\lambda=m\lambda^2$. Since $\lambda\neq0$, we obtain (3.23). Notice also that $m_1\neq m_2$. $\square$

The case $|H|=\frac{m-2}{m}$ was solved in Theorem 3.11, thus we shall consider now only the case $|H|\in(0,\frac{m-2}{m})$. Since $|H|=\frac{|m_1-m_2|}{m}$, $m_1\neq m_2$, we have $m\geqslant5$ and $m_1\geqslant2$, $m_2\geqslant2$. We are able to prove the following result.

Theorem 3.16. *Let M^m, $m\geqslant5$, be a proper biharmonic submanifold in $\mathbb{S}^n$ with $\nabla^\perp H=0$, $\nabla A_H=0$ and $|H|\in(0,\frac{m-2}{m})$. Then, locally,*

$$M=M_1^{m_1}\times M_2^{m_2}\subset\mathbb{S}^{n_1}(1/\sqrt2)\times\mathbb{S}^{n_2}(1/\sqrt2)\subset\mathbb{S}^n,$$

where M_i is a minimal submanifold of $\mathbb{S}^{n_i}(1/\sqrt2)$, $m_i\geqslant2$, $i=1,2$, $m_1+m_2=m$, $m_1\neq m_2$, $n_1+n_2=n-1$.

Proof. We are in the hypotheses of Corollary 3.15, thus A_H has two distinct eigenvalues $\lambda=\frac{m_1-m_2}{m}$ and $\mu=-\frac{m_1-m_2}{m}$. Consider the distributions

$$T_\lambda=\big\{X\in TM:\ A_H(X)=\lambda X\big\},\qquad \dim T_\lambda=m_1,$$
$$T_\mu=\big\{X\in TM:\ A_H(X)=\mu X\big\},\qquad \dim T_\mu=m_2.$$

As A_H is parallel, T_λ and T_μ are mutually orthogonal, smooth, involutive and parallel, and from the de Rham decomposition theorem follows that for every $p_0\in M$ there exists a neighborhood $U\subset M$ which is isometric to a product $\tilde M_1^{m_1}\times\tilde M_2^{m_2}$, such that the submanifolds which are parallel to $\tilde M_1$ in $\tilde M_1\times\tilde M_2$ correspond to integral submanifolds for T_λ and the submanifolds which are parallel to $\tilde M_2$ correspond to integral submanifolds for T_μ.

Consider the immersions

$$\phi:\tilde M_1\times\tilde M_2\to\mathbb{S}^n,$$

and

$$\tilde\phi=\jmath\circ\phi:\tilde M_1\times\tilde M_2\to\mathbb{R}^{n+1}.$$

It can be easily verified that $\tilde{B}(X,Y)=B(X,Y)$, for all $X\in C(T\tilde{M}_1)$ and $Y\in C(T\tilde{M}_2)$. Since $A_H\circ A_\eta=A_\eta\circ A_H$ for all $\eta\in C(NM)$, we have that T_λ and T_μ are invariant subspaces for A_η, for all $\eta\in C(NM)$, thus

$$\langle B(X,Y),\eta\rangle=\langle A_\eta(X),Y\rangle=0,\quad \forall\eta\in C(NM).$$

Thus $\tilde{B}(X,Y)=0$, for all $X\in C(T\tilde{M}_1)$ and $Y\in C(T\tilde{M}_2)$, and we can apply Lemma 2.4. In this way we have an orthogonal decomposition $\mathbb{R}^{n+1}=\mathbb{R}^{n_0}\oplus\mathbb{R}^{n_1+1}\oplus\mathbb{R}^{n_2+1}$ and $\tilde{\phi}$ is a product immersion. From Corollary 3.6, since $|H|\neq 1$, follows that $n_0=0$. Thus

$$\tilde{\phi}=\tilde{\phi}_1\times\tilde{\phi}_2:\tilde{M}_1\times\tilde{M}_2\to\mathbb{R}^{n_1+1}\oplus\mathbb{R}^{n_2+1}.$$

We denote by $M_1=\tilde{\phi}_1(\tilde{M}_1)\subset\mathbb{R}^{n_1+1}$, $M_2=\tilde{\phi}_2(\tilde{M}_2)\subset\mathbb{R}^{n_2+1}$ and we have $U=M_1\times M_2\subset\mathbb{S}^n$.

Consider now $\{X_\alpha\}_{\alpha=1}^{m_1}$ an orthonormal frame field in T_λ and $\{Y_\ell\}_{\ell=1}^{m_2}$ an orthonormal frame field in T_μ, on U. From (2.2), by using the fact that $\lambda=-\mu=\frac{m_1-m_2}{m}$, we obtain

$$\sum_{\alpha=1}^{m_1}B(X_\alpha,X_\alpha)=\frac{m_1}{\lambda}H,\qquad \sum_{\ell=1}^{m_2}B(Y_\ell,Y_\ell)=-\frac{m_2}{\lambda}H. \tag{3.24}$$

Since $\nabla^\perp H=0$, from (3.24) follows that $M_1\times\{p_2\}$ is pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{R}^{n+1}$, for any $p_2\in M_2$. But $M_1\times\{p_2\}$ is included in $\mathbb{R}^{n_1+1}\times\{p_2\}$ which is totally geodesic in $\mathbb{R}^{n+1}$, thus M_1 is pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{R}^{n_1+1}$. This implies that M_1 is minimal in $\mathbb{R}^{n_1+1}$ or minimal in a hypersphere of $\mathbb{R}^{n_1+1}$. The first case leads to a contradiction, since $M_1\times\{p_2\}\subset\mathbb{S}^n$ and cannot be minimal in $\mathbb{R}^{n+1}$. Thus M_1 is minimal in a hypersphere $\mathbb{S}^{n_1}_{c_1}(r_1)\subset\mathbb{R}^{n_1+1}$, where $c_1\in\mathbb{R}^{n_1+1}$ denotes the center of the hypersphere.

In the following we will show that $c_1=0$. Since $U\subset\mathbb{S}^n$ and $M_1\subset\mathbb{S}^{n_1}_{c_1}(r_1)$, we get $|p_1|^2+|p_2|^2=1$ and $|p_1-c_1|^2=r_1^2$, for all $p_1\in M_1$. This implies $\langle p_1,c_1\rangle=$ constant for all $p_1\in M_1$. Thus $\langle u_1,c_1\rangle=0$, for all $u_1\in T_{p_1}M_1$ and for all $p_1\in M_1$. From Lemma 2.4 follows that $c_1=0$, thus $M_1\subset\mathbb{S}^{n_1}(r_1)\subset\mathbb{R}^{n_1+1}$.

From (3.24) also follows that the mean curvature of $M_1\times\{p_2\}$ in $\mathbb{S}^n$ is 1, so its mean curvature in $\mathbb{R}^{n+1}$ is $\sqrt{2}$. As $\mathbb{R}^{n_1+1}\times\{p_2\}$ is totally geodesic in $\mathbb{R}^{n+1}$ it follows that the mean curvature of M_1 in $\mathbb{R}^{n_1+1}$ is $\sqrt{2}$ too. Further, as M_1 is minimal in $\mathbb{S}^{n_1}(r_1)$, we get $r_1=1/\sqrt{2}$.

Analogously, M_2 is minimal in a hypersphere $\mathbb{S}^{n_2}(1/\sqrt{2})$ in $\mathbb{R}^{n_2+1}$, and we conclude the proof. □

Corollary 3.17. *Let M^m, $m\geqslant 5$, be a complete proper biharmonic submanifold in $\mathbb{S}^n$ with $\nabla^\perp H=0$, $\nabla A_H=0$ and $|H|\in(0,\frac{m-2}{m})$. Then,*

$$M=M_1^{m_1}\times M_2^{m_2}\subset\mathbb{S}^{n_1}(1/\sqrt{2})\times\mathbb{S}^{n_2}(1/\sqrt{2})\subset\mathbb{S}^n,$$

where M_i is a complete minimal submanifold of $\mathbb{S}^{n_i}(1/\sqrt{2})$, $m_i\geqslant 2$, $i=1,2$, $m_1+m_2=m$, $m_1\neq m_2$, $n_1+n_2=n-1$.

Remark 3.18. In the case of a non-minimal hypersurface the hypotheses $\nabla^\perp H=0$ and $\nabla A_H=0$ are equivalent to $\nabla^\perp B=0$, i.e. the hypersurface is parallel. Such hypersurfaces have at most two principal curvatures and the proper biharmonic hypersurfaces with at most two principal curvatures in $\mathbb{S}^n$ are those given by (2.4) and (2.5) (see [5]).

If one searches for a relaxation of the hypothesis $\nabla A_H=0$ in Theorem 3.16, natural candidates would be $RA_H=0$ (see, for example, [24]), or M has at most two distinct principal curvatures in the direction of H everywhere. But the following can be proved.

Proposition 3.19. *Let M^m be a proper biharmonic submanifold in $\mathbb{S}^n$ with $\nabla^\perp H=0$. The following statements are equivalent:*

(i) *$RA_H=0$, where $(RA_H)(X,Y,Z)=(R(X,Y)A_H)(Z)$,*
(ii) *M has at most two distinct principal curvatures in the direction of H everywhere,*
(iii) *$\nabla A_H=0$.*

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