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The non-parametrizability of the word equation $xyz=zvx$: A short proof

Elena Czeizler

Department of Mathematics, University of Turku and Turku Centre for Computer Science, Turku 20520, Finland

Abstract

Although Makanin proved the problem of satisfiability of word equations to be decidable, the general structure of solutions is difficult to describe. In particular, Hmelevskii proved that the set of solutions of $xyz=zvx$ cannot be described using only finitely many parameters, contrary to the case of equations in three unknowns. In this paper we give a short, elementary proof of Hmelevskii's result.

Keywords: Word equations; Parametrizability; Fibonacci numbers; Fibonacci word

1. Introduction

The theory of *word equations*, a central subfield of *Combinatorics on Words*, was initiated in 1954 by A.A. Markov. He proposed in [18] the problem of satisfiability of word equations: decide whether or not a given word equation has solutions. The problem remained open for quite a long time and it was solved by Makanin who proved it to be decidable for free semigroups in [15], and for free groups in [16,17], see also [6] for a recent survey. However, Makanin's algorithm is considered as one of the most involved results in the literature. More recently, Plandowski found a new way to solve word equations and gave an algorithm with polynomial space complexity for the satisfiability problem, see [21] and [22]. Nevertheless, neither Makanin's nor Plandowski's results can be used to characterize the general structure of solutions of word equations. However, Razborov gave in [23] an algorithm which generates all solutions of a given word equation; see also [6].

E-mail address: elenac@it.utu.fi.

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There are several results in the literature describing the general structure of solutions of different types of word equations in terms of parametric functions. Lyndon in [14], proposed a general pattern for approaching this problem. He introduced the notion of *parametric words* and he proved that, using them, the set of solutions of arbitrary one variable equations in free groups can be finitely characterized. His result was subsequently strengthened and simplified in [1,11,12]. Also, some analysis of the case of *quadratic equations*, i.e., equations where every unknown appears twice, are given in [4,7,24]. As a direct consequence of the defect theorem, constant-free word equations with two unknowns may have only periodic solutions. For constant-free equations in three unknowns, Hmelevskii proved in [8] that the solutions can be expressed using only a finite number of *parametric formulas*, i.e., formulas involving word parameters and numerical parameters. Perhaps more importantly, he also proved that this is a *boundary point*—equations with four unknowns need not be finitely parametrizable. In the same paper, Hmelevskii gave a concrete example of such an equation for which the set of solutions cannot be finitely characterized: $xyz = zvx$. Nevertheless, he did characterize all solutions of this equation, but using an infinite number of parameters. This characterization has been recently simplified by Weinbaum in [25]. Hmelevskii's results are also discussed in a chapter, [2], in Lothaire [13].

In this paper, we present a short, elementary proof for the nonparametrizability of the equation $xyz = zvx$, simplifying Hmelevskii's proof. Our approach uses only elementary techniques on word equations and the well-known property of the Fibonacci word of being fourth power free, see [9].

The paper is organized as follows. In Section 2 we fix our terminology and introduce some basic notions and results. In Section 3 we present the main result of this paper, the fact that the equation $xyz = zvx$ is not finitely parametrizable. In Section 4 we give some concluding remarks.

The conference version of this paper was published in [20].

2. Preliminaries

In this section we give basic definitions that we need later on, some already known results and also one preliminary result we will use in the main proof. For more details on combinatorics on words we refer to [3] and [13].

Let $\mathbb{N}$ denote the set of all nonnegative integers. For a finite alphabet Σ let us denote by Σ^* the set of all finite words over Σ, by 1 the empty word, and by Σ^+ the set of all nonempty finite words over Σ, $\Sigma^+ = \Sigma^* \backslash \{1\}$. A word *u* is a *factor* (resp. *left factor* or *prefix*, *right factor* or *suffix*) of *w* if we can write $w = xuy$ (resp. $w = uy$, $w = xu$) for some words $x, y \in \Sigma^*$. We use the notation $pref_k(w)$ (resp. $suf_k(w)$) to denote the prefix (resp. the suffix) of length *k* of the word *w*. For a word *w*, let $Alph(w)$ denote the set of all distinct letters appearing in it and $|w|$ its *length*, i.e., the number of letters in *w*. Two words *u* and *v* are said to be *conjugates* if there exist words $x, y \in \Sigma^*$ such that $u = xy$ and $v = yx$. The following lemma is a well-known characterization for the conjugacy of two words, see [10].

Lemma 1. *Let $u, v \in \Sigma^+$. The following conditions are equivalent*:
(1) *u and v are conjugates,*

(2) *there exists a word z such that* $uz = zv$,
(3) *there exist words* z, p, q *and a nonnegative integer i such that* $u = pq$, $v = qp$, *and* $z = (pq)^i p = p(qp)^i$.

The *Fibonacci sequence* of numbers is a recursive sequence where the first two values are 1 and each successive term is obtained by adding together the two previous ones. However, to simplify future notations we overlook the first term of this sequence and denote:

$$f_0 = 1, \quad f_1 = 2, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \geqslant 2. \tag{1}$$

By analogy, we define now the *Fibonacci word* as the limit of the sequence of words given by the following recurrence formula:

$$w_0 = a, \quad w_1 = ab, \quad w_n = w_{n-1}w_{n-2} \quad \text{for all } n \geqslant 2 \tag{2}$$

and notice that for every $n \geqslant 0$, $|w_n| = f_n$ and

$$suf_2(w_n) = \begin{cases} ab & \text{if } n \text{ is odd} \\ ba & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \geqslant 1. \tag{3}$$

One of the important properties of this word, see [9], is that it is 4-*free*, i.e., for any $w \in \Sigma^+$, w^4 does not appear as one of its factors. In fact, in [19] it is shown that the Fibonacci word is $(2 + \phi)^-$-free, where $\phi = \frac{1}{2}(\sqrt{5} + 1)$ is the golden number, but in our considerations we need only the 4-freeness property.

Let us consider next, the words given by the following formula:

$$G_n = pref_{(f_n-2)} w_n \quad \text{for all } n \geqslant 1, \tag{4}$$

where f_n's are the numbers in the sequence (1). Thus, for all indexes *n*, the words G_n are prefixes of the Fibonacci word.

Let Σ be a finite alphabet and *X* a finite set of unknowns, with $\Sigma \cap X = \emptyset$. An *equation* over the alphabet Σ, with *X* as the set of unknowns is a pair

$$(u, v) \in (\Sigma \cup X)^* \times (\Sigma \cup X)^*.$$

Normally, an equation is written as $u = v$. We say that an equation is *constant-free* if both *u* and *v* contain only elements from *X*. The *total length* of an equation $u = v$ is the sum of the lengths of *u* and *v*. An equation $u = v$ is called *reduced* if $pref_1(u) \neq pref_1(v)$ and $suf_1(u) \neq suf_1(v)$. A *solution* of an equation $u = v$ is a morphism $\varphi : (X \cup \Sigma)^* \to \Sigma^*$ such that $\varphi(u) = \varphi(v)$ and $\varphi(a) = a$ for every $a \in \Sigma$. Consequently, a solution is a $|X|$-tuple of words over the alphabet Σ.

We define *word parameters* and *numerical parameters* as parameters whose values are words over the alphabet Σ, and nonnegative integers, respectively. Let Δ be a new alphabet. A *parametric word* over Δ is defined inductively as follows:
(1) Every letter in Δ is a parametric word.
(2) If δ is a parametric word, and *k* is a numerical parameter, then δ^k is a parametric word.
(3) If δ_1 and δ_2 are parametric words, then also $\delta_1\delta_2$ is a parametric word, where $\delta_1\delta_2$ is obtained by concatenating δ_1 and δ_2.

We define the concatenation of parametric words as the natural extension of concatenation of words.

Let us consider now some examples of parametric words. For instance, Lemma 1 states that the set of solutions of the equation $uz = zv$ is expressed by a triple of parametric words $(u, v, z) = (pq, qp, (pq)^i p)$, with p and q word parameters, and i numerical parameter. Then, let us consider the parametric word $(x^k y)^l z$, with x, y, and z word parameters and k and l numerical parameters. If we fix some values for all numerical parameters, e.g. $k = 2$ and $l = 3$, we obtain the word *xxyxxyxxyz* belonging to the free monoid generated by x, y, and z.

Given a parametric word δ, every assignment φ of values in Σ^* to the letters of Δ, and of values in $\mathbb{N}$ to the numerical parameters, defines a unique word in Σ^*, called the *value* of δ under φ, and is denoted by $\varphi(\delta)$.

We say that an equation over Σ and with n unknowns is *parametrizable* if there exists a finite number of n-tuples of parametric words $F_1, \ldots, F_k$ over an alphabet Δ such that every value of these n-tuples is a solution of the equation, and every solution is a value of at least one of these n-tuples. Every F_i, $1 \leqslant i \leqslant k$, is a *parametric solution* of the considered equation.

Example 1. An example of a parametrizable equation is $(xy)^n x = (uv)^m u$, with $n, m \geqslant 2$, which was solved in [5,26]. The set of solutions of this equation is characterized by the following parametric solution:

$$(x, y, u, v) = ((t_1t_2)^i t_1,\ t_2(t_1t_2)^j,\ (t_1t_2)^r t_1,\ t_2(t_1t_2)^s),$$

where i, j, r, s are numerical parameters and t_1, t_2 are word parameters.

The following technical result is useful for our later considerations.

Lemma 2. *Let $u = v$ be a constant-free equation with n unknowns over the alphabet Σ with $|\Sigma| \geqslant 2$ and $(T_1, \ldots, T_n)$ be a parametric solution. Let $(V_1, \ldots, V_n) \in (\Delta^*)^n$ be the n-tuple obtained from $(T_1, \ldots, T_n)$ by assigning fixed values to all numerical parameters. Then, $(V_1, \ldots, V_n)$ is a solution of the equation $u = v$ over Δ.*

Proof. Since $(T_1, \ldots, T_n)$ is a parametric solution of the equation $u = v$, for any assignment φ, $(\varphi(T_1), \ldots, \varphi(T_n))$ is a solution of $u = v$ over the alphabet Σ.

Let $u' = v'$ be the relation over Δ obtained by substituting $(V_1, \ldots, V_n)$ in $u = v$. Suppose now that $u' = v'$ is not an identity over Δ. Then, up to cancelling a common prefix, we may assume that

$$u' = \alpha u'' \quad \text{and} \quad v' = \beta v'' \quad \text{with } u'', v'' \in \Delta^*,\ \ \alpha, \beta \in \Delta \text{ and } \alpha \neq \beta. \tag{5}$$

Consider now any assignment φ for $(T_1, \ldots, T_n)$ that assumes the numerical values fixed in $(V_1, \ldots, V_n)$ and takes $\varphi(\alpha) = a$ and $\varphi(\beta) = b$, where $a, b \in \Sigma$, $a \neq b$. Then $(\varphi(T_1), \ldots, \varphi(T_n)) = (\varphi(V_1), \ldots, \varphi(V_n))$ and consequently, it follows from (5) that $(\varphi(T_1), \ldots, \varphi(T_n))$ is not a solution of $u = v$ over Σ. But, this is a contradiction. So, $u' = v'$ is an identity over Δ, i.e., $(V_1, \ldots, V_n)$ is a solution of the equation $u = v$ over Δ. $\square$

One of the referees of this paper proposed a second proof of this result using the fact that $(V_1, \dots, V_n) \in (\Delta^*)^n$ and for every assignment $\varphi : \Delta \to \Sigma^*$, $(\varphi(V_1), \dots, \varphi(V_n))$ is a solution over Σ; hence if we choose φ to be injective we obtain that $(V_1, \dots, V_n)$ is a solution over Δ.

3. Main result

Hmelevskii in [8] proved that the set of solutions of any constant-free equation in three unknowns is finitely parametrizable. However, the situation changes when the number of unknowns increases. In the same paper, Hmelevskii also gives a concrete example of an equation with four unknowns which is not parametrizable: $xyz = zvx$. Here, we reprove this result. Our proof becomes shorter and easier to understand due to an efficient use of some basic techniques on word equations and of the property of the Fibonacci word of being 4th power free.

Theorem 3. *The set of solutions of the equation $xyz = zvx$ over an alphabet with at least two distinct letters is not parametrizable.*

Proof. Let us begin by supposing that the equation $xyz = zvx$ is parametrizable. By definition, this means that we have a finite number of 4-tuples (T_1, T_2, T_3, T_4), where all T_i's, $1 \leqslant i \leqslant 4$ are parametric words, from which we can obtain all the solutions, and also any solution matches at least one of the patterns (T_1, T_2, T_3, T_4). Thus, we also have a finite number of word and numerical parameters in the parametric words of the parametrization.

Let (T_1, T_2, T_3, T_4) be one of the parametric solutions of the equation $xyz = zvx$, and let Δ and Λ be the sets of word parameters and numerical parameters respectively, which appear in formulas T_i, $1 \leqslant i \leqslant 4$. Let (V_1, V_2, V_3, V_4) be the 4-tuple obtained from (T_1, T_2, T_3, T_4) by assigning some fixed values to each numerical parameter. Now, note that for every $1 \leqslant i \leqslant 4$, we have $V_i \in \Delta^*$ and we can define the length, prefix, suffix, and alphabet for such a word like in Section 2 and denote them as usual by $|V_i|$, $pref_k(V_i)$, $suf_k(V_i)$, and $Alph(V_i)$, respectively. Moreover, from Lemma 2, (V_1, V_2, V_3, V_4) is a solution of $xyz = zvx$ over the alphabet Δ, i.e., $V_1V_2V_3 = V_3V_4V_1$ is an identity over the set of word parameters. In particular we see that the words V_2 and V_4 have the same length, i.e., $|V_2| = |V_4|$.

We also notice that, due to the (almost) symmetric form of the equation $xyz = zvx$, if (V_1, V_2, V_3, V_4) is a solution over Δ, then also (V_3, V_4, V_1, V_2) is a solution over Δ. So, we can suppose, without loss of generality that $|V_1| \geqslant |V_3|$.

We prove now that for such (V_1, V_2, V_3, V_4) we must have

$$V_2 = V_4 \quad \text{or} \quad Alph(V_1V_3) \subseteq Alph(V_2V_4).$$

We discuss here three cases, depending on the length of V_1.

Case 1: $|V_1| = |V_3|$. Since $V_1V_2V_3 = V_3V_4V_1$ is an identity over Δ, we obtain $V_1 = V_3$ and $V_2 = V_4$. Thus, in this case (V_1, V_2, V_3, V_4) is such that $V_2 = V_4$.

Case 2: $|V_3| < |V_1| \leqslant |V_3V_4| = |V_2V_3|$. In this case, we can write $V_1 = V_3P$ and $V_1 = QV_3$, where $P \in \Delta^+$ is a prefix of V_4 and $Q \in \Delta^+$ is a suffix of V_2. Thus, $QV_3 = V_3P$

which implies, by Lemma 1, that we have

$$Q = SW, \quad P = WS, \quad V_3 = (SW)^i S, \quad V_1 = (SW)^{i+1} S,$$

where S, W are two words over the alphabet Δ, and $i \geqslant 0$ is a nonnegative integer. Now, let $\gamma \in Alph(V_1V_3) \subseteq \Delta$. But, this means that γ appears in at least one of the words S or W, so it appears both in P and in Q, i.e., $\gamma \in Alph(V_2V_4)$.

Thus, in this case, (V_1, V_2, V_3, V_4) is such that $Alph(V_1V_3) \subseteq Alph(V_2V_4)$.

Case 3: $|V_1| > |V_3V_4|$. In this case, we can write $V_1 = V_3V_4P$ and $V_1 = QV_2V_3$, with $P, Q \in \Delta^+$. Substituting these relations in the identity $V_1V_2V_3 = V_3V_4V_1$, we obtain that $P = Q$. So, in this case (V_1, V_2, V_3, V_4) is a solution of the equation $xyz = zvx$ over Δ if and only if

$$PV_2V_3 = V_3V_4P \quad \text{and} \quad V_1 = PV_2V_3,$$

i.e., (P, V_2, V_3, V_4) is a solution of the equation $xyz = zvx$ over Δ and $V_1 = PV_2V_3$. If $V_1 = P$, i.e., $V_2V_3 = 1$, then we immediately obtain that $V_2 = V_3 = V_4 = 1$. So, in this case the solution (V_1, V_2, V_3, V_4) is such that $V_2 = V_4$. Otherwise (i.e., $V_1 \neq P$), we reduced the solution (V_1, V_2, V_3, V_4) to (P, V_2, V_3, V_4), with $|P| < |V_1|$, and $V_1 = PV_2V_3$. Now, we can repeat this reduction step until $|P| \leqslant |V_3V_4|$, which means that we can apply *Case* 1 or *Case* 2 for the new solution (P, V_2, V_3, V_4). So, for (P, V_2, V_3, V_4) we have $V_2 = V_4$ or $Alph(PV_3) \subseteq Alph(V_2V_4)$. But, since $V_1 = PV_2V_3$, this implies that for the solution (V_1, V_2, V_3, V_4) we have $V_2 = V_4$ or $Alph(V_1V_3) \subseteq Alph(V_2V_4)$.

Thus, for any parametric solution (V_1, V_2, V_3, V_4) containing only word parameters, i.e., obtained from some parametric solution (T_1, T_2, T_3, T_4) by fixing some values for all numerical parameters, we must have $V_2 = V_4$ or $Alph(V_1V_3) \subseteq Alph(V_2V_4)$.

Now, we claim that, for the words G_k defined by formula (4), the 4-tuple (G_k, ab, G_{k-1}, ba) is a solution for the equation $xyz = zvx$ over the alphabet Σ for any odd index k. To prove our claim, it is enough to verify that $G_kabG_{k-1} = G_{k-1}baG_k$, for any odd index k. Using formulas (3) and (4), this identity is equivalent to $w_kG_{k-1} = w_{k-1}G_k$. Now, using formula (2), we obtain $w_{k-2}G_{k-1} = G_k$, and this can be proved by induction on k using formulas (2) and (4).

Consider now an assignment φ and a parametric solution (T_1, T_2, T_3, T_4) such that $(\varphi(T_1), \varphi(T_2), \varphi(T_3), \varphi(T_4)) = (G_k, ab, G_{k-1}, ba)$ for some odd index k. We prove now that the length of $\varphi(T_1)$ is bounded by some constant.

First, since every G_k is a prefix of the Fibonacci word, which is 4-free, and $\varphi(T_1) = G_k$ for some k odd, we must have that every power of a factor in $\varphi(T_1)$ is less than 4. Consequently, for every numerical parameter λ appearing in the parametric word T_1, we must have $\varphi(\lambda) < 4$.

Second, consider the 4-tuple (V_1, V_2, V_3, V_4) over Δ obtained from the parametric solution (T_1, T_2, T_3, T_4) by substituting every numerical parameter λ with its value $\varphi(\lambda)$. Since $\varphi(T_i) = \varphi(V_i)$ for every $1 \leqslant i \leqslant 4$, we obtain the following relations:

$$G_k = \varphi(V_1), \quad G_{k-1} = \varphi(V_3), \tag{6}$$

$$ab = \varphi(V_2), \quad ba = \varphi(V_4). \tag{7}$$

Notice now that the values of formulas V_2 and V_4 under the assignment φ must be *ab* and *ba*, respectively, so $V_2 \neq V_4$. Thus, the first part of this proof implies that this 4-tuple (V_1, V_2, V_3, V_4) must be such that $Alph(V_1V_3) \subseteq Alph(V_2V_4)$. Moreover, from relations (7) we observe that $|\varphi(\alpha)| \leqslant 2$, for any word parameter $\alpha \in Alph(V_2V_4)$, which implies that $|\varphi(\alpha)| \leqslant 2$ also for any word parameter $\alpha \in Alph(V_1V_3)$.

So, what we obtained is that for any numerical parameter λ which appears in T_1, $\varphi(\lambda) < 4$, and for any word parameter α which appears in T_1, $|\varphi(\alpha)| \leqslant 2$. Consequently, $|\varphi(T_1)|$ is bounded by some positive constant, i.e., we cannot generate arbitrarily large solutions (G_k, ab, G_{k-1}, ba), with k odd. But this is a contradiction since words G_k can be arbitrarily large.

Thus, the equation $xyz = zvx$ is not parametrizable.

4. Conclusions

Although the existence of solutions of a word equation is decidable due to Makanin's result, the general structure of solutions is difficult to find. Hmelevskii in [8], proved that the solutions of constant-free word equations with three unknowns can be expressed using only finitely many parameters. He also proved, in the same paper, that this result is no longer valid for equations with four unknowns and he gave as a concrete example the equation $xyz = zvx$.

In this paper, we give a short elementary proof for the nonparametrizability of the equation $xyz = zvx$. The "simplicity" of our solution comes from the fact that we only use some elementary techniques on word equations and some basic properties of the Fibonacci word. Moreover, this elementary solution gives us hope that there may exists also a simpler proof for Hmelevskii's result on the parametrizability of constant-free word equations in three unknowns.

It is worth noting that in the conference version of the paper, [20], we also investigated the connection between the graph associated to an equation and the parametrizability of its solutions and we succeeded to give a simple, necessary, (though nonsufficient) condition for an equation to be nonparametrizable.

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