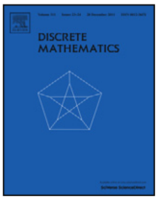
Discrete Mathematics 312 (2012) 2550–2560

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc



Proper connection of graphs

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ARTICLE INFO

Article history:
Received 31 August 2010
Accepted 1 September 2011
Available online 1 October 2011

Keywords:
Proper coloring
Proper connection

ABSTRACT

An edge-colored graph G is k-proper connected if every pair of vertices is connected by k internally pairwise vertex-disjoint proper colored paths. The k-proper connection number of a connected graph G, denoted by $pc_k(G)$, is the smallest number of colors that are needed to color the edges of G in order to make it k-proper connected. In this paper we prove several upper bounds for $pc_k(G)$. We state some conjectures for general and bipartite graphs, and we prove them for the case when $k = 1$. In particular, we prove a variety of conditions on G which imply $pc_1(G) = 2$.

1. Introduction and notation

The notion of proper edge colorings has been very important over the years since the classical work of Vizing [15]. More recent works like [1,9,16] have considered proper colored subgraphs as opposed to looking at the entire graph. There is even a survey of work concerning alternating cycles [2] in which the authors collect results concerning colorings of graphs and multigraphs. Here *alternating* means the colors of the edges alternate as you traverse the cycle thus making it proper colored. The problem of finding an alternating cycle is precisely the problem of finding a proper colored cycle when only two colors are available.

Similarly, some researchers have considered rainbow colored subgraphs (meaning that every edge has a distinct color). In fact, our definition of the proper connection number $pc(G)$ is a natural extension of the rainbow connection number $rc(G)$ as defined in [5] and studied in [3,4,6,11,14]. Many of the conditions we assume in this work are much weaker than those needed to produce upper bounds on the rainbow connection number $rc(G)$. This can be explained by the fact that it takes far fewer colors to make a path properly colored than are needed to make it rainbow colored.

A path in an edge-colored graph is said to be *properly edge-colored* (or *proper*), if every two adjacent edges differ in color. An edge-colored graph G is *k-proper connected* if any two vertices are connected by k internally pairwise vertex-disjoint proper paths. We define the *k-proper connection number* of a k-connected graph G, denoted by $pc_k(G)$, as the smallest number of colors that are needed in order to make Gk-proper connected. Clearly, if a graph is k-proper connected, then it is also k-connected. Conversely, any k-connected graph has an edge coloring that makes it k-proper connected; the number of colors is easily bounded by the edge chromatic number which is well known to be at most $\Delta(G)$ or $\Delta(G) + 1$ by Vizing's

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doi:10.1016/j.disc.2011.09.003

Theorem [15] (where $\Delta(G)$, or simply Δ, is the maximum degree of a vertex in G over all its vertices). Thus $pc_k(G) \le \Delta + 1$ for any k-connected graph G.

In this work, all graphs considered are simple, without loops or multiedges. The edge between the vertices v and w is denoted by vw, and its color by $c(v, w)$. The *rainbow degree* of a vertex v, denoted by $rd(x)$, equals the maximum number of distinct colors presented on edges incident to v. The *length* of a path or of a cycle is the number of its edges. An edge-colored graph is connected if the underlying non-colored graph is connected. We denote the *connectivity* of a graph by $\kappa(G)$. Throughout this paper, all edge-colored graphs are considered to be connected unless otherwise specified. Given a colored path $P = v_1v_2 \dots v_{s-1}v_s$ between any two vertices v_1 and v_s, we denote by start(P) the color of the first edge in the path, i.e. $c(v_1, v_2)$, and by end(P) the last color, i.e. $c(v_{s-1}, v_s)$. If P is just the edge v_1v_s then start$(P) =$ end$(P) = c(v_1, v_s)$.

This paper is organized as follows: In Section 2 we study $pc_k(G)$ for bipartite graphs. We state a conjecture, prove several small results and finally we prove the conjecture for $k = 1$, that is, for $pc(G)$. In Section 3, we study $pc(G)$ for general graphs and prove non-trivial bounds, improving Vizing's trivial bound of $\Delta+1$. Then, motivated by both of these sections, we state a conjecture regarding $pc_k(G)$ for general graphs. In Section 4 we prove a bound concerning the minimum degree of G. Finally we present the conclusions of the work and some open problems.

2. Bipartite graphs

In this section, we study proper connection numbers in bipartite graphs. We state a general conjecture for $pc_k(G)$ where G is a bipartite graph with some specific connectivity that depends on k. Following that, we show that this conjecture is best possible in the sense of connectivity. Later, we prove some results for specific classes of graphs such as complete bipartite graphs with weaker connectivity assumptions than that which is required for the conjecture. Then we prove that the conjecture is true for complete bipartite graphs. Finally, we study the case $k = 1$ and obtain results for trees and other graphs depending on their connectivity. We end the section by obtaining, as main result, the proof of the conjecture for the special case $k = 1$ and some corollaries stemming from it.

Conjecture 1. *If G is a $2k$-connected bipartite graph with $k \ge 1$, then $pc_k(G) = 2$.*

If true, Conjecture 1 is the best possible in the sense of connectivity. In the following we present a family of bipartite graphs which are $(2k - 1)$-connected with the property that $pc_k(G) > 2$. It is also clear that we cannot exchange the vertex connectivity for edge connectivity since it is easy to find graphs with connectivity 1 which have edge connectivity $2k$.

Consider the complete bipartite graph $G = K_{p,q}$ with $p = 2k - 1$ $(k \ge 1)$ and $q > 2^p$ where $G = V \cup W$, $V = \{v_1, v_2, \dots, v_p\}$ and $W = \{w_1, w_2, \dots, w_q\}$. Clearly G is $(2k - 1)$-connected. We will show that $pc_k(G) > 2$.

Proposition 1. *Let $p = 2k - 1$ $(k \ge 1)$ and $q > 2^p$. Then $pc_k(K_{p,q}) > 2$.*

Proof. Suppose that $pc_k(G) = 2$ and consider a k-proper connected coloring of G with 2 colors. For each vertex $w_i \in W$, there exists a p-tuple $C_i = (c_1, c_2, \dots, c_p)$ so that $c(v_j, w_i) = c_j$ for $1 \le j \le p$. Therefore, each vertex $w_i \in W$ has 2^p different ways of coloring its incident edges using 2 colors. Since $q > 2^p$, there exist at least two vertices $w_i, w_j \in W$ such that $C_i = C_j$. As $pc_k(G) = 2$, there exist k internally disjoint proper paths in G between w_i, w_j. Using this, we will arrive at a contradiction. First observe that one of these paths between w_i, w_j (say P) must have only one intermediate vertex $v_l \in V$ since otherwise, if all the paths have at least two intermediate vertices in V, we would have $|V| \ge 2k$, which is a contradiction. Hence, as $C_i = C_j$ we have $c(v_l, w_i) = c(v_l, w_j)$ and therefore the path P is not properly colored, leading to a contradiction. □

Based on the previous result we prove the following. The proof methods used for Theorem 1 are similar to the concept of color coding, as applied in [5] for proving results about multipartite graphs.

Theorem 1. *Let $G = K_{n,3}$ then*

$$pc_2(G) = \begin{cases} 2 & \text{if } 3 \le n \le 6 \\ 3 & \text{if } 7 \le n \le 8 \\ \lceil \sqrt[3]{n} \rceil & \text{if } n \ge 9 \end{cases}$$

Proof. It is easy to check that $pc_2(G) = 2$ for $3 \le n \le 6$ and $pc_2(G) = 3$ for $7 \le n \le 8$. Now let $n \ge 9$. We will give a 2-proper coloring of G using $c = \lceil \sqrt[3]{n} \rceil$ colors and we will also show that this is the best possible. Consider the bipartition of $G = V \cup W$ such that $|V| = n$ and $|W| = 3$. Let $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, w_2, w_3\}$. For each vertex $v_i \in V$, we consider a 3-tuple $C_i = (c_1, c_2, c_3)$ so that $c(v_i, w_j) = c_j$ for $1 \le j \le 3$. Therefore, each vertex $v_i \in V$ has c^3 different ways of coloring its incident edges using c colors. We then color the edges of G as follows. If $c \ge 4$ then we color the edges of $(c - 1)^3$ vertices of V with all the different triples of $c - 1$ colors and, for the remaining vertices, we choose different triples but this time using the cth color. If $c = 3$, we just choose different triples of colors but first choosing from the $c!$ colorings in which all three colors differ. Under this coloring, for each pair of vertices $v_i, v_j \in V$, we have that $C_i \ne C_j$ for all $1 \le i \ne j \le n$.

Before proving that this coloring is 2-proper, it is easy to see that G cannot be colored to make it 2-proper connected using fewer than c colors by following the same argument as in Proposition 1. That is, if we use fewer than c colors, there must exist at least two vertices $v_i, v_j \in V$ such that $C_i = C_j$, a contradiction.

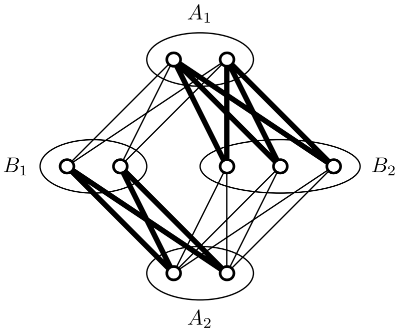

Fig. 1. Coloring of $K_{4,5}$. Thin edges represent color 1 and bold edges color 2.

Now consider two vertices $v_i, v_j \in V$ and we would like show the existence of 2-proper paths between them. Since $C_i \neq C_j$, we know that at least one of the three colors is different. If two or three are different, then we have 2-proper paths of the form v_i, w_k, v_j and v_i, w_l, v_j such that $c(v_i, w_k) \neq c(v_j, w_k)$ and $c(v_i, w_l) \neq c(v_j, w_l)$. Suppose now that exactly one of the three colors is different, say c_1 without losing generality, then v_i, w_1, v_j is a proper path. For the second path, there exists a vertex $v_k \in V$ such that, by construction of the coloring, $c(v_i, w_2) \neq c(v_k, w_2)$, $c(v_j, w_3) \neq c(v_k, w_3)$ and $c(v_k, w_2) \neq c(v_k, w_3)$. Therefore v_i, w_2, v_k, w_3, v_j is a proper path between v_i and v_j.

Next consider $w_i, w_j \in W$, it is clear that there exist two vertices $v_k, v_l \in V$ such that C_k and C_l have both colors different to w_i, w_j. Therefore w_i, v_k, w_j and w_i, v_l, w_j are proper paths. Finally, we consider the case where $v_i \in V$ and $w_j \in W$. The edge v_iw_j provides a trivial proper path. For the second path, simply choose other appropriate vertices $v_k \in V$ and $w_l \in W$ such that v_i, w_l, v_k, w_k results in a proper path. These vertices exist by the constructed coloring of G. As no cases are left, the theorem holds. □

Now we prove the conjecture for complete bipartite graphs.

Theorem 2. *Let $G = K_{n,m}$, $m \geq n \geq 2k$ for $k \geq 1$. Then $pc_k(G) = 2$.*

Proof. Take the bipartition of $G = A \cup B$. Then split each set A and B into the sets A_1, A_2, B_1, B_2 such that $|A_i|, |B_i| \geq k$ for $i = 1, 2$. This is clearly possible since $|A|, |B| \geq 2k$. Now color the graph in the following way. Put $c(v, w) = 1$ for all $v \in A_1$ and $w \in B_1$, and for all $v \in A_2$ and $w \in B_2$. Finally put color 2 to the rest of the edges, that is, $c(v, w) = 2$ for all $v \in A_1$ and $w \in B_2$, and for all $v \in A_2$ and $w \in B_1$ (see Fig. 1). Now we prove that this coloring produces k proper paths between each pair of vertices of G. First, consider two vertices $v, w \in A_1$ (an identical argument holds for pairs in other sets). Since the cardinality of each set is at least k, we form k proper paths v, b_1, a_2, b_2, w choosing $b_1 \in B_1$, $a_2 \in A_2$ and $b_2 \in B_2$. If $v \in A_1$ and $w \in A_2$ (similarly for $v \in B_1$ and $w \in B_2$) we have at least $2k$ proper paths formed as v, b, w for each choice of $b \in B$. The final case is when $v \in A_1$ and $w \in B_1$ (that is, v and w are adjacent). Here we have at least $k+1$ proper paths, as follows. One path is simply the edge vw while the k that remain are of the form v, b_2, a_2, w for each choice of $b_2 \in B_2$ and $a_2 \in A_2$. This completes the proof. □

Now we will study the case $k = 1$, that is $pc(G)$. By König's Bipartite Theorem [10] we have that the edge chromatic number is Δ for bipartite graphs and therefore Δ is a trivial upper bound for $pc(G)$ for any bipartite graph G. Then we obtain this trivial corollary.

Corollary 1. *If G is a tree then $pc(G) = \Delta$.*

We present now the following proposition.

Proposition 2. *If $pc(G) = 2$ then $pc(G \cup v) = 2$ as long as $d(v) \geq 2$.*

Proof. Let u, w be two neighbors of v in G. Since we have assumed there is a 2-coloring of G so that G is properly connected, there is a properly colored path P from u to w in G. Color the edge uv so that $c(u, v) \neq \text{start}(P)$ and color vw so that $c(v, w) \neq \text{end}(P)$. Since every vertex of G has a properly colored path to a vertex of P, every vertex has a properly colored path to v through either u or w, thereby completing the proof. □

The following theorem is the main result of the section. It improves upon the upper bound of Δ by König to the best possible whenever the graph is bipartite and 2-edge-connected.

Theorem 3. *Let G be a graph. If G is bipartite and 2-connected then $pc(G) = 2$ and there exists a 2-coloring of G that makes it properly connected with the following strong property. For any pair of vertices v, w there exists two paths P_1, P_2 between them (not necessarily disjoint) such that* $\text{start}(P_1) \neq \text{start}(P_2)$ *and* $\text{end}(P_1) \neq \text{end}(P_2)$.

Given a 2-connected graph G, let G_1 be an instance of the graph $G \setminus P$ where P is the set of internal vertices of the last ear of an ear decomposition of a G. Similarly, if the graph is 2-edge-connected, there is a (closed) ear decomposition in which an ear may attach to the previous structure at a single vertex. Therefore, using the same argument, one could easily show the result also holds for a 2-edge-connected graph G.

Proof. Suppose G is 2-connected and bipartite and consider a spanning minimally 2-connected subgraph (meaning that the removal of any edge would leave G 1-connected). For the sake of simplicity, we call this subgraph G. This proof is by induction on the number of ears in an ear decomposition of G. The base case of this induction is when G is simply an even cycle and we alternate colors on the edges.

Let P be the last ear added where the ends u and v of P are in G_1 and all internal vertices of P are in $G \setminus G_1$. Since G is minimally 2-edge-connected, we know that the length of P is at least 2.

By induction on the number of ears, we obtain a 2-coloring of G_1 so that G_1 has the strong property. Color P with alternating colors.

Finally we show that this coloring of G is properly connected with the strong property. Every pair of vertices in C has the strong property since C is an alternating even cycle. Also, by induction, every pair of vertices in G_1 has the strong property. Let $x \in G \setminus C$ and let $y \in P$. The pair xu has the strong property so there exists a path Q_u from x to u so that xQ_uuPy forms a proper path Q'_u. Similarly the pair xv has the strong property so there exists a path Q_v from x to v so that xQ_vvPy is a proper path Q'_v. Since C is a proper cycle, Q'_u and Q'_v must have different colors on the edges incident to y. Note also that, since G is bipartite, the parity of the length of Q'_u is the same as the parity of the length of Q'_v. Hence, Q'_u and Q'_v must also have different colors on the edges incident to x. This shows that x and y have the strong property, thereby completing the proof. □

As a result of Theorem 3 we obtain the following corollaries.

Corollary 2. *Let G be a graph. If G is 3-connected and noncomplete, then $pc(G) = 2$ and there exists a 2-edge-coloring of G that makes it proper connected with the following strong property. For any pair of vertices v, w there exist two paths P_1, P_2 between them (not necessarily disjoint) such that $\text{start}(P_1) \neq \text{start}(P_2)$ and $\text{end}(P_1) \neq \text{end}(P_2)$.*

Proof. By [13], any 3-connected graph has a spanning 2-connected bipartite subgraph. Then the result holds by Theorem 3. □

3. General graphs

We begin this section by studying $pc(G)$ for a general graph G. We show some easy results for specific classes such as complete graphs and cycles. Following this, we prove a result analogous to that obtained in the previous section for 2-connected graphs but using 3 colors instead of 2. We also show that this bound is sharp by presenting a 2-connected graph for which 2 colors are not enough to make it proper connected. As a main result of the section, we state an upper bound for $pc(G)$ for general graphs that can be possibly reached as we saw in the previous section. Based on the results of 2-connected graphs we extend Conjecture 1 to general graphs and finally we prove this for complete graphs.

By Vizing's Theorem [15], we have that the edge chromatic number of any graph is at most $\Delta + 1$ and therefore $\Delta + 1$ is a trivial upper bound for $pc(G)$ for any graph G. First we present some easy results.

Fact 1. *A graph G has $pc(G) = 1$ if and only if G is complete.*

By using alternating colors, it is easy to see that any path of length at least 2 and any cycle of length at least 4 has proper connection number 2.

Also it is clear that the addition of an edge to G cannot increase $pc_k(G)$.

Fact 2. *For $n \geq 3$, $pc(P_n) = 2$ and if $n \geq 4$, $pc(C_n) = 2$. Furthermore, pc_k is monotone decreasing with respect to edge addition.*

The following theorem improves the Vizing's $\Delta + 1$ upper bound whenever the graph is 2-connected. This result is a natural extension of Theorem 3.

Theorem 4. *Let G be a graph. If G is 2-connected, then $pc(G) \leq 3$ and there exists a 3-edge-coloring of G that makes it proper connected with the following strong property. For any pair of vertices v, w there exist two paths P_1, P_2 between them (not necessarily disjoint) such that $\text{start}(P_1) \neq \text{start}(P_2)$ and $\text{end}(P_1) \neq \text{end}(P_2)$.*

As in Theorem 3, we note that an edge-connected version of this result is immediate from the proof.

Proof. Suppose G is a 2-connected graph and consider a spanning minimally 2-connected subgraph (meaning that the removal of any edge would leave G 1-connected). For the sake of simplicity, we call this subgraph G. This proof is by induction on the number of ears in an ear decomposition of G. The base case of this induction is when G is simply a cycle and we properly color the edges with at most 3 colors.

Let P be the last ear added in an ear decomposition of G and let G_1 be the graph after removal of the internal vertices of P. Since G is assumed to be minimally 2-connected, we know that P has at least one internal vertex. Let u and v be the vertices

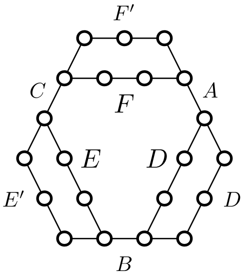

Fig. 2. Smallest 2-connected graph with $pc(G) = 3$.

of $P \cap G_1$ so $P = uu_1u_2 \dots u_pv$. By induction, there is a 3-coloring of G_1 which is proper connected with the strong property. Color the edges of G_1 as such.

Within this coloring, there exist two paths P_1 and P_2 from u to v such that $\text{start}(P_1) \neq \text{start}(P_2)$ and $\text{end}(P_1) \neq \text{end}(P_2)$. If possible, properly color P so that $c(u, u_1) \notin \{\text{start}(P_1), \text{start}(P_2)\}$ and $c(u_p, v) \notin \{\text{end}(P_1), \text{end}(P_2)\}$. Note that this is always possible if either P has at least 2 internal vertices or $\{\text{start}(P_1), \text{start}(P_2)\} \cup \{\text{end}(P_1), \text{end}(P_2)\} = \{1, 2, 3\}$. It will become clear that this is the easier case so will assume this is not the case, namely that P has only one internal vertex x and $\{\text{start}(P_1), \text{start}(P_2)\} \cup \{\text{end}(P_1), \text{end}(P_2)\} = \{1, 2\}$.

Color the edge xu with color 3 and xv with color 2 (supposing that $\text{end}(P_2) = 2$). We will show that this coloring of G is proper connected with the strong property. For any pair of vertices in G_1, there is a pair of proper paths connecting them with the strong property by induction. Since $P \cup P_1$ forms a proper cycle, any pair of vertices in this cycle also have the desired paths. Let $y \in G_1 \setminus P_1$ and note that our goal is to find two proper paths from x to y with the strong property.

Since y and u are both in G_1, there exist a pair of paths P_{u_1} and P_{u_2} starting at y and ending at u with the strong property. Similarly, there exist two paths P_{v_1} and P_{v_2} starting at y and ending at v with the strong property. Since these paths have the strong property, we know that $Q_1 = xuP_{u_i}y$ (note that the implied orientation on P_{u_i} is reversed when traversing the path from u to y) is a proper path for some $i \in \{1, 2\}$ (suppose $i = 1$) and similarly $Q_2 = xvP_{v_j}y$ is a proper path for some $j \in \{1, 2\}$ (suppose $j = 1$). These paths form the desired pair if $\text{end}(Q_1) \neq \text{end}(Q_2)$ so suppose $\text{start}(P_{v_1}) = \text{start}(P_{u_1})$.

Next consider walk $R_1 = xuP_1vP_{v_2}y$ and the path $R_2 = Q_2$. If R_1 is a path, then R_1 and R_2 are the desired pair of paths since $\text{end}(P_1) \neq c(x, v) = \text{end}(P_{v_2})$, meaning that R_1 is a proper walk. Hence, suppose R_1 is not a path and let z be the vertex closest to y on P_{v_2} which is in $P_1 \cap P_{v_2}$. Now if the path $R_1' = xuP_1zP_{v_2}y$ is a proper path, then R_1' and R_2 are the desired pair of paths so we may assume that $\text{end}(uP_1z) = \text{start}(zP_{v_2}y)$.

Finally we show that the paths $S_1 = xvP_1zP_{v_2}y$ and $S_2 = Q_1 = xuP_{u_1}y$ are proper paths from x to y with the strong property. Certainly, as noted above, S_2 is a proper path. Also, S_1 is a proper path since P_1 is proper so $\text{end}(vP_1z) \neq \text{end}(uP_1z) = \text{start}(zP_{v_2}y)$. Finally since $\text{end}(zP_{v_2}y) = \text{start}(P_{v_2}) \neq \text{start}(P_{v_1}) = \text{start}(P_{u_1})$, we see that S_1 and S_2 have the strong property. □

It is important to mention that there exist 2-connected graphs with $pc(G) = 3$ and therefore the bound obtained by Theorem 4 is reached.

Now we give an example (see Fig. 2) of such a graph and prove why two colors are not enough.

Proposition 3. *Any graph G consisting of an even cycle with the addition of three ears creating disjoint odd cycles such that each uninterrupted segment has at least 4 edges has $pc(G) = 3$.*

The assumption that each uninterrupted segment has length at least 4 is mostly for convenience. Note that the graph G (in Fig. 2) does not satisfy this condition but it can still be shown that $pc(G) = 3$ by a similar argument.

Proof. By Theorem 4, we know that $pc(G) \leq 3$ so it suffices to show that $pc(G) \neq 2$. Suppose we have a 2-coloring of G which is properly connected. Label the segments of G as in Fig. 2. Note that we may assume there are no three edges in a row of the same color within an uninterrupted segment since we could switch the color of the middle edge (making that subsegment alternating) without disturbing the proper connectivity.

We would first like to show that the segments A, B and C are all alternating. If two of these segments are not alternating, suppose A and B, then any vertex in D cannot be properly connected to any vertex of C so this is clearly not the case. This means that at most one segment, suppose A, is non-alternating. Suppose the edges uv and vw have the same color for some $u, v, w \in A$ (see Fig. 3).

There must exist a proper path from u to w so suppose there is such a path using the segments $FCEBD$. Since the following argument does not rely on the parity of this path, this assumption, as opposed to using any of D', E' or F', does not lose any generality.

Let x be a vertex in the interior of B. We already know there is a proper path from x to v using D. Since $D \cup D'$ forms an odd cycle, there can be no proper path from x to v through D'. Let $y \in E'$. In order for y to have a proper path to w, it must use the segments BD (as opposed to BD') and similarly to reach u, it must use CF (as opposed to CF'). Since $E \cup E'$ forms an

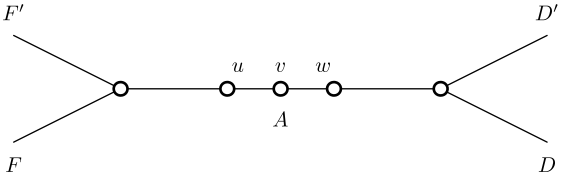

Fig. 3. Placement of vertices.

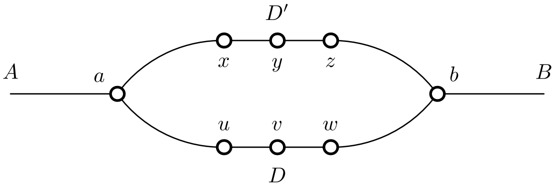

Fig. 4. Placement of vertices.

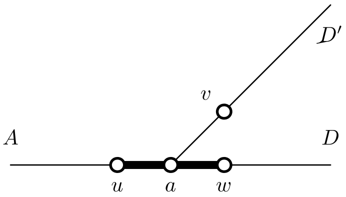

Fig. 5. Placement of vertices.

odd cycle, and yet y can reach both u and w, we know that the edges on either side of y must have the same color. This holds for all $y \in E'$, clearly a contradiction. Therefore we know that A, B and C are all alternating segments.

Next we would like to show that at least one of D or D' must be alternating (and similarly at least one of E or E' and one of F or F'). Suppose D and D' are both non-alternating. Let v be an interior vertex in D which has two edges of the same color and let y be a vertex of D' with two edges of the same color. Let u and w be the neighbors of v and let x and z be the neighbors of y (see Fig. 4). Clearly there can be at most one pair (in this case D and D') in which neither segment is alternating since there must be an alternating path from u to w and it must pass through the other segments. Also, there can be no other pairs of adjacent monochromatic edges within D and D' since u, v and w (likewise x, y and z) must have alternating paths out of the segment and we have assumed that there are no three edges of the same color in a row. Note that, in the figure, possibly $x = a, u = a, z = b$ or $w = b$.

Let $Q = D \cup D'$ and let a and b be the vertices in $D \cap D' \cap A$ and $D \cap D' \cap B$ respectively. If we let $c \in C$, then each of u, w, x and z must have an alternating path to c. Suppose the edge of A incident to a has color 1. Then both edges incident to a in Q must have color 2. This means that both edges of Q which are incident to a must be the same color (and similarly both edges of Q incident to b must have the same color). Therefore, there are exactly 4 vertices in Q for which both edges of Q have the same color. Unless $x = a$ (or possibly $z = b$, $u = a$ or $w = b$), this means that Q is even, a contradiction. Suppose $x = a$ so, in order for $z \neq b$ to have a proper path to w, we must also have $w = b$, meaning that $u \neq a$ and z so again Q is even for a contradiction. Hence, we know that at least one of D or D' must be alternating (and similarly for the other odd ears). Without loss of generality, suppose D, E and F are all alternating.

Our next goal is to show that $Q = A \cup B \cup C \cup D \cup E \cup F$ forms an alternating cycle (with the possible replacement of D with D', E with E' or F with F'). As we have shown, the only places where we can have a problem is at the intersections so let a and b be (as before) the end-vertices of D (the same argument may be applied for E or F) and suppose a is between two edges of the same colors (suppose color 1) on Q. Let u, v, w be the neighbors of a with $u \in A$, $v \in D'$ and $w \in D$ so we have assumed the edges au and aw both have color 1 (see Fig. 5 where the darker edges represent edges that must have color 1).

In order for an alternating path to get from u to w, we must either use $D' \cup D$ or Q (with the possible replacements noted above). If the path uses D', then $D \cup D'$ forms an alternating (and hence even) cycle, a contradiction. Hence, we may assume there is an alternating path from u to w through $BECFA$ (recall again that E may be replaced with E' or F with F' in this argument).

Let $x \in E'$. There is an alternating path from u to x and from w to x. Since $E \cup E'$ forms an odd cycle but x has an alternating path through B (to get to w) and through C and A (to get to u), we know that x must have two edges of the same color within E'. Since x was chosen arbitrarily, this is clearly a contradiction. This means that Q is an alternating (and hence even) cycle.

Now we simply consider one vertex in each of D', E' and F'. Since these ears form odd cycles, there exists a vertex in each segment from which (and to which) an alternating path can only go one direction on Q. By the pigeon hole principle, at least two of them must go the same direction, meaning there is no alternating path between them. This completes the proof of Proposition 3. □

If the diameter is small, then the proper connection number is also small. More formally, we get the following result.

Theorem 5. *If* $\operatorname{diam}(G) = 2$ *and* G *is* 2*-connected, then* $pc(G) = 2$.

Proof. If G is 3-connected, Corollary 2 implies that $pc(G) = 2$ so we may assume $\kappa(G) = 2$. Let $C = \{c_1, c_2\}$ be a (minimum) 2-cut of G and let $H_1, \ldots, H_t$ be the components of $G \setminus C$. Order components so that there is an integer $0 \leq s \leq t$ such that every vertex of H_i is adjacent to both c_1 and c_2 for $i > s$. Note that if $s = 0$, we have all edges from C to $G \setminus C$ so G contains a spanning 2-connected bipartite graph and by Theorem 3, $pc(G) = 2$.

For each component H_i with $i \leq s$, define subsets $H_{i,1} = N(c_1) \cap H_i$ and $H_{i,2} = N(c_2) \cap H_i$. Since each component is connected and C is a minimum cut, there must be an edge from $H_{i,1}$ to $H_{i,2}$. Let $e_i = v_{i,1}v_{i,2}$ be one such edge in each component H_i. Define the graph $G_0 = G[C \cup (\bigcup_{i=1}^{s} \{v_{i,1}, v_{i,2}\})]$. This graph is 2-connected and bipartite so $pc(G_0) = 2$ and notice that $|G_0| = 2 + 2s$.

Let G_1 be a subgraph of G obtained by adding a vertex to G_0 which has at least 2 edges into G_0. Furthermore, let G_i be a subgraph of G obtained by adding a vertex to G_{i-1} which has at least 2 edges into G_{i-1}. By Proposition 2, $pc(G_i) = 2$ for all i. We claim that there exists such a sequence of subgraphs of G such that $G_{n-(2+2s)}$ is a spanning subgraph of G. In order to prove this, suppose that G_i is the largest such subgraph of G and suppose there exists a vertex $v \in G \setminus G_i$. Certainly every vertex which is adjacent to both c_1 and c_2 is in G_i. This means $v \in H_j$ for some $1 \leq j \leq s$. Since H_j is connected, there exists a path from $v_{i,1}$ to v within H_j.

Let w be the first vertex on this path which is not in G_i. Since $\operatorname{diam}(G) = 2$, we know that w must be adjacent to at least one vertex of C. This means that $d_{G_i}(w) \geq 2$ so we may set $G_{i+1} = G_i \cup w$ for a contradiction. This completes the proof. □

Finally we prove an upper bound for $pc(G)$ for general graphs which is best possible as we saw before.

Theorem 6. *Let* G *be a connected graph. Consider* $\widetilde{\Delta}(G)$ *as the maximum degree of a vertex which is an endpoint of a bridge in* G*. Then* $pc(G) \leq \widetilde{\Delta}(G)$ *if* $\widetilde{\Delta}(G) \geq 3$ *and* $pc(G) \leq 3$ *otherwise.*

Proof. Let $B_1, B_2, \ldots, B_s$ be the blocks of G with at least 3 vertices. For each block of B_i we have the following cases.

- B_i is bipartite or 3-connected: Then by Theorem 3 and Corollary 2, B_i can be colored with 2 colors having the strong property. We color B_i in such a way.
- $\kappa(B_i) = 2$: Then by Theorem 4, B_i can be colored with 3 colors having the strong property. We color B_i in such a way.

It is easy to see that G is proper connected if there are no more uncolored edges in G since each B_i has the strong property. Thus, suppose that there remain uncolored edges in G. It is clear that these edges induce a forest F in G. We color them as follows. Take one of the blocks, say B_1, which contains a vertex $v \in B_1$ which is incident with some uncolored edges. Clearly, v is an endpoint of a bridge in G. We color these uncolored edges incident to v with different colors starting with color $rd_{B_1}(v) + 1$. Then, we have that $rd_G(v) \leq \widetilde{\Delta}(G)$. We do the same for the rest of the vertices incident to bridges in B_1. Then, we extend our coloring for each tree going out from B_1 in a Breadth First Search (BFS) way, coloring its edges with different colors (observe from Corollary 1 that $rd_G(w) \leq \widetilde{\Delta}(G) \leq \Delta$ for each vertex w in the interior of a tree) until we reach the rest of the blocks. And finally, for each of these blocks (in this order), we repeat the previous step. Before proving that this coloring makes G proper connected, it is important to mention that, if we reach a block B_i with some color $c \geq rd_{B_i}(w) + 1$, and the corresponding vertex, say w, of B_i has more than $c - rd_{B_i}(w)$ uncolored incident edges, then, when we color these edges, we do not repeat color c. Also, it is important to remark that, by coloring F in this way, we have that in any path that traverses some block from one tree in F to another, at least one of the colors before or after traversing the block is not used in the block.

We now prove that G is proper connected. Let v, w be vertices of G. It is clear that if both belong to the same block B_i, then there exists a proper path between them and the same happens if they belong to the same tree outside the blocks. If $v \in B_i$, $w \in B_j$ and $B_i \cap B_j = \{u\}$, then there exist two paths P_1, P_2 between v and u in B_i, and two paths P_3, P_4 between u and w in B_j with the strong property. Suppose without losing generality that $\operatorname{end}(P_1) \neq \operatorname{start}(P_3)$ and $\operatorname{end}(P_2) \neq \operatorname{start}(P_4)$, then we obtain the paths P_1P_3 and P_2P_4 between v and w. It is clear that $\operatorname{start}(P_1P_3) \neq \operatorname{start}(P_2P_4)$ and $\operatorname{end}(P_1P_3) \neq \operatorname{end}(P_2P_4)$ since $\operatorname{start}(P_1P_3) = \operatorname{start}(P_1) \neq \operatorname{start}(P_2) = \operatorname{start}(P_2P_4)$ and $\operatorname{end}(P_1P_3) = \operatorname{end}(P_3) \neq \operatorname{end}(P_4) = \operatorname{end}(P_2P_4)$. Therefore, these paths are proper. Now, if $B_i \cap B_j = \emptyset$ and there is a tree T in F such that $B_i \cap T = \{u_1\}$ and $B_j \cap T = \{u_2\}$, we form a proper path between v and w as follows. Let P_1 be the unique (proper) path in the tree T between u_1 and u_2. Let P_2 be the proper path in B_i between v and u_1 such that $\operatorname{end}(P_2) \neq \operatorname{start}(P_1)$. This path exists since we have the strong property in each block. Analogously, let P_3 be the proper path in B_j between u_2 and w such that $\operatorname{end}(P_1) \neq \operatorname{start}(P_3)$. Finally the path $P = P_2P_1P_3$ is proper between v and w. The same idea applies if v is in a block B_i and w is in a tree T in F such that $B_i \cap T = \{u\}$. The idea also applies in the case that v is in a tree T_i in F, w is in a tree T_j in F and there is a block B such that $T_i \cap B = \{u_1\}$ and $T_j \cap B = \{u_2\}$. Finally, the result holds by induction on the number of trees and blocks between vertices v and w using the remark stated before to guarantee the paths always traverse the blocks. Therefore, $pc(G) \leq \widetilde{\Delta}(G)$ if $\widetilde{\Delta}(G) \geq 3$ and $pc(G) \leq 3$ otherwise. □

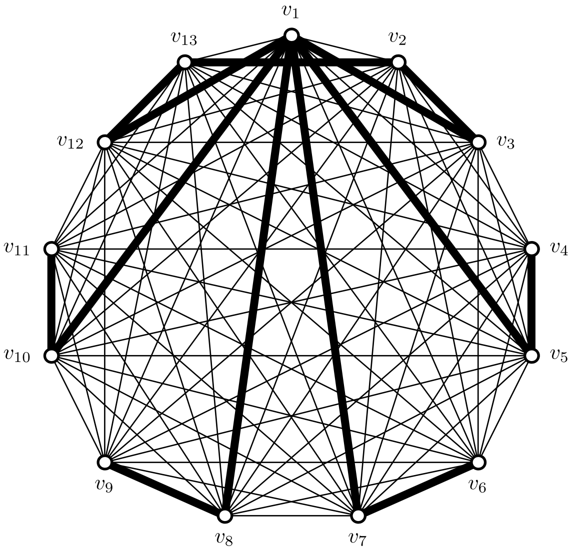

Fig. 6. Coloring of K_{13}. Normal edges represent color 1 and bold edges color 2.

To end the section, based on the Theorem 4 and the previous section, we extend the Conjecture 1 to general graphs.

Conjecture 2. *If G is a $2k$-connected graph with $k \geq 1$, then $pc_k(G) \leq 3$.*

This conjecture is proved for $k = 1$ in Theorem 4. Now we prove a stronger result for complete graphs.

Theorem 7. *Let $G = K_n$, $n \geq 4$, and $k > 1$. If $n \geq 2k$ then $pc_k(G) = 2$*

Proof. *Case* 1.

$n = 2p$ for $p \geq 2$.

Take a Hamiltonian cycle $C = v_1, v_2, \dots, v_{2p}$ of G and alternate colors on the edges using colors 1 and 2 starting with color 1. Color the rest of the edges using color 1. It is clear that there are $p \geq k$ edges with color 2. We will prove that this coloring gives us k proper paths between each pair of vertices of G. Take two vertices v, w such that $c(v, w) = 2$. This edge colored with color 2 is one proper path between v and w. Now, since there are at least other $p - 1 \geq k - 1$ edges colored with color 2 and the rest of the edges are colored with color 1, we have at least $k - 1$ proper paths between v and w using these edges. That is, for each vertices v', w' such that $c(v', w') = 2$ we form the proper path v, v', w', w. The case where $c(v, w) = 1$ is similar. *Case* 2.

$n = 2p - 1$ for $p \geq 2$.

Take a Hamiltonian cycle $C = v_1, v_2, \dots, v_{2p-1}$ of G and alternate colors on the edges using colors 1 and 2 starting with color 1. We have p edges with color 1 and $p - 1$ edges with color 2 so far since $c(v_1, v_2) = 1$ and $c(v_1, v_{2p-1}) = 1$. Now, put $c(v_2, v_{2p-1}) = 2$, $c(v_1, v_3) = 2$, $c(v_1, v_{2p-2}) = 2$ and for each edge with color 2, different from v_2, v_3 and v_{2p-2}, v_{2p-1}, choose one of the endpoints, say v', and put $c(v_1, v') = 2$ (see Fig. 6). Finally, color the rest of the edges with color 1. We now show that this coloring gives k proper paths between each pair of vertices v and w of G. First, take $v = v_1$ and $w = v_2$ (or similarly taking $w = v_{2p-1}$). We have the edge v_1v_2 and the path v_1, v_{2p-1}, v_2. Now since $n = 2p - 1 \geq 2k$ we have at least $(p - 1) - 2 \geq k - 2$ edges in the cycle C with color 2 different from v_2, v_3 and v_{2p-2}, v_{2p-1} and therefore we form the following $k - 2$ proper paths between v_1 and v_2 of the form v_1, v', v_2 where v' is an endpoint of each of these edges such that $c(v_1, v') = 2$. Now take $v = v_1$ and $w = v_3$ (analog taking $w = v_{2p-2}$). This case is similar to the previous except changing the second formed path to v_1, v_2, v_3. Suppose now that $v = v_1$ and $w = w'$ with $w' \notin \{v_2, v_3, v_{2p-2}, v_{2p-1}\}$. We take the edge v_1w' and now, since there are at least $i(p - 1) - 1 \geq k - 1$ edges in the cycle C with color 2 with endpoints different from v', we form the following $k - 1$ proper paths between v_1 and w' of the form v_1, v', w' where v' is an endpoint of each of these edges such that $c(v_1, v') = 2$. The rest of the cases are similar to those described before in the case $n = 2p$ forming most of the proper paths with length 3. □

4. Minimum degree

In this section, we prove the following result concerning minimum degrees.

Theorem 8. *If G is a connected non-complete graph with $n \geq 68$ vertices and $\delta(G) \geq n/4$, then $pc(G) = 2$.*

The minimum degree condition is best possible. To see this, we construct the following graph. Let G_i be a complete graph with $n/4$ vertices for $i = 1, 2, 3, 4$, and take a vertex $v_i \in G_i$ for each $1 \leq i \leq 4$. Let G be a graph obtained from $G_1 \cup G_2 \cup G_3 \cup G_4$ by joining v_1 and v_j with an edge for each $2 \leq j \leq 4$. Then the resulting graph G is connected and it has $\delta(G) = n/4 - 1$ and $pc(G) = 3$. To prove Theorem 8, we will make use of the following theorems.

Theorem 9 ([7]). *Let G be a graph with n vertices. If* $\delta(G) \geq \frac{n-1}{2}$, *then G has a Hamiltonian path. Moreover, if* $\delta(G) \geq n/2$, *then G has a Hamiltonian cycle. Also, if* $\delta(G) \geq \frac{n+1}{2}$, *then G is Hamilton-connected.*

Theorem 10 ([17]). *Let G be a graph with n vertices. If* $\delta(G) \geq \frac{n+2}{2}$ *then G is panconnected meaning that, between any pair of vertices in G, there is a path of every length from 2 up to* $n - 1$.

Theorem 11 ([12]). *Let G be a 3-connected graph with n vertices and* $\delta(G) \geq n/4 + 2$. *Then, for any longest cycle C in G, every component of* $G - C$ *has at most two vertices.*

Theorem 12 ([8]). *Let G be a connected graph with n vertices and* $\delta(G) \geq n/3$. *Then one of the following holds:*

(i) *G contains a Hamiltonian path.*
(ii) *For any longest cycle C of G,* $G - C$ *has no edge.*

Also we use the following easy fact as a matter of course.

Fact 3. *Every 2-connected graph G with* $\delta(G) \geq 2$ *is either Hamiltonian or contains a cycle C with at least* $2\delta(G)$ *vertices.*

For this statement, we use the following notation. For a path $P = v_1v_2 \cdots v_\ell$, we let endpoints$(P) = \{v_1, v_\ell\}$.

Lemma 1. *The following graphs* H_i, *for* $(i = 1, 2, \ldots, 6)$, *have* $pc(H_i) = 2$.

(1) *The graph* H_1 *obtained from a path P with* $|P| \geq 2$ *and* $m \geq 0$ *isolated vertices* $v_1, \ldots, v_m$ *by joining each* v_i *for* $(i \leq m)$ *within P with at least two edges.*
(2) *The graph* H_2 *obtained from a path P with* $|P| \geq 1$ *and even cycle C by identifying exactly one vertex (i.e.,* $|P \cap C| = 1$*).*
(3) *The graph* H_3 *obtained from* H_2 *and* $m \geq 0$ *isolated vertices* $v_1, \ldots, v_m$ *by joining each* v_i *for* $(i \leq m)$ *with at least two edges to either* $P - C$ *or* $C - P$ *in* H_2.
(4) *The graph* H_4 *obtained from an even cycle C and two paths* P_1 *and* P_2 *by identifying an end of each path to a vertex of C. As in* H_3, *we may also join vertices each with at least 2 edges to either a path* P_i *or C.*
(5) *The graph* H_5 *obtained from the union of two disjoint cycles which are connected by two disjoint paths to form a 2-connected graph. Furthermore, we may also add vertices each with at least 2 edges to this structure.*
(6) *The graph* H_6 *obtained from* H_5 *by removing an edge from one of the cycles. Again we may add vertices each with at least 2 edges to this structure.*

Proof. One can easily get a 2-coloring of H_i which forces $pc(H_i) = 2$ for $i = 1, 2, \ldots, 6$. For example, as for H_1, by Fact 2 and Proposition 2, there is a 2-coloring of H_1 that is properly connected. □

Proof of Theorem 8. If $\kappa(G) \geq 3$, then by Corollary 2, we have $pc(G) = 2$. So we may assume that $\kappa(G) = 1$ or 2. We divide the proof into two cases according to the value of $\kappa(G)$.

Case 1. $\kappa(G) = 1$.

Let v be a cutvertex of G and let $C_1, \ldots, C_\ell$ be the components of $G \setminus v$ such that $|C_1| \leq \cdots \leq |C_\ell|$.

By the minimum degree condition, we see that $\ell = 2$ or 3 and $|C_1| \geq n/4$. We further divide the proof into two subcases:

Subcase 1.1: $\ell = 2$.

In this case note that $|C_1| \leq (n-1)/2$ and, by the minimum degree condition, $|C_2| \leq 3n/4 - 1$. Utilizing Theorem 9 and the minimum degree condition, it is easy to check that $\langle \{v\} \cup C_1 \rangle$ contains a Hamiltonian path P_1 such that $v \in$ endpoints(P_1).

If $\kappa(C_2) \geq 3$, then let C be a longest cycle of C_2. Since G is connected, there is a path P' from v to C. Now $H = P_1 \cup P' \cup C$ satisfies the conditions of H_2 in Lemma 1. This means that $pc(H) = 2$. By Theorem 11, every component of $C_2 \setminus C$ has at most 2 vertices. By the minimum degree condition and since we assume $n \geq 12$, for each $x \in C_2 \setminus H$, we have $|E(x, H)| \geq \frac{n}{4} - 1 \geq 2$. Hence, G contains a spanning subgraph which satisfies the properties of H_3 in Lemma 1 so $pc(G) = 2$.

Thus we may assume that $\kappa(C_2) = 1$ or 2. Let S be a cutset in C_2 with $1 \leq |S| \leq 2$. By the minimum degree condition, it is easy to check that there are exactly two components C_{21}, C_{22} with $|C_{21}| \leq |C_{22}|$ in $C_2 - S$. Note that $n/4 - |S| \leq |C_{21}| \leq |C_{22}| \leq (3n/4 - 1) - |S| - (n/4 - |S|) = n/2 - 1$ because $\delta(G) \geq n/4$ and $|C_{21}| \leq (3n/4 - 1 - |S|)/2 = 3n/8 - (|S| + 1)/2 \leq 3n/8 - 1$. Hence by Theorem 9, C_{21} contains a Hamiltonian cycle C'_{21}.

Since $\delta(C_{22}) \geq n/4 - 3$, C_{22} is either Hamiltonian or contains a cycle C'_{22} with $|C'_{22}| \geq n/2 - 6$. Now take a path P_2 with $v \in$ endpoints(P_2) so that

(1) P_2 contains a longer segment of C'_{2j} for each $j = 1, 2$, and subject to condition (1),
(2) $|P_2|$ is as large as possible.

By the choice of P_2, note that $P_2 \cap S \neq \emptyset$. Let P be a path joining P_1 and P_2 at the common vertex v. Then, utilizing P and the assumption $\delta(G) \geq n/4$, we will find a spanning subgraph which has a property of H_1 in Lemma 1. In order to show this, we need only show that each vertex in $G \setminus P$ has at least 2 edges to P. As previously discussed, we know that all vertices in C_1 have at least 2 edges to P_1 so we need only check vertices $x \in C_2 \setminus P_2$. If $x \in C_{21}$ then since $|P \cap C_{21}| \geq |C_{21}|/2$ and $|C_{21}| \leq 3n/8 - 1$, by the minimum degree condition, x has at least $n/4 - 3n/16 \geq 2$ edges to P since $n \geq 32$. For $x \in C_{22}$, we know $|C_{22}| \leq n/2 - 2$ and either C_{22} is Hamiltonian or contains a cycle of length at least $n/2 - 6$. In either case, the same arguments easily show that x has at least 2 edges to P, meaning that $pc(G) = 2$.

Subcase 1.2: $\ell = 3$.

In this case, by the minimum degree condition, we see that $n/4 \leq |C_1| \leq (n-1)/3 \leq |C_3| \leq n/2 - 1$, and $|C_2| \leq 3n/8 - 1/2$. Hence by Theorem 9, each C_i with $i = 1, 2$ is Hamilton-connected. Also, by the minimum degree condition and since $n \geq 36$, we see that $\delta(C_i) \geq (|C_i| + 2)/2$ for $i = 1, 2$ so for any vertex $z \in C_i$, $C_i - z$ is Hamilton-connected. By Theorem 9, C_3 is Hamiltonian so it contains a spanning path P with $v \in \text{endpoints}(P)$.

If $|E(v, C_i)| \geq 2$ holds for $i = 1$ or 2 (suppose $i = 1$), then we can find an even cycle C in $C_1 \cup v$ such that $v \in C$ and $|C_1| \leq |C| \leq |C_1| + 1$. Using a Hamiltonian path of C_2 ending at v, together with the path P and the even cycle C, we can easily find a spanning subgraph which satisfies the property of H_3 in Lemma 1, and hence $pc(G) = 2$.

Thus we may assume that $|E(v, C_1)| = |E(v, C_2)| = 1$. This implies $|C_1| \geq n/4 + 1$, because there is a vertex of C_1 which is not adjacent to v. Then we get $|C_3| \leq n/2 - 3$ so $\delta(C_3) \geq n/4 - 1 \geq (|C_3| + 1)/2$. If $|C_3|$ is odd, then by Theorem 9, C_3 is Hamiltonian connected. Hence, we can find an even cycle using all of C_3 and v and a single path through v using all of C_1 and $C2$. This provides a spanning subgraph satisfying the properties of H_3 in Lemma 1.

If $|C_3|$ is even, then $\delta(C_3) \geq \left\lceil \frac{|C_3|+1}{2} \right\rceil = \frac{|C_3|+2}{2}$ so, by Theorem 10, C_3 is panconnected. Thus we can find an even cycle through $v \cup C_3$ which avoids exactly 1 vertex of C_3 again easily providing a subgraph satisfying the conditions of H_3 in Lemma 1. This shows that $pc(G) = 2$ and completes the proof of this case.

Case 2. $\kappa(G) = 2$.

Let u and v be a minimum cutset of G. Again we let $C_1, C_2, \ldots, C_\ell$ be the components of $G \setminus \{u, v\}$ with $|C_i| \leq |C_j|$ for $i \leq j$ and break the rest of the argument into cases based on the value of ℓ. Note that, since $\delta(G) \geq n/4$, we have $2 \leq \ell \leq 4$.

Subcase 2.1: $\ell = 4$.

Since $\delta(G) \geq n/4$, we know that $n/4 - 1 \leq |C_1| \leq (n-2)/4 \leq |C_4| \leq n/4 + 1$. This means that $\delta(C_i) \geq |C_i| - 2$ for all i. The graph G is 2-connected so there are two independent edges from $\{u, v\}$ to each component C_i. With $n \geq 26$, we see that $|C_i| \geq 6$ so the minimum degree condition $\delta(C_i) \geq |C_i| - 2$ implies, by Theorem 10, that each component C_i is panconnected. This means that, if $|C_3 \cup C_4|$ is even, we may find a cycle through $\{u, v\} \cup C_3 \cup C_4$ using all the vertices, and if $|C_3 \cup C_4|$ is odd, we may find a similar cycle which misses exactly one vertex $w \in C_4$. This cycle, along with a spanning path of $u \cup C_1 \cup C_2$ and possibly w provides a spanning subgraph of G satisfying the properties of H_3 from Lemma 1, meaning that $pc(G) = 2$.

Subcase 2.2: $\ell = 3$.

Since $\delta(G) \geq n/4$, we have $n/4 - 1 \leq |C_1| \leq |C_2| \leq (5n-4)/12$ and $\delta(C_i) \geq n/4 - 2$ so $\delta(C_i) \geq \frac{3|C_i|+1}{5} - 2$ for $i = 1, 2$. Since $n \geq 23$, this implies that $\delta(C_i) \geq \frac{|C_i|+1}{2}$ for $i = 1, 2$ so C_1 and C_2 are both Hamiltonian-connected by Theorem 9. This means we may create a single cycle D_{12} using all of $C_1 \cup C_2$.

If $\kappa(C_3) \geq 2$, then let D_3 be a longest cycle in C_3. Since $\delta(C_3) \geq \frac{n}{4} - 2$, we know $|D_3| \geq \min\{|C_3|, \frac{n}{2} - 4\}$. In either case every vertex of C_3 has at least 2 edges to H_3.

Now since G is 2-connected, there exist two disjoint paths from D_{12} to D_3 meaning there is a spanning subgraph of G satisfying the conditions of the graph H_5. By Lemma 1, we have $pc(G) = 2$.

If $\kappa(C_3) = 1$, then by Theorem 12, there is a spanning path P of C_3. The vertices u and v must each have at least one edge to P so $P \cup D_{12}$ forms a spanning subgraph of G satisfying the conditions of the graph H_6 in Lemma 1. Hence, $pc(G) = 2$.

Subcase 2.3: $\ell = 2$.

If C_1 and C_2 are both 3-connected, then by Corollary 2, there is a 2-coloring of each with that strong property. Along with these colorings, we also color all edges between $\{u, v\}$ and C_i with color i. This coloring clearly shows that $PC(G) = 2$ so we may assume that at least one component C_i has $1 \leq \kappa(C_i) \leq 2$. Next we will suppose that $1 \leq \kappa(C_i) \leq 2$ for both $i = 1, 2$. In this case, by the minimum degree condition and the fact that G is 2-connected, we may easily show that each component is Hamiltonian connected (since n is large) so G is Hamiltonian. This means $pc(G) = 2$.

Finally, if we suppose C_1 is 3-connected while $1 \leq \kappa(C_2) \leq 2$, each possible case contains a large (almost spanning) subgraph with the properties of H_4 from Lemma 1, meaning that $pc(G) = 2$. This completes the proof of Theorem 8. □

5. Conclusion

From Theorem 8, it is clear that if G is 2-connected and $\delta(G) \geq \frac{n}{4}$, then $pc(G) = 2$. The authors believe this degree condition can be greatly improved in the 2-connected case. In particular, we propose the following conjecture.

Conjecture 3. *If* $\kappa(G) = 2$ *and* $\delta(G) \geq 3$, *then* $pc(G) = 2$.

As observed in the graph of Fig. 2 satisfying $\kappa(G) = 2$, $\delta(G) = 2$ and $pc(G) = 3$, the bound on $\delta(G)$ in Conjecture 3 would be sharp if the conjecture is true. By the proof of Theorem 4 and the standard ear decomposition of a 2-connected graph, it is easy to produce a linear-time algorithm to 3-color any 2-connected graph to be proper connected with the strong property. Also since there is an $O(n + m)$ algorithm for finding a block decomposition of a graph G with $\kappa(G) = 1$ on n vertices with m edges, we can find an $O(n + m)$ algorithm to produce a proper connected coloring of such graphs. Therefore, in practice, these colorings are not difficult to find.

Acknowledgments

The second author's work was supported by JSPS Grant No. 20740068. This research was partly supported by the Everett Pitcher Fund.

The last author's work was supported in part by the Hungarian Scientific Research Fund, OTKA grant T-81493.

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