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# Time-dependent variational inequalities for viscoelastic contact problems

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**Abstract**

We consider a class of abstract evolutionary variational inequalities arising in the study of contact problems for viscoelastic materials. We prove an existence and uniqueness result, using standard arguments of time-dependent elliptic variational inequalities and Banach's fixed point theorem. We then consider numerical approximations of the problem. We use the finite element method to discretize the spatial domain and we introduce spatially semi-discrete and fully discrete schemes. For both schemes, we show the existence of a unique solution, and derive error estimates. Finally, we apply the abstract results to the analysis and numerical approximations of a viscoelastic contact problem with normal compliance and friction. © 2001 Elsevier Science B.V. All rights reserved.

*Keywords:* Variational inequality; Viscoelastic material; Contact; Normal compliance; Friction; Finite-element method; Semi-discrete scheme; Fully discrete scheme; Error estimates

## 1. Introduction

We investigated recently a number of quasistatic problems related to frictional contact for viscoelastic materials. In particular, a model for bilateral contact with friction was considered in [4], a model for frictional contact with normal compliance was analyzed in [18], and the problem of contact with a general damped response and friction was studied in [10]. In these papers friction was modeled with versions of the Coulomb law and the material was assumed to have nonlinear viscoelastic constitutive relation of the form

$$\boldsymbol{\sigma} = \mathscr{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathscr{B}\varepsilon(\boldsymbol{u}), \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field and $\boldsymbol{\sigma}$ and $\varepsilon(\boldsymbol{u})$ denote the stress and linearized strain tensor, respectively. Here $\mathscr{A}$ and $\mathscr{B}$ are nonlinear constitutive functions and the dot above a variable

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represents its time derivative. The results obtained in [4,10] deal with the variational and numerical analysis of the mechanical problems. They involve existence and uniqueness of weak solutions, i.e., solutions which satisfy variational formulations of the corresponding mechanical problems, and error estimates for the approximate solutions. The study of the contact problem in [10] was made using a general result on a class of abstract evolutionary variational inequalities with strongly monotone operators. The results obtained in [18] concern the existence and uniqueness of a weak solution, and its continuous dependence on the data. The wear of the contacting surface was taken into account, but numerical analysis of the mechanical problem was not included in that paper.

The present paper represents a continuation of [18] and parallels [10]. Indeed, we will provide the variational analysis of an abstract problem which includes as a special case the contact problem with normal compliance and friction studied in [18]. We will then analyze numerical approximations of the abstract problem. To this end, we introduce here abstract evolutionary problems of the form:

**Problem P.** Find $u : [0,T] \to V$ such that for $t \in [0,T]$,

$$(A\dot{u}(t), v - \dot{u}(t))_V + (Bu(t), v - \dot{u}(t))_V + j(u(t), v) - j(u(t), \dot{u}(t)) \geqslant (f(t), v - \dot{u}(t))_V \quad \forall v \in V, \tag{1.2}$$

$$u(0) = u_0. \tag{1.3}$$

Here $V$ is a real Hilbert space, $A$ and $B$ are given nonlinear operators on $V$, $[0,T]$ is the time interval of interest.

We provide variational and numerical analysis for the abstract Cauchy problem (1.2)–(1.3). The results are applied to the study of the viscoelastic problem with normal compliance and friction. We then analyze semi-discrete and fully discrete approximation schemes and derive error estimates. The literature is abundant in numerical treatment of elliptic or evolution variational inequalities, see for instance [8,9,11]. Here we follow the style of [9].

The rest of the paper is structured as follows. In Section 2 we show an existence and uniqueness result to the problem (1.2)–(1.3). The result is proved based on standard arguments for time-dependent elliptic variational inequalities followed by applying Banach's fixed point theorem twice. In Sections 3 and 4 we analyze spatially semi-discrete and fully discrete schemes, respectively. We use the finite element method to discretize the spatial domain. For both schemes, we show the existence of a unique solution. We also derive error estimates under suitable solution regularities. Finally, in Section 5 we present the quasistatic viscoelastic problem with normal compliance and friction which is modeled by an evolutionary variational inequality of the form (1.2)–(1.3). We apply all the abstract results obtained in Sections 2–4 to this mechanical problem.

## 2. An existence and uniqueness result

In this section we list the assumptions on the data and present an existence and uniqueness result in the study of the Cauchy problem (1.2)–(1.3).

We suppose in the sequel that $V$ is a real Hilbert space endowed with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$. Let $T > 0$. We denote by $C([0,T];V)$ and $C^1([0,T];V)$ the space of continuous and continuously differentiable functions from $[0,T]$ to $V$, with norms

$$\|u\|_{C([0,T];V)} = \max_{t\in[0,T]} \|u(t)\|_V$$

and

$$\|u\|_{C^1([0,T];V)} = \max_{t\in[0,T]} \|u(t)\|_V + \max_{t\in[0,T]} \|\dot{u}(t)\|_V,$$

respectively. We use the standard notation for $L^p(0,T;V)$ and Sobolev spaces $W^{k,p}(0,T;V)$, $k\in\mathbb{N}$, $1\leqslant p\leqslant\infty$. When $p=2$, we write $H^k(0,T;V)$ instead of $W^{k,2}(0,T;V)$.

We assume that $A: V\to V$ is a strongly monotone Lipschitz continuous operator, i.e.,

$$\begin{aligned}
&\text{(a) there exists } M>0 \text{ such that}\\
&\qquad (Au_1-Au_2,u_1-u_2)_V \geqslant M\|u_1-u_2\|_V^2 \quad \forall u_1,u_2\in V;\\
&\text{(b) there exists } L_A>0 \text{ such that}\\
&\qquad \|Au_1-Au_2\|_V \leqslant L_A\|u_1-u_2\|_V \quad \forall u_1,u_2\in V.
\end{aligned} \tag{2.1}$$

The nonlinear operator $B: V\to V$ is Lipschitz continuous, i.e.,

$$\exists L_B>0, \text{ such that } \|Bu_1-Bu_2\|_V \leqslant L_B\|u_1-u_2\|_V \quad \forall u_1,u_2\in V. \tag{2.2}$$

The functional $j: V\times V\to\mathbb{R}$ satisfies

$$\begin{aligned}
&\text{(a) for all } g\in V,\ j(g,\cdot) \text{ is convex and lower semicontinuous on } V;\\
&\text{(b) there exists } m>0 \text{ such that}\\
&\qquad j(g_1,v_2)-j(g_1,v_1)+j(g_2,v_1)-j(g_2,v_2)\leqslant m\|g_1-g_2\|_V\|v_1-v_2\|_V\\
&\qquad \forall g_1,g_2,v_1,v_2\in V.
\end{aligned} \tag{2.3}$$

Finally, we assume that

$$f\in C([0,T];V) \tag{2.4}$$

and

$$u_0\in V. \tag{2.5}$$

The main result of this section is the following.

**Theorem 2.1.** *Let* (2.1)–(2.5) *hold. Then, there exists a unique solution* $u\in C^1([0,T];V)$ *to the problem* (1.2)–(1.3).

The proof of Theorem 2.1 is based on fixed point arguments similar to those used in [10,18]. The result will be established in several steps. We assume in the sequel that (2.1)–(2.5) hold and, to simplify the notation, sometimes we will not indicate explicitly the dependence of various functions on the time variable $t$.

In the first step let $\eta\in C([0,T];V)$ and $g\in C([0,T];V)$ be given and we consider the following variational inequality of finding $v_{\eta g}:[0,T]\to V$, such that for $t\in[0,T]$,

$$\begin{aligned}
&(Av_{\eta g}(t),v-v_{\eta g}(t))_V+(\eta(t),v-v_{\eta g}(t))_V+j(g(t),v)-j(g(t),v_{\eta g}(t))\\
&\quad\geqslant(f(t),v-v_{\eta g}(t))_V \quad \forall v\in V.
\end{aligned} \tag{2.6}$$

**Lemma 2.2.** *There exists a unique solution* $v_{\eta g} \in C([0,T];V)$ *of problem* (2.6).

**Proof.** For each $t \in [0,T]$, it follows from classical results for elliptic variational inequalities (see, e.g., [3]) that there exists a unique element $v_{\eta g}(t) \in V$ that solves (2.6). It remains to prove the time continuity of the solution. Let $t_1, t_2 \in [0,T]$. For the sake of simplicity in writing we denote $v_{\eta g}(t_i) = v_i$, $\eta(t_i) = \eta_i$, $g(t_i) = g_i$, $f(t_i) = f_i$, for $i = 1,2$. Using (2.6) we easily derive the relation

$$(Av_1 - Av_2, v_1 - v_2)_V \leqslant (f_1 - f_2, v_1 - v_2)_V + (\eta_1 - \eta_2, v_2 - v_1)_V$$
$$+ j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2).$$

Then we use the assumptions (2.1)(a) and (2.3)(b) to obtain

$$M\|v_1 - v_2\|_V \leqslant \|f_1 - f_2\|_V + \|\eta_1 - \eta_2\|_V + m\|g_1 - g_2\|_V. \tag{2.7}$$

We deduce from (2.7) that $t \mapsto v_{\eta g}(t) : [0,T] \to V$ is a continuous function which concludes the proof. □

For each $\eta \in C([0,T];V)$, we consider now the operator $\Lambda_\eta : C([0,T];V) \to C([0,T];V)$ defined by

$$\Lambda_\eta g(t) = \int_0^t v_{\eta g}(s)\,\mathrm{d}s + u_0 \quad \forall g \in C([0,T];V), \;\; t \in [0,T]. \tag{2.8}$$

We have the following result.

**Lemma 2.3.** *The operator* $\Lambda_\eta$ *has a unique fixed point* $g_\eta \in C([0,T];V)$.

**Proof.** Let $g_1, g_2 \in C([0,T];V)$, $\eta \in C([0,T];V)$ and let $v_i$, $i = 1,2$, denote the solution of (2.6) for $g = g_i$, i.e., $v_i = v_{\eta g_i}$. From the definition (2.8) we have

$$\|\Lambda_\eta g_1(t) - \Lambda_\eta g_2(t)\|_V \leqslant \int_0^t \|v_1(s) - v_2(s)\|_V \,\mathrm{d}s \quad \forall t \in [0,T]. \tag{2.9}$$

An argument similar to that in the proof of Lemma 2.2 shows

$$M\|v_1(s) - v_2(s)\|_V \leqslant m\|g_1(s) - g_2(s)\|_V \quad \forall s \in [0,T]. \tag{2.10}$$

Combine (2.9) and (2.10),

$$\|\Lambda_\eta g_1(t) - \Lambda_\eta g_2(t)\|_V \leqslant c\int_0^t \|g_1(s) - g_2(s)\|_V \,\mathrm{d}s \quad \forall t \in [0,T].$$

Here and everywhere in Sections 2–4, the symbol $c$ denotes a positive constant which may depend on $A$, $B$, $j$ and $T$ and whose value may change from place to place. Reiterating the last inequality $n$ times, we infer

$$\|\Lambda_\eta^n g_1(t) - \Lambda_\eta^n g_2(t)\|_{C([0,T];V)} \leqslant \frac{c^n}{n!}\|g_1 - g_2\|_{C([0,T];V)}.$$

This inequality shows that for a sufficiently large $n$, the operator $\Lambda_\eta^n$ is a contraction on $C([0,T];V)$. Thus, there exists a unique $g_\eta \in C([0,T];V)$ such that $\Lambda_\eta^n g_\eta = g_\eta$ and moreover $g_\eta$ is the unique fixed point of $\Lambda_\eta$ too. □

In the sequel for each $\eta \in C([0,T];H)$, we denote by $g_\eta$ the fixed point given in Lemma 2.2 and let $u_\eta \in C^1([0,T];V)$ be the function given by

$$u_\eta(t) = \int_0^t v_{\eta g_\eta}(s)\,\mathrm{d}s + u_0 \quad \forall t \in [0,T]. \tag{2.11}$$

We have $\Lambda_\eta g_\eta = g_\eta$ and using (2.8) and (2.11) it follows that

$$u_\eta = g_\eta. \tag{2.12}$$

Therefore, taking $g = g_\eta$ in (2.6) and using (2.11) and (2.12) we see that $u_\eta$ satisfies

$$(A\dot{u}_\eta(t), v - \dot{u}_\eta(t))_V + (\eta(t), v - \dot{u}_\eta(t))_V + j(u_\eta(t), v) - j(u_\eta(t), \dot{u}_\eta(t))$$
$$\geqslant (f(t), v - \dot{u}_{\eta g}(t))_V \quad \forall v \in V,\ t \in [0,T]. \tag{2.13}$$

We define now the operator $\Lambda : C([0,T];H) \to C([0,T];V)$ by

$$\Lambda\eta = Bu_\eta \quad \forall \eta \in C([0,T];V). \tag{2.14}$$

We have the following result.

**Lemma 2.4.** *The operator $\Lambda$ has a unique fixed point $\eta^* \in C([0,T];V)$.*

**Proof.** Let $\eta_1, \eta_2 \in C([0,T];V)$. We use the notation $u_i = u_{\eta_i}$ and $v_i = \dot{u}_{\eta_i}$ for $i = 1,2$. By (2.11) we have

$$\|u_1(t) - u_2(t)\|_V \leqslant \int_0^t \|v_1(s) - v_2(s)\|_V\,\mathrm{d}s \quad \forall t \in [0,T]. \tag{2.15}$$

Moreover, using (2.13) and estimates similar to those in the proof of Lemma 2.2 (see (2.7)) we find

$$M\|v_1(s) - v_2(s)\|_V \leqslant \|\eta_1(s) - \eta_2(s)\|_V + m\|u_1(s) - u_2(s)\|_V, \quad \forall s \in [0,T]. \tag{2.16}$$

Combining (2.15) and (2.16) and using a Gronwall-type inequality we obtain

$$\|u_1(t) - u_2(t)\|_V \leqslant c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V\,\mathrm{d}s \quad \forall t \in [0,T].$$

Taking (2.14) and (2.2) into account, the last inequality leads to

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leqslant c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V\,\mathrm{d}s \quad \forall t \in [0,T]. \tag{2.17}$$

The results of the lemma follows from (2.17) and an application of Banach's fixed point theorem as in the proof of Lemma 2.2. □

We have now all the ingredients to prove Theorem 2.1.

**Proof of Theorem 2.1** (Existence). Let $\eta^* \in C([0,T];V)$ be the fixed point of $\Lambda$ and let $u_{\eta^*} \in C^1([0,T];V)$ be the function defined by (2.11) for $\eta = \eta^*$. Using (2.13) for $\eta = \eta^*$ it follows that for any $t \in [0,T]$,

$$(A\dot{u}_{\eta^*}(t), v - \dot{u}_{\eta^*}(t))_V + (\eta^*(t), v - \dot{u}_{\eta^*}(t))_V + j(u_{\eta^*}(t), v) - j(u_{\eta^*}(t), \dot{u}_{\eta^*}(t))$$
$$\geqslant (f(t), v - \dot{u}_{\eta^*}(t))_V \quad \forall v \in V. \tag{2.18}$$

Inequality (1.2) follows now from (2.18) and (2.14) since $\eta^* = \Lambda\eta^* = Bu_{\eta^*}$ and (1.3) results form (2.11). We conclude that $u_{\eta^*}$ is a solution of (1.2)–(1.3).

(*Uniqueness*). To prove the uniqueness of the solution let $u_i \in C^1([0,T];V)$ be two solutions of (1.2)–(1.3), $i = 1,2$. We have

$$(A\dot{u}_1(t), v - \dot{u}_1(t))_V + (Bu_1(t), v - \dot{u}_1(t))_V + j(u_1(t), v) - j(u_1(t), \dot{u}_1(t)) \geqslant (f(t), v - \dot{u}_1(t))_V,$$

$$(A\dot{u}_2(t), v - \dot{u}_2(t))_V + (Bu_2(t), v - \dot{u}_2(t))_V + j(u_2(t), v) - j(u_2(t), \dot{u}_2(t)) \geqslant (f(t), v - \dot{u}_2(t))_V,$$

for all $v \in V$ and $t \in [0,T]$. We take $v = \dot{u}_2(t)$ in the first inequality, $v = \dot{u}_1(t)$ in the second inequality and add the two inequalities to obtain

$$\begin{aligned}(A\dot{u}_1(t) - A\dot{u}_2(t), \dot{u}_1(t) - \dot{u}_2(t))_V \leqslant{} & (Bu_1(t) - Bu_2(t), \dot{u}_2(t) - \dot{u}_1(t))_V \\ & + j(u_1(t), \dot{u}_2(t)) - j(u_1(t), \dot{u}_1(t)) \\ & + j(u_2(t), \dot{u}_1(t)) - j(u_2(t), \dot{u}_2(t)) \quad \forall t \in [0,T].\end{aligned}$$

Using (2.1)–(2.3) we deduce

$$M\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leqslant (L_B + m)\|u_1(t) - u_2(t)\|_V \quad \forall t \in [0,T]$$

which implies

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leqslant c \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V \, ds \quad \forall t \in [0,T]. \tag{2.19}$$

The uniqueness part in Theorem 2.1 is now a consequence of (2.19) and (1.3). □

We finish this section with a regularity result that will be used in Section 4, in the study of fully discrete approximation of Problem **P**.

**Proposition 2.5.** *Under the conditions stated in Theorem* 2.1, *if* $f \in W^{1,1}(0,T;V)$, *then* $\dot{u} \in W^{1,1}(0,T;V)$, *and*

$$\|\dot{u}\|_{W^{1,1}(0,T;V)} \leqslant c\,(\|f\|_{W^{1,1}(0,T;V)} + \|u\|_{C^1([0,T];V)}).$$

**Proof.** For any $t_1, t_2 \in [0,T]$, we apply inequality (2.7) to inequality (1.2) to obtain

$$M\|\dot{u}(t_1) - \dot{u}(t_2)\|_V \leqslant \|f(t_1) - f(t_2)\|_V + \|Bu(t_1) - Bu(t_2)\|_V + m\,\|u(t_1) - u(t_2)\|_V.$$

Using now the assumption (2.2), we find

$$\|\dot{u}(t_1) - \dot{u}(t_2)\|_V \leqslant c(\|f(t_1) - f(t_2)\|_V + \|u(t_1) - u(t_2)\|_V),$$

from which the result of the proposition follows. □

From the proof of Proposition 2.5, we observe that actually a more general result holds:

$$f \in W^{k,p}(0,T;V) \Rightarrow u \in W^{k+1,p}(0,T;V).$$

In the next two sections, we will assume all the conditions stated in Theorem 2.1 are satisfied, so that the Problem **P** has a unique solution.

## 3. Spatially semi-discrete approximation

In this section we consider an approximation of Problem (1.2)–(1.3) by discretizing only the space $V$. Let $V^h \subset V$ be a finite-dimensional space which for example can be constructed by the finite element method. Then a spatially semi-discrete scheme can be formed as following.

**Problem $\mathbf{P}^h$.** Find $u^h : [0,T] \to V^h$ such that for $t \in [0,T]$,

$$(A\dot{u}^h(t), v^h - \dot{u}^h(t))_V + (Bu^h(t), v^h - \dot{u}^h(t))_V + j(u^h(t), v^h) - j(u^h(t), \dot{u}^h(t))$$
$$\geqslant (f(t), v^h - \dot{u}^h(t))_V \quad \forall v^h \in V^h, \tag{3.1}$$

$$u^h(0) = u_0^h. \tag{3.2}$$

Here, $u_0^h \in V^h$ is an appropriate approximation of $u_0$.

Using the arguments presented in the previous section, we see that under the conditions stated in Theorem 2.1, Problem $\mathbf{P}^h$ has a unique solution $u^h \in C^1([0,T]; V^h)$. Our main purpose here is to derive estimates for the errors $u - u^h$.

Let $t \in [0,T]$. To simplify the writing, we introduce the velocity variable

$$w(t) = \dot{u}(t). \tag{3.3}$$

Then by using the initial value condition (1.3), we have the relation

$$u(t) = \int_0^t w(s)\,\mathrm{d}s + u_0. \tag{3.4}$$

Similarly, we introduce the discrete velocity variable

$$w^h(t) = \dot{u}^h(t). \tag{3.5}$$

With the initial value condition (3.2), we have

$$u^h(t) = \int_0^t w^h(s)\,\mathrm{d}s + u_0^h. \tag{3.6}$$

Now the variational inequalities (1.2) and (3.1) can be rewritten as

$$(Aw(t), v - w(t))_V + (Bu(t), v - w(t))_V + j(u(t), v) - j(u(t), w(t))$$
$$\geqslant (f(t), v - w(t))_V \quad \forall v \in V, \tag{3.7}$$

and

$$(Aw^h(t), v^h - w^h(t))_V + (Bu^h(t), v^h - w^h(t))_V + j(u^h(t), v^h) - j(u^h(t), w^h(t))$$
$$\geqslant (f(t), v^h - w^h(t))_V \quad \forall v^h \in V^h. \tag{3.8}$$

We now take $v = w^h(t)$ in (3.7) and add the inequality to (3.8) with $v^h = v^h(t) \in V^h$. After some manipulations, we have

$$(Aw(t) - Aw^h(t), w(t) - w^h(t))_V$$
$$\leqslant (Aw^h(t), v^h(t) - w(t))_V + (Bu(t), w^h(t) - w(t))_V + (Bu^h(t), v^h(t) - w^h(t))_V$$

$$
\begin{aligned}
&+ j(u(t),w^h(t)) - j(u(t),w(t)) + j(u^h(t),v^h(t)) - j(u^h(t),w^h(t))\\
&+ (f(t),w(t)-v^h(t))_V\\
\leqslant\ & (Aw^h(t)-Aw(t),v^h(t)-w(t))_V + (Bu(t)-Bu^h(t),w^h(t)-v^h(t))_V\\
&+ R(t;v^h(t),w(t)) + j(u(t),w^h(t)) - j(u(t),v^h(t)) + j(u^h(t),v^h(t)) - j(u^h(t),w^h(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
R(t;v^h(t),w(t)) = {} & (Aw(t),v^h(t)-w(t))_V + (Bu(t),v^h(t)-w(t))_V\\
& + j(u(t),v^h(t)) - j(u(t),w(t)) - (f(t),v^h(t)-w(t))_V
\end{aligned}
\tag{3.9}
$$

represents a residual quantity. Using assumptions (2.1)–(2.3), we have

$$
\begin{aligned}
M\|w(t)-w^h(t)\|_V^2 &\leqslant L_A\|w(t)-w^h(t)\|_V\|w(t)-v^h(t)\|_V + |R(t;v^h(t),w(t))|\\
&\quad + (L_B+m)\|u(t)-u^h(t)\|_V\|w^h(t)-v^h(t)\|_V\\
&\leqslant L_A\|w(t)-w^h(t)\|_V\,\|w(t)-v^h(t)\|_V + |R(t;v^h(t),w(t))|\\
&\quad + (L_B+m)\|u(t)-u^h(t)\|_V(\|w(t)-w^h(t)\|_V + \|w(t)-v^h(t)\|_V).
\end{aligned}
$$

Thus, we have the inequality

$$
\|w(t)-w^h(t)\|_V^2 \leqslant c(\|w(t)-v^h(t)\|_V^2 + \|u(t)-u^h(t)\|_V^2 + |R(t;v^h(t),w(t))|). \tag{3.10}
$$

By (3.4) and (3.6), we have

$$
u(t)-u^h(t) = \int_0^t (w(s)-w^h(s))\,\mathrm{d}s + u_0 - u_0^h,
$$

and so

$$
\|u(t)-u^h(t)\|_V^2 \leqslant c\left(\int_0^t \|w(s)-w^h(s)\|_V^2\,\mathrm{d}s + \|u_0-u_0^h\|_V^2\right). \tag{3.11}
$$

Then inequality (3.10) can be rewritten as

$$
\begin{aligned}
&\|w(t)-w^h(t)\|_V^2\\
&\quad\leqslant c\left(\|w(t)-v^h(t)\|_V^2 + \int_0^t \|w(s)-w^h(s)\|_V^2\,\mathrm{d}s + \|u_0-u_0^h\|_V^2 + |R(t;v^h(t),w(t))|\right).
\end{aligned}
$$

Applying the Gronwall inequality, we have

$$
\|w-w^h\|_{C([0,T];V)} \leqslant c \inf_{v^h\in C([0,T];V)} (\|w-v^h\|_{C([0,T];V)} + \|R(\cdot;v^h(\cdot),w(\cdot))\|_{C([0,T])}^{1/2}) + c\|u_0-u_0^h\|_V. \tag{3.12}
$$

Summarizing, with (3.11) and (3.12), we have proved the following result.

**Theorem 3.1.** *Assume conditions* (2.1)–(2.5). *Then for the error of the spatially semi-discrete solution of* (3.1)–(3.2), *we have the estimate*

$$\|u - u^h\|_{C^1([0,T];V)} c \inf_{v^h \in C([0,T];V)} (\|\dot{u} - v^h\|_{C([0,T];V)} + \|R(\cdot; v^h(\cdot), \dot{u}(\cdot))\|^{1/2}_{C([0,T])}) + c\|u_0 - u_0^h\|_V, \quad (3.13)$$

*where* $R(\cdot; v^h(\cdot), \dot{u}(\cdot))$ *is defined in* (3.9).

Inequality (3.11) is the basis for error analysis, as is shown in Section 5 in the context of a viscoelastic frictional contact problem.

## 4. Fully discrete approximation

In this section we consider a fully discrete approximation of problem (1.2)−(1.3). In addition to the finite dimensional space $V^h$ introduced in the preceding section, we need a partition of the time interval $[0,T]$: $0 = t_0 < t_1 < \cdots < t_N = T$. We denote the time step size $k_n = t_n - t_{n-1}$ for $n = 1,\dots,N$. We allow nonuniform partition of the time interval, and let $k = \max_n k_n$ be the maximal step size. For a function $w(t)$ continuous in $t$ we use the notation $w_n = w(t_n)$. For a sequence $\{w_n\}_{n=0}^N$, we denote $\Delta w_n = w_n - w_{n-1}$ for the difference, and $\delta w_n = \Delta w_n / k_n$ the corresponding divided difference. No summation is implied over the repeated index $n$.

The fully discrete approximation method we will analyze is the following.

**Problem $\mathbf{P}^{hk}$.** Find $\{u_n^{hk}\}_{n=0}^N \subset V^h$, such that for $n = 1,\dots,N$,

$$(A\delta u_n^{hk}, v^h - \delta u_n^{hk})_V + (Bu_{n-1}^{hk}, v^h - \delta u_n^{hk})_V + j(u_{n-1}^{hk}, v^h) - j(u_{n-1}^{hk}, \delta u_n^{hk})$$
$$\geqslant (f_n, v^h - \delta u_n^{hk})_V \quad \forall v^h \in V^h, \quad (4.1)$$

$$u_0^{hk} = u_0^h. \quad (4.2)$$

Here, $u_0^h \in V^h$ is an appropriate approximation of $u_0$. Again to simplify the notation, we introduce the discrete velocity

$$w_n^{hk} = \delta u_n^{hk}, \quad n = 1,\dots,N. \quad (4.3)$$

Then using the initial value condition (4.2), we have the relation

$$u_n^{hk} = \sum_{j=1}^n w_j^{hk} k_j + u_0^h. \quad (4.4)$$

We can rewrite (4.1) in the form

$$(Aw_n^{hk}, v^h - w_n^{hk})_V + j(u_{n-1}^{hk}, v^h) - j(u_{n-1}^{hk}, w_n^{hk})$$
$$\geqslant (f_n, v^h - w_n^{hk})_V - (Bu_{n-1}^{hk}, v^h - w_n^{hk})_V \quad \forall v^h \in V^h. \quad (4.5)$$

Given $u_{n-1}^{hk} \in V^h$, it is easy to see that inequality (4.5) has a unique solution $w_n^{hk} \in V^h$. Note that $u_0^{hk} = u_0^h$ is given and we have relation (4.4) between $\{u_n^{hk}\}_{n=1}^N$ and $\{w_n^{hk}\}_{n=1}^N$. A mathematical induction argument yields the existence and uniqueness of a solution of the problem $\mathbf{P}^{hk}$. Our main objective of the section is to derive an error estimate for the fully discrete solution. For this purpose, we will assume in this section the regularity $w \in W^{1,1}(0,T;V)$. By Lemma 2.5, this regularity condition follows from the assumption $f \in W^{1,1}(0,T;V)$, and we also have

$$\|w\|_{W^{1,1}(0,T;V)} \leqslant c(\|f\|_{W^{1,1}(0,T;V)} + \|u\|_{C^1([0,T];V)}).$$

Recall that $W^{1,1}(0,T;V) \subset C([0,T];V)$ and

$$\|w\|_{C([0,T];V)} \leqslant c\|w\|_{W^{1,1}(0,T;V)}.$$

Take $v = w_n^{hk}$ in (3.7) at $t = t_n$,

$$(Aw_n, w_n^{hk} - w_n)_V + (Bu_n, w_n^{hk} - w_n) + j(u_n, w_n^{hk}) - j(u_n, w_n) \geqslant (f_n, w_n^{hk} - w_n)_V. \tag{4.6}$$

We now add (4.5) with $v^h = v_n^h \in V^h$ and (4.6) to obtain an error relation

$$\begin{aligned}
&(Aw_n - Aw_n^{hk}, w_n - w_n^{hk})_V \\
&\quad \leqslant (Aw_n^{hk}, v_n^h - w_n)_V + (Bu_{n-1}^{hk}, v_n^h - w_n^{hk})_V + (Bu_n, w_n^{hk} - w_n)_V - (f_n, v_n^h - w_n)_V \\
&\qquad + j(u_{n-1}^{hk}, v_n^h) - j(u_{n-1}^{hk}, w_n^{hk}) + j(u_n, w_n^{hk}) - j(u_n, w_n) \\
&\quad = (Aw_n^{hk} - Aw_n, v_n^h - w_n)_V + (Bu_{n-1}^{hk} - Bu_n, v_n^h - w_n^{hk})_V + R_n(v_n^h, w_n) \\
&\qquad + j(u_{n-1}^{hk}, v_n^h) - j(u_n, v_n^h) + j(u_n, w_n^{hk}) - j(u_{n-1}^{hk}, w_n^{hk}),
\end{aligned}$$

where

$$R_n(v_n^h, w_n) = (Aw_n, v_n^h - w_n)_V + (Bu_n, v_n^h - w_n)_V + j(u_n, v_n^h) - j(u_n, w_n) - (f_n, v_n^h - w_n)_V. \tag{4.7}$$

By assumptions (2.1)–(2.3), we have

$$\begin{aligned}
M\|w_n - w_n^{hk}\|_V^2 &\leqslant L_A\|w_n - w_n^{hk}\|_V \, \|w_n - v_n^h\|_V \\
&\qquad + (L_B + m)\,\|u_{n-1}^{hk} - u_n\|_V \, \|v_n^h - w_n^{hk}\|_V + |R_n(v_n^h, w_n)|.
\end{aligned}$$

Here $\|v_n^h - w_n^{hk}\|_V$ will be bounded as follows:

$$\|v_n^h - w_n^{hk}\|_V \leqslant \|v_n^h - w_n\|_V + \|w_n - w_n^{hk}\|_V.$$

Then we get the relation

$$\|w_n - w_n^{hk}\|_V^2 \leqslant c\{\|v_n^h - w_n\|_V^2 + \|u_{n-1}^{hk} - u_n\|_V^2 + |R_n(v_n^h, w_n)|\},$$

or,

$$\|w_n - w_n^{hk}\|_V \leqslant c\{\|v_n^h - w_n\|_V + |R_n(v_n^h, w_n)|^{1/2} + \|u_{n-1}^{hk} - u_n\|_V\}. \tag{4.8}$$

Let us bound the term $\|u_{n-1}^{hk} - u_n\|_V$. We have

$$u_{n-1}^{hk} - u_n = \sum_{j=1}^{n-1} w_j^{hk} k_j + u_0^h - \int_0^{t_n} w(s)\,\mathrm{d}s - u_0$$

$$= \sum_{j=1}^{n-1} (w_j^{hk} - w_j) k_j + u_0^h - u_0$$

$$+ \sum_{j=1}^{n-1} \left( w_j k_j - \int_{t_{j-1}}^{t_j} w(s)\,\mathrm{d}s \right) - \int_{t_{n-1}}^{t_n} w(s)\,\mathrm{d}s.$$

Now

$$\left\| \sum_{j=1}^{n-1} \left( w_j k_j - \int_{t_{j-1}}^{t_j} w(s)\,\mathrm{d}s \right) \right\|_V = \left\| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (w_j - w(s))\,\mathrm{d}s \right\|_V$$

$$= \left\| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \int_s^{t_j} \dot{w}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \right\|_V$$

$$\leqslant \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \int_s^{t_j} \|\dot{w}(\tau)\|_V\,\mathrm{d}\tau\,\mathrm{d}s$$

and also

$$\left\| \int_{t_{n-1}}^{t_n} w(s)\,\mathrm{d}s \right\|_V \leqslant \int_{t_{n-1}}^{t_n} \|w(s)\|_V\,\mathrm{d}s \leqslant k \|w\|_{C([0,T];V)}.$$

Hence,

$$\|u_{n-1}^{hk} - u_n\|_V \leqslant \sum_{j=1}^{n-1} \|w_j^{hk} - w_j\|_V k_j + \|u_0^h - u_0\|_V + ck\|w\|_{W^{1,1}(0,T;V)}. \tag{4.9}$$

Therefore, from (4.8), we have

$$\|w_n - w_n^{hk}\|_V \leqslant c \Bigg\{ \|v_n^h - w_n\|_V + |R_n(v_n^h, w_n)|^{1/2} + \|u_0^h - u_0\|_V$$

$$+ k(\|f\|_{W^{1,1}(0,T;V)} + \|u\|_{C^1([0,T];V)}) + \sum_{j=1}^{n-1} \|w_j^{hk} - w_j\|_V k_j \Bigg\}. \tag{4.10}$$

To proceed further, we need the following result proved in [10].

**Lemma 4.1.** *Assume $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of non-negative numbers, satisfying*

$$e_n \leqslant c g_n + c \sum_{j=1}^{n-1} k_j e_j.$$

*Then*

$$e_n \leqslant c\left(g_n + \sum_{j=1}^{n-1} k_j g_j\right), \quad n = 1,\ldots,N.$$

*Therefore,*

$$\max_{1\leqslant n\leqslant N} e_n \leqslant c \max_{1\leqslant n\leqslant N} g_n.$$

Applying Lemma 4.1 to inequality (4.10), we obtain the following estimate:

$$\max_n \|w_n - w_n^{hk}\|_V \leqslant c \max_n\{\|v_n^h - w_n\|_V + |R_n(v_n^h, w_n)|^{1/2}\} + c\|u_0^h - u_0\|_V$$
$$+ ck(\|f\|_{W^{1,1}(0,T;V)} + \|u\|_{C^1([0,T];V)}).$$

Similar to (4.9), we have

$$\|u_n^{hk} - u_n\|_V \leqslant \sum_{j=1}^{n-1} \|w_j^{hk} - w_j\|_V k_j + \|u_0^h - u_0\|_V + ck(\|f\|_{W^{1,1}(0,T;V)} + \|u\|_{C^1([0,T];V)}).$$

Therefore, we have proved the following result.

**Theorem 4.2.** *Assume conditions* (2.1)–(2.5) *and let* $f \in W^{1,1}(0,T;V)$. *Then for the error of the fully discrete solution of* (4.1)–(4.2), *we have the estimate*

$$\max_{1\leqslant n\leqslant N}(\|u_n^{hk} - u_n\|_V + \|\dot{u}_n - \delta u_n^{hk}\|_V)$$
$$\leqslant c \max_{1\leqslant n\leqslant N}\{\|v_n^h - \dot{u}_n\|_V + |R_n(v_n^h, \dot{u}_n)|^{1/2}\} + c\|u_0^h - u_0\|_V$$
$$+ ck(\|f\|_{W^{1,1}(0,T;V)} + \|u\|_{C^1([0,T];V)}), \tag{4.11}$$

*where* $R_n(v_n^h, \dot{u}_n)$ *is defined by* (4.7).

## 5. Applications in frictional contact problems for viscoelastic materials

The abstract results of Sections 2–4 may be used in the study of various contact problems for visco-elastic materials. If the friction is modeled with Tresca's law or with a simplified version of Coulomb's law, then the corresponding variational formulations for the displacement field are of the form (1.2)–(1.3) in which the functional $j$ does not depend on $u$ (see for instance [2]). In this section we apply the abstract results of Sections 2–4 in the study of a frictional contact problem for viscoelastic materials with normal compliance and friction. In this case both the displacement field $u$ and the velocity field $\dot{u}$ are involved in the functional $j$.

### *5.1. The contact problem*

The physical setting is the following. A viscoelastic body occupies a regular domain $\Omega$ of $\mathbb{R}^d$ $(d = 2,3)$ with the boundary $\Gamma$ partitioned into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ such that $\text{meas}(\Gamma_1) > 0$. We are interested in the evolution process of the mechanical state of the body in the time interval $[0, T]$ with $T > 0$. The body is clamped on $\Gamma_1$ and so the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2$ and volume forces of density $\boldsymbol{f}_0$ act in $\Omega$. We assume that the forces and tractions change slowly in time so that the acceleration of the system is negligible. The body may come in contact with an obstacle, called the foundation. There is a gap $g$ between the potential contact surface $\Gamma_3$ and the foundation, measured along the outward normal. The constitutive law and the contact conditions on the contact surface are assumed as in [18] and will be discussed below.

Under these conditions, the classical formulation of the mechanical problem of frictional contact of the viscoelastic body is the following: find the displacement $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ and the stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$ such that

$$\boldsymbol{\sigma} = \mathscr{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathscr{B}\varepsilon(\boldsymbol{u}) \quad \text{in } \Omega \times (0, T), \tag{5.1}$$

$$\text{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0, T), \tag{5.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{5.3}$$

$$\boldsymbol{\sigma}\boldsymbol{v} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{5.4}$$

$$-\sigma_v = p_v(u_v - g) \quad \text{on } \Gamma_3 \times (0, T), \tag{5.5}$$

$$\left.\begin{array}{l} |\boldsymbol{\sigma}_\tau| \leqslant p_\tau(u_v - g) \\ |\boldsymbol{\sigma}_\tau| < p_\tau(u_v - g) \Rightarrow \dot{\boldsymbol{u}}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = p_\tau(u_v - g) \Rightarrow \boldsymbol{\sigma}_\tau = -\lambda\dot{\boldsymbol{u}}_\tau, \ \lambda \geqslant 0 \end{array}\right\} \quad \text{on } \Gamma_3 \times (0, T), \tag{5.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{5.7}$$

Here $S_d$ represents the space of second-order symmetric tensors on $\mathbb{R}^d$. Relation (5.1) is the viscoelastic constitutive law in which $\mathscr{A}$ and $\mathscr{B}$ are given nonlinear operators, called the *viscosity* operator and *elasticity* operator, respectively. As usual, $\varepsilon(\boldsymbol{u})$ is the infinitesimal strain tensor. Relation (5.2) represents the equilibrium equation in which Div denotes the divergence operator, (5.3) and (5.4) are the displacement-traction boundary conditions in which $\boldsymbol{v}$ represents the unit outward normal vector on $\Gamma$. The function $\boldsymbol{u}_0$ in (5.7) denotes the initial displacement.

Equality (5.5) represents the *normal compliance* contact condition where $u_v$ represents the normal displacement, $\sigma_v$ denotes the normal stress, and $p_v$ is a prescribed function. Here, $u_v - g$, when positive, represents the penetration of the surface asperities into those of the foundation. As an example of normal compliance function $p_v$ we may consider

$$p_v(r) = c_v r_+ \tag{5.8}$$

where $c_v$ is a positive constant and $r_+ = \max\{0, r\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_v \to \infty$. The normal compliance contact condition was proposed in [15] and used in a large number of papers, see, e.g., [1,12–14] and the references therein.

Relations (5.6) represent a version of Coulomb's law of dry friction. Here $\boldsymbol{\sigma}_\tau$ denotes the tangential stress, $\dot{\boldsymbol{u}}_\tau$ represents the tangential velocity and $p_\tau$ is a prescribed nonnegative function, called the *friction bound*. Law (5.6) states that the tangential shear cannot exceed the maximal frictional resistance $p_\tau(u_\nu - g)$. Then, if the strict inequality holds, the surface adheres to the foundation and is in the so-called *stick* state, and when the equality holds there is relative sliding, the so-called *slip* state. Therefore, at each time instant the contact surface $\Gamma_3$ is divided into three zones: the stick zone, the slip zone and the zone of separation, in which $u_\nu < g$ and there is no contact. The boundaries of these zones are unknown a priori and are part of the problem. When we choose

$$p_\tau = \mu p_\nu, \tag{5.9}$$

we obtain the usual Coulomb law of dry friction where $\mu \geqslant 0$ represents the coefficient of friction (see, e.g., [7] or [17]). Recently a modified version of the Coulomb friction law has been derived in [19,20] from thermodynamic considerations. It consists of using the friction law (5.6) with

$$p_\tau = \mu p_\nu (1 - \delta p_\nu)_+, \tag{5.10}$$

where $\delta$ is a small positive material constant related to the wear and hardness of the surface and $\mu \geqslant 0$ is the coefficient of friction.

We denote in the sequel by " $\cdot$ " and $|\cdot|$ the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $S_d$ and we introduce the spaces

$$V = \{\boldsymbol{v} = (v_i) \in (H^1(\Omega))^d \colon \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\},$$

$$Q = \{\boldsymbol{\tau} = (\tau_{ij}) \in (L^2(\Omega))^{d\times d} \colon \tau_{ij} = \tau_{ji},\ 1 \leqslant i,j \leqslant d\},$$

$$Q_1 = \{\boldsymbol{\tau} \in Q \colon \operatorname{Div} \boldsymbol{\tau} \in (L^2(\Omega))^d\}.$$

These are real Hilbert spaces with their canonical inner products. Since meas$(\Gamma_1) > 0$, Korn's inequality holds:

$$\|\boldsymbol{v}\|_{(H^1(\Omega))^d} \leqslant c_K \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \quad \forall \boldsymbol{v} \in V. \tag{5.11}$$

Here $c_K > 0$ is a constant depending only on $\Omega$ and $\Gamma_1$ and $\boldsymbol{\varepsilon} \colon H^1(\Omega)^d \to Q$ is the deformation operator. A proof of Korn's inequality can be found in, for instance, [16, p. 79].

Over the space $V$, we use the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{5.12}$$

It follows from (5.11) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on $V$ and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space.

Finally, for all $\boldsymbol{v} \in V$ we denote by $v_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on $\Gamma$ given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

In the study of the mechanical problem (5.1)–(5.7) we assume that the *viscosity operator* $\mathscr{A}$ and the *elasticity operator* $\mathscr{B}$ have the following properties.

(5.13)

(a) $\mathscr{A} \colon \Omega \times S_d \to S_d$.
(b) There exists $\mathscr{L}_A > 0$ such that
$|\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leqslant \mathscr{L}_A |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1 \in S_d$, a.e. $\boldsymbol{x} \in \Omega$.
(c) There exists $M > 0$ such that
$(\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant M |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1 \in S_d$,
a.e. $\boldsymbol{x} \in \Omega$.
(d) For any $\boldsymbol{\varepsilon} \in S_d$, $\boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is Lebesgue measurable on $\Omega$.
(e) The mapping $\boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, \boldsymbol{0}) \in Q$.

(a) $\mathscr{B}: \Omega \times S_d \to S_d$.
(b) There exists an $\mathscr{L}_B > 0$ such that
$$|\mathscr{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leqslant \mathscr{L}_B |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d \text{ a.e. in } \Omega. \tag{5.14}$$
(c) For any $\boldsymbol{\varepsilon} \in S_d$, $\boldsymbol{x} \mapsto \mathscr{B}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is measurable.
(d) The mapping $\boldsymbol{x} \mapsto \mathscr{B}(\boldsymbol{x}, \boldsymbol{0}) \in Q$.

The *normal compliance functions* $p_r$ $(r = \nu, \tau)$ satisfy

(a) $p_r: \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$.
(b) There exists an $\mathscr{L}_r > 0$ such that
$$|p_r(\boldsymbol{x}, u_1) - p_r(\boldsymbol{x}, u_2)| \leqslant \mathscr{L}_r |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R} \text{ a.e. in } \Omega. \tag{5.15}$$
(c) For any $u \in \mathbb{R}$, $\boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u)$ is measurable.
(d) The mapping $\boldsymbol{x} \mapsto p_r(\boldsymbol{x}, 0) \in L^2(\Gamma_3)$.

We observe that assumptions (5.15) on the functions $p_\nu$ and $p_\tau$ are pretty general except assumption (5.15)(b) which, roughly speaking, requires the functions to grow at most linearly. Certainly the function defined in (5.8) satisfies condition (5.15)(b). We also observe that if the functions $p_\nu$ and $p_\tau$ are related by (5.9) or (5.10) and $p_\nu$ satisfies the condition (5.15)(b), then $p_\tau$ also satisfies the condition (5.15)(b) with $\mathscr{L}_\tau = \mu \mathscr{L}_\nu$. We conclude that our results below are valid for the boundary value problems related to each of these examples.

We also assume that the force and traction densities satisfy
$$\boldsymbol{f}_0 \in C([0,T]; (L^2(\Omega))^d), \qquad \boldsymbol{f}_2 \in C([0,T]; (L^2(\Gamma_2))^d), \tag{5.16}$$
and the gap function satisfies
$$g \in L^2(\Omega), \quad g \geqslant 0 \text{ a.e. on } \Gamma_3. \tag{5.17}$$
Finally, the initial displacement has the regularity
$$\boldsymbol{u}_0 \in V. \tag{5.18}$$
We denote by $\boldsymbol{f}(t)$ the element of $V$ given by
$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, \mathrm{d}a \tag{5.19}$$
for all $\boldsymbol{v} \in V$ and $t \in [0,T]$, and we note that conditions (5.16) imply
$$\boldsymbol{f} \in C([0,T]; V). \tag{5.20}$$
Let $j: V \times V \to \mathbb{R}$ be the functional
$$j(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Gamma_3} p_\nu(v_\nu - g) w_\nu \, \mathrm{d}a + \int_{\Gamma_3} p_\tau(v_\nu - g) |\boldsymbol{w}_\tau| \, \mathrm{d}a \quad \forall \boldsymbol{v}, \boldsymbol{w} \in V. \tag{5.21}$$
Using assumptions (5.15) and (5.17) if follows that integrals in (5.21) are well defined. With these notations, it follows from [18] that if $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ are sufficiently regular functions satisfying (5.1)–(5.6), then for all $t \in [0,T]$, $\boldsymbol{u}(t) \in V$ and
$$(\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_V + (\mathscr{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_V$$
$$+ j(\boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V. \tag{5.22}$$
Thus, we obtain the following variational formulation of problem (5.1)–(5.7) in terms of displacements.

**Problem $\mathbf{P}_0$.** Find a displacement $\boldsymbol{u} : \Omega \times [0,T] \to V$ which satisfies (5.22) and (5.7).

The well-posedness of Problem $\mathbf{P}_0$ follows from an application of Theorem 2.1.

**Theorem 5.1.** *Assume that* (5.13)–(5.18) *hold. Then Problem* $\mathbf{P}_0$ *has a unique solution* $\boldsymbol{u} \in C^1([0,T]; V)$.

**Proof.** Let $A : V \to V$ and $B : V \to V$ be the operators defined by

$$(A\boldsymbol{v}, \boldsymbol{w})_V = (\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q, \quad (B\boldsymbol{v}, \boldsymbol{w})_V = (\mathscr{B}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \tag{5.23}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in V$. Using (5.13) and (5.14) it follows that $A$ and $B$ are Lipschitz continuous operators. Using (5.13) and (5.12) we deduce that $A$ is strongly monotone on $V$:

$$(A\boldsymbol{v}_1 - A\boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_V \geqslant M\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2 \quad \forall \boldsymbol{v}_1\boldsymbol{v}_2 \in V. \tag{5.24}$$

Moreover, from (5.15), (5.11) and (5.12) it follows that the function $j$ defined by (5.21) satisfies (2.3) and

$$j(\boldsymbol{g}_1, \boldsymbol{v}_2) - j(\boldsymbol{g}_1, \boldsymbol{v}_1) + j(\boldsymbol{g}_2, \boldsymbol{v}_1) - j(\boldsymbol{g}_2, \boldsymbol{v}_2)$$

$$\leqslant c_0(L_\nu + L_\tau)\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \quad \forall \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V,$$

where $c_0 > 0$ depends only on $\Omega$ and $\Gamma_1$. Applying Theorem 2.1, we conclude that Problem $\mathbf{P}_0$ has a unique solution $\mathbf{u} \in C^1([0,T]; V)$.

Let now $\mathbf{u} \in C^1([0,T]; V)$ be the solution of the problem $\mathbf{P}_0$ and let $\boldsymbol{\sigma}$ be the stress field defined by (5.1). Using (5.22) and (5.16) it can be shown that $\operatorname{Div} \boldsymbol{\sigma} \in C([0,T]; L^2(\Omega)^d)$; therefore $\boldsymbol{\sigma} \in C([0,T]; Q_1)$. A pair of functions $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ which satisfies (5.1), (5.7) and (5.22) is called a *weak solution* of problem (5.1)–(5.7). We conclude that problem (5.1)–(5.7) has a unique weak solution which represents a result already obtained in [18].

### *5.2. Numerical approximations*

Now we state some sample results on error estimates for numerical approximations of Problem $\mathbf{P}_0$.

We first briefly describe how to construct the finite dimensional space $V^h$ via the finite element method. Details can be found in [5]. For simplicity, we assume that $\Omega$ is polygonal. Let $\mathscr{T}^h$ be a regular finite element partition of $\Omega$ in such a way that if one side of an element lies on the boundary, the side belongs entirely to one of the subsets $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$. Let $h$ be the maximal diameter of the elements, and $V^h \subset V$ be the finite element space consisting of piecewise polynomials of degree less than or equal to $l$, corresponding to the partition $\mathscr{T}^h$. Then the spatially semi-discrete approximation of Problem $\mathbf{P}_0$ is

**Problem $\mathbf{P}_0^h$.** Find the displacement field $\boldsymbol{u}^h : [0,T] \to V^h$, such that for $t \in [0,T]$,

$$(\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^h(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^h(t)))_V + (\mathscr{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^h(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^h(t)))_V$$

$$+ j(\boldsymbol{u}^h(t), \boldsymbol{v}^h) - j(\boldsymbol{u}^h(t), \dot{\boldsymbol{u}}^h(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v}^h - \dot{\boldsymbol{u}}^h(t))_V \quad \forall \boldsymbol{v}^h \in V^h, \tag{5.25}$$

$$\boldsymbol{u}^h(0) = \boldsymbol{u}_0^h, \tag{5.26}$$

where $\boldsymbol{u}_0^h \in V^h$ is a suitable approximation of $\boldsymbol{u}_0$.

From the discussions in Section 3, Problem $\mathbf{P}_0^h$ has a unique solution, and we have the following estimate from (3.13):

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{C^1([0,T];V)}$$

$$\leqslant c \inf_{\boldsymbol{v}^h\in C([0,T];V)} (\|\dot{\boldsymbol{u}}-\boldsymbol{v}^h\|_{C([0,T];V)} + \|R(\cdot\,;\boldsymbol{v}^h(\cdot),\dot{\boldsymbol{u}}(\cdot))\|^{1/2}_{C([0,T])}) + c\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V. \tag{5.27}$$

Assume

$$\sigma_\nu \in C([0,T];L^2(\Gamma_3)), \quad \boldsymbol{\sigma}_\tau \in C([0,T];(L^2(\Gamma_3))^d).$$

By the definitions (3.9) and (5.19) and using a Green-type formula, we have

$$R(t;\boldsymbol{v}^h(t),\dot{\boldsymbol{u}}(t))$$

$$= (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}^h(t))-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_V + (\mathscr{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}^h(t))-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_V$$

$$+ j(\boldsymbol{u}(t),\boldsymbol{v}^h(t)) - j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) - (\boldsymbol{f}(t),\boldsymbol{v}^h(t)-\dot{\boldsymbol{u}}(t))_V$$

$$= \int_\Omega \boldsymbol{\sigma}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}^h(t)-\dot{\boldsymbol{u}}(t))\,\mathrm{d}x - \int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}^h(t)-\dot{\boldsymbol{u}}(t))\,\mathrm{d}x - \int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot(\boldsymbol{v}^h(t)-\dot{\boldsymbol{u}}(t))\,\mathrm{d}a$$

$$+ \int_{\Gamma_3} p_\nu(u_\nu(t)-g)(v_\nu^h(t)-\dot{u}_\nu(t))\,\mathrm{d}a + \int_{\Gamma_3} p_\tau(u_\nu(t)-g)(|\boldsymbol{v}_\tau^h(t)|-|\dot{\boldsymbol{u}}_\tau(t)|)\,\mathrm{d}a$$

$$= \int_{\Gamma_3} (\boldsymbol{\sigma}_\tau(t)\cdot(\boldsymbol{v}_\tau^h(t)-\dot{\boldsymbol{u}}_\tau(t)) + p_\tau(u_\nu(t)-g)(|\boldsymbol{v}_\tau^h(t)|-|\dot{\boldsymbol{u}}_\tau(t)|))\,\mathrm{d}a.$$

Thus,

$$|R(t;\boldsymbol{v}^h(t),\dot{\boldsymbol{u}}(t))| \leqslant c\|\boldsymbol{v}_\tau^h(t)-\dot{\boldsymbol{u}}_\tau(t)\|_{(L^2(\Gamma_3))^d}. \tag{5.28}$$

Here and in the remaining part of the section, the constant $c$ may depend on the solution of the problem. Therefore, estimate (5.27) in this case reduces to

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{C^1([0,T];V)}$$

$$\leqslant c \inf_{\boldsymbol{v}^h\in C([0,T];V)} (\|\dot{\boldsymbol{u}}-\boldsymbol{v}^h\|_{C([0,T];V)} + \|\dot{\boldsymbol{u}}_\tau-\boldsymbol{v}_\tau^h\|^{1/2}_{C([0,T];L^2(\Gamma_3)^d)}) + c\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V.$$

When $\dot{\boldsymbol{u}}(t)\in C(\bar{\Omega})$, we use $\Pi^h\dot{\boldsymbol{u}}(t)$ to denote the standard finite element interpolant of $\dot{\boldsymbol{u}}(t)$ (cf. [5]); while if $\dot{\boldsymbol{u}}(t)\notin C(\bar{\Omega})$, we use $\Pi^h\dot{\boldsymbol{u}}(t)$ to denote Clément's interpolant introduced in [6]. Also we use the same symbol $\Pi^h$ for the interpolation on $\Gamma_3$. Under the additional regularity conditions

$$\dot{\boldsymbol{u}}\in C([0,T];H^{l+1}(\Omega)^d), \quad \dot{\boldsymbol{u}}_\tau\in C([0,T];H^{l+1}(\Gamma_3)^d), \quad \boldsymbol{u}_0\in H^{l+1}(\Omega)^d, \tag{5.29}$$

we have the following interpolation error estimates:

$$\|\dot{\boldsymbol{u}}(t)-\Pi^h\dot{\boldsymbol{u}}(t)\|_V \leqslant ch^l|\dot{\boldsymbol{u}}(t)|_{H^{l+1}(\Omega)^d}, \quad t\in[0,T], \tag{5.30}$$

$$\|\dot{\boldsymbol{u}}_\tau(t)-\Pi^h\dot{\boldsymbol{u}}_\tau(t)\|_{L^2(\Gamma_3)^d} \leqslant ch^l|\dot{\boldsymbol{u}}_\tau(t)|_{H^{l+1}(\Gamma_3)^d}, \quad t\in[0,T], \tag{5.31}$$

$$\|\boldsymbol{u}_0-\Pi^h\boldsymbol{u}_0\|_V \leqslant ch^l|\boldsymbol{u}_0|_{H^{l+1}(\Omega)^d}. \tag{5.32}$$

In conclusion, under the above stated additional assumptions, if we take $\boldsymbol{u}_0^h = \Pi^h \boldsymbol{u}_0$, then using the estimates (5.27), (5.28) and (5.30)–(5.32), we have

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{C^1([0,T];V)} \leqslant c h^{\min\{l,(l+1)/2\}}(|\dot{\boldsymbol{u}}|_{C([0,T];H^{l+1}(\Omega)^d)} + |\dot{\boldsymbol{u}}_\tau|^{1/2}_{C([0,T];H^{l+1}(\Gamma_3)^d)} + |\boldsymbol{u}_0|_{H^{l+1}(\Omega)^d}). \tag{5.33}$$

In particular, when $l=1$, i.e., when we use linear elements for the approximation, we get the optimal order error estimate

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{C^1([0,T];V)} \leqslant c h(|\dot{\boldsymbol{u}}|_{C([0,T];H^2(\Omega)^d)} + |\dot{\boldsymbol{u}}_\tau|^{1/2}_{C([0,T];H^2(\Gamma_3)^d)} + |\boldsymbol{u}_0|_{H^2(\Omega)^d}).$$

For fully discrete approximations, we need the partition of the time interval introduced in Section 4. Then a fully discrete approximation for Problem $\mathbf{P}_0$ is

**Problem $\mathbf{P}_0^{hk}$.** Find the displacement field $\mathbf{u}^{hk} = \{\mathbf{u}_n^{hk}\}_{n=0}^N \subset V^h$, such that for $n = 1,\dots,N$,

$$(\mathscr{A}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}))_V + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}^h) - j(\boldsymbol{u}_{n-1}^{hk}, \delta\boldsymbol{u}_n^{hk})$$
$$\geqslant (\boldsymbol{f}(t), \boldsymbol{v}^h - \delta\boldsymbol{u}_n^{hk})_V - (\mathscr{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}))_V \quad \forall \boldsymbol{v}^h \in V^h, \tag{5.34}$$

$$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h, \tag{5.35}$$

where again $\boldsymbol{u}_0^h \in V^h$ is a suitable approximation of $\boldsymbol{u}_0$.

From the discussions in Section 4, Problem $\mathbf{P}_0^h$ has a unique solution, and we have the following estimate from (4.11):

$$\max_{1\leqslant n\leqslant N}(\|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_n\|_V + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n^{hk}\|_V)$$
$$\leqslant c \max_{1\leqslant n\leqslant N}\{\|\boldsymbol{v}_n^h - \dot{\boldsymbol{u}}_n\|_V + |R_n(\boldsymbol{v}_n^h, \dot{\boldsymbol{u}}_n)|^{1/2}\} + c\|\boldsymbol{u}_0^h - \boldsymbol{u}_0\|_V$$
$$+ck(\|\boldsymbol{f}\|_{W^{1,1}(0,T;V)} + \|\boldsymbol{u}\|_{C^1([0,T];V)}). \tag{5.36}$$

The term $R_n(\boldsymbol{v}_n^h, \dot{\boldsymbol{u}}_n)$ is defined in (4.7). Similar to (5.28), we have

$$|R_n(\boldsymbol{v}_n^h, \dot{\boldsymbol{u}}_n)| \leqslant c\|(\boldsymbol{v}_n^h)_\tau - (\dot{\boldsymbol{u}}_n)_\tau\|_{(L^2(\Gamma_3))^d}.$$

Under the additional regularity conditions (5.29), we have interpolation error estimates similar to (5.30)–(5.32). And finally, for the fully discrete solution, the error estimate (5.33) is modified to

$$\max_{1\leqslant n\leqslant N}(\|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_n\|_V + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n^{hk}\|_V)$$
$$\leqslant c h^{\min\{l,(l+1)/2\}}(|\dot{\boldsymbol{u}}|_{C([0,T];H^{l+1}(\Omega)^d)} + |\dot{\boldsymbol{u}}_\tau|^{1/2}_{C([0,T];H^{l+1}(\Gamma_3)^d)} + |\boldsymbol{u}_0|_{H^{l+1}(\Omega)^d})$$
$$+ck(\|\boldsymbol{f}\|_{W^{1,1}(0,T;V)} + \|\boldsymbol{u}\|_{C^1([0,T];V)}). \tag{5.37}$$

We emphasize that the error estimates (5.33) and (5.37) are only sample results under the stated regularity conditions. If the regularity conditions are different, the error estimates need to be changed accordingly, but that follows easily from (5.27) and (5.36).

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