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Critical branching random walks with small drift

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Abstract

We study critical branching random walks (BRWs) $U^{(n)}$ on $\mathbb{Z}_+$ where the displacement of an offspring from its parent has drift $2\beta/\sqrt{n}$ towards the origin and reflection at the origin. We prove that for any $\alpha > 1$, conditional on survival to generation $[n^\alpha]$, the maximal displacement is $\sim (\alpha - 1)/(4\beta)\sqrt{n}\log n$. We further show that for a sequence of critical BRWs with such displacement distributions, if the number of initial particles grows like yn^α for some $y > 0$, $\alpha > 1$, and the particles are concentrated in $[0, O(\sqrt{n})]$, then the measure-valued processes associated with the BRWs converge to a measure-valued process, which, at any time $t > 0$, distributes its mass over $\mathbb{R}_+$ like an exponential distribution.

MSC: 60J80; 60G57

Keywords: Branching random walk; Maximal displacement; Galton–Watson process; Feller diffusion; Dawson–Watanabe process

1. Introduction

Durrett et al. [4] and Kesten [10] studied the maximal displacement of critical branching random walks (BRWs) on the real line conditioned to survive for a large number of generations. When the spatial displacement distribution has drift $\mu > 0$, the results in [4] imply that conditional on the event that the BRW survives for n generations, the maximal displacement of a particle from the position of the initial particle will be of order $O_P(n)$. The main result in [10] asserts that if the spatial displacement distribution has mean 0 and finite $(4 + \delta)$th moment, then

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conditional on the event that the BRW survives for n generations, the maximal displacement will be of order $O_P(\sqrt{n})$. The sharp difference between these two results gives rise to the following natural question: What happens if the spatial motions have "small drift"?

In this paper we supplement these results by showing what happens for BRWs on the nonnegative integers $\mathbb{Z}_+$ with small negative drift and reflection at 0. Assume that $U^{(n)}$ is a sequence of critical BRWs on the half line $\mathbb{Z}_+ = \{x \in \mathbb{Z} : x \geq 0\}$, each started by one particle at the origin, that evolve as follows: (A) At each time $t = 1, 2, \ldots$, particles produce offspring particles as in a standard Galton–Watson process with a mean 1, finite variance σ^2 offspring distribution $\mathcal{Q}$. (B) Each offspring particle then moves from the location of its parent according to the transition probabilities $\mathbb{P} = \mathbb{P}^{(\beta,n)}$, where $\beta \geq 0$,

$$\begin{aligned} &\mathbb{P}(x, x+1) = \frac{1}{2} - \frac{\beta}{\sqrt{n}} \quad \text{for } x \geq 1; \\ &\mathbb{P}(x, x-1) = \frac{1}{2} + \frac{\beta}{\sqrt{n}} \quad \text{for } x \geq 1; \\ &\mathbb{P}(0, 1) = 1. \end{aligned} \tag{1}$$

The spatial motion is hence slightly biased towards the origin, which serves as a reflecting barrier. Such a BRW can be used to model, for example, a branching process occurring in a V-shaped valley, where the particles, due to gravity, have a slight tendency to move towards the bottom. In [9] the aforementioned slightly biased random walk is used to model the motion of "heavy Brownian particles" in a container with its bottom as a reflecting barrier. [9] also states about the reflecting barrier that "the elucidation of its influence on the Brownian motion is of considerable theoretical interest". In this article we will study the influence of the barrier on the BRW.

Denote by $U_t^{(n)}(x)$ the number of particles in the nth BRW $U^{(n)}$ at location x at time t, and by $R_t^{(n)}$ the location of the rightmost particle at time t. Our main interest is in the conditional distribution of $R_{[n^\alpha]}^{(n)}$ given that the process $U^{(n)}$ survives for $[n^\alpha]$ generations. For $\alpha < 1$, the effect of the drift $-2\beta/\sqrt{n}$ will be negligible compared to diffusion effects over this time interval, and for $\alpha = 1$ it is just large enough to match the diffusion effects. Thus, we will focus on the case when $\alpha > 1$.

Theorem 1. *When $\beta > 0$, for each $\alpha > 1$ and $\varepsilon > 0$, the range $R_{[n^\alpha]}^{(n)}$ at time $[n^\alpha]$ satisfies*

$$\lim_{n\to\infty} P\left(\left|\frac{R_{[n^\alpha]}^{(n)}}{\sqrt{n}\log n} - \frac{\alpha-1}{4\beta}\right| \geq \varepsilon \mid G_{[n^\alpha]}^{(n)}\right) = 0, \tag{2}$$

where for any $k \in \mathbb{Z}_+$,

$$G_k^{(n)} = \{U^{(n)} \text{ survives to generation } k\}. \tag{3}$$

It is natural to consider in connection with the behavior of the maximal displacement the process-level scaling behavior of the BRWs. To this end, consider a series of BRWs $\{X^{(n)}\}$ on the set $\mathbb{Z}_+$ of nonnegative integers that evolve by the rules described above, but with arbitrary initial states $X_0^{(n)}$. (In Theorem 1 the initial state consisted of a single particle located at the origin 0.) For integers $x, k \geq 0$, set

$$X_k^{(n)}(x) = \# \quad \text{particles at } x \text{ at time } k. \tag{4}$$

For any subset $\mathrm{I} \subseteq \mathbb{R}_+$, let

$$X_k^{(n)}(\mathrm{I}) = \sum_{x \in \mathrm{I}} X_k^{(n)}(x).$$

Finally, let

$$Z_k^{(n)} = X_k^{(n)}(\mathbb{Z}_+) = \sum_x X_k^{(n)}(x).$$

Recall that by Kolmogorov's estimate for critical Galton–Watson processes (see relation (21) below), if the nth BRW $X^{(n)}$ is initiated by $O(n^\alpha)$ particles, then the total lifetime of the process will be on the order of $O_P(n^\alpha)$ generations. If $\alpha < 1$, then the effect of the drift over a time interval $[0, O(n^\alpha)]$ is too small to be felt. If $\alpha = 1$ then the drift will be just large enough to be felt, and so for large n the BRW $X^{(n)}$, suitably rescaled, will look like a Dawson–Watanabe process on the half line $[0, \infty)$ with drift -2β and reflection at 0 (for the convergence of ordinary BRWs to Dawson–Watanabe processes; see [15], or [5,13]). The case we will focus on is again when $\alpha > 1$, as in this case the effect of the reflecting barrier at 0 dominates the diffusion effects over the lifetime of the branching process, and the result is an entirely different scaling behavior:

Theorem 2. *When $\beta > 0$, assume that for some $\alpha > 1$,*

$$\frac{Z_0^{(n)}}{n^\alpha} \to y > 0, \quad \textit{as } n \to \infty, \tag{5}$$

and $\{X_0^{(n)}(\sqrt{n}\cdot)/n^\alpha\}_{n\geq 1}$ is tight, i.e., for any $\varepsilon > 0$ there exists $C > 0$ such that for all n,

$$\frac{X_0^{(n)}([C\sqrt{n}, \infty))}{n^\alpha} \leq \varepsilon. \tag{6}$$

Then the measure-valued processes $\left(X_{[n^\alpha t]}^{(n)}(\sqrt{n}\cdot)/n^\alpha : t > 0\right)$ converge, in the sense of convergence of finite-dimensional distributions, to a process $(X_t : t > 0)$, where $(X_t)_{t\geq 0}$ is such that for all $t \geq 0$ and $0 \leq a < b$,

$$X_t((a,b)) = Y_t \cdot (\exp(-4\beta a) - \exp(-4\beta b)) := Y_t \cdot \pi((a,b)). \tag{7}$$

Here Y_t is the Feller diffusion:

$$\mathrm{d}Y_t = \sigma\sqrt{Y_t}\,\mathrm{d}W_t, \quad Y_0 = y. \tag{8}$$

Observe that we *do not* require the initial measures $X_0^{(n)}(\sqrt{n}\cdot)/n^\alpha$ to converge; what we only require are (i) the total mass converges, and (ii) the particles are not too spread out. In particular, we cannot guarantee that $X_0^{(n)}(\sqrt{n}\cdot)/n^\alpha \Longrightarrow X_0$. Theorem 2 says that one has finite-dimensional convergence on $(0, \infty)$.

The Feller diffusion (Y_t) defined by (8) is the limit of $(Z_{[n^\alpha t]}^{(n)}/n^\alpha)$:

$$\left(\frac{Z_{[n^\alpha t]}^{(n)}}{n^\alpha}\right) \Rightarrow (Y_t) \quad \text{on } D([0,\infty); \mathbb{R}), \tag{9}$$

(see [6,7]). See Chapter XI of [14] for some basic properties of the Feller diffusion. The limiting process X_t hence can be described in this way: its total mass evolves like the Feller diffusion Y_t, but the *distribution* of the mass Y_t at any time $t > 0$ is always the exponential distribution π. As is proved in [9], the exponential distribution π is the stationary distribution of a diffusion process obtained by suitably normalizing the RWs defined by (1) and taking limit as $n \to \infty$.

The following elementary relation between the expected number of particles at a site y in generation m for a critical BRW and the m-step transition probability $P(S_m = y)$ of the random walk will be frequently used: if the critical BRW is started by one particle at site x, and $U_m(y)$ stands for the number of particles at site y in generation m, then

$$EU_m(y) = P(S_m = y \mid S_0 = x). \tag{10}$$

This is easily proved by induction on m, by conditioning on the first generation and using the fact the offspring distribution has mean 1.

The structure of this article is as follows: in Section 2 we prove some properties of the random walks on the half line, in Section 3 we prove Theorem 1; Theorem 2 is proved in Section 4.

Notation. We follow the custom of writing $f \sim g$ to mean that the ratio f/g converges to 1. For any $a, b \in \mathbb{R}$, $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. Throughout the paper, c, C etc. denote generic constants whose values may change from line to line. For any $x \ge 0$, $[x]$ denotes its integer part, i.e., the greatest integer no greater than x. The notation $Y_n = o_P(f(n))$ means that $Y_n/f(n) \to 0$ in probability; and $Y_n = O_P(f(n))$ means that the sequence $|Y_n|/f(n)$ is tight.

2. Random walks

Throughout this article we use the notation $\{S_m\}_{m\ge0} = \{S_m^{(\beta,n)}\}$ to denote a random walk with transition probabilities $\mathbb{P} = \mathbb{P}^{(\beta,n)}$ defined by Eq. (1); use $\{\widetilde{S}_m\}_{m\ge0}$ to denote the simple random walk on $\mathbb{Z}_+$ with reflection at 0; and use $\{\widehat{S}_m\}_{m\ge0}$ to denote the simple random walk on $\mathbb{Z}$. Furthermore, for any such random walks, e.g., $\{S_m\}$, for any $x, y \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, $P^x(S_m = y) = P(S_m = y \mid S_0 = x)$ is the probability that S_m started at x finds its way to site y in m steps.

The following lemma says that the random walk S_m which has drift towards the origin is stochastically dominated by the reflected simple random walk $\widetilde{S}_m$.

Lemma 3. *For any $\beta > 0$, $n \in \mathbb{N}$ and $x \in \mathbb{Z}_+$, we can build random walks $\{S_m\}_{m\ge0} \sim \mathbb{P}^{(\beta,n)}$ and $\{\widetilde{S}_m\}_{m\ge0} \sim \widetilde{\mathbb{P}}$ on a common probability space so that*

$$S_0 = \widetilde{S}_0 = x, \quad \text{and} \quad S_m \le \widetilde{S}_m, \quad \text{for all } m.$$

Proof. It suffices to prove the result for the case $x > 0$; the case $x = 0$ then follows since $S_1 = \widetilde{S}_1 = 1$.

Let $S_0 = \widetilde{S}_0 = x$. At time 1 sample a $U_1 \sim$ Unif $(0, 1)$. If $U_1 \le 1/2 + \beta/\sqrt{n}$, then let $S_1 = x - 1$, otherwise let $S_1 = x + 1$. In the meanwhile, if $U_1 \le 1/2$, then let $\widetilde{S}_1 = x - 1$, otherwise let $\widetilde{S}_1 = x+1$. Clearly $\{S_0, S_1\}$ and $\{\widetilde{S}_0, \widetilde{S}_1\}$ follow their laws respectively and $S_1 \le \widetilde{S}_1$. Now suppose that we have built $\{S_m\}$ and $\{\widetilde{S}_m\}$ up to time m, and we have $S_m \le \widetilde{S}_m$. If $S_m < \widetilde{S}_m$, we must have $S_m \le \widetilde{S}_m - 2$ since at each step the difference between the jumps is either 0 or 2; now because at each step the random walks can at most jump 1, at time $m + 1$, we must still have $S_{m+1} \le \widetilde{S}_{m+1}$. In the other case when $S_m = \widetilde{S}_m$, if $S_m > 0$ then we can build $S_{m+1} \le \widetilde{S}_{m+1}$ just as at time 0; otherwise $S_m = 0$, then necessarily $S_{m+1} = \widetilde{S}_{m+1} = 1$. Thus, we have proved that we can build $\{S_m\}$ and $\{\widetilde{S}_m\}$ up to time $m + 1$. By induction, the conclusion holds. $\square$

Lemma 4. *For any $k \in \mathbb{N}$, any $x \geq k$, and any $m \geq 0$,*

$$P^x(\widetilde{S}_m \geq x + k) \leq P^0\left(\max_{i \leq m} |\widehat{S}_i| \geq k\right). \tag{11}$$

Moreover, there exist $C > 0$ and $b > 0$ such that

$$P^0\left(\max_{i \leq m} |\widehat{S}_i| \geq k\right) \leq C \exp\left(-\frac{bk^2}{m}\right), \quad \text{for all } m. \tag{12}$$

Proof. Inequality (11) holds because in order that $\widetilde{S}_m \geq x + k$, either the random walk $\{\widetilde{S}_i\}_{i \leq m}$ has never visited 0, in which case it just evolves like a simple random walk whose maximal deviation from x is no less than $\widetilde{S}_m - \widetilde{S}_0 \geq k$, or the random walk $\{\widetilde{S}_i\}_{i \leq m}$ has visited 0 in which case it evolves like a simple random walk before hitting 0, and the maximal deviation from x before time m is no less than $x \geq k$.

Now let us prove (12). First recall the fact that for the simple random walk $\{\widehat{S}_m \mid \widehat{S}_0 = 0\}$, there exists $b > 0$ such that

$$\sup_m E \exp\left(b \frac{|\widehat{S}_m|^2}{m}\right) := C < \infty, \tag{13}$$

(see, e.g., Exercise 2.6 in [12]). Now by the submartingale maximal inequality, we get

$$P^0\left(\max_{i \leq m} |\widehat{S}_i| \geq k\right) = P^0\left(\max_{i \leq m} \exp(\theta|\widehat{S}_i|^2) \geq \exp(\theta k^2)\right) \leq \frac{E \exp(\theta|\widehat{S}_m|^2)}{\exp(\theta k^2)}.$$

Inequality (12) follows by taking θ to be b/m and using (13). □

Next lemma indicates that if two random walks S_m^1 and S_m^2 have the same drift $2\beta/\sqrt{n}$ towards the origin, and are such that $S_0^2 - S_0^1$ is a positive even number, then S_m^1 is stochastically dominated by S_m^2.

Lemma 5. *For any fixed $\beta > 0$, $n \in \mathbb{N}$, $0 \leq i_1 \neq i_2$, and a random walk $\{S_m^1\}_{m \geq 0} \sim \mathbb{P}^{(\beta,n)}$ with $S_0^1 = 2i_1$, we can build a coupling random walk $\{S_m^2\}_{m \geq 0} \sim \mathbb{P}^{(\beta,n)}$ with $S_0^2 = 2i_2$ on a possibly extended probability space such that*

$$\begin{cases} S_m^1 \leq S_m^2, & \text{for all } m, \text{ if } i_1 < i_2 \\ S_m^1 \geq S_m^2, & \text{for all } m, \text{ if } i_1 > i_2. \end{cases} \tag{14}$$

Similar conclusion holds if we change the initial positions of $\{S_m^1\}$ and $\{S_m^2\}$ to $S_0^1 = 2i_1 + 1$, $S_0^2 = 2i_2 + 1$.

Proof. We shall only prove for the case where $S_0^1 = 2i_1$, $S_0^2 = 2i_2$ and $i_1 < i_2$. We will build $\{S_m^2\}$ step by step: if $S_m^1 > 0$, then S_{m+1}^2 moves in the same direction away from S_m^2 as S_{m+1}^1 does, i.e.,

$$S_{m+1}^2 = S_m^2 + (S_{m+1}^1 - S_m^1);$$

otherwise if $S_m^1 = 0$, then choose S_{m+1}^2 according to distribution (1). Since $S_0^2 - S_0^1 = 2(i_2 - i_1)$ is even and at each step the difference between the jumps is either 0 or 2, the two random walks cannot cross each other and will either never meet, or merge after they meet. The dominance (14) follows. □

We now look more closely at the random walks $\{S_m\} \sim \mathbb{P} = \mathbb{P}^{(\beta,n)}$. Based on the results in [9] we show the following.

Proposition 6. *For any fixed $\beta > 0$, $a \geq 0$, and any nonnegative integer sequences $\{s_n\}$, $\{m_n\}$ with $s_n = O(\sqrt{n})$ and $\liminf_n m_n/(n(\log n)^2) > 0$, the random walks $\{S_m^{(n)} \mid S_0^{(n)} = s_n\} \sim \mathbb{P}^{(\beta,n)}$ satisfy*

$$\lim_{n\to\infty} P\left(S_{m_n}^{(n)} \geq a\sqrt{n} \mid S_0^{(n)} = s_n\right) = \exp(-4\beta a), \tag{15}$$

and

$$\lim_{n\to\infty} \frac{P\left(S_{m_n}^{(n)} \geq a\sqrt{n}\log n \mid S_0^{(n)} = s_n\right)}{n^{-4\beta a}} = 1. \tag{16}$$

Proof. When $a = 0$, (15) and (16) clearly hold. So below we assume that $a > 0$.

Let

$$q = q^{(n)} = \frac{1}{2} - \frac{\beta}{\sqrt{n}}, \quad \text{and} \quad p = p^{(n)} = \frac{1}{2} + \frac{\beta}{\sqrt{n}}.$$

By (41) in [9], for any $k > 0$,

$$\begin{aligned} P(S_{m_n}^{(n)} = k \mid S_0^{(n)} = s_n) &= \frac{p-q}{2pq}\left(\frac{q}{p}\right)^k \left(1 + (-1)^{s_n+k+m_n}\right) \\ &+ \frac{2}{\pi}\left(\frac{p}{q}\right)^{s_n/2}\left(\frac{q}{p}\right)^{k/2}\left(2\sqrt{pq}\right)^{m_n}\int_0^\pi \cos^{m_n}\theta\,\frac{\tan^2\theta}{(p-q)^2+\tan^2\theta} f_{s_n}(\theta) f_k(\theta)\,\mathrm{d}\theta \\ &:= p_{m_n}^*(k) + R_{m_n}(k), \end{aligned} \tag{17}$$

where for any $i \geq 1$,

$$f_i(\theta) = \cos i\theta - 2\frac{\beta}{\sqrt{n}}\frac{\sin i\theta}{\sin\theta}, \quad \theta \in [0,\pi].$$

We first estimate the main term $p_{m_n}^*(k)$. Depending on whether $s_n + m_n$ is even or odd, $S_{m_n}^{(n)}$ only takes even or odd values. We shall only deal with the case when $s_n + m_n$ is even. In this case,

$$\sum_{k\geq a\sqrt{n}} p_{m_n}^*(k) = 2\frac{p-q}{2pq}\sum_{k\geq a\sqrt{n},\ k \text{ even}}\left(\frac{q}{p}\right)^k.$$

Using the sum formula for geometric series and noting that

$$\frac{q}{p} = \frac{\frac{1}{2}-\frac{\beta}{\sqrt{n}}}{\frac{1}{2}+\frac{\beta}{\sqrt{n}}} \sim 1 - \frac{4\beta}{\sqrt{n}},$$

one can easily show that

$$\lim_{n\to\infty}\sum_{k\geq a\sqrt{n}} p_{m_n}^*(k) = \exp(-4\beta a). \tag{18}$$

Similarly,

$$\lim_{n\to\infty} \frac{\sum_{k\geq a\sqrt{n}\log n} p^*_{m_n}(k)}{n^{-4\beta a}} = 1. \tag{19}$$

It remains to show that the remainder terms $R_{m_n}(k)$ decay rapidly as $n \to \infty$. In fact, by the simple bound

$$|\sin i\theta| \leq i \sin\theta, \quad \text{for all } \theta \in [0, \pi],$$

we get

$$|f_i(\theta)| \leq 1 + \frac{2\beta}{\sqrt{n}} i.$$

Hence, since $s_n = O(\sqrt{n})$,

$$\begin{aligned} |R_{m_n}(k)| &\leq C \left(2\sqrt{pq}\right)^{m_n} \cdot \left(\frac{q}{p}\right)^{k/2} (1 + 2k\beta) \\ &\leq C \exp(-2\beta^2 m_n/n) \cdot \exp(-k\beta/\sqrt{n})(1 + 2k\beta). \end{aligned}$$

As

$$\sum_{k=1}^{\infty} \exp(-k\beta/\sqrt{n})(1 + 2k\beta) = O(\sqrt{n}),$$

and $\liminf_n m_n/(n(\log n)^2) > 0$, (15) and (16) follow from (18) and (19). □

3. Proof of Theorem 1

We first recall some well known facts about critical Galton–Watson processes. Let $\sigma^2 < \infty$ be the variance of the offspring distribution $\mathcal{Q}$. Then, if Z_m is the number of particles at time m with $Z_0 = 1$, and $G_m = \{Z_m > 0\}$ is the event that the Galton–Watson process survives to generation m, then

$$\mathrm{Var}(Z_m) = m\sigma^2, \tag{20}$$

$$\rho_m := P(G_m) \sim \frac{2}{m\sigma^2}, \tag{21}$$

$$E(Z_m \mid G_m) = \frac{1}{\rho_m} \sim \frac{\sigma^2 m}{2}, \quad \text{and} \tag{22}$$

$$\mathcal{L}\left(\frac{Z_m}{m}\middle| G_m\right) \Longrightarrow \mathrm{Exp}(\sigma^2/2), \quad \text{as } m \to \infty, \tag{23}$$

(see, e.g., Sections I.2 and I.9 of [2]). Relation (21) is known as Kolmogorov's estimate; (23) is Yaglom's theorem.

We will decompose the proof of Theorem 1 into two steps. In Proposition 9 we show that for any $\varepsilon > 0$, $(\alpha - 1 - \varepsilon)\sqrt{n}\log n/(4\beta)$ is an asymptotic lower bound for $R^{(n)}_{[n^\alpha]}$. Proposition 10 says that $(\alpha - 1 + \varepsilon)\sqrt{n}\log n/(4\beta)$ is an asymptotic upper bound. Theorem 1 follows by combining these two propositions.

To prove Theorem 1, we will follow the strategy used to prove Theorems 4 and 5 in [11], namely, changing the conditional event G_k to some event defined with respect to a generation $m(k) \leq k$.

The following two lemmas are Lemmas 17 and 18 in [11].

Lemma 7. *Suppose that on some probability space $(\Omega, \mathcal{F}, P)$ there are two events E_1, E_2 with $P(E_1)P(E_2) > 0$ such that*

$$\frac{P(E_1 \Delta E_2)}{P(E_1)} \leq \varepsilon, \tag{24}$$

where $E_1 \Delta E_2$ is the symmetric difference of E_1 and E_2. Then

$$\|P(\cdot|E_1) - P(\cdot|E_2)\|_{TV} \leq 2\varepsilon, \tag{25}$$

where $P(\cdot|E_i)$ denotes the conditional probability measure given the event E_i, and $\|\cdot\|_{TV}$ denotes the total variation distance.

Lemma 8. *Let $m(k) \leq k$ be integers and $\varepsilon_k > 0$ be real numbers such that $m(k)/k \to 1$ and $\varepsilon_k \to 0$ as $k \to \infty$. Then*

$$\lim_{k\to\infty} \frac{P(G_k \Delta H_k)}{P(G_k)} = 0, \tag{26}$$

where

$$G(k) = \{Z_k > 0\} \quad \text{and} \quad H(k) = \{Z_{m(k)} \geq k\varepsilon_k\}.$$

By Lemmas 7 and 8, we can change the conditioning event $G_k = \{Z_k > 0\}$ to $H_k = \{Z_{m(k)} \geq k\varepsilon_k\}$, and it suffices to prove the convergence in Theorem 1 when the conditioning event is H_k rather than G_k. The advantage of this is that, conditional on the state of the BRW at time $m(k)$, the next $k - m(k)$ generations are gotten by running *independent* BRWs for time $k - m(k)$ starting from the locations of the particles in generation $m(k)$.

We now show that $(\alpha - 1 - \varepsilon)\sqrt{n}\log n/(4\beta)$ is an asymptotic lower bound for $R^{(n)}_{[n^\alpha]}$.

Proposition 9. *For any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P\left(R^{(n)}_{[n^\alpha]} \geq \frac{\alpha - 1 - \varepsilon}{4\beta} \cdot \sqrt{n}\log n \,\middle|\, G^{(n)}_{[n^\alpha]} \right) = 1.$$

Proof. By Lemmas 7 and 8, we can change the conditioning event from $G^{(n)}_{[n^\alpha]}$ to $\{Z^{(n)}_{[n^\alpha]-nL(n)} > [n^\alpha/L(n)]\}$ for $L(n) := [(\log n)^2]$, where for any $k \geq 0$, $Z^{(n)}_k$ is the number of particles at generation k for the nth BRW $U^{(n)}$. Conditioning on $\{Z^{(n)}_{[n^\alpha]-nL(n)} > [n^\alpha/L(n)]\}$, there will be at least $X \sim \text{Bin}([n^\alpha/L(n)], \rho_{nL(n)})$ number of particles at time $[n^\alpha] - nL(n)$ whose families will survive to time $[n^\alpha]$. For any such particle, among its descendants at time $[n^\alpha]$ we uniformly pick one, then the trajectory of the chosen particle from time $[n^\alpha] - nL(n)$ to $[n^\alpha]$ will be a random walk following the law $\mathbb{P}^{\beta,n}$, starting at the location of its ancestor at time $[n^\alpha] - nL(n)$. In this way we get at least $\text{Bin}([n^\alpha/L(n)], \rho_{nL(n)})$ number of independent random walks. We would like to show the probability that the maximum of the end positions of these random walks is bigger than $(\alpha - 1 - \varepsilon)\sqrt{n}\log n/(4\beta)$ is asymptotically 1. By Lemma 5, this probability is not increased if we assume that all these random walks are started at 0 or 1, depending on whether

$[n^\alpha] - nL(n)$ is even or odd. But since the random walks have $nL(n)$ steps to go, by relation (16), no matter whether the starting point is 0 or 1, for large n, the probability that each random walk is to the right of $(\alpha - 1 - \varepsilon)/(4\beta) \cdot \sqrt{n} \log n$ is asymptotically $n^{-(\alpha-1-\varepsilon)}$. However we have at least $X \sim \text{Bin}([n^\alpha/L(n)], \rho_{nL(n)})$ number of i.i.d. trials, and by relation (21) and Chernoff bound ([3] or [1]), for all n sufficiently large,

$$P\left(X \leq \frac{1}{2} \cdot \frac{n^\alpha}{L(n)} \frac{2}{nL(n)\sigma^2}\right) \leq \exp\left(-\frac{n^\alpha}{L(n)} \frac{2}{nL(n)\sigma^2} \cdot \frac{1}{9}\right) \to 0. \tag{27}$$

It follows that the probability for the maximum of the end positions of these random walks to be bigger than $(\alpha - 1 - \varepsilon)/(4\beta) \cdot \sqrt{n} \log n$ is asymptotically 1. □

Proposition 9 gives the desired lower bound. We now prove the upper bound.

Proposition 10. *For any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P\left(R^{(n)}_{[n^\alpha]} \leq \frac{\alpha - 1 + \varepsilon}{4\beta} \cdot \sqrt{n} \log n \,\middle|\, G^{(n)}_{[n^\alpha]}\right) = 1.$$

Proof. For any $\varepsilon_n \to 0$, define $H^{(n)}_{[n^\alpha]} = \{Z^{(n)}_{[n^\alpha]-n} \geq ([n^\alpha] - n) \cdot \varepsilon_n\}$. Applying Lemmas 7 and 8 once we see that we can change the conditioning event from $G^{(n)}_{[n^\alpha]}$ to $H^{(n)}_{[n^\alpha]}$; applying these lemmas again we see that we can change the conditioning event to $G^{(n)}_{[n^\alpha]-n}$. Since $\alpha > 1$, by relation (16), the probability that the random walk $\{S_m\}_m = \{S^{(\beta,n)}_m\}_m$ is to the right of $(\alpha - 1 + \varepsilon/2)/(4\beta) \cdot \sqrt{n} \log n$ at time $[n^\alpha] - n$ is asymptotically $n^{-(\alpha-1+\varepsilon/2)}$. Thus, using relations (10) and (22), the conditional expectation of the number of particles to the right of $(\alpha - 1 + \varepsilon/2)\sqrt{n} \log n/(4\beta)$ in generation $[n^\alpha] - n$ is

$$E\left(Z^{(n)}_{[n^\alpha]-n} \mid G^{(n)}_{[n^\alpha]-n}\right) \cdot P\left(S_{[n^\alpha]-n} \geq (\alpha - 1 + \varepsilon/2)/(4\beta) \cdot \sqrt{n} \log n\right)$$
$$\sim \frac{\sigma^2([n^\alpha] - n)}{2} \cdot n^{-(\alpha-1+\varepsilon/2)} \sim \frac{n^{1-\varepsilon/2}\,\sigma^2}{2}.$$

However, by relation (21), the probability that a Galton–Watson process survives to time n is $\sim 2/(n\sigma^2)$, hence the number of particles to the *right* of $(\alpha-1+\varepsilon/2)/(4\beta)\cdot\sqrt{n} \log n$ in generation $[n^\alpha] - n$ whose families survive to time $[n^\alpha]$ has expectation asymptotically equivalent to $n^{-\varepsilon/2}$, which goes to 0. Therefore if we denote by

$R'^{(n)}_{[n^\alpha]}$ = the rightmost location in generation $[n^\alpha]$ of the descendants of the particles which are to the *left* of $(\alpha - 1 + \varepsilon/2)/(4\beta) \cdot \sqrt{n} \log n$ in generation $[n^\alpha] - n$,

then it suffices to show further that

$$P\left(R'^{(n)}_{[n^\alpha]} \geq (\alpha - 1 + \varepsilon)/(4\beta) \cdot \sqrt{n} \log n \,\middle|\, G^{(n)}_{[n^\alpha]-n}\right) \to 0. \tag{28}$$

By Lemma 5, this probability is not decreased if we assume all the particles to the left of $(\alpha - 1 + \varepsilon/2)/(4\beta) \cdot \sqrt{n} \log n$ at time $[n^\alpha] - n$ are located at M_n, where

$$M_n := \begin{cases} \text{the biggest even number } \leq (\alpha - 1 + \varepsilon/2)/(4\beta) \cdot \sqrt{n} \log n, \\ \quad \text{if } [n^\alpha] - n \text{ is even;} \\ \text{the biggest odd number } \leq (\alpha - 1 + \varepsilon/2)/(4\beta) \cdot \sqrt{n} \log n, \\ \quad \text{if } [n^\alpha] - n \text{ is odd.} \end{cases}$$

In either case, in order that $R'^{(n)}_{[n^\alpha]} \geq (\alpha - 1 + \varepsilon)/(4\beta) \cdot \sqrt{n} \log n$, since the ancestors are to the left of $(\alpha - 1 + \varepsilon/2)/(4\beta) \cdot \sqrt{n} \log n$, at least one descendant will have to travel to the right at least $\varepsilon/(8\beta) \cdot \sqrt{n} \log n$ distance. Hence, since the BRW is critical, we get

$$P\left(R'^{(n)}_{[n^\alpha]} \geq (\alpha - 1 + \varepsilon)/(4\beta) \cdot \sqrt{n} \log n \mid G^{(n)}_{[n^\alpha]-n}\right) \leq E(Z^{(n)}_{[n^\alpha]-n} \mid G^{(n)}_{[n^\alpha]-n}) \cdot P^{M_n}(S_n \geq M_n + \varepsilon/(8\beta) \cdot \sqrt{n} \log n). \tag{29}$$

By Lemma 3,

$$P^{M_n}\left(S_n \geq M_n + \varepsilon/(8\beta) \cdot \sqrt{n} \log n\right) \leq P^{M_n}\left(\widetilde{S}_n \geq M_n + \varepsilon/(8\beta) \cdot \sqrt{n} \log n\right). \tag{30}$$

When n is sufficiently large, M_n will be bigger than $\varepsilon/(8\beta) \cdot \sqrt{n} \log n$, so by Lemma 4 we get that the probability on the right side of (30) is bounded by $C \exp\left(-b\varepsilon^2/(64\beta^2) \cdot (\log n)^2\right)$. Using (29), noting that $E(Z^{(n)}_{[n^\alpha]-n} | G^{(n)}_{[n^\alpha]-n}) = O(n^\alpha)$ only grows polynomially in n, we get (28). □

4. Proof of Theorem 2

We start with a simple observation. The following lemma about the probabilities of survival is a supplement to the convergence in (9).

Lemma 11. *For the total mass processes $(Z^{(n)}_{[n^\alpha t]})_{t \geq 0}$ and the Feller diffusion $(Y_t)_{t \geq 0}$, the following convergence holds:*

$$P\left(Z^{(n)}_{[n^\alpha t]} > \delta n^\alpha\right) \to P\left(Y_t > \delta\right), \quad \textit{for all } \delta \geq 0 \textit{ and for all } t > 0. \tag{31}$$

Proof. For any $t > 0$, the convergence in (31) when $\delta > 0$ follows from the marginal convergence $Z^{(n)}_{[n^\alpha t]}/n^\alpha \Longrightarrow Y_t$ and that $P(Y_t = \delta) = 0$ (for any fixed $t > 0$, by (21) and (23) it is easy to show that the marginal distribution of Y_t can be described as a Poisson sum of exponentials, see, e.g., page 136 in [13], hence is continuous on $(0, \infty)$; see also page 441 in [14] for an explicit density formula). It remains to show

$$P\left(Z^{(n)}_{[n^\alpha t]} > 0\right) \to P(Y_t > 0).$$

In fact, by the independence between the BRWs engendered by different initial particles,

$$P\left(Z^{(n)}_{[n^\alpha t]} = 0\right) = (1 - \rho_{[n^\alpha t]})^{Z_0^{(n)}},$$

where ρ_m, as defined in (21), is the probability that a Galton–Watson process started by a single particle survives to generation m. By (5) and (21),

$$(1 - \rho_{[n^\alpha t]})^{Z_0^{(n)}} \sim \exp\left(-\frac{2}{n^\alpha t \sigma^2} \cdot Z_0^{(n)}\right) \to \exp\left(-\frac{2y}{t\sigma^2}\right).$$

The right side equals $P(Y_t = 0)$; see, e.g., Equation (II.5.12) in [13]. □

Proof of Theorem 2. A. Convergence of Marginal distributions. We will show that for any fixed $t > 0$, on the Skorokhod space $D([0, \infty); \mathbb{R})$,

$$\left(\frac{X^{(n)}_{[n^\alpha t]}([0, \sqrt{n}a])}{n^\alpha}\right)_{a \geq 0} \Longrightarrow (X_t([0, a]) = Y_t \cdot \pi([0, a]))_{a \geq 0}. \tag{32}$$

Let $L(n) := [(\log n)^2]$, and write

$$\frac{X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a])}{n^\alpha} = \frac{Z^{(n)}_{[n^\alpha t]-nL(n)}}{n^\alpha} \cdot \frac{X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a])}{Z^{(n)}_{[n^\alpha t]-nL(n)}} \cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>0\}}.$$

For any $a \geq 0$ and $\delta > 0$, we will show the following law of large numbers:

$$\left(\frac{X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a])}{Z^{(n)}_{[n^\alpha t]-nL(n)}} - \pi([0,a])\right) \cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}} \to 0. \tag{33}$$

Claim: If this holds, then we have the finite-dimensional convergence below: for any $k \in \mathbb{N}$ and any $0 \leq a_1 \leq \cdots \leq a_k < \infty$,

$$\left(\frac{X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a_i])}{n^\alpha}\right)_{a_1,\ldots,a_k} \Longrightarrow (Y_t \cdot \pi([0,a_i]))_{a_1,\ldots,a_k}. \tag{34}$$

Note that the LHS and RHS of (32) are both increasing processes and the RHS is continuous, by Theorem VI.3.37 in [8], the above finite-dimensional convergence implies the convergence (32) as processes on $[0,\infty)$.

We now prove the claim, which is a direct consequence of Lemma 11, (9), Slutsky's theorem and (33). We shall only prove the convergence for any single $a \geq 0$; the joint convergence can be proved similarly. Let $f : \mathbb{R} \to \mathbb{R}$ be any bounded Lipschitz continuous function. We want to show that

$$Ef\left(\frac{X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a])}{n^\alpha}\right) \to Ef(Y_t \cdot \pi[0,a]). \tag{35}$$

In fact, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$P(0 < Y_t \leq \delta) \leq \varepsilon.$$

By Lemma 11, for all n sufficiently large,

$$P(0 < Z^{(n)}_{[n^\alpha t]-nL(n)} \leq \delta n^\alpha) \leq 2\varepsilon.$$

Hence, denote by $M = \max_x |f(x)|$,

$$\begin{aligned}
&\left|Ef\left(X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a])/n^\alpha\right) - Ef(Y_t \cdot \pi[0,a])\right| \\
&\quad \leq \left|f(0)\cdot P\left(Z^{(n)}_{[n^\alpha t]-nL(n)} = 0\right) - f(0)\cdot P(Y_t = 0)\right| + 3M\varepsilon \\
&\qquad + \left|E\left(f\left(\frac{Z^{(n)}_{[n^\alpha t]-nL(n)}}{n^\alpha} \cdot \frac{X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a]}{Z^{(n)}_{[n^\alpha t]-nL(n)}}\right)\mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)\right. \\
&\qquad \left. - E\left(f\left(\frac{Z^{(n)}_{[n^\alpha t]-nL(n)}}{n^\alpha}\cdot \pi[0,a]\right)\mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)\right| \\
&\qquad + \left|E\left(f\left(\frac{Z^{(n)}_{[n^\alpha t]-nL(n)}}{n^\alpha}\cdot \pi[0,a]\right)\mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)\right.
\end{aligned}$$

$$- \; E\left(f(Y_t \cdot \pi[0,a]) \cdot \mathbf{1}_{\{Y_t>\delta\}}\right)\Bigg|$$

$$:= \mathrm{I} + 3M\varepsilon + \mathrm{II} + \mathrm{III}.$$

By Lemma 11, I $\to 0$. By (9) and Slutsky's theorem, III $\to 0$. Finally, II $\to 0$ by the Lipschitz continuity of f, (9), (33) and the dominated convergence theorem.

We now prove the law of large numbers (33), by using a mean-variance calculation. Let $\mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}$ be the configuration of the BRW at time $[n^\alpha t]-nL(n)$, $\mathcal{Z}^{(n)}_{[n^\alpha t]-nL(n)}$ be the set of particles at time $[n^\alpha t]-nL(n)$, and for each particle $u_i = u_i^{(n)} \in \mathcal{Z}^{(n)}_{[n^\alpha t]-nL(n)}$, let $x_i = x_i^{(n)}$ be its location (at time $[n^\alpha t]-nL(n)$), $U_k^{u_i}(x)$ be its number of descendants at site x at time $k+[n^\alpha t]-nL(n)$, and $Z_k^{u_i}$ be its total number of descendants at time $k+[n^\alpha t]-nL(n)$.

We start with the mean calculation.

$$E\left(\frac{X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a]}{Z^{(n)}_{[n^\alpha t]-nL(n)}} \cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)$$
$$= E\left(\frac{E\left(X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a] \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}\right)}{Z^{(n)}_{[n^\alpha t]-nL(n)}} \cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)$$

By relation (10),

$$E\left(X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a] \mid \mathcal{F}^{(n)}_{[n^\alpha t-nL(n)]}\right) = \sum_{i=1}^{Z^{(n)}_{[n^\alpha t]-nL(n)}} P(S_{nL(n)} \in [0,\sqrt{n}a] \mid S_0 = x_i).$$

By Lemma 5, if we let

$$\begin{cases} p^0_{nL(n)} := P(S_{nL(n)} \in [0,\sqrt{n}a] \mid S_0 = 0) \\ p^1_{nL(n)} := P(S_{nL(n)} \in [0,\sqrt{n}a] \mid S_0 = 1), \end{cases}$$

then

$$P(S_{nL(n)} \in [0,\sqrt{n}a] \mid S_0 = x_i) \le \begin{cases} p^0_{nL(n)}, & \text{if } x_i \text{ is even} \\ p^1_{nL(n)}, & \text{if } x_i \text{ is odd.} \end{cases} \tag{36}$$

Therefore, by Proposition 6 and Lemma 11,

$$E\left(\frac{E\left(X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a] \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}\right)}{Z^{(n)}_{[n^\alpha t]-nL(n)}} \cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)$$
$$\le E\left(p^0_{nL(n)} \vee p^1_{nL(n)} \cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right) \to \pi[0,a]\cdot P(Y_t>\delta). \tag{37}$$

On the other hand, by relation (10) again, for any $C>0$,

$$E\left(\frac{1}{n^\alpha} X^{(n)}_{[n^\alpha t]-nL(n)}([C\sqrt{n},\infty))\right) = \frac{1}{n^\alpha}\sum_{i=1}^{Z_0^{(n)}} P\left(S_{[n^\alpha t]-nL(n)} \ge C\sqrt{n} \mid S_0 = x^{(n)}_{0;i}\right),$$

where $Z_0^{(n)}$ is the total number of particles at time 0, and $x_{0;i}^{(n)}$ is the location of the ith initial particle. By the tightness of $\{X_0^{(n)}(\sqrt{n}\cdot)/n^\alpha\}$ (6) and (15) we see for any $\varepsilon > 0$, there exists $C > 0$ such that for all n sufficiently large,

$$E\left(\frac{1}{n^\alpha} X_{[n^\alpha t]-nL(n)}^{(n)}([C\sqrt{n}, \infty))\right) \le \varepsilon. \tag{38}$$

Therefore by Markov's inequality,

$$P\left(\frac{1}{n^\alpha} X_{[n^\alpha t]-nL(n)}^{(n)}([C\sqrt{n}, \infty)) \ge \sqrt{\varepsilon}\right) \le \sqrt{\varepsilon}. \tag{39}$$

Now by Lemma 5 again, for those particles u_i at time $[n^\alpha t] - nL(n)$ which are *to the left* of $C\sqrt{n}$, if we let

$$\begin{cases} M_{n;even} = \text{the biggest even number } \le C\sqrt{n}; \\ M_{n;odd} = \text{the biggest odd number } \le C\sqrt{n}, \end{cases}$$

and

$$\begin{cases} p_{nL(n)}^{even} := P(S_{nL(n)} \in [0, \sqrt{n}a] \mid S_0 = M_{n;even}) \\ p_{nL(n)}^{odd} := P(S_{nL(n)} \in [0, \sqrt{n}a] \mid S_0 = M_{n;odd}), \end{cases}$$

then

$$P(S_{nL(n)} \in [0, \sqrt{n}a] \mid S_0 = x_i) \ge \begin{cases} p_{nL(n)}^{even}, & \text{if } x_i \text{ is even} \\ p_{nL(n)}^{odd}, & \text{if } x_i \text{ is odd.} \end{cases} \tag{40}$$

Hence, by Proposition 6, Lemma 11 and (39),

$$\begin{aligned}
&\liminf_n E\left(\frac{E\left(X_{[n^\alpha t]}^{(n)}[0, \sqrt{n}a] \mid \mathcal{F}_{[n^\alpha t]-nL(n)}^{(n)}\right)}{Z_{[n^\alpha t]-nL(n)}^{(n)}} \cdot \mathbf{1}_{\{Z_{[n^\alpha t]-nL(n)}^{(n)} > \delta n^\alpha\}}\right) \\
&\ge \liminf_n E\Bigg(\frac{E\left(X_{[n^\alpha t]}^{(n)}[0, \sqrt{n}a] \mid \mathcal{F}_{[n^\alpha t]-nL(n)}^{(n)}\right)}{Z_{[n^\alpha t]-nL(n)}^{(n)}} \\
&\qquad \times \mathbf{1}_{\{Z_{[n^\alpha t]-nL(n)}^{(n)} > \delta n^\alpha;\ X_{[n^\alpha t]-nL(n)}^{(n)}([C\sqrt{n}, \infty)) \le n^\alpha \sqrt{\varepsilon}\}}\Bigg) \\
&\ge \liminf_n E\Bigg(\frac{Z_{[n^\alpha t]-nL(n)}^{(n)} - n^\alpha \sqrt{\varepsilon}}{Z_{[n^\alpha t]-nL(n)}^{(n)}}\ p_{nL(n)}^{even} \wedge p_{nL(n)}^{odd} \\
&\qquad \times \mathbf{1}_{\{Z_{[n^\alpha t]-nL(n)}^{(n)} > \delta n^\alpha;\ X_{[n^\alpha t]-nL(n)}^{(n)}([C\sqrt{n}, \infty)) \le n^\alpha \sqrt{\varepsilon}\}}\Bigg) \\
&\ge \left(1 - \frac{\sqrt{\varepsilon}}{\delta}\right) \cdot \pi[0, a] \cdot \left(P(Y_t > \delta) - \sqrt{\varepsilon}\right).
\end{aligned} \tag{41}$$

By the arbitrariness of ε, we get the desired lower bound

$$\liminf_n E\left(\frac{E\left(X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a]) \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}\right)}{Z^{(n)}_{[n^\alpha t]-nL(n)}}\cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right) \geq \pi[0,a]\cdot P(Y_t>\delta).$$

So, combining it with (37), we get the convergence of expectation

$$\lim_n E\left(\left(\frac{X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a]}{Z^{(n)}_{[n^\alpha t]-nL(n)}}-\pi[0,a]\right)\cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)=0.$$

It remains to show that

$$\lim_n \mathrm{Var}\left(\left(\frac{X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a]}{Z^{(n)}_{[n^\alpha t]-nL(n)}}-\pi[0,a]\right)\cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)=0. \tag{42}$$

By conditioning on $\mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}$, we get

$$\begin{aligned}
&\mathrm{Var}\left(\left(\frac{X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a]}{Z^{(n)}_{[n^\alpha t]-nL(n)}}-\pi[0,a]\right)\cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)\\
&\quad = E\left(\left(\frac{\mathrm{Var}\left(X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a] \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}\right)}{\left(Z^{(n)}_{[n^\alpha t]-nL(n)}\right)^2}\right)\cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)\\
&\qquad + \mathrm{Var}\left(\left(\frac{E\left(X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a] \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}\right)}{Z^{(n)}_{[n^\alpha t]-nL(n)}}-\pi[0,a]\right)\cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)\\
&\quad := \mathrm{I}+\mathrm{II}.
\end{aligned}$$

We will show that both terms converge to 0.

We start with term I. Recall that for each particle $u_i \in \mathcal{Z}^{(n)}_{[n^\alpha t]-nL(n)}$, $U^{u_i}_k(x)$ denotes its number of descendants at site x at time $k+[n^\alpha t]-nL(n)$, and $Z^{u_i}_k$ is its total number of descendants at time $k+[n^\alpha t]-nL(n)$. By the independence between the BRWs U^{u_i} and (20),

$$\begin{aligned}
\mathrm{Var}\left(X^{(n)}_{[n^\alpha t]}[0,\sqrt{n}a] \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}\right) &= \sum_{u_i\in \mathcal{Z}^{(n)}_{[n^\alpha t]-nL(n)}} \mathrm{Var}\left(\sum_{x\in[0,\sqrt{n}a]} U^{u_i}_{nL(n)}(x)\right)\\
&\leq \sum_{u_i\in \mathcal{Z}^{(n)}_{[n^\alpha t]-nL(n)}} E\left(Z^{u_i}_{nL(n)}\right)^2\\
&= Z^{(n)}_{[n^\alpha t]-nL(n)}(1+nL(n)\sigma^2).
\end{aligned}$$

Hence

$$\mathrm{I}\leq E\left(\frac{1+nL(n)\sigma^2}{Z^{(n)}_{[n^\alpha t]-nL(n)}}\cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)}>\delta n^\alpha\}}\right)\to 0.$$

As to term II, by (36),

$$E\left(X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a]) \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}\right) \le Z^{(n)}_{[n^\alpha t]-nL(n)} \cdot p^0_{nL(n)} \vee p^1_{nL(n)}$$

furthermore, on the event $\{X^{(n)}_{[n^\alpha t]-nL(n)}([C\sqrt{n},\infty)) \le n^\alpha\sqrt{\varepsilon}\}$, by (40),

$$E\left(X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a]) \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)}\right) \ge \left(Z^{(n)}_{[n^\alpha t]-nL(n)} - n^\alpha\sqrt{\varepsilon}\right) \cdot p^{even}_{nL(n)} \wedge p^{odd}_{nL(n)}.$$

Hence,

$$\begin{aligned}
\mathrm{II} &\le E\left(\left(\frac{E(X^{(n)}_{[n^\alpha t]}([0,\sqrt{n}a]) \mid \mathcal{F}^{(n)}_{[n^\alpha t]-nL(n)})}{Z^{(n)}_{[n^\alpha t]-nL(n)}} - \pi[0,a]\right)^2 \cdot \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)} > \delta n^\alpha\}}\right) \\
&\le \sqrt{\varepsilon} + E\Bigg(\max\Bigg(\left(p^0_{nL(n)} \vee p^1_{nL(n)} - \pi[0,a]\right)^2, \\
&\qquad ((1-\sqrt{\varepsilon}/\delta)\; p^{even}_{nL(n)} \wedge p^{odd}_{nL(n)} - \pi[0,a])^2\Bigg) \\
&\qquad \times \mathbf{1}_{\{Z^{(n)}_{[n^\alpha t]-nL(n)} > \delta n^\alpha;\; X^{(n)}_{[n^\alpha t]-nL(n)}([C\sqrt{n},\infty)) \le n^\alpha\sqrt{\varepsilon}\}}\Bigg) \\
&= O(\sqrt{\varepsilon}),
\end{aligned}$$

where the term $\sqrt{\varepsilon}$ in the second inequality comes from (39), and in the last equation we used Proposition 6. By the arbitrariness of ε, II $\to 0$ and hence (42) holds.

B. Convergence of finite-dimensional distributions. This follows from the Markov property and similar calculations as in Part A. □

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