Journal of Algebra 348 (2011) 233–263

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The group of automorphisms of the algebra of polynomial integro-differential operators

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ARTICLE INFO

Article history:
Received 18 August 2010
Available online 19 October 2011
Communicated by J.T. Stafford

MSC:
16W20
14E07
14H37
14R10
14R15

Keywords:
Algebra of polynomial integro-differential operators
Group of automorphisms
Stabilizer
Weyl algebras
Jacobian algebras
Inversion formula
Prime spectrum

ABSTRACT

The group G_n of automorphisms of the algebra $\mathbb{I}_n := K\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \int_1, \ldots, \int_n\rangle$ of polynomial integro-differential operators is found:

$$\mathrm{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n) \supseteq S_n \ltimes \mathbb{T}^n \ltimes \underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes GL_\infty(K)}_{2^n-1 \text{ times}},$$

$$\mathrm{G}_1 \simeq \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K),$$

where S_n is the symmetric group, $\mathbb{T}^n$ is the n-dimensional algebraic torus, $\mathrm{Inn}(\mathbb{I}_n)$ is the group of inner automorphisms of $\mathbb{I}_n$ (which is huge). It is proved that each automorphism $\sigma \in \mathrm{G}_n$ is uniquely determined by the elements $\sigma(x_i)$'s or $\sigma(\frac{\partial}{\partial x_i})$'s or $\sigma(\int_i)$'s. The stabilizers in G_n of all the ideals of $\mathbb{I}_n$ are found, they are subgroups of *finite* index in G_n. It is shown that the group G_n has trivial centre, $\mathbb{I}_n^{\mathrm{G}_n} = K$ and $\mathbb{I}_n^{\mathrm{Inn}(\mathbb{I}_n)} = K$, the (unique) maximal ideal of $\mathbb{I}_n$ is the *only* nonzero prime G_n-invariant ideal of $\mathbb{I}_n$, and there are precisely $n+2$ G_n-invariant ideals of $\mathbb{I}_n$. For each automorphism $\sigma \in \mathrm{G}_n$, an *explicit inversion formula* is given via the elements $\sigma(\frac{\partial}{\partial x_i})$ and $\sigma(\int_i)$.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.09.025

1. Introduction

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; K is a field of characteristic zero and K^* is its group of units; $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra over K; $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n; $\mathrm{End}_K(P_n)$ is the algebra of all K-linear maps from P_n to P_n and $\mathrm{Aut}_K(P_n)$ is its group of units (i.e. the group of all the invertible linear maps from P_n to P_n); the subalgebra $A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ of $\mathrm{End}_K(P_n)$ is called the nth *Weyl* algebra.

Definition. (See [2].) The *Jacobian algebra* $\mathbb{A}_n$ is the subalgebra of $\mathrm{End}_K(P_n)$ generated by the Weyl algebra A_n and the elements $H_1^{-1}, \dots, H_n^{-1} \in \mathrm{End}_K(P_n)$ where

$$H_1 := \partial_1 x_1, \quad \dots, \quad H_n := \partial_n x_n.$$

Clearly, $\mathbb{A}_n = \bigotimes_{i=1}^n \mathbb{A}_1(i) \simeq \mathbb{A}_1^{\otimes n}$ where

$$\mathbb{A}_1(i) := K\langle x_i, \partial_i, H_i^{-1}\rangle \simeq K\langle x_i, H_i^{\pm 1}, y_i := H_i^{-1}\partial_i\rangle \simeq \mathbb{A}_1.$$

The algebra $\mathbb{A}_n$ contains all the integrations $\int_i : P_n \to P_n$, $p \mapsto \int p\, dx_i$, since

$$\int_i = x_i H_i^{-1} : x^\alpha \mapsto (\alpha_i + 1)^{-1} x_i x^\alpha.$$

In particular, the algebra $\mathbb{A}_n$ contains the *algebra* $\mathbb{I}_n := K\langle x_1, \dots, x_n, \ \partial_1, \dots, \partial_n, \int_1, \dots, \int_n\rangle$ *of polynomial integro-differential operators*. Note that $\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i) \simeq \mathbb{I}_1^{\otimes n}$ where $\mathbb{I}_1(i) := K\langle x_i, \partial_i, \int_i\rangle$. Let $\mathrm{G}_n := \mathrm{Aut}_{K\text{-alg}}(\mathbb{I}_n)$ and $\mathbb{G}_n := \mathrm{Aut}_{K\text{-alg}}(\mathbb{A}_n)$.

The Jacobian algebra $\mathbb{A}_n$ is a (two-sided) localization $\mathbb{A}_n = S^{-1}\mathbb{I}_n$ of the algebra $\mathbb{I}_n$ at a countably generated commutative monoid $S \simeq \mathbb{N}^{(\mathbb{N})}$, each element of S is a regular element of the algebra $\mathbb{I}_n$ [10]. In general, there is no connection between the groups of automorphisms of an algebra and its localization. As a rule, the latter is smaller than the former, and an automorphism of the algebra cannot be extended to an automorphism of its localization (e.g., the group $\mathrm{Aut}_{K\text{-alg}}(P_n)$ is huge but $\mathrm{Aut}_{K\text{-alg}}(K[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ is a tiny group). Completely the opposite is true for the pair $\mathbb{I}_n$, $\mathbb{A}_n = S^{-1}\mathbb{I}_n$: each automorphism of the algebra $\mathbb{I}_n$ can be extended to an automorphism of the algebra $\mathbb{A}_n$ (this is not straightforward since $\mathrm{G}_n S \nsubseteq S$). Moreover, the group G_n can be seen as a subgroup of $\mathbb{G}_n$ (Theorem 5.4), and the group $\mathbb{G}_n$ is bigger than G_n. This fact, i.e. $\mathrm{G}_n \subseteq \mathbb{G}_n$, is one of the key moments in finding the group G_n as the group $\mathbb{G}_n$ was already found in [9].

The algebras $P_{2n} = P_2^{\otimes n}$, $A_n = A_1^{\otimes n}$, $\mathbb{S}_n = \mathbb{S}_1^{\otimes n}$, $\mathbb{I}_n = \mathbb{I}_1^{\otimes n}$ and $\mathbb{A}_n = \mathbb{A}_1^{\otimes n}$ have similar defining relations:

$$P_2 = K\langle x, y\rangle: \quad yx - xy = 0;$$

$$A_1 = K\langle x, \partial\rangle: \quad \partial x - x\partial = 1;$$

$$\mathbb{S}_1 = K\langle x, y\rangle: \quad yx = 1;$$

$$\mathbb{I}_1 = K\left\langle \partial, H, \int \right\rangle: \quad \partial \int = 1, \quad \left[H, \int\right] = \int, \quad [H, \partial] = -\partial,$$
$$H\left(1 - \int \partial\right) = \left(1 - \int \partial\right)H = 1 - \int \partial;$$
$$\mathbb{A}_1 = K\langle x, H^{\pm 1}, y\rangle: \quad yx = 1, \quad [H, x] = x, \quad [H, y] = -y, \quad H(1 - xy) = (1 - xy)H = 1 - xy;$$

where $[a, b] := ab - ba$ is the commutator of elements a and b. It is reasonable to believe that they should have similar groups of automorphisms. This is exactly the case when $n = 1$: the groups of automorphisms of the algebras P_2, A_1, $\mathbb{S}_1$, $\mathbb{I}_1$ and $\mathbb{A}_1$ have almost identical structure (when properly interpreted). Namely, each of the groups is a 'product' (in the last three cases it is even the semi-direct product) of an obvious subgroup of affine automorphisms and a non-obvious subgroup generated by 'transvections.'

The group $\mathrm{Aut}_{K\text{-alg}}(P_2)$ was found by Jung [15] in 1942 and van der Kulk [17] in 1953. In 1968, Dixmier [12] found the group of automorphisms of the first Weyl algebra A_1 (in prime characteristic the group of automorphism of the first Weyl algebra A_1 was found by Makar-Limanov [18] in 1984, see also [4] for a different approach and for further developments). In 2000, Gerritzen [13] found generators for the group $\mathrm{Aut}_{K\text{-alg}}(\mathbb{S}_1)$. For the higher Weyl algebras A_n, $n \geqslant 2$, and the polynomial algebras P_n, $n \geqslant 3$, to find their groups of automorphisms and generators are old open problems, and solutions to the Jacobian Conjecture and the Problem/Conjecture of Dixmier would be an important (easier) part in finding the groups. (Positive solutions to these two problems would define the groups as infinite dimensional varieties, i.e. they would give defining equations of the varieties but not generators. To find generators one would have to find the solutions of the equations. A finite dimensional analogue of this situation is the group SL_n: the defining equation $\det = 1$ tells nothing about generators of the group.)

The Jacobian algebras $\mathbb{A}_n$ arose in my study of the group of polynomial automorphisms and the Jacobian Conjecture, which is a conjecture that makes sense *only* for polynomial algebras in the class of all commutative algebras [1]. In order to solve the Jacobian Conjecture, it is reasonable to believe that one should create a technique which makes sense *only* for polynomials; the Jacobian algebras are a step in this direction (they exist for polynomials but make no sense even for Laurent polynomials).

The Jacobian algebras $\mathbb{A}_n$ were invented to deal with polynomial automorphisms. A study of these algebras led to study of 'simpler' algebras $\mathbb{S}_n$ [5], the so-called *algebras of one-sided inverses of polynomial algebras*. This ended up in finding their groups of automorphisms $\mathrm{Aut}_{K\text{-alg}}(\mathbb{S}_n)$, $n \geqslant 1$, and their explicit generators in the series of three papers [6–8]. Recently, the groups $\mathrm{Aut}_{K\text{-alg}}(\mathbb{A}_n)$, $n \geqslant 1$, are found in [9]. Finally, in the present paper the groups $\mathrm{G}_n := \mathrm{Aut}_{K\text{-alg}}(\mathbb{I}_n)$, $n \geqslant 1$, are found.

- (Theorem 5.5.(1)) $\mathrm{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ *where* S_n *is the symmetric group,* $\mathbb{T}^n$ *is the n-dimensional algebraic torus and* $\mathrm{Inn}(\mathbb{I}_n)$ *is the group of inner automorphisms of the algebra* $\mathbb{I}_n$.
- (Theorem 3.1.(2)) *The map* $(1 + \mathfrak{a}_n)^* \to \mathrm{Inn}(\mathbb{I}_n)$, $u \mapsto \omega_u$, *is a group isomorphism where* $\omega_u(a) := uau^{-1}$, $(1 + \mathfrak{a}_n)^* := \mathbb{I}_n^* \cap (1 + \mathfrak{a}_n)$, $\mathbb{I}_n^*$ *is the group of units of the algebra* $\mathbb{I}_n$, *and* $\mathfrak{a}_n$ *is the only maximal ideal of the algebra* $\mathbb{I}_n$.

The paper proceeds as follows. In Section 2, some known results about the algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ are collected that are used freely in the paper.

One of the key ideas in finding the group G_n is the fact that the polynomial algebra P_n is the only (up to isomorphism) faithful simple $\mathbb{I}_n$-module [10, Proposition 3.8]. This enables us to visualize the group G_n as a subgroup of $\mathrm{Aut}_K(P_n)$ (Corollary 3.3.(2)):

$$\mathrm{G}_n = \left\{\sigma_\varphi \;\middle|\; \varphi \in \mathrm{Aut}_K(P_n),\ \varphi\mathbb{I}_n\varphi^{-1} = \mathbb{I}_n\right\} \quad \text{where } \sigma_\varphi(a) := \varphi a \varphi^{-1},\ a \in \mathbb{I}_n.$$

In Section 3, two 'rigidity theorems' are proved for the group G_n: Corollary 3.7 and

- (Theorem 3.6) (Rigidity of the group G_n) *Let $\sigma, \tau \in G_n$. Then the following statements are equivalent.*
  1. $\sigma = \tau$.
  2. $\sigma(\int_1) = \tau(\int_1), \dots, \sigma(\int_n) = \tau(\int_n)$.
  3. $\sigma(\partial_1) = \tau(\partial_1), \dots, \sigma(\partial_n) = \tau(\partial_n)$.
  4. $\sigma(x_1) = \tau(x_1), \dots, \sigma(x_n) = \tau(x_n)$.

In Section 4, the group G_1 and its explicit generators are found (Theorem 4.3). The key ingredients of the proof of Theorem 4.3 are the Fredholm operators, their indices and the Rigidity of the group G_1 (Theorem 3.6). It is proved that each algebra endomorphism of the algebra $\mathbb{I}_1$ is a monomorphism (Theorem 4.5), and no proper prime factor algebra of the algebra $\mathbb{I}_n$ can be embedded into the algebra $\mathbb{I}_n$ (Theorem 5.19). These two results have bearing of the Jacobian Conjecture and the Problem/Conjecture of Dixmier (*each algebra endomorphism of the Weyl algebra is an isomorphism*).

Section 5 contains the main results of the paper, a proof that $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ (Theorem 5.5) is given.

- (Theorem 5.4)
  1. $G_n = \{\sigma \in \mathbb{G}_n \mid \sigma(\mathbb{I}_n) = \mathbb{I}_n\}$ *and G_n is a subgroup of $\mathbb{G}_n$.*
  2. *Each automorphism of the algebra $\mathbb{I}_n$ has a unique extension to an automorphism of the algebra $\mathbb{A}_n$.*
- (Theorem 5.15) *The centre of the group G_n is $\{e\}$.*
- (Theorem 5.17) $\mathbb{I}_n^{G_n} = K$ *and* $\mathbb{I}_n^{\mathrm{Inn}(\mathbb{I}_n)} = K$, *the algebras of invariants.*

Each automorphism $\sigma \in G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ is a unique product $\sigma = st_\lambda \omega_\varphi$ which is called the *canonical form* of σ where $s \in S_n$, $t_\lambda \in \mathbb{T}^n$, $\omega_\varphi \in \mathrm{Inn}(\mathbb{I}_n)$ and $\varphi \in (1 + \mathfrak{a}_n)^*$ (φ is unique).

- (Corollary 5.10) *Let $\sigma \in G_n$ and $\sigma = st_\lambda \omega_\varphi$ be its canonical form. Then the automorphisms s, t_λ, and ω_φ can be effectively* (*in finitely many steps*) *found from the action of the automorphism σ on the elements $\{H_i, \partial_i, \int_i \mid i = 1, \dots, n\}$:*

$$\sigma(H_i) \equiv H_{s(i)} \mod \mathfrak{a}_n, \qquad \sigma(\partial_i) \equiv \lambda_i^{-1} \partial_{s(i)} \mod \mathfrak{a}_n, \qquad \sigma\left(\int_i\right) \equiv \lambda_i \int_{s(i)} \mod \mathfrak{a}_n,$$

and the elements φ and φ^{-1} are given by the formulae (26) *and* (27) *respectively for the automorphism $(st_\lambda)^{-1}\sigma \in \mathrm{Inn}(\mathbb{I}_n)$.*

The explicit formulae (27) and (26) are too complicated to reproduce them in the Introduction.

- (Corollary 5.11) (A criterion of being inner automorphism) *Let $\sigma \in G_n$. The following statements are equivalent.*
  1. $\sigma \in \mathrm{Inn}(\mathbb{I}_n)$.
  2. $\sigma(\partial_i) \equiv \partial_i \mod \mathfrak{a}_n$ *for* $i = 1, \dots, n$.
  3. $\sigma(\int_i) \equiv \int_i \mod \mathfrak{a}_n$ *for* $i = 1, \dots, n$.

1.1. An inversion formula for $\sigma \in G_n$

The next theorem gives an inversion formula for $\sigma \in G_n$ via the elements $\{\sigma(\partial_i), \sigma(\int_i) \mid i = 1, \dots, n\}$.

- (Theorem 5.14) *Let $\sigma \in G_n$ and $\sigma = st_\lambda \omega_\varphi$ be its canonical form where $s \in S_n$, $t_\lambda \in \mathbb{T}^n$ and $\omega_\varphi \in \mathrm{Inn}(\mathbb{I}_n)$ for a unique element $\varphi \in (1 + \mathfrak{a}_n)^*$. Then $\sigma^{-1} = s^{-1} t_{s(\lambda^{-1})} \omega_{st_\lambda(\varphi^{-1})}$ is the canonical form of the automorphism σ^{-1} where the elements φ^{-1} and φ are given by the formulae* (27) *and* (26) *respectively for the automorphism $(st_\lambda)^{-1}\sigma \in \mathrm{Inn}(\mathbb{I}_n)$.*

In Section 6, the stabilizers in the group G_n of all the ideals of the algebra $\mathbb{I}_n$ are computed (Theorem 6.2). In particular, the stabilizers of all the prime ideals of $\mathbb{I}_n$ are found (Corollary 6.4.(2)).

- (Corollary 6.4.(3)) *The ideal $\mathfrak{a}_n$ is the only nonzero, prime, G_n-invariant ideal of the algebra $\mathbb{I}_n$.*
- (Corollary 6.4) *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{I}_n$. Then its stabilizer $\mathrm{St}_{\mathbb{I}_n}(\mathfrak{p})$ is a maximal subgroup of the group G_n iff $n > 1$ and $\mathfrak{p}$ is of height 1, and, in this case, $[G_n : \mathrm{St}_{G_n}(\mathfrak{p})] = n$.*
- (Corollary 6.3) *Let $\mathfrak{a}$ be a proper ideal of $\mathbb{I}_n$. Then its stabilizer $\mathrm{St}_{G_n}(\mathfrak{a})$ has finite index in the group G_n.*
- (Corollary 6.5) *If $\mathfrak{a}$ is a generic ideal of $\mathbb{I}_n$ then its stabilizer can be written via the wreath products of the symmetric groups*:

$$\mathrm{St}_{G_n}(\mathfrak{a}) = \left(S_m \times \prod_{i=1}^{t} (S_{h_i} \wr S_{n_i}) \right) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n),$$

where $\wr$ stands for the wreath *product of groups.*

Corollary 6.6 classifies all the proper G_n-invariant ideals of the algebra $\mathbb{I}_n$, there are exactly n of them.

2. The algebras $\mathbb{I}_n$ and $\mathbb{A}_n$

In this section, for the reader's convenience we collect some known results about the algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ from the papers [2,9,10] that are used later in the paper.

The algebra $\mathbb{I}_n$ is a prime, central, catenary, non-Noetherian algebra of classical Krull dimension n and of Gelfand–Kirillov dimension $2n$ [10]. Since $x_i = \int_i H_i$, the algebra $\mathbb{I}_n$ is generated by the elements $\{\partial_i, H_i, \int_i \mid i = 1, \dots, n\}$, and $\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i)$ where

$$\mathbb{I}_1(i) := K\left\langle \partial_i, H_i, \int_i \right\rangle = K\left\langle \partial_i, x_i, \int_i \right\rangle \simeq \mathbb{I}_1.$$

When $n = 1$ we usually drop the subscript '1' in ∂_1, $\int_1$, H_1, and x_1. The following elements of the algebra $\mathbb{I}_1 = K\langle \partial, H, \int \rangle$,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \quad i, j \in \mathbb{N}, \tag{1}$$

satisfy the relations: $e_{ij} e_{kl} = \delta_{jk} e_{il}$ where δ_{ij} is the Kronecker delta. The matrices of the linear maps $e_{ij} \in \mathrm{End}_K(K[x])$ with respect to the basis $\{x^{[s]} := \frac{x^s}{s!}\}_{s \in \mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e.

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

The direct sum $F := \bigoplus_{i,j \in \mathbb{N}} K e_{ij}$ is the only proper (hence maximal) ideal of the algebra $\mathbb{I}_1$. As an algebra without 1 it is isomorphic to the algebra without 1 of infinite dimensional matrices $M_\infty(K) := \varinjlim M_d(K) = \bigoplus_{i,j \in \mathbb{N}} K E_{ij}$ via $e_{ij} \mapsto E_{ij}$ where E_{ij} are the matrix units. For all $i, j \in \mathbb{N}$,

$$\int e_{ij} = e_{i+1,j}, \qquad e_{ij} \int = e_{i,j-1}, \qquad \partial e_{ij} = e_{i-1,j}, \qquad e_{ij} \partial = e_{i,j+1}, \tag{2}$$

where $e_{-1,j} := 0$ and $e_{i,-1} := 0$. The algebra $\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i)$ contains the ideal $F_n := F^{\otimes n} = \bigotimes_{i=1}^n F(i) = \bigoplus_{\alpha,\beta\in\mathbb{N}^n} Ke_{\alpha\beta}$ where $e_{\alpha\beta} := \prod_{i=1}^n e_{\alpha_i\beta_i}(i)$, $e_{\alpha_i\beta_i}(i) := \int_i^{\alpha_i}\partial_i^{\beta_i} - \int_i^{\alpha_i+1}\partial_i^{\beta_i+1}$ and $F(i) = \bigoplus_{s,t\in\mathbb{N}} Ke_{st}(i)$.

Proposition 2.1. *(See [10, Proposition 2.2].)*

1. *The algebra $\mathbb{I}_n$ is generated by the elements $\{\partial_i, \int_i, H_i \mid i = 1, \dots, n\}$ that satisfy the following defining relations*:

$$\forall i:\quad \partial_i\int_i = 1,\quad \left[H_i, \int_i\right] = \int_i,\quad [H_i, \partial_i] = -\partial_i,$$
$$H_i\left(1-\int_i\partial_i\right) = \left(1-\int_i\partial_i\right)H_i = 1-\int_i\partial_i,$$
$$\forall i\neq j:\quad a_ia_j = a_ja_i \quad \text{where } a_k \in \left\{\partial_k, \int_k, H_k\right\}.$$

2. *The algebra $\mathbb{I}_n = \bigotimes_{i=1}^n D_1(i)(\sigma_i, 1) = D_n((\sigma_1, \dots, \sigma_n), (1, \dots, 1))$ is a generalized Weyl algebra ($\int_i \leftrightarrow x_i$, $\partial_i \leftrightarrow y_i$, $H_i \leftrightarrow H_i$) where $D_n := \bigotimes_{i=1}^n D_1(i)$, $D_1(i) := K[H_i] \oplus \bigoplus_{j\geqslant 0} Ke_{jj}(i)$, $H_ie_{jj}(i) = e_{jj}(i)H_i = (j+1)e_{jj}(i)$, and the K-algebra endomorphisms σ_i are given by the rule $\sigma_i(a) := \int_i a\partial_i$ ($\sigma_i(H_i) = H_i - 1$, $\sigma_i(e_{jj}(i)) = e_{j+1,j+1}(i)$). Moreover, the algebra $\mathbb{I}_n = \bigoplus_{\alpha\in\mathbb{Z}^n}\mathbb{I}_{n,\alpha}$ is $\mathbb{Z}^n$-graded where $\mathbb{I}_{n,\alpha} = D_nv_\alpha = v_\alpha D_n$ for all $\alpha\in\mathbb{Z}^n$ where $v_\alpha := \prod_{i=1}^n v_{\alpha_i}(i)$ and $v_j(i) := \begin{cases} \int_i^j & \text{if } j>0,\\ 1 & \text{if } j=0,\\ \partial_i^{-j} & \text{if } j<0.\end{cases}$*

Remark. Note that $\sigma_i(1) = \int_i\partial_i = 1 - e_{00}(i) \neq 1$ for all $i = 1, \dots, n$.

Definition. Let A and B be algebras, let $\mathcal{J}(A)$ and $\mathcal{J}(B)$ be their lattices of ideals. We say that a bijection $f : \mathcal{J}(A) \to \mathcal{J}(B)$ is an *isomorphism* if $f(\mathfrak{a} * \mathfrak{b}) = f(\mathfrak{a}) * f(\mathfrak{b})$ for $* \in \{+, \cdot, \cap\}$, and in this case we say that the algebras A and B are *ideal equivalent*.

The ideal equivalence is an equivalence relation on the class of algebras (introduced in [10]). The next theorem shows that the Jacobian algebra $\mathbb{A}_n$ and the algebra $\mathbb{I}_n$ are ideal equivalent.

Theorem 2.2. *(See [10, Theorem 3.1].) The restriction map $\mathcal{J}(\mathbb{A}_n) \to \mathcal{J}(\mathbb{I}_n)$, $\mathfrak{a} \mapsto \mathfrak{a}^r := \mathfrak{a} \cap \mathbb{I}_n$, is an isomorphism (i.e. $(\mathfrak{a}_1 * \mathfrak{a}_2)^r = \mathfrak{a}_1^r * \mathfrak{a}_2^r$ for $* \in \{+, \cdot, \cap\}$) and its inverse is the extension map $\mathfrak{b} \mapsto \mathfrak{b}^e := \mathbb{A}_n\mathfrak{b}\mathbb{A}_n$.*

The next corollary shows that the ideal theory of $\mathbb{I}_n$ is 'very arithmetic.' In some sense, it is the best and the simplest possible ideal theory one can imagine. Let $\mathcal{B}_n$ be the set of all functions $f : \{1, 2, \dots, n\} \to \{0, 1\}$. For each function $f \in \mathcal{B}_n$, $I_f := I_{f(1)} \otimes \cdots \otimes I_{f(n)}$ is the ideal of $\mathbb{I}_n$ where $I_0 := F$ and $I_1 := \mathbb{I}_1$. Let $\mathcal{C}_n$ be the set of all subsets of $\mathcal{B}_n$ all distinct elements of which are incomparable (two distinct elements f and g of $\mathcal{B}_n$ are *incomparable* if neither $f(i) \leqslant g(i)$ nor $f(i) \geqslant g(i)$ for all i). For each $C \in \mathcal{C}_n$, let $I_C := \sum_{f\in C} I_f$ be the ideal of $\mathbb{I}_n$. The number $\mathfrak{d}_n$ of elements in the set $\mathcal{C}_n$ is called the *Dedekind number*. It appeared in the paper of Dedekind [11]. An asymptotic of the Dedekind numbers was found by Korshunov [16].

Recall that a submodule of a module that intersects non-trivially each nonzero submodule of the module is called an *essential* submodule.

Corollary 2.3. *(See [10, Corollary 3.3].)*

1. *The set of height one prime ideals of the algebra* $\mathbb{I}_n$ *is* $\{\mathfrak{p}_1 := F \otimes \mathbb{I}_{n-1}, \mathfrak{p}_1 := \mathbb{I}_1 \otimes F \otimes \mathbb{I}_{n-2}, \dots, \mathfrak{p}_n := \mathbb{I}_{n-1} \otimes F\}$.
2. *Each ideal of the algebra* $\mathbb{I}_n$ *is an idempotent ideal* ($\mathfrak{a}^2 = \mathfrak{a}$).
3. *The ideals of the algebra* $\mathbb{I}_n$ *commute* ($\mathfrak{ab} = \mathfrak{ba}$).
4. *The lattice* $\mathcal{J}(\mathbb{I}_n)$ *of ideals of the algebra* $\mathbb{I}_n$ *is distributive.*
5. [10, Lemma 5.2.(1)] *Each nonzero ideal of the algebra* $\mathbb{I}_n$ *is an essential left and right submodule of* $\mathbb{I}_n$.
6. $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$ *for all ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of the algebra* $\mathbb{I}_n$.
7. *The ideal* $\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$ *is the largest* (*hence, the only maximal*) *ideal of* $\mathbb{I}_n$ *distinct from* $\mathbb{I}_n$, *and* $F_n = F^{\otimes n} = \bigcap_{i=1}^n \mathfrak{p}_i$ *is the smallest nonzero ideal of* $\mathbb{I}_n$.
8. (A classification of ideals of $\mathbb{I}_n$) *The map* $\mathcal{C}_n \to \mathcal{J}(\mathbb{I}_n)$, $C \mapsto I_C := \sum_{f \in C} I_f$ *is a bijection where* $I_\emptyset := 0$. *The number of ideals of* $\mathbb{I}_n$ *is equal to the Dedekind number* $\mathfrak{d}_n$. *For* $n = 1$, F *is the unique proper ideal of the algebra* $\mathbb{I}_1$.
9. (A classification of prime ideals of $\mathbb{I}_n$) *Let* Sub_n *be the set of all subsets of* $\{1, \dots, n\}$. *The map* $\mathrm{Sub}_n \to \mathrm{Spec}(\mathbb{I}_n)$, $I \mapsto \mathfrak{p}_I := \sum_{i \in I} \mathfrak{p}_i$, $\emptyset \mapsto 0$, *is a bijection, i.e. any nonzero prime ideal of* $\mathbb{I}_n$ *is a unique sum of primes of height* 1; $|\mathrm{Spec}(\mathbb{I}_n)| = 2^n$; *the height of* $\mathfrak{p}_I$ *is* $|I|$; *and*
10. $\mathfrak{p}_I \subset \mathfrak{p}_J$ *iff* $I \subset J$.

2.1. The involution $*$ *on the algebra* $\mathbb{I}_n$

Using the defining relations in Proposition 2.1.(1), we see that the algebra $\mathbb{I}_n$ admits the involution:

$$* : \mathbb{I}_n \to \mathbb{I}_n, \quad \partial_i \mapsto \int_i, \quad \int_i \mapsto \partial_i, \quad H_i \mapsto H_i, \quad i = 1, \dots, n, \tag{3}$$

i.e. it is a K-algebra *anti-isomorphism* ($(ab)^* = b^*a^*$) such that $* \circ * = \mathrm{id}_{\mathbb{I}_n}$. Therefore, the algebra $\mathbb{I}_n$ is *self-dual*, i.e. is isomorphic to its *opposite* algebra $\mathbb{I}_n^{op}$. As a result, the left and the right properties of the algebra $\mathbb{I}_n$ are the same. For all elements $\alpha, \beta \in \mathbb{N}^n$,

$$e_{\alpha\beta}^* = e_{\beta\alpha}. \tag{4}$$

The involution $*$ can be extended to an involution of the algebra $\mathbb{A}_n$ by setting

$$x_i^* = H_i \partial_i, \qquad \partial_i^* = \int_i, \qquad \left(H_i^{\pm 1}\right)^* = H_i^{\pm 1}, \quad i = 1, \dots, n.$$

Note that $y_i^* = (H_i^{-1}\partial_i)^* = \int_i H_i^{-1} = x_i H_i^{-2}$, $A_n^* \not\subseteq A_n$, but $\mathcal{I}_n^* = \mathcal{I}_n$ where

$$\mathcal{I}_n := K\left\langle \partial_1, \dots, \partial_n \int_1, \dots, \int_n \right\rangle$$

is the algebra of integro-differential operators with constant coefficients.

For a subset S of a ring R, the sets $\mathrm{l.ann}_R(S) := \{r \in R \mid rS = 0\}$ and $\mathrm{r.ann}_R(S) := \{r \in R \mid Sr = 0\}$ are called the *left* and the *right annihilators* of the set S in R. Using the fact that the algebra $\mathbb{I}_n$ is a GWA and its $\mathbb{Z}^n$-grading, we see that

$$\mathrm{l.ann}_{\mathbb{I}_n}\left(\int_i\right) = \bigoplus_{k \in \mathbb{N}} K e_{k0}(i) \otimes \bigotimes_{i \neq j} \mathbb{I}_1(j), \qquad \mathrm{r.ann}_{\mathbb{I}_n}\left(\int_i\right) = 0. \tag{5}$$

$$\mathrm{r.ann}_{\mathbb{I}_n}(\partial_i) = \bigoplus_{k \in \mathbb{N}} K e_{0k}(i) \otimes \bigotimes_{i \neq j} \mathbb{I}_1(j), \qquad \mathrm{l.ann}_{\mathbb{I}_n}(\partial_i) = 0. \tag{6}$$

Let $\mathfrak{a}$ be an ideal of the algebra $\mathbb{I}_n$. The factor algebra $\mathbb{I}_n/\mathfrak{a}$ is a Noetherian algebra iff $\mathfrak{a}=\mathfrak{a}_n$ [10, Proposition 4.1]. The factor algebra $B_n := \mathbb{I}_n/\mathfrak{a}_n$ is isomorphic to the skew Laurent polynomial algebra

$$\bigotimes_{i=1}^{n} K[H_i]\left[\partial_i, \partial_i^{-1}; \tau_i\right] = \mathcal{P}_n\left[\partial_1^{\pm 1}, \dots, \partial_n^{\pm 1}; \tau_1, \dots, \tau_n\right],$$

via $\partial_i \mapsto \partial_i$, $\int_i \mapsto \partial_i^{-1}$, $H_1 \mapsto H_i$ (and $x_i \mapsto \partial_i^{-1} H_i$) where $\mathcal{P}_n := K[H_1, \dots, H_n]$ and $\tau_i(H_i) = H_i + 1$. We identify these two algebras via this isomorphism. It is obvious that

$$B_n = \bigotimes_{i=1}^{n} K[H_i]\left[z_i, z_i^{-1}; \sigma_i\right] = \mathcal{P}_n\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}; \sigma_1, \dots, \sigma_n\right],$$

where $z_i := \partial_i^{-1}$ and $\sigma_i = \tau_i^{-1} : H_i \mapsto H_i - 1$. We use this alternative presentation of the algebra B_n in order to avoid awkward expressions like $\frac{\partial}{\partial \partial_i}$ later. By Theorem 2.2, $\mathfrak{a}_n^e$ is the only maximal ideal of the Jacobian algebra $\mathbb{A}_n$. The factor algebra $\mathcal{A}_n := \mathbb{A}_n/\mathfrak{a}_n^e$ is the skew Laurent polynomial algebra

$$\mathcal{A}_n = \mathcal{L}_n\left[\partial_1^{\pm 1}, \dots, \partial_n^{\pm 1}; \tau_1, \dots, \tau_n\right] = \mathcal{L}_n\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}; \sigma_1, \dots, \sigma_n\right] = \mathcal{L}_n\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}; \sigma_1, \dots, \sigma_n\right]$$

where $\mathcal{L}_n := K[H_1^{\pm 1}, (H_1 \pm 1)^{-1}, (H_1 \pm 2)^{-1}, \dots, H_n^{\pm 1}, (H_n \pm 1)^{-1}, (H_n \pm 2)^{-1}, \dots]$, $\tau_i(H_j) = H_j + \delta_{ij}$ where δ_{ij} is the Kronecker delta and $\sigma_i = \tau_i^{-1}$. By Theorem 2.2, $\mathfrak{a}_n^{er} = \mathfrak{a}_n$, hence the algebra B_n is a subalgebra of $\mathcal{A}_n$. Moreover, the algebra $\mathcal{A}_n$ is the localization of the algebra B_n at the multiplicatively closed set $\{(H_1+\alpha_1)^{m_1} \cdots (H_n+\alpha_n)^{m_n} \mid (\alpha_i) \in \mathbb{Z}^n,\ (m_i) \in \mathbb{N}^n\}$. The algebra B_n is also the left (but not right) localization of the algebra $\mathbb{I}_n$ at the multiplicatively closed set $S_{\partial_1,\dots,\partial_n} := \{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \mid (\alpha_i) \in \mathbb{N}^n\}$, $B_n \simeq S_{\partial_1,\dots,\partial_n}^{-1} \mathbb{I}_n$.

3. A description of the group G_n and two criteria

In this section, the key ingredients of the group G_n are introduced, namely, the groups S_n, $\mathbb{T}^n$ and $\mathrm{Inn}(\mathbb{I}_n)$. The subgroup of G_n they generate is their semi-direct product $S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$. An important description of the group G_n is given (Corollary 3.3.(2)), and two criteria of equality of two automorphisms of the algebra $\mathbb{I}_n$ are obtained (Theorem 3.6 and Corollary 3.7).

3.1. *The group of inner automorphisms* $\mathrm{Inn}(\mathbb{I}_n)$ *of the algebra* $\mathbb{I}_n$

For a group G, let $Z(G)$ denote its centre. Since $\mathfrak{a}_n$ is an ideal of the algebra $\mathbb{I}_n$, the intersection $(1+\mathfrak{a}_n)^* := \mathbb{I}_n^* \cap (1+\mathfrak{a}_n)$ is a subgroup of the group $\mathbb{I}_n^*$ of units of the algebra $\mathbb{I}_n$.

Theorem 3.1.

1. [10, Theorem 5.6] $\mathbb{I}_n^* = K^* \times (1+\mathfrak{a}_n)^*$ *and* $Z(\mathbb{I}_n^*) = K^*$.
2. *The map* $(1+\mathfrak{a}_n)^* \to \mathrm{Inn}(\mathbb{I}_n)$, $u \mapsto \omega_u$, *is a group isomorphism where* $\omega_u(a) := uau^{-1}$, *i.e.* $\mathrm{Inn}(\mathbb{I}_n) = \{\omega_u \mid u \in (1+\mathfrak{a}_n)^*\}$.
3. $\mathbb{I}_1^* = K^* \times (1+F)^* \simeq K^* \times \mathrm{GL}_\infty(K)$.

Proof. Statement 2 follows from statement 1. Statement 3 follows from statement 1 and the fact that $(1+F)^* \simeq \mathrm{GL}_\infty(K)$. $\square$

3.2. The algebraic torus $\mathbb{T}^n$

The *n-dimensional algebraic torus* $\mathbb{T}^n := \{t_\lambda \mid \lambda = (\lambda_1, \dots, \lambda_n) \in K^{*n}\} \simeq K^{*n}$ is a subgroup of the group G_n where

$$t_\lambda\left(\int_i\right) = \lambda_i \int_i, \qquad t_\lambda(\partial_i) = \lambda_i^{-1}\partial_i, \qquad t_\lambda(H_i) = H_i, \quad i = 1, \dots, n.$$

Note that $t_\lambda(x_i) = \lambda_i x_i$ since $x_i = \int_i H_i$. $\mathbb{T}^n = \prod_{i=1}^n \mathbb{T}^1(i)$ where $\mathbb{T}^1(i) := \{t_\lambda(i) := t_{(1,\dots,1,\lambda,1,\dots,1)} \mid \lambda \in K^*\} \simeq K^*$ and the scalar λ is on ith place.

3.3. The symmetric group S_n

The symmetric group S_n is a subgroup of the group G_n where, for $\tau \in S_n$,

$$\tau\left(\int_i\right) = \int_{\tau(i)}, \qquad \tau(\partial_i) = \partial_{\tau(i)}, \qquad \tau(H_i) = H_{\tau(i)}, \quad i = 1, \dots, n.$$

The subgroup of G_n generated by S_n and $\mathbb{T}^n$ is the semi-direct product $S_n \ltimes \mathbb{T}^n$ since $S_n \cap \mathbb{T}^n = \{e\}$ and

$$\tau t_\lambda \tau^{-1} = t_{\tau(\lambda)} \quad \text{where } \tau(\lambda) := (\lambda_{\tau^{-1}(1)}, \dots, \lambda_{\tau^{-1}(n)}), \tag{7}$$

for all $\tau \in S_n$ and $t_\lambda \in \mathbb{T}^n$.

Since $\mathfrak{a}_n$ is the only maximal ideal of the algebra $\mathbb{I}_n$, $\sigma(\mathfrak{a}_n) = \mathfrak{a}_n$ for all $\sigma \in \mathrm{G}_n$. There is the group homomorphism (recall that $B_n = \mathbb{I}_n/\mathfrak{a}_n$):

$$\xi : \mathrm{G}_n \to \mathrm{Aut}_{K\text{-alg}}(B_n), \quad \sigma \mapsto \big(\overline{\sigma} : a + \mathfrak{a}_n \mapsto \sigma(a) + \mathfrak{a}_n\big). \tag{8}$$

The subgroup $S_n \ltimes \mathbb{T}^n$ of G_n maps isomorphically to its image and $\xi(\mathrm{Inn}(\mathbb{I}_n)) = \{e\}$, by Theorem 3.1.(2). Therefore, the subgroup G_n' of G_n generated by the subgroups S_n, $\mathbb{T}^n$, and $\mathrm{Inn}(\mathbb{I}_n)$ is equal to their semi-direct product,

$$\mathrm{G}_n' = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n). \tag{9}$$

The goal of the paper is to prove that $\mathrm{G}_n = \mathrm{G}_n'$ (Theorem 5.5).

3.4. A description of the group G_n

Let A be an algebra and σ be its automorphism. For an A-module M, the *twisted* A-module ${}^\sigma M$, as a vector space, coincides with the module M but the action of the algebra A is given by the rule: $a \cdot m := \sigma(a)m$ where $a \in A$ and $m \in M$. The next lemma is useful in finding the group of automorphisms of algebras that have a *unique faithful* module satisfying some isomorphism-invariant properties.

Lemma 3.2. *Suppose that an algebra A has a unique* (*up to isomorphism*) *faithful A-module M that satisfies an isomorphism-invariant property, say $\mathcal{P}$. Then*

$$\mathrm{Aut}_{K\text{-alg}}(A) = \left\{\sigma_\varphi \;\middle|\; \varphi \in \mathrm{Aut}_K(M),\ \varphi A \varphi^{-1} = A\right\}$$

where $\sigma_\varphi(a) := \varphi a \varphi^{-1}$ for $a \in A$, and the algebra A is identified with its isomorphic copy in $\mathrm{End}_K(M)$ via the algebra monomorphism $a \mapsto (m \mapsto am)$.

Proof. Let $\sigma \in \mathrm{Aut}_{K\text{-alg}}(A)$. The twisted A-module ${}^\sigma M$ is faithful and satisfies the property $\mathcal{P}$. By the uniqueness of M, the A-modules M and ${}^\sigma M$ are isomorphic. So, there exists an element $\varphi \in \mathrm{Aut}_K(M)$ such that $\varphi a = \sigma(a)\varphi$ for all $a \in A$, and so $\sigma(a) = \varphi a \varphi^{-1}$, as required. □

Example. The matrix algebra $M_d(K)$ has a unique (up to isomorphism) simple module which is K^n. Then, by Lemma 3.2, every automorphism of $M_d(K)$ is inner.

Recall that the polynomial algebra P_n is a unique (up to isomorphism) faithful, simple module for the algebra $\mathbb{I}_n$ (Proposition 3.4.(1)) and the algebra $\mathbb{A}_n$ [2, Corollary 2.7.(10)].

Corollary 3.3.

1. [9, Corollary 4.8.(1)] $\mathbb{G}_n = \{\sigma_\varphi \mid \varphi \in \mathrm{Aut}_K(P_n),\ \varphi \mathbb{A}_n \varphi^{-1} = \mathbb{A}_n\}$ *where* $\sigma_\varphi(a) := \varphi a \varphi^{-1}$, $a \in \mathbb{A}_n$.
2. $\mathrm{G}_n = \{\sigma_\varphi \mid \varphi \in \mathrm{Aut}_K(P_n),\ \varphi \mathbb{I}_n \varphi^{-1} = \mathbb{I}_n\}$ *where* $\sigma_\varphi(a) := \varphi a \varphi^{-1}$, $a \in \mathbb{I}_n$.

In [9], Corollary 3.3.(1) was used in finding the group $\mathbb{G}_n$.

3.5. *The automorphism* $\widehat{*} \in \mathrm{Aut}(\mathrm{G}_n)$

The involution $*$ of the algebra $\mathbb{I}_n$ induces the automorphism $\widehat{*}$ of the group G_n by the rule

$$\widehat{*} : \mathrm{G}_n \to \mathrm{G}_n, \quad \sigma \mapsto * \circ \sigma \circ *^{-1}. \tag{10}$$

3.6. *The* $\mathbb{I}_n$*-module* P_n

By the very definition of the algebra $\mathbb{I}_n$ as a subalgebra of $\mathrm{End}_K(P_n)$, the $\mathbb{I}_n$-module P_n is faithful. For the Weyl algebra A_n, the A_n-module $A_n / \sum_{i=1}^n A_n \partial_i$ is isomorphic to P_n via $1 + \sum_{i=1}^n A_n \partial_i \mapsto 1$. The same statement is true for the algebra $\mathbb{I}_n$ (Proposition 3.4.(3)).

Proposition 3.4. *(See [10, Proposition 3.8 and Proposition 6.1].)*

1. *The polynomial algebra P_n is the only (up to isomorphism) faithful, simple $\mathbb{I}_n$-module.*
2. $\mathbb{I}_1 = \mathbb{I}_1 \partial \oplus \mathbb{I}_1 e_{00}$ *and* $\mathbb{I}_1 = \int \mathbb{I}_1 \oplus e_{00} \mathbb{I}_1$.
3. ${}_{\mathbb{I}_n} P_n \simeq \mathbb{I}_n / \sum_{i=1}^n \mathbb{I}_n \partial_i$.

The $\mathbb{I}_n$-module P_n is a very special module for the algebra $\mathbb{I}_n$. Its properties, especially the uniqueness, are used often in this paper. The polynomial algebra $P_n = \bigoplus_{\alpha \in \mathbb{N}^n} K x^\alpha$ is a naturally $\mathbb{N}^n$-graded algebra. This grading is compatible with the $\mathbb{Z}^n$-grading of the algebra $\mathbb{I}_n$, i.e. the polynomial algebra P_n is a $\mathbb{Z}^n$-graded $\mathbb{I}_n$-module. Each element $\partial_i \in \mathbb{I}_n \subseteq \mathrm{End}_K(P_n)$ is a *locally nilpotent* map, that is $P_n = \bigcup_{j \geqslant 1} \ker_{P_n}(\partial_i^j)$. Moreover,

$$\bigcap_{i=1}^n \ker_{P_n}(\partial_i) = K.$$

Each element $\int_i \in \mathbb{I}_n \subseteq \mathrm{End}_K(P_n)$ is an injective (but not a surjective) map. Each element $H_i \in \mathbb{I}_n \subseteq \mathrm{End}_K(P_n)$ is a *semi-simple* map (that is $P_n = \bigoplus_{\lambda \in K} \ker_{P_n}(H_i - \lambda)$) with the set of eigenvalues $\mathbb{Z}_+ := \{1, 2, \dots\}$ since $H_i * x^\alpha = (\alpha_i + 1) x^\alpha$ for all $\alpha \in \mathbb{N}^n$. Moreover,

$$\bigcap_{i=1}^{n} \ker_{P_n}\big(H_i - (\alpha_i + 1)\big) = Kx^{\alpha}, \quad \alpha \in \mathbb{N}^n. \tag{11}$$

In particular, the $K[H_1, \dots, H_n]$-module $P_n = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^{\alpha}$ is the sum of simple, non-isomorphic, one-dimensional submodules Kx^{α}, and so P_n is a semi-simple $K[H_1, \dots, H_n]$-module.

Corollary 3.5.

1. *Let M be an $\mathbb{I}_n$-module. Then* $\mathrm{Hom}_{\mathbb{I}_n}(P_n, M) \simeq \bigcap_{i=1}^{n} \ker(\partial_{i,M})$, $f \mapsto f(1)$, *where* $\partial_{i,M} : M \to M$, $m \mapsto \partial_i m$. *In particular,* $\mathrm{End}_{\mathbb{I}_n}(P_n) \simeq K$.
2. *By Proposition* 3.4.(1), (3), *for each automorphism $\sigma \in \mathrm{G}_n$, the $\mathbb{I}_n$-modules P_n and ${}^{\sigma}P_n$ are isomorphic, and each isomorphism $f : P_n \to {}^{\sigma}P_n$ is given by the rule*: $f(p) = \sigma(p) * v$, *where $v = f(1)$ is any nonzero element of the* 1*-dimensional vector space* $\bigcap_{i=1}^{n} \ker(\sigma(\partial_i)_{P_n})$.

As an application of these results to the $\mathbb{I}_n$-module P_n, we have two useful criteria of equality of two elements in the group G_n. They are used in many proofs in this paper.

For an algebra A and a subset $S \subseteq A$, $\mathrm{Cen}_A(S) := \{a \in A \mid as = sa \text{ for all } s \in S\}$ is the *centralizer* of the set S in A. It is a subalgebra of A. It follows from the presentation of the algebra $\mathbb{I}_n$ as a GWA that

$$\mathrm{Cen}_{\mathbb{I}_n}(x_1, \dots, x_n) = P_n, \qquad \mathrm{Cen}_{\mathbb{I}_n}(\partial_1, \dots, \partial_n) = K[\partial_1, \dots, \partial_n],$$
$$\mathrm{Cen}_{\mathbb{I}_n}\bigg(\int_1, \dots, \int_n\bigg) = K\bigg[\int_1, \dots, \int_n\bigg]. \tag{12}$$

In more detail, since $\mathbb{I}_n = \bigotimes_{i=1}^{n} \mathbb{I}_1(i)$ it suffices to prove the equalities for $n = 1$, but in this case the equalities are obvious.

Let $\mathbb{E}_n := \mathrm{End}_{K\text{-alg}}(\mathbb{I}_n)$ be the monoid of all the K-algebra endomorphisms of $\mathbb{I}_n$. The group of units of this monoid is G_n. The automorphism $\widehat{*} \in \mathrm{Aut}(\mathrm{G}_n)$ can be extended to an automorphism $\widehat{*} \in \mathrm{Aut}(\mathbb{E}_n)$ of the monoid $\mathbb{E}_n$:

$$\widehat{*} : \mathbb{E}_n \to \mathbb{E}_n, \quad \sigma \mapsto * \circ \sigma \circ *^{-1}. \tag{13}$$

For each element $\alpha \in \mathbb{N}^n$, let $x^{[\alpha]} := \int^{\alpha} *1$. Then $x^{[\alpha]} := \frac{x^{\alpha}}{\alpha!} := \prod_{i=1}^{n} \frac{x_i^{\alpha_i}}{\alpha_i!}$ and the set $\{x^{[\alpha]} \mid \alpha \in \mathbb{N}^n\}$ is a K-basis for the polynomial algebra P_n. The next result is instrumental in finding the group of automorphisms of the algebra $\mathbb{I}_n$.

Theorem 3.6 *(Rigidity of the group G_n). Let $\sigma, \tau \in \mathrm{G}_n$. Then the following statements are equivalent.*

1. $\sigma = \tau$.
2. $\sigma(\int_1) = \tau(\int_1), \dots, \sigma(\int_n) = \tau(\int_n)$.
3. $\sigma(\partial_1) = \tau(\partial_1), \dots, \sigma(\partial_n) = \tau(\partial_n)$.
4. $\sigma(x_1) = \tau(x_1), \dots, \sigma(x_n) = \tau(x_n)$.

Remark. It is not true that $\sigma(H_i) = \tau(H_i)$ for all $i = 1, \dots, n$ implies $\sigma = \tau$ (Corollary 5.9.(2)).

Proof of Theorem 3.6. Without loss of generality we may assume that $\tau = e$, the identity automorphism. The proof consists of two parts: $(1 \Leftrightarrow 2 \Leftrightarrow 3)$ and $(4 \Rightarrow 1)$. Consider the following two subgroups of G_n, the stabilizers of the sets $\{\int_1, \dots, \int_n\}$ and $\{\partial_1, \dots, \partial_n\}$:

$$\mathrm{St}_{\mathrm{G}_n}\left(\int_1,\dots,\int_n\right):=\left\{g\in \mathrm{G}_n \;\Big|\; g\left(\int_1\right)=\int_1,\dots,g\left(\int_n\right)=\int_n\right\},$$
$$\mathrm{St}_{\mathrm{G}_n}(\partial_1,\dots,\partial_n):=\left\{g\in \mathrm{G}_n \;\big|\; g(\partial_1)=\partial_1,\dots,g(\partial_n)=\partial_n\right\}.$$

Then

$$\widehat{*}\left(\mathrm{St}_{\mathrm{G}_n}\left(\int_1,\dots,\int_n\right)\right)=\mathrm{St}_{\mathrm{G}_n}(\partial_1,\dots,\partial_n),\qquad \widehat{*}\big(\mathrm{St}_{\mathrm{G}_n}(\partial_1,\dots,\partial_n)\big)=\mathrm{St}_{\mathrm{G}_n}\left(\int_1,\dots,\int_n\right).$$

Therefore, to prove that $(1 \Leftrightarrow 2 \Leftrightarrow 3)$ (where $\tau = e$) is equivalent to show that $\mathrm{St}_{\mathrm{G}_n}(\int_1,\dots,\int_n)=\{e\}$. So, let $\sigma\in \mathrm{St}_{\mathrm{G}_n}(\int_1,\dots,\int_n)$. We have to show that $\sigma=e$, i.e. $\sigma(\partial_i)=\partial_i$ and $\sigma(H_i)=H_i$ for all i. For each $i=1,\dots,n$, $1=\sigma(\partial_i\int_i)=\sigma(\partial_i)\int_i$ and $1=\partial_i\int_i$. By taking the difference of these equalities we see that $a_i:=\sigma(\partial_i)-\partial_i\in \mathrm{l.ann}_{\mathbb{I}_n}(\int_i)$. By (5), $a_i=\sum_{j\geqslant 0}\lambda_{ij}e_{j0}(i)$ for some elements $\lambda_{ij}\in\bigotimes_{k\neq i}\mathbb{I}_1(k)$, and so

$$\sigma(\partial_i)=\partial_i+\sum_{j\geqslant 0}\lambda_{ij}e_{j0}(i).$$

The element $\sigma(\partial_i)$ commutes with the elements $\sigma(\int_k)=\int_k$, $k\neq i$, hence all $\lambda_{ij}\in K[\int_1,\dots,\widehat{\int_i},\dots,\int_n]$, by (12). Since $e_{j0}(i)=\int_i^j e_{00}(i)$, we can write

$$\sigma(\partial_i)=\partial_i+p_ie_{00}(i)\quad\text{for some } p_i\in K\left[\int_1,\dots,\int_n\right].$$

We have to show that all $p_i=0$. Suppose that this is not the case. Then $p_i\neq 0$ for some i. We seek a contradiction. Note that $\sigma^{-1}\in \mathrm{St}_{\mathrm{G}_n}(\int_1,\dots,\int_n)$, and so $\sigma^{-1}(\partial_i)=\partial_i+q_ie_{00}(i)$ for some $q_i\in K[\int_1,\dots,\int_n]$. Recall that $e_{00}(i)=1-\int_i\partial_i$. Then $\sigma^{-1}(e_{00}(i))=1-\int_i(\partial_i+q_ie_{00}(i))=(1-\int_i q_i)e_{00}(i)$, and

$$\partial_i=\sigma^{-1}\sigma(\partial_i)=\sigma^{-1}\big(\partial_i+p_ie_{00}(i)\big)=\partial_i+\left(q_i+p_i\left(1-\int_i q_i\right)\right)e_{00}(i),$$

and so $q_i+p_i=\int_i p_iq_i$ since the map $K[\int_1,\dots,\int_n]\to K[\int_1,\dots,\int_n]e_{00}$, $p\mapsto pe_{00}$, is a bijection, by (2). This is impossible by comparing the total degrees (with respect to the integrations) of the elements on both sides of the equality. Therefore, $\sigma(\partial_i)=\partial_i$ for all i.

By Corollary 3.5.(2), there is an $\mathbb{I}_n$-module isomorphism $\varphi: P_n\to {}^{\sigma}P_n$, $p\mapsto\sigma(p)*v$, where

$$v:=\varphi(1)\in\bigcap_{i=1}^n \ker_{{}^{\sigma}P_n}\big(\sigma(\partial_i)\big)=\bigcap_{i=1}^n \ker_{P_n}(\partial_i)=K1.$$

Without loss of generality we may assume that $v=1$. Then $1=\varphi(1)=\varphi(H_i*1)=\sigma(H_i)*1$ for all i. For $i=1,\dots,n$,

$$\sigma(H_i)*x^{[\alpha]}=\sigma(H_i)\int^\alpha *1=\sigma(H_i)\sigma\left(\int^\alpha\right)*1=\sigma\left(H_i\int^\alpha\right)*1=\sigma\left(\int^\alpha(H_i+\alpha_i)\right)*1$$
$$=\sigma\left(\int^\alpha\right)\big(\sigma(H_i)+\alpha_i\big)*1=\int^\alpha(\alpha_i+1)*1=(\alpha_i+1)x^{[\alpha]}.$$

This means that the linear maps $\sigma(H_i)$, $H_i \in \mathrm{End}_K(P_n)$ coincide. Therefore, $\sigma(H_i) = H_i$ for all i since the $\mathbb{I}_n$-module P_n is faithful. This proves that $\sigma = e$.

$(4 \Rightarrow 1)$ Suppose that $\sigma(x_i) = x_i$ for all i. Then $\sigma(p) = p$ for all polynomials $p \in P_n$. We have to show that $\sigma = e$. By Corollary 3.5.(2), there exists the $\mathbb{I}_n$-module isomorphism $f : P_n \to {}^{\sigma}P_n$, $p \mapsto \sigma(p) * v = pv$. The map f is a bijection hence $v \in K^*$. Without loss of generality we may assume that $v = 1$, then $f = \mathrm{id}_{P_n}$. Let $a \in \{\partial_i, H_i, \int_i\}$ and $b \in P_n$. Then

$$a * b = f(a * b) = \sigma(a) * f(b) = \sigma(a) * b,$$

and so $\sigma(a) = a$ since the $\mathbb{I}_n$-module P_n is faithful. This means that $\sigma = e$. The proof of the theorem is complete. □

Theorem 3.6 means that $\mathrm{St}_{\mathrm{G}_n}(\int_1, \dots, \int_n) = \mathrm{St}_{\mathrm{G}_n}(\partial_1, \dots, \partial_n) = \mathrm{St}_{\mathrm{G}_n}(x_1, \dots, x_n) = \{e\}$.

In zero characteristic, the Weyl algebra A_n is the ring $\mathcal{D}(P_n)$ of differential operators on the polynomial algebra P_n. In prime characteristic, the Weyl algebra A_n and the algebra $\mathcal{D}(P_n)$ are distinct, and the algebra $\mathcal{D}(P_n)$ is much more complicated object than the Weyl algebra A_n. An analogue of Theorem 3.6 does not hold for the algebra $\mathcal{D}(P_n)$ in characteristic zero, but does hold in prime characteristic [3, Theorem 1.1]. Also, the Rigidity Theorem is true for the Jacobian algebra $\mathbb{A}_n$ [9, Theorem 4.12] and for the algebra $\mathbb{S}_n$ of one-sided inverses of the polynomial algebra P_n [6, Theorem 3.7] but the Rigidity Theorem fails for the polynomial algebra P_n.

The ideal F_n is the smallest nonzero ideal of the algebra $\mathbb{I}_n$. Therefore, $\sigma(F_n) = F_n$ for all $\sigma \in \mathrm{G}_n$. The next corollary shows that the action of the group G_n on the ideal F_n is faithful. This result is used in the proof of the fact that the group G_n has trivial centre (Theorem 5.15).

Corollary 3.7. *Let $\sigma, \tau \in \mathrm{G}_n$. Then $\sigma = \tau$ iff $\sigma(e_{\alpha 0}) = \tau(e_{\alpha 0})$ for all $\alpha \in \mathbb{N}^n$ iff $\sigma(e_{0\alpha}) = \tau(e_{0\alpha})$ for all $\alpha \in \mathbb{N}^n$ iff $\sigma(e_{\alpha\beta}) = \tau(e_{\alpha\beta})$ for all $\alpha, \beta \in \mathbb{N}^n$.*

Proof. The last 'iff' follows from the previous two. The second 'iff' follows from the first one by using the automorphism $\widehat{*}$ of the group G_n: $\sigma = \tau$ iff $\widehat{*}(\sigma) = \widehat{*}(\tau)$ iff $\widehat{*}(\sigma)(e_{\alpha 0}) = \widehat{*}(\tau)(e_{\alpha 0})$ for all $\alpha \in \mathbb{N}^n$ (by the first 'iff') iff $\sigma(e_{0\alpha})^* = \tau(e_{0\alpha})^*$ for all $\alpha \in \mathbb{N}^n$ (since $e_{\alpha 0}^* = e_{0\alpha}$) iff $\sigma(e_{0\alpha}) = \tau(e_{0\alpha})$ for all $\alpha \in \mathbb{N}^n$.

So, it remains to prove that if $\sigma(e_{\alpha 0}) = \tau(e_{\alpha 0})$ for all $\alpha \in \mathbb{N}^n$ then $\sigma = \tau$. Without loss of generality we may assume that $\tau = e$, the identity of the group G_n. So, we have to prove that if $\sigma(e_{\alpha 0}) = e_{\alpha 0}$ for all $\alpha \in \mathbb{N}^n$ then $\sigma = e$. For each number $i = 1, \dots, n$,

$$0 = \big(1 - e_{00}(i)\big) * 1 = \sigma\big(1 - e_{00}(i)\big) * 1 = \sigma\left(\int_i \partial_i\right) * 1 = \sigma\left(\int_i\right)\sigma(\partial_i) * 1,$$

and so $0 = \sigma(\partial_i)\sigma(\int_i)\sigma(\partial_i) * 1 = \sigma(\partial_i \int_i)\sigma(\partial_i) * 1 = \sigma(\partial_i) * 1$, i.e. $\bigcap_{i=1}^n \ker(\sigma(\partial_i)_{P_n}) = K$, by Corollary 3.5.(2). By Corollary 3.5.(2), the map $f : P_n \to {}^{\sigma}P_n$, $p \mapsto \sigma(p) * 1$, is an $\mathbb{I}_n$-module isomorphism. Now, $f(x^\alpha) = f(\alpha! e_{\alpha 0} * 1) = \sigma(\alpha! e_{\alpha 0}) * 1 = \alpha! e_{\alpha 0} * 1 = x^\alpha$ for all $\alpha \in \mathbb{N}^n$ where $\alpha! := \alpha_1! \cdots \alpha_n!$. This means that f is the identity map. For all $a \in \mathbb{I}_n$ and $p \in P_n$, $a * p = f(a * p) = \sigma(a) * f(p) = \sigma(a) * p$, and so $\sigma(a) = a$ since the $\mathbb{I}_n$-module P_n is faithful. That is $\sigma = e$, as required. □

Corollary 3.8.

1. *Let $\mathfrak{a}$ be a nonzero ideal of the algebra $\mathbb{I}_n$ and $\sigma, \tau \in \mathrm{G}_n$. Then $\sigma = \tau$ iff $\sigma(a) = \tau(a)$ for all $a \in \mathfrak{a}$.*
2. *Let $\mathfrak{a}$ be a nonzero ideal of the algebra $\mathbb{A}_n$ and $\sigma, \tau \in \mathbb{G}_n$. Then $\sigma = \tau$ iff $\sigma(a) = \tau(a)$ for all $a \in \mathfrak{a}$.*

Proof. 1. Since $F = \bigoplus_{\alpha,\beta \in \mathbb{N}^n} K e_{\alpha\beta} \subseteq \mathfrak{a}$, statement 1 follows from Corollary 3.7.

2. Similarly, since $F = \bigoplus_{\alpha,\beta \in \mathbb{N}^n} K e_{\alpha\beta} \subseteq \mathfrak{a}$, statement 2 follows from [9, Corollary 4.6]. □

4. The group $\mathrm{Aut}_{K\text{-alg}}(\mathbb{I}_1)$

In this section, the group G_1 and its explicit generators are found (Theorem 4.3). The key idea in finding the group G_1 of automorphisms of the algebra $\mathbb{I}_1$ is to use Theorem 3.6, some of the properties of the index of linear maps in the vector space $P_1 = K[x]$, and the explicit structure of the group $\mathrm{Aut}_{K\text{-alg}}(\mathbb{I}_1/F)$ (Theorem 4.1). It is proved that any algebra endomorphism of the algebra $\mathbb{I}_1$ is a monomorphism (Theorem 4.5); note that the algebra $\mathbb{I}_1$ is not a simple algebra.

4.1. The group $\overline{\mathrm{G}}_1 := \mathrm{Aut}_{K\text{-alg}}(B_1)$

Recall that $B_1 = K[H][x, x^{-1}; \sigma]$ and $\sigma(H) = H - 1$. Consider the following automorphisms of the algebra B_1:

$$t_\lambda : x \mapsto \lambda x, \quad H \mapsto H \quad \bigl(\lambda \in K^*\bigr),$$
$$s_p : x \mapsto x, \quad H \mapsto H + p \quad \bigl(p \in K\bigl[x, x^{-1}\bigr]\bigr),$$
$$\zeta : x \mapsto x^{-1}, \quad H \mapsto -H,$$

and the subgroups they generate in the group $\overline{\mathrm{G}}_1$:

$$\mathbb{T}^1 := \bigl\{t_\lambda \bigm| \lambda \in K^*\bigr\} \simeq K^*, \qquad \mathrm{Sh}_1 := \bigl\{s_p \bigm| p \in K\bigl[x, x^{-1}\bigr]\bigr\} \simeq K\bigl[x, x^{-1}\bigr], \qquad \langle\zeta\rangle \simeq \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}.$$

We can easily check that

$$\zeta t_\lambda \zeta^{-1} = t_\lambda^{-1}, \qquad \zeta s_p \zeta^{-1} = s_{-\zeta(p)}, \qquad t_\lambda s_p t_\lambda^{-1} = s_{t_\lambda(p)}. \tag{14}$$

It follows that the subgroup of $\overline{\mathrm{G}}_1$ generated by the three subgroups above is, in fact, their semidirect product,

$$\langle\zeta\rangle \ltimes \mathbb{T}^1 \ltimes \mathrm{Sh}_1 \simeq \mathbb{Z}_2 \ltimes K^* \ltimes K\bigl[x, x^{-1}\bigr],$$

since, for $\varepsilon = 0, 1$; $\lambda \in K^*$; and $p \in K[x, x^{-1}]$:

$$\zeta^\varepsilon t_\lambda s_p : x \mapsto \lambda x^{1-2\varepsilon}, \quad H \mapsto (-1)^\varepsilon H + \zeta^\varepsilon t_\lambda(p), \tag{15}$$

and $\zeta^\varepsilon t_\lambda s_p = e$ iff $\varepsilon = 0$, $\lambda = 1$, and $p = 0$. Theorem 4.1 shows that this is the whole group of automorphisms of the algebra B_1.

For a group G, $[G, G]$ denotes its *commutant*, i.e. the subgroup of G generated by all the group *commutators* $[a, b] := aba^{-1}b^{-1}$ of the elements $a, b \in G$. The centre of a group G is denoted by $Z(G)$. For subgroups A and B of the group G, let $[A, B]$ be the subgroup of G generated by all the commutators $[a, b]$ where $a \in A$ and $b \in B$. Given a semi-direct product $A \ltimes \prod_{i=1}^m B_i$ of groups such that $aB_ia^{-1} \subseteq B_i$ for all $a \in A$ and $i = 1, \dots, m$; then its commutant is equal to $[A, A] \ltimes \prod_{i=1}^m ([A, B_i] \cdot [B_i, B_i])$ [9, Lemma 5.4.(2)]. This fact is used in the proof of the following theorem.

Theorem 4.1.

1. $\overline{\mathrm{G}}_1 = \langle\zeta\rangle \ltimes \mathbb{T}^1 \ltimes \mathrm{Sh}_1$.
2. $Z(\overline{\mathrm{G}}_1) = \{e\}$.
3. $[\overline{\mathrm{G}}_1, \overline{\mathrm{G}}_1] = \{t_{\lambda^2} \mid \lambda \in K^*\} \ltimes \mathrm{Sh}_1$ *and* $\overline{\mathrm{G}}_1/[\overline{\mathrm{G}}_1, \overline{\mathrm{G}}_1] \simeq \mathbb{Z}_2 \times K^*/K^{*2}$.

Proof. 1. Let $\sigma \in \bar{\mathrm{G}}_1$ and $\bar{\mathrm{G}}_1'$ be the semi-direct product. It suffices to show that $\sigma \in \bar{\mathrm{G}}_1'$ since $\bar{\mathrm{G}}_1' \subseteq \bar{\mathrm{G}}_1$. The automorphism σ of the algebra B_1 induces an automorphism of its group of units $B_1^* = \bigcup_{i\in\mathbb{Z}} K^*x^i$. Then $\sigma(x) = \lambda x^{\pm 1}$ for a nonzero scalar $\lambda \in K^*$. Multiplying σ on the left by a suitable element of the group $\langle\zeta\rangle \ltimes \mathbb{T}^1$ we may assume that $\sigma(x) = x$. Then

$$\big[\sigma(H) - H, x\big] = \sigma\big([H, x]\big) - [H, x] = \sigma(x) - x = 0.$$

Therefore, $p := \sigma(H) - H \in \mathrm{Cen}_{B_1}(x) = K[x, x^{-1}]$, and so $\sigma = h_p \in \bar{\mathrm{G}}_1'$. This proves that $\bar{\mathrm{G}}_1 = \bar{\mathrm{G}}_1'$.

2. Let $z \in Z(\bar{\mathrm{G}}_1)$. By statement 1, $z = \zeta^\varepsilon t_\lambda s_p$ for some elements $\varepsilon = 0, 1$; $\lambda \in K^*$; and $p \in K[x, x^{-1}]$. By (14),

$$\zeta^\varepsilon t_{\lambda\mu} s_{t_{\mu^{-1}}(p)} = zt_\mu = t_\mu z = \zeta^\varepsilon t_{\lambda\mu^{1-2\varepsilon}} s_p,$$

hence $\varepsilon = 0$ and $p \in K$. Next, $\zeta t_\lambda s_p = \zeta z = z\zeta = \zeta t_{\lambda^{-1}} s_{-p}$, hence $\lambda = \pm 1$ and $p = 0$, i.e. $z = t_{\pm 1}$. Since $s_x t_{-1} \neq t_{-1} s_x$, $z = t_1 = e$. Therefore, $Z(\bar{\mathrm{G}}_1) = \{e\}$.

3. It suffices to prove only that the equality holds since then the isomorphism is obvious, by statement 1. Let R be the RHS of the equality. Then $R \subseteq [\bar{\mathrm{G}}_1, \bar{\mathrm{G}}_1]$ since

$$t_{\lambda^2} = [\zeta, t_{\lambda^{-1}}], \qquad s_{\mu x^j} = [t_2, s_{\frac{\mu x^j}{2^j - 1}}], \qquad s_\mu = [\zeta, s_{-\frac{\mu}{2}}],$$

where $0 \neq j \in \mathbb{Z}$, $\lambda \in K^*$, and $\mu \in K$. It suffices to show that $[\bar{\mathrm{G}}_1/\mathrm{Sh}_1, \bar{\mathrm{G}}_1/\mathrm{Sh}_1] \subseteq R'$ where $R' := \{t_{\lambda^2} \mid \lambda \in K^*\}$ is treated as a subgroup of the factor group $\bar{\mathrm{G}}_1/\mathrm{Sh}_1 \simeq \langle\zeta\rangle \ltimes \mathbb{T}^1$. By Lemma 5.4 of [9], $[\langle\zeta\rangle \ltimes \mathbb{T}^1, \langle\zeta\rangle \ltimes \mathbb{T}^1] = [\langle\zeta\rangle, \mathbb{T}^1] = R'$. $\square$

4.2. *The index* ind *of linear maps and the Fredholm operators*

Let $\mathcal{C} = \mathcal{C}(K)$ be the family of all K-linear maps with finite dimensional kernel and cokernel (such maps are called the *Fredholm linear maps/operators*). So, $\mathcal{C}$ is the family of *Fredholm* linear maps/operators. For vector spaces V and U, let $\mathcal{C}(V, U)$ be the set of all the linear maps from V to U with finite dimensional kernel and cokernel. So, $\mathcal{C} = \bigcup_{V,U} \mathcal{C}(V, U)$ is the disjoint union.

Definition. For a linear map $\varphi \in \mathcal{C}$, the integer

$$\mathrm{ind}(\varphi) := \dim\ker(\varphi) - \dim\mathrm{coker}(\varphi)$$

is called the *index* of the map φ.

Example. Note that $\partial, \int \in \mathbb{I}_1 \subset \mathrm{End}_K(P_1)$. Then

$$\mathrm{ind}\big(\partial^i\big) = i \quad \text{and} \quad \mathrm{ind}\bigg(\int^i\bigg) = -i, \quad i \geqslant 1. \tag{16}$$

Each nonzero element u of the skew Laurent polynomial algebra $\mathcal{A}_1 = \mathcal{L}_1[x, x^{-1}; \sigma_1]$ (where $\sigma_1(H) = H - 1$) is a unique sum $u = \lambda_s x^s + \lambda_{s+1} x^{s+1} + \cdots + \lambda_d x^d$ where all $\lambda_i \in \mathcal{L}_1$, $\lambda_d \neq 0$, and $\lambda_d x^d$ is the *leading term* of the element u. Recall that $\mathcal{L}_1 := K[H^{\pm 1}, (H \pm 1)^{-1}, (H \pm 2)^{-1}, \ldots]$, $B_1 \subset \mathcal{A}_1$, and $\mathbb{I}_1 \subset \mathbb{A}_1$. The integer $\deg_x(u) = d$ is called the *degree* of the element u, $\deg_x(0) := -\infty$. For all $u, v \in \mathcal{A}_1$, $\deg_x(uv) = \deg_x(u) + \deg_x(v)$ and $\deg_x(u + v) \leqslant \max\{\deg_x(u), \deg_x(v)\}$. The next lemma explains how to compute the index of the elements $\mathbb{A}_1 \backslash F$ (resp. $\mathbb{I}_1 \backslash F$) via the degree function $\deg_x$ and proves that the index is a $\mathbb{G}_1$-invariant (resp. a G_1-invariant) concept. Note that $F \cap \mathcal{C} = \emptyset$.

Lemma 4.2.

1. [9, Lemma 5.3.(1)] $\mathcal{C} \cap \mathbb{A}_1 = \mathbb{A}_1 \backslash F$ (*recall that* $\mathbb{A}_1 \subset \mathrm{End}_K(P_1)$) *and, for each element* $a \in \mathbb{A}_1 \backslash F$, $\mathrm{ind}(a) = -\deg_x(\bar{a})$ *where* $\bar{a} = a + F \in \mathbb{A}_1/F = \mathcal{A}_1$.
2. [9, Lemma 5.3.(2)] $\mathrm{ind}(\sigma(a)) = \mathrm{ind}(a)$ *for all* $\sigma \in \mathbb{G}_1$ *and* $a \in \mathbb{A}_1 \backslash F$.
3. $\mathcal{C} \cap \mathbb{I}_1 = \mathbb{I}_1 \backslash F$; *for each element* $a \in \mathbb{I}_1 \backslash F$, $\mathrm{ind}(a) = -\deg_x(\bar{a})$ *where* $\bar{a} = a + F \in B_1$; *and* $\mathrm{ind}(\sigma(a)) = \mathrm{ind}(a)$ *for all* $\sigma \in \mathrm{G}_1$ *and* $a \in \mathbb{I}_1 \backslash F$.

Proof. Since $\mathbb{I}_n \subseteq \mathbb{A}_n$, statement 3 follows from statements 1, 2 and Corollary 3.3.(2). □

The next theorem presents the group G_1 and its explicit generators.

Theorem 4.3.

1. $\mathrm{G}_1 = \mathbb{T}^1 \ltimes \mathrm{Inn}(\mathbb{I}_1)$.
2. $\mathrm{G}_1 \simeq K^* \ltimes \mathrm{GL}_\infty(K)$.
3. $[\mathrm{G}_1, \mathrm{G}_1] = \{\omega_u \mid u \in \mathrm{SL}_\infty(K)\}$ *and* $\mathrm{G}_1/[\mathrm{G}_1, \mathrm{G}_1] \simeq \mathbb{T}^1 \times \mathbb{T}^1$.
4. *The group* G_1 *is generated by the elements* t_λ, $\omega_{1+\lambda e_{ij}}$ *where* $i \neq j$ *and* $\lambda \in K^*$, *and* $\omega_{1+\mu e_{11}}$ *where* $\mu \in K \backslash \{-1\}$.

Proof. 1. Let $\sigma \in \mathrm{G}_1$. By (9), $\mathrm{G}_1' = \mathbb{T}^1 \ltimes \mathrm{Inn}(\mathbb{I}_1) \subseteq \mathrm{G}_1$. It remains to show that the reverse inclusion holds, that is $\sigma \in \mathrm{G}_1'$. The ideal F of the algebra $\mathbb{I}_1$ is the only maximal ideal. Therefore, $\sigma(F) = F$ and $\overline{\sigma} := \xi(\sigma) \in \overline{\mathrm{G}}_1$, see (8). By Theorem 4.1.(1) and (15), either $\overline{\sigma}(\bar{\partial}) = \lambda x^{-1}$ or, otherwise, $\overline{\sigma}(\bar{\partial}) = \lambda x$ for some element $\lambda \in K^*$. Equivalently, either $\sigma(\partial) = \lambda\partial + f$ or $\sigma(\partial) = \lambda \int + f$ for some element $f \in F$. Recall that $\mathbb{I}_1 \subset \mathbb{A}_1$. By Lemma 4.2.(3) and Corollary 3.3.(2), the second case is impossible as we have the contradiction:

$$1 = \mathrm{ind}(\partial) = \mathrm{ind}\big(\sigma(\partial)\big) = \mathrm{ind}\left(\lambda \int + f\right) = \mathrm{ind}\big(\lambda x H^{-1} + f\big) = -\deg_x\big(\lambda x H^{-1}\big) = -1.$$

Therefore, $\sigma(\partial) = \lambda\partial + f$. Replacing σ by $t_\lambda \sigma$ we may assume that $\sigma(\partial) = \partial + g$ where $g := t_\lambda(f) \in F$ (as F is the only maximal ideal of the algebra $\mathbb{I}_1$, hence $\tau(F) = F$ for all $\tau \in \mathrm{G}_1$). Fix a natural number m such that $g \in \sum_{i,j=0}^m K e_{ij}$. Then the finite dimensional vector spaces

$$V := \bigoplus_{i=0}^{m} K x^{[i]} \subset V' := \bigoplus_{i=0}^{m+1} K x^{[i]}$$

are ∂'-invariant where $\partial' := \sigma(\partial) = \partial + g$, $x^{[i]} := \frac{x^i}{i!}$, and $x^{[0]} := 1$. Note that $\partial' * x^{[m+1]} = \partial * x^{[m+1]} = x^{[m]}$ since $g * x^{[m+1]} = 0$. Note that $P_1 = \bigcup_{i \geqslant 1} \ker(\partial^i)$ and $\dim \ker_{P_1}(\partial) = 1$. Since the $\mathbb{I}_1$-modules P_1 and ${}^\sigma P_1$ are isomorphic (Corollary 3.5.(2)), $P_1 = \bigcup_{i \geqslant 1} \ker(\partial'^i)$ and $\dim \ker_{P_1}(\partial') = 1$. This implies that the elements $x'^{[0]}, x'^{[1]}, \dots, x'^{[m]}, x^{[m+1]}$ are a K-basis for the vector space V' where

$$x'^{[i]} := \partial'^{m+1-i} * x^{[m+1]}, \quad i = 0, 1, \dots, m;$$

and the elements $x'^{[0]}, x'^{[1]}, \dots, x'^{[m]}$ are a K-basis for the vector space V. Then the elements

$$x'^{[0]}, x'^{[1]}, \dots, x'^{[m]}, x^{[m+1]}, x^{[m+2]}, \dots$$

are a K-basis for the vector space P_1. The K-linear map

$$\varphi : P_1 \to P_1, \quad x^{[i]} \mapsto x'^{[i]} \quad (i = 0, 1, \ldots, m), \qquad x^{[j]} \mapsto x^{[j]} \quad (j > m), \tag{17}$$

belongs to the group $(1 + F)^* = \mathrm{GL}_\infty(K) \simeq \mathrm{Inn}(\mathbb{I}_1)$ (by Theorem 3.1.(2)) and satisfies the property that $\partial'\varphi = \varphi\partial$, the equality is in $\mathrm{End}_K(P_1)$. This equality can be rewritten as follows:

$$\omega_{\varphi^{-1}}\sigma(\partial) = \partial \quad \text{where } \omega_{\varphi^{-1}} \in \mathrm{Inn}(\mathbb{I}_1).$$

By Theorem 3.6, $\omega_{\varphi^{-1}}\sigma = e$, hence $\sigma \in \mathrm{G}_1'$.

2. Statement 2 follows from statement 1 and the fact that $\mathrm{Inn}(\mathbb{I}_1) \simeq (1 + F)^* \simeq \mathrm{GL}_\infty(K)$ (by Theorem 3.1.(2)).

3. $[\mathrm{G}_1, \mathrm{G}_1] = [\mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K), \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)] = [\mathbb{T}^1, \mathrm{GL}_\infty(K)][\mathrm{GL}_\infty(K), \mathrm{GL}_\infty(K)] = \mathrm{SL}_\infty(K)$ since $[\mathbb{T}^1, \mathrm{GL}_\infty(K)] \subseteq \mathrm{SL}_\infty(K)$ and $\mathrm{SL}_\infty(K) = [\mathrm{GL}_\infty(K), \mathrm{GL}_\infty(K)]$. Now, $\mathrm{G}_1/[\mathrm{G}_1, \mathrm{G}_1] \simeq \mathbb{T}^1 \times \mathrm{GL}_\infty(K)/\mathrm{SL}_\infty(K) \simeq \mathbb{T}^1 \times \mathbb{T}^1$.

4. Statement 4 follows from statements 1 and 2 and the fact that the group $\mathrm{GL}_\infty(K)$ is generated by the elements $1 + \lambda e_{ij}$ and $1 + \mu e_{11}$ where $i \neq j$, $\lambda \in K^*$ and $\mu \in K\backslash\{-1\}$. □

Corollary 4.4. $\xi(\mathrm{G}_1) = \mathbb{T}^1$ *and* $\ker(\xi) = \mathrm{Inn}(\mathbb{I}_1)$.

Proof. The homomorphism ξ maps isomorphically the torus $\mathbb{T}^1$ onto its image $\mathbb{T}^1$, and $\xi(\mathrm{Inn}(\mathbb{I}_1)) = \{e\}$ (by Theorem 3.1.(2)). Therefore, $\xi(\mathrm{G}_1) = \mathbb{T}^1$ and $\ker(\xi) = \mathrm{Inn}(\mathbb{I}_1)$ since $\mathrm{G}_1 = \mathbb{T}^1 \ltimes \mathrm{Inn}(\mathbb{I}_1)$. □

Every algebra endomorphism of a *simple* algebra is a *monomorphism*. The algebra $\mathbb{I}_1$ is not simple but the same result holds.

Theorem 4.5. *Every algebra endomorphism of the algebra* $\mathbb{I}_1$ *is a monomorphism.*

Proof. Recall that F is the only proper ideal of the algebra $\mathbb{I}_1$, and $\mathbb{I}_1/F = B_1 := K[H][x, x^{-1}; \sigma_1]$ is a simple algebra where $\sigma_1(H) = H - 1$. Suppose that γ is an algebra endomorphism of $\mathbb{I}_1$ which is not a monomorphism, then necessarily $\gamma(F) = 0$, and the endomorphism γ induces the algebra monomorphism $\overline{\gamma} : B_1 \to \mathbb{I}_1$, $a + F \mapsto \gamma(a)$. We seek a contradiction. Since $\partial \int = 1$ and $\int \partial = 1 - e_{00}$, we have the equalities $\gamma(\partial)\gamma(\int) = 1$ and $\gamma(\int)\gamma(\partial) = 1$, i.e. the elements $\gamma(\partial)$ and $\gamma(\int)$ are units of the algebra $\mathbb{I}_1$. Therefore, the images of the elements $\gamma(\partial)$ and $\gamma(\int)$ in the algebra B_1 under the epimorphism $\pi : \mathbb{I}_1 \to B_1$ belong to the group of units of the algebra B_1 which is K^*, hence $\pi(\mathrm{im}(\gamma)) \subseteq K\langle \pi\gamma(H)\rangle$, a commutative algebra. This is impossible since the algebra $\mathrm{im}(\gamma) \simeq B_1$ is a simple non-commutative algebra. This contradiction proves the theorem. □

Question. Is an algebra endomorphism of the algebra $\mathbb{I}_1$ an isomorphism? The same question we can ask for $\mathbb{I}_n$, see Theorem 5.19.

This question has flavour of the Question/Conjecture of Dixmier [12]: *is an algebra endomorphism of the Weyl algebra an isomorphism*?

5. The group of automorphisms of the algebra $\mathbb{I}_n$

In this section, it is proved that $\mathrm{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ (Theorem 5.5), the groups $\mathrm{Inn}(\xi)$ and $\ker(\xi)$ are found (Theorem 5.5.(3), (4)). The group G_n has trivial centre (Theorem 5.15). For each automorphism σ of the algebra $\mathbb{I}_n$, an explicit inversion formula is given via the elements $\{\sigma(\partial_i), \sigma(\int_i) \mid i = 1, \ldots, n\}$ (Theorem 5.14). It is proved that no proper prime factor algebra of the algebra $\mathbb{I}_n$ can be embedded into $\mathbb{I}_n$ (Theorem 5.19). It is shown that each automorphism of the algebra $\mathcal{I}_n$ of scalar integro-differential operators can be uniquely extended to an automorphism of the algebra $\mathbb{I}_n$ (Theorem 5.21).

5.1. The group $\overline{\mathrm{G}}_n := \mathrm{Aut}_{K\text{-alg}}(B_n)$

Recall that $B_n = K[H_1, \dots, H_n][z_1^{\pm 1}, \dots, z_n^{\pm 1}; \sigma_1, \dots, \sigma_n]$, $\sigma_i(H_j) = H_j - \delta_{ij}$. The algebra B_n contains the Laurent polynomial algebra $L_n := K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. The set $L'_n := \{(p_i) \in L_n^n \mid z_i \frac{\partial p_j}{\partial z_i} = z_j \frac{\partial p_i}{\partial z_j},\ \forall i \neq j\}$ is a K-subspace of L_n^n. Clearly, $K^n \subseteq L'_n$ and $L'_n \supseteq \bigoplus_{i=1}^n K[z_i, z_i^{-1}] = \{(p_i) \mid p_i \in K[z_i, z_i^{-1}]\}$. *The elements* $H_1 + p_1, \dots, H_n + p_n$ *(where* $p_i \in L_n$*) of the algebra* B_n *commute iff* $(p_i) \in L'_n$.

Consider the following automorphisms of the algebra B_n: for $i = 1, \dots, n$,

$$a : z_i \mapsto \prod_{j=1}^n z_j^{a_{ij}}, \quad H_i \mapsto \sum_{j=1}^n H_j b_{ji} \quad \big(a = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Z}),\ (b_{ij}) = a^{-1}\big),$$

$$t_\lambda : z_i \mapsto \lambda_i z_i, \quad H_i \mapsto H_i \quad \big(\lambda = (\lambda_i) \in K^{*n}\big),$$

$$s_p : z_i \mapsto z_i, \quad H_i \mapsto H_i + p_i \quad \big(p = (p_i) \in L'_n\big),$$

and the subgroups they generate in the group $\overline{\mathrm{G}}_n$:

$$\Omega_n := \big\{a \;\big|\; a \in \mathrm{GL}_n(\mathbb{Z})\big\} \simeq \mathrm{GL}_n(\mathbb{Z})^{op}, \qquad \mathbb{T}^n := \big\{t_\lambda \;\big|\; \lambda \in K^{*n}\big\} \simeq K^{*n}, \qquad \mathrm{Sh}_n := \big\{s_p \;\big|\; p \in L'_n\big\} \simeq L'_n.$$

The group Ω_n is isomorphic to the *opposite group* $\mathrm{GL}_n(\mathbb{Z})^{op}$ of the general linear group $\mathrm{GL}_n(\mathbb{Z})$ via $a \mapsto a$. Recall that as a set $\mathrm{GL}_n(\mathbb{Z})^{op} = \mathrm{GL}_n(\mathbb{Z})$ but the group structure on $\mathrm{GL}_n(\mathbb{Z})^{op}$ is given by the rule $a \circ b = ba$, the matrix multiplication. The group $\mathrm{GL}_n(\mathbb{Z})^{op}$ is isomorphic to the group $\mathrm{GL}_n(\mathbb{Z})$ via $a \mapsto a^{-1}$.

For each nonzero element $\alpha \in \mathbb{Z}^n$, the set $\mathrm{Supp}(\alpha) := \{i \mid \alpha_i \neq 0\}$ is called the *support* of α, and $\min(\alpha)$ denotes the minimal number in the support of α. The following lemma gives a K-basis for the vector space L'_n. Recall that $K^n \subseteq L'_n$.

Lemma 5.1. $L'_n = K^n \oplus \bigoplus_{0 \neq \alpha \in \mathbb{Z}^n} K b_\alpha$ *where* $b_\alpha = (\lambda_i z^\alpha)$, $\lambda_i = \begin{cases} 0 & \text{if } i \notin \mathrm{Supp}(\alpha), \\ 1 & \text{if } i = \min(\alpha), \\ \frac{\alpha_i}{\alpha_{\min(\alpha)}} & \text{if } i \in \mathrm{Supp}(\alpha). \end{cases}$

Proof. It is obvious that $L'_n \supseteq R$ where R is the RHS of the equality. Each direct summand Kz^α of the Laurent polynomial algebra $K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \mathbb{Z}^n} K z^\alpha$ is invariant under the actions of the K-derivations $z_1 \frac{\partial}{\partial z_1}, \dots, z_n \frac{\partial}{\partial z_n}$ since $z_i \frac{\partial (z^\alpha)}{\partial z_i} = \alpha_i z^\alpha$. Therefore, a K-basis for the vector space L'_n can be chosen in such a way that every element of the basis is of the type $(\lambda_i z^\alpha)$ where $0 \neq (\lambda_i) \in K^n$, $\alpha \in \mathbb{Z}^n$ and

$$\alpha_i \lambda_j = \alpha_j \lambda_i \quad \text{for all } i \neq j. \tag{18}$$

If $\alpha = 0$ then there is no restriction on the scalars λ_j and we get the vector space K^n.

If $\alpha \neq 0$ then either $\lambda_{\min(\alpha)} \neq 0$ or $\lambda_{\min(\alpha)} = 0$. In the first case, the space of solutions to the system of linear equations (18) is $K(\lambda_i)$ where the vector $(\lambda_i) \in K^n$ is as in the lemma. If $\lambda_{\min(\alpha)} = 0$ then $\lambda_i = 0$ for all $i = 1, \dots, n$. Now, the lemma is obvious. □

We can easily verify that

$$a t_\lambda a^{-1} = t_{\lambda^{a^{-1}}}, \qquad a s_p a^{-1} = s_{a(p)a}, \qquad t_\lambda s_p t_\lambda^{-1} = s_{t_\lambda(p)}, \tag{19}$$

where $\lambda^{a^{-1}} = (\lambda'_i)$, $\lambda'_i := \prod_{j=1}^n \lambda_j^{b_{ij}}$, $(b_{ij}) = a^{-1}$; $a(p)a = (p'_i)$, $p'_i = \sum_{j=1}^n a(p_j)a_{ji}$; $t_\lambda(p) = (t_\lambda(p_i))$. It follows that the subgroup of the group $\bar{\mathrm{G}}_n$ generated by the three subgroups above is, in fact, the semi-direct product $\Omega_n \ltimes \mathbb{T}^n \ltimes \mathrm{Sh}_n$ since

$$at_\lambda s_p : z_i \mapsto \lambda_i \prod_{j=1}^n z_j^{a_{ij}}, \quad H_i \mapsto \sum_{j=1}^n H_j b_{ji} + at_\lambda(p_i), \tag{20}$$

and so $at_\lambda s_p = e$ iff $a = e$, all $\lambda_i = 1$, and $p = 0$. Theorem 5.2.(1) shows that this is the whole group of automorphisms of the algebra B_n.

Theorem 5.2.

1. $\bar{\mathrm{G}}_n = \Omega_n \ltimes \mathbb{T}^n \ltimes \mathrm{Sh}_n \simeq \mathrm{GL}_n(\mathbb{Z})^{op} \ltimes K^{*n} \ltimes L'_n$.
2. $Z(\bar{\mathrm{G}}_n) = \{e\}$.
3. *Let* $n \geqslant 2$. *Then* $[\bar{\mathrm{G}}_n, \bar{\mathrm{G}}_n] = [\Omega_n, \Omega_n] \ltimes \mathbb{T}^n \ltimes \mathrm{Sh}_n$ *and* $\bar{\mathrm{G}}_n/[\bar{\mathrm{G}}_n, \bar{\mathrm{G}}_n] \simeq \mathbb{Z}_2$.

Proof. 1. Let $\sigma \in \bar{\mathrm{G}}_n$ and $\bar{\mathrm{G}}'_n$ be the semi-direct product $\Omega_n \ltimes \mathbb{T}^n \ltimes \mathrm{Sh}_n$. Recall that $\bar{\mathrm{G}}'_n \subseteq \bar{\mathrm{G}}_n$. It remains to show that the reverse inclusion holds. The automorphism σ of the algebra B_n induces an automorphism of its group of units $B_n^* = \bigcup_{\alpha \in \mathbb{Z}^n} K^* z^\alpha$. Then

$$\sigma(z_i) = \lambda_i \prod_{j=1}^n z_j^{a_{ij}}, \quad i = 1, \dots, n,$$

where $\lambda_i \in K^*$ and $a = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$. Replacing the automorphism σ with $t_\mu a^{-1}\sigma$ for some $\mu \in K^{*n}$ we may assume that $\sigma(z_i) = z_i$ for all $i = 1, \dots, n$. Then, for all indices $i, j = 1, \dots, n$,

$$[\sigma(H_i) - H_i, z_j] = \sigma([H_i, z_j]) - [H_i, z_j] = \sigma(\delta_{ij} z_j) - \delta_{ij} z_j = \delta_{ij} z_j - \delta_{ij} z_j = 0.$$

Therefore, $p_i := \sigma(H_i) - H_i \in \mathrm{Cen}_{B_n}(z_1, \dots, z_n) = L_n$. The elements $\sigma(H_1), \dots, \sigma(H_n)$ commute

$$0 = [\sigma(H_i), \sigma(H_j)] = [H_i + p_i, H_j + p_j] = [H_i, p_j] - [H_j, p_i] = z_i \frac{\partial p_j}{\partial z_i} - z_j \frac{\partial p_i}{\partial z_j}.$$

Therefore, $(p_i) \in L'_n$, i.e. $\sigma = s_p \in \bar{\mathrm{G}}'_n$, and so $\bar{\mathrm{G}}_n \subseteq \bar{\mathrm{G}}'_n$.

2. By Theorem 4.1.(2), we may assume that $n \geqslant 2$. Let $z \in Z(\bar{\mathrm{G}}_n)$. By statement 1, $z = at_\lambda s_p$. By (19), for all elements $b \in \Omega_n$,

$$bat_\lambda s_p = bz = zb = abt_{\lambda^b} s_{b^{-1}(p)b^{-1}}.$$

Then $\lambda = (1, \dots, 1)$ and $b \in Z(\Omega_n) = \{\pm e\}$ where $-e : z_i \mapsto z_i^{-1}$, $H_i \mapsto -H_i$, for all $i = 1, \dots, n$. Since $(-e)s_p s_{(z_1,\dots,z_n)} \neq s_{(z_1,\dots,z_n)}(-e)s_p$, we have $z = s_p$. The equalities $s_{t_\lambda(p)} = t_\lambda s_p t_\lambda^{-1} = s_p$ for all $t_\lambda \in \mathbb{T}^n$ imply that $p \in K^n$. The equalities

$$s_{pa} = s_{a(p)a} = as_p a^{-1} = s_p$$

for all elements $a \in \Omega_n$ imply that $p = 0$, and so $z = e$. Therefore, $\mathbb{Z}(\Omega_n) = \{e\}$.

3. Let R be the RHS of the first equality in statement 3. Then $R \subseteq [\bar{\mathrm{G}}_n, \bar{\mathrm{G}}_n]$ since, for all $i \neq j$,

$$[E - E_{ij}, t_\lambda(j)] = t_\lambda(i)$$

where $t_\lambda(i) \in \mathbb{T}^1(i)$, $E \in M_n(K)$ is the identity matrix, $E_{ij} \in M_n(K)$ are the matrix units, and

$$[t_2(i), s_{\frac{b_\alpha}{2^{\alpha_i}-1}}] = s_{b_\alpha}, \qquad [(-e), s_{-\frac{\mu}{2}e_j}] = s_{\mu e_j},$$

where $\alpha \in \mathbb{Z}^n$ with $\alpha_i \neq 0$, the elements b_α are as in Lemma 5.1, and the set $\{e_1, \dots, e_n\}$ is the standard K-basis for the vector space K^n. The reverse inclusion $R \supseteq [\bar{\mathrm{G}}_n, \bar{\mathrm{G}}_n]$ is obvious since the factor group

$$\bar{\mathrm{G}}_n/R \simeq \Omega_n/[\Omega_n, \Omega_n] \simeq \mathrm{GL}_n(\mathbb{Z})/[\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Z})] \simeq \mathbb{Z}_2$$

is abelian. Now, it is obvious that $\bar{\mathrm{G}}_n/[\bar{\mathrm{G}}_n, \bar{\mathrm{G}}_n] \simeq \mathbb{Z}_2$. $\square$

5.2. A characterization of the elements of the group G_n

For each automorphism σ of the algebra $\mathbb{I}_n$, the next lemma gives explicitly the map $\varphi \in \mathrm{Aut}_K(P_n)$ such that $\sigma = \sigma_\varphi$ (see Corollary 3.3.(2)). Lemma 5.3 is used at the final stage of the proof of Theorem 5.5.

Lemma 5.3. *For each automorphism σ of the algebra $\mathbb{I}_n$, there exists a K-basis $\{x'^{[\alpha]}\}_{\alpha\in\mathbb{N}^n}$ of the polynomial algebra P_n such that $\sigma(H_i) * x'^{[\alpha]} = (\alpha_i + 1)x'^{[\alpha]}$ and $\sigma(\partial_i) * x'^{[\alpha]} = x'^{[\alpha-e_i]}$ for all $i = 1, \dots, n$ (where $x'^{[\beta]} := 0$ if $\beta \in \mathbb{Z}^n \backslash \mathbb{N}^n$). Moreover,*

1. *$\sigma = \sigma_\varphi$ where the map $\varphi \in \mathrm{Aut}_K(P_n)$: $x^{[\alpha]} \mapsto x'^{[\alpha]}$ is the change-of-the-basis map,*
2. *$\sigma(\int_i) * x'^{[\alpha]} = x'^{[\alpha+e_i]}$ for all $i = 1, \dots, n$, and*
3. *the basis $\{x'^{[\alpha]}\}_{\alpha\in\mathbb{N}^n}$ is unique up to a simultaneous multiplication of each element of the basis by the same nonzero scalar.*

Proof. Recall that the polynomial algebra $P_n = \bigoplus_{\alpha\in\mathbb{N}^n} Kx^{[\alpha]}$ (where $x^{[\alpha]} := \prod_{i=1}^n \frac{x_i^{\alpha_i}}{i!}$) is the direct sum of non-isomorphic, one-dimensional, simple $K[H_1, \dots, H_n]$-modules (see (11)) such that $\partial_i * x^{[\alpha]} = x^{[\alpha-e_i]}$ for all $\alpha \in \mathbb{N}^n$ and $i = 1, \dots, n$ (where $x^{[\beta]} := 0$ if $\beta \in \mathbb{Z}^n \backslash \mathbb{N}^n$). Recall that $\sigma = \sigma_\varphi$ for some linear map $\varphi \in \mathrm{Aut}_K(P_n)$, the linear map $\varphi : P_n \to {}^\sigma P_n$ is an $\mathbb{I}_n$-module isomorphism (Corollary 3.3.(2)), and the map φ is unique up to a multiplication by a nonzero scalar since $\mathrm{End}_{\mathbb{I}_n}(P_n) \simeq K$ (Corollary 3.5.(1)). Let

$$x'^{[\alpha]} := \varphi(x^{[\alpha]}), \quad \alpha \in \mathbb{N}^n.$$

Then the fact that the map φ is an $\mathbb{I}_n$-module homomorphism is equivalent to the fact that the following equations hold:

$$\begin{aligned}
\sigma(H_i) * x'^{[\alpha]} &= \varphi H_i \varphi^{-1} \varphi * x^{[\alpha]} = (\alpha_i + 1)\varphi * x^{[\alpha]} = (\alpha_i + 1)x'^{[\alpha]}, \\
\sigma(\partial_i) * x'^{[\alpha]} &= \varphi \partial_i \varphi^{-1} \varphi * x^{[\alpha]} = \varphi * x^{[\alpha-e_i]} = x'^{[\alpha_i - e_i]} \quad (x'^{[\beta]} := 0,\ \beta \in \mathbb{Z}^n \backslash \mathbb{N}^n), \\
\sigma\left(\int_i\right) * x'^{[\alpha]} &= \varphi \int_i \varphi^{-1} \varphi * x^{[\alpha]} = \varphi * x^{[\alpha+e_i]} = x'^{[\alpha_i + e_i]}.
\end{aligned}$$

Note that the last equality follows from the previous two: by the first equality, the polynomial algebra $P_n = \bigoplus_{\alpha\in\mathbb{N}^n} Kx'^{[\alpha]}$ is the direct sum of non-isomorphic, one-dimensional, simple $K[\sigma(H_1), \dots, \sigma(H_n)]$-modules. Since $\sigma(H_i)\sigma(\int_i) * x'^{[\alpha]} = \sigma(H_i \int_i) * x'^{[\alpha]} = \sigma(\int_i (H_i + 1)) * x'^{[\alpha]} = \sigma(\int_i)(\sigma(H_i) + 1) * x'^{[\alpha]} = (\alpha_i + 2)\sigma(\int_i) * x'^{[\alpha]}$ for all i, we have $\sigma(\int_i) * x'^{[\alpha]} = \lambda_{i,\alpha} x'^{[\alpha+e_i]}$ for a scalar $\lambda_{i,\alpha}$ which is necessarily equal to 1 since

$$x'^{[\alpha]} = \sigma(\partial_i)\sigma\left(\int_i\right) * x'^{[\alpha]} = \lambda_{i,\alpha}\sigma(\partial_i) * x'^{[\alpha+e_i]} = \lambda_{i,\alpha}x'^{[\alpha]}.$$

Since the isomorphism φ is unique up to a multiplication by a nonzero scalar, the basis $\{x'^{[\alpha]}\}$ is unique up to a simultaneous multiplication of each element of it by the same nonzero scalar. The proof of the lemma is complete. □

5.3. The group G_n is a subgroup of $\mathbb{G}_n$

In [10], it is proved that the Jacobian algebra $\mathbb{A}_n = S^{-1}\mathbb{I}_n$ is the two-sided localization of the algebra $\mathbb{I}_n$ at the multiplicatively closed subset

$$S := \left\{ \prod_{i=1}^{n} (H_i + \alpha_i)_*^{n_i} \;\middle|\; (\alpha_i) \in \mathbb{Z}^n,\ (n_i) \in \mathbb{N}^n \right\}$$

of $\mathbb{I}_n$ where $(H_i+\alpha_i)_* := \begin{cases} H_i+\alpha_i & \text{if } \alpha_i \geqslant 0, \\ (H_i+\alpha_i)_1 & \text{if } \alpha_i < 0, \end{cases}$ and $(H_i - j)_1 := H_i - j + e_{j-1,j-1}(i)$ for $j \geqslant 1$. The elements of the set $S \subseteq \mathrm{End}_K(P_n)$ are invertible linear maps in P_n, i.e. $S \subseteq \mathrm{Aut}_K(P_n)$, and therefore are regular elements of the algebra $\mathbb{I}_n$ since $\mathbb{I}_n \subseteq \mathrm{End}_K(P_n)$.

Theorem 5.4.

1. *$\mathrm{G}_n = \{\sigma \in \mathbb{G}_n \mid \sigma(\mathbb{I}_n) = \mathbb{I}_n\}$ and G_n is a subgroup of $\mathbb{G}_n$.*
2. *Each automorphism of the algebra $\mathbb{I}_n$ has a unique extension to an automorphism of the algebra $\mathbb{A}_n$.*

Proof. 1. Statement 1 follows from statement 2: the set $\{\sigma \in \mathbb{G}_n \mid \sigma(\mathbb{I}_n) = \mathbb{I}_n\}$ is a subgroup of the group $\mathbb{G}_n$ that is mapped isomorphically onto the group G_n via $\sigma \mapsto \sigma|_{\mathbb{I}_n}$, by statement 2.

2. It suffices to prove that each automorphism σ of the algebra $\mathbb{I}_n$ can be extended to an automorphism of the algebra $\mathbb{A}_n$, since then its uniqueness is obvious as $\mathbb{A}_n = S^{-1}\mathbb{I}_n$. By Corollary 3.5.(2) and Lemma 5.3, there exists an $\mathbb{I}_n$-module isomorphism $\varphi : P_n \to {}^{\sigma}P_n$ which is unique up to K^*, and $\sigma(a) = \varphi a \varphi^{-1}$ for all $a \in \mathbb{I}_n$. Using the K-basis $\{x'^{[\alpha]}\}_{\alpha\in\mathbb{N}^n}$ of Lemma 5.3 we see that all the elements $\{\sigma(s) \mid s \in S\}$ are invertible in $\mathrm{End}_K({}^{\sigma}P_n) = \mathrm{End}_K(P_n)$. By the universal property of localization, the algebra monomorphism $\mathbb{I}_n \to \mathrm{End}_K({}^{\sigma}P_n)$, $a \mapsto (p \mapsto \sigma(a)p)$, can be extended uniquely to the algebra homomorphism $\mathbb{A}_n \to \mathrm{End}_K({}^{\sigma}P_n)$, $s^{-1}a \mapsto (p \mapsto \sigma(s)^{-1}\sigma(a)p)$ where $s \in S$ and $a \in \mathbb{I}_n$. It is obvious that the extension is an algebra monomorphism since $\sigma(S) \subseteq \mathrm{Aut}_K({}^{\sigma}P_n)$. Therefore, the $\mathbb{A}_n$-module ${}^{\sigma}P_n$ is simple and faithful (since the $\mathbb{I}_n$-module ${}^{\sigma}P_n$ is simple and $\mathbb{I}_n \subseteq \mathbb{A}_n$). The $\mathbb{A}_n$-module P_n is the only (up to isomorphism) simple and faithful $\mathbb{A}_n$-module [2, Corollary 2.7.(10)], and $\mathrm{End}_{\mathbb{A}_n}(P_n) \simeq K$. Therefore, there exists a unique (up to K^*) $\mathbb{A}_n$-module isomorphism $\psi : P_n \to {}^{\sigma}P_n$ such that $\sigma(a) = \psi a \psi^{-1}$ for all elements $a \in \mathbb{A}_n$. In particular, the map ψ is an $\mathbb{I}_n$-module isomorphism. Therefore, $K^*\psi = K^*\varphi$ since $\mathrm{End}_{\mathbb{I}_n}(P_n) \simeq K$. Without loss of generality we may assume that $\psi = \varphi$. Therefore, the automorphism $\sigma \in \mathrm{G}_n$ can be uniquely extended to an automorphism of the algebra $\mathbb{A}_n$. □

For each natural number $d \geqslant 1$, there is the decomposition $K[x_i] = (\bigoplus_{j=0}^{d-1} Kx_i^j) \oplus (\bigoplus_{k\geqslant d} Kx_i^k)$. The idempotents of the algebra $\mathbb{I}_n$, $p(i,d) := \sum_{j=0}^{d-1} e_{jj}(i)$ and $q(i,d) := 1 - p(i,d)$, are the projections onto the first and the second summand correspondingly. For a subset I of the set $\{1,\dots,n\}$, CI denotes its complement. Since $P_n = \bigotimes_{i=1}^n K[x_i]$, the identity map $1 = \mathrm{id}_{P_n}$ on the vector space P_n is the sum

$$1 = \bigotimes_{i=1}^{n} \bigl(p(i,d) + q(i,d)\bigr) = \sum_{I\subseteq\{1,\dots,n\}} p(I,d)q(CI,d) \tag{21}$$

of orthogonal idempotents where $p(I,d) := \prod_{i\in I} p(i,d)$, $q(CI,d) := \prod_{i\in CI} q(i,d)$, $p(\emptyset,d) := 1$, and $q(\emptyset,d) := 1$. Each idempotent $p(I,d)q(CI,d) \in \mathbb{I}_n \subset \mathrm{End}_K(P_n)$ is the projection onto the summand $P_n(I,d)$ in the following decomposition of the vector space P_n,

$$P_n = \bigoplus_{I\subseteq\{1,\dots,n\}} P_n(I,d), \quad P_n(I,d) := \bigoplus \{ Kx^{[\alpha]} \mid \alpha_i < d, \text{ if } i \in I;\ \alpha_j \geqslant d, \text{ if } j \in CI \}. \tag{22}$$

In particular, the idempotent $q(\{1,\dots,n\},d)$ is the projection onto the subspace $P_n(\{1,\dots,n\},d) = \bigoplus\{Kx^{[\alpha]} \mid \text{all } \alpha_i \geqslant d\}$.

5.4. The group G_n and a formula for the map φ such that $\sigma = \sigma_\varphi$

By Theorem 3.6, each element $\sigma = \sigma_\varphi \in \mathrm{G}_n$ (Corollary 3.3.(2)) is uniquely determined by the elements $\sigma(\partial_1),\dots,\sigma(\partial_n)$. In the proof of Theorem 5.5, an explicit formula for the map φ is given, (26), via the elements $\sigma(\partial_1),\dots,\sigma(\partial_n)$.

By the very definition, the group $\ker(\xi)$ (see (8)) contains precisely all the automorphisms $\sigma \in \mathrm{G}_n$ such that

$$\sigma\left(\int_i\right) \equiv \int_i \mod \mathfrak{a}_n, \qquad \sigma(\partial_i) \equiv \partial_i \mod \mathfrak{a}_n, \qquad \sigma(H_i) \equiv H_i \mod \mathfrak{a}_n, \quad i = 1,\dots,n. \tag{23}$$

Theorem 5.5.

1. $\mathrm{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$.
2. $\mathrm{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \ker(\xi)$.
3. $\mathrm{im}(\xi) = S_n \ltimes \mathbb{T}^n$.
4. $\ker(\xi) = \mathrm{Inn}(\mathbb{I}_n)$.

Proof. 1. Statement 1 follows from statements 2 and 4.

2. Statement 2 follows from statement 3: suppose that $\mathrm{im}(\xi) = S_n \ltimes \mathbb{T}^n$, then the homomorphism ξ maps isomorphically the subgroup $S_n \ltimes \mathbb{T}^n$ of G_n onto its image $S_n \ltimes \mathbb{T}^n$, and so statement 2 follows from the short exact sequence of groups: $1 \to \ker(\xi) \to \mathrm{G}_n \to \mathrm{im}(\xi) \to 1$ which is obviously a split one.

3. Let $\sigma \in \mathrm{G}_n$. We have to show that there exists an element $\sigma' \in S_n \ltimes \mathbb{T}^n$ such that the automorphism $\sigma'\sigma$ satisfies the conditions (23). By Theorem 5.4, $\sigma \in \mathbb{G}_n$. By [9, Corollary 7.5],

$$\sigma(H_i) \equiv H_{\tau(i)} \mod \mathfrak{a}_n^e \quad \text{for all } i = 1,\dots,n$$

and for some element $\tau \in S_n$ where $\mathfrak{a}_n^e$ is the only maximal ideal of the algebra $\mathbb{A}_n$ (Theorem 2.2). Then $\tau^{-1}\sigma(H_i) \equiv H_i \mod \mathfrak{a}_n^e$ for all $i = 1,\dots,n$, and so $\tau^{-1}\sigma(H_i) - H_i \in \mathbb{I}_n \cap \mathfrak{a}_n^e = \mathfrak{a}_n^{er} = \mathfrak{a}_n$ (Theorem 2.2). For the automorphism $\xi(\tau^{-1}\sigma) \in \bar{\mathrm{G}}_n$, we have the action (20),

$$\xi(\tau^{-1}\sigma) : z_i \mapsto \lambda_i z_i, \quad H_i \mapsto H_i, \quad i = 1,\dots,n,$$

for some element $(\lambda_i) \in K^{*n}$. Then $t_\lambda^{-1}\tau^{-1}\sigma \in \ker(\xi)$ where $\lambda = (\lambda_i)$. Therefore, $\mathrm{im}(\xi) = S_n \ltimes \mathbb{T}^n$.

4. By Theorem 3.1.(2), $\ker(\xi) \supseteq \mathrm{Inn}(\mathbb{I}_n)$. Let $\sigma \in \ker(\xi)$. It remains to show that $\sigma \in \mathrm{Inn}(\mathbb{I}_n)$. Fix a natural number d such that

$$\sigma(H_i) - H_i, \sigma(\partial_i) - \partial_i, \sigma\left(\int_i\right) - \int_i \in \sum_{k=1}^n \mathbb{I}_{n-1,k} \otimes \left(\sum_{s,t=0}^{d-1} Ke_{st}(k)\right) \tag{24}$$

for all $i=1,\dots,n$ where $\mathbb{I}_{n-1,k} := \bigotimes_{j\neq k}\mathbb{I}_1(j)$. By Lemma 5.3, $\sigma = \sigma_\varphi : a \mapsto \varphi a \varphi^{-1}$ where $\varphi \in \mathrm{Aut}_K(P_n) : x^{[\alpha]} \mapsto x'^{[\alpha]}$ is the change-of-the-basis map (see Lemma 5.3). By the choice of the number d above, for each element $x^{[\alpha]} \in P(\{1,\dots,n\},d)$,

$$\sigma(H_i) * x^{[\alpha]} = H_i * x^{[\alpha]} = (\alpha_i+1)x^{[\alpha]} \quad \text{and} \quad \sigma(\partial_i) * x^{[\alpha]} = \partial_i * x^{[\alpha]} = x^{[\alpha - e_i]} \quad \text{for all } i=1,\dots,n.$$

By multiplying the map φ by a nonzero scalar, by Lemma 5.3, we may assume that

$$x'^{[\alpha]} = x^{[\alpha]} \quad \text{for all } \alpha = (\alpha_i) \in \mathbb{N}^n \text{ such that } \alpha_1 \geqslant d, \dots, \alpha_n \geqslant d. \tag{25}$$

It suffices to show that $\varphi \in \mathbb{I}_n$ (since then $\varphi^{-1} \in \mathbb{I}_n$ as $\sigma^{-1} = \sigma_{\varphi^{-1}}$). This is obvious since

$$\varphi = q\big(\{1,\dots,n\},d\big) + \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} \left(\sum_{\alpha \in C_d(I)} \prod_{j\in I} \sigma\big(\partial_j^{d-\alpha_j}\big) \cdot \prod_{i\in I} \int_i^{d-\alpha_i} \cdot e_{\alpha\alpha}(I) \right) p(I,d) q(CI,d) \tag{26}$$

where $C_d(I) := \{(\alpha_i)_{i\in I} \in \mathbb{N}^I \mid \text{all } \alpha_i < d\}$, $e_{\alpha\alpha}(I) := \prod_{i\in I} e_{\alpha_i\alpha_i}(i)$, and d is as in (24). To prove that this formula holds for the map φ we have to show that $\varphi * x^{[\alpha]} = x'^{[\alpha]}$ for all $\alpha \in \mathbb{N}^n$. For each α, let $I := \{i \mid \alpha_i < d\}$. Then $x^{[\alpha]} = \prod_{i\in I} x_i^{[\alpha_i]} \cdot \prod_{k\in CI} x_k^{[\alpha_k]}$. If $I \neq \emptyset$ then

$$\begin{aligned}\varphi * x^{[\alpha]} &= \prod_{j\in I} \sigma\big(\partial_j^{d-\alpha_j}\big) \cdot \prod_{i\in I} \int_i^{d-\alpha_i} * \prod_{i\in I} x_i^{[\alpha_i]} \cdot \prod_{k\in CI} x_k^{[\alpha_k]} = \prod_{j\in I} \sigma\big(\partial_j^{d-\alpha_j}\big) * \prod_{i\in I} x_i^{[d]} \cdot \prod_{k\in CI} x_k^{[\alpha_k]} \\ &= \prod_{j\in I} \sigma\big(\partial_j^{d-\alpha_j}\big) * x'^{[\sum_{i\in I} de_i + \sum_{k\in CI}\alpha_k e_k]} = x'^{[\sum_{i\in I}\alpha_i e_i + \sum_{k\in CI}\alpha_k e_k]} = x'^{[\alpha]}.\end{aligned}$$

If $I = \emptyset$ then $\varphi * x^{[\alpha]} = q(\{1,\dots,n\},d) * x^{[\alpha]} = x^{[\alpha]} = x'^{[\alpha]}$. The proof of the theorem is complete. □

Corollary 5.6. *Let $\sigma \in \mathrm{Inn}(\mathbb{I}_n)$. Then there is a unique element $\varphi \in (1+\mathfrak{a}_n)^*$ such that $\sigma(a) = \varphi a \varphi^{-1}$ for all elements $a \in \mathbb{I}_n$, and the element φ is given by the formula* (26)*.*

Proof. The element $\varphi \in (1+\mathfrak{a}_n)^*$ such that $\sigma(a) = \varphi a \varphi^{-1}$ for all $a \in \mathbb{I}_n$ is unique by Theorem 3.1.(2). By the very definition, the element $\varphi \in \mathbb{I}_n^*$ from (26) satisfies $\varphi \equiv 1 \bmod \mathfrak{a}_n$ and $\sigma(a) = \varphi a \varphi^{-1}$ for all $a \in \mathbb{I}_n$. Therefore, both φ's coincide. □

Corollary 5.7. $\mathrm{Out}(\mathbb{I}_n) \simeq S_n \ltimes \mathbb{T}^n$.

Proof. $\mathrm{Out}(\mathbb{I}_n) = \mathrm{G}_n / \mathrm{Inn}(\mathbb{I}_n) = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)/\mathrm{Inn}(\mathbb{I}_n) \simeq S_n \ltimes \mathbb{T}^n$. □

Recall that $\mathcal{H}_1 := \{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$ is the set of height one prime ideals of the algebra $\mathbb{I}_n$. The next corollary describes its stabilizer $\mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1) := \{\sigma \in \mathrm{G}_n \mid \sigma(\mathfrak{p}_1) = \mathfrak{p}_1, \dots, \sigma(\mathfrak{p}_n) = \mathfrak{p}_n\}$.

Corollary 5.8. $\mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1) = \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$.

Proof. It is obvious that $\mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1) \supseteq R := \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ and $S_n \cap \mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1) = \{e\}$. Now,

$$\mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1) = \mathrm{G}_n \cap \mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1) = (S_n \ltimes R) \cap \mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1) = \big(S_n \cap \mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1)\big) \ltimes R = R. \qquad \square$$

The algebra $\mathbb{I}_n = \bigoplus_{\alpha\in\mathbb{Z}^n} \mathbb{I}_{n,\alpha}$ is a $\mathbb{Z}^n$-graded subalgebra of the Jacobian algebra $\mathbb{A}_n = \bigoplus_{\alpha\in\mathbb{Z}^n} \mathbb{A}_{n,\alpha}$, see [10]. The group

$$\mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{A}_n) := \{\sigma \in \mathbb{G}_n \mid \sigma(\mathbb{A}_{n,\alpha}) = \mathbb{A}_{n,\alpha} \text{ for all } \alpha \in \mathbb{Z}^n\}$$

of $\mathbb{Z}^n$-grading preserving automorphisms of the algebra $\mathbb{A}_n$ is equal to

$$\mathrm{St}_{\mathbb{G}_n}(H_1,\dots,H_n) := \{\sigma \in \mathbb{G}_n \mid \sigma(H_1) = H_1,\dots,\sigma(H_n) = H_n\}$$

and $\mathrm{St}_{\mathbb{G}_n}(H_1,\dots,H_n) = \mathbb{T}^n \times \mathbb{U}_n$ [9, Corollary 7.10] where the subgroup $\mathbb{U}_n$ of $\mathbb{G}_n$ is defined as follows. Recall that $\mathbb{A}_n = S^{-1}\mathbb{I}_n$. Let $\mathcal{H}_n$ be the subgroup of $\mathbb{A}_n^*$ generated by the commutative monoid $S \subseteq \mathbb{A}_n^*$. Then the group $\mathcal{H}_n = \prod_{i=1}^n \mathcal{H}_1(i)$ is the direct product of its subgroups

$$\mathcal{H}_1(i) := \left\{\prod_{j\geqslant 0}(H_i + j)^{n_j} \cdot \prod_{j\geqslant 1}(H_i - j)_1^{n_{-j}} \;\Big|\; (n_k)_{k\in\mathbb{Z}} \in \mathbb{Z}^{(\mathbb{Z})}\right\} \simeq \mathbb{Z}^{(\mathbb{Z})},$$

and so $\mathcal{H}_n \simeq (\mathbb{Z}^n)^{(\mathbb{Z})}$. Each element $u = u_1 \cdots u_n \in \mathcal{H}_n$, where $u_i \in \mathcal{H}_1(i)$, determines the automorphism μ_u of the algebra $\mathbb{A}_n$ (see (35) in [9] for details),

$$\mu_u : x_i \mapsto x_i u_i, \quad y_i \mapsto u_i^{-1} y_i, \quad H_i^{\pm 1} \mapsto H_i^{\pm 1}, \quad i = 1,\dots,n.$$

Then $\mathbb{U}_n := \{\mu_u \mid u \in \mathcal{H}_n\} \simeq (\mathbb{Z}^n)^{(\mathbb{Z})}$. Let $\mathrm{St}_{\mathrm{G}_n}(H_1,\dots,H_n) := \{\sigma \in \mathrm{G}_n \mid \sigma(H_1) = H_1,\dots,\sigma(H_n) = H_n\}$.

Corollary 5.9.

1. $\mathrm{St}_{\mathrm{G}_n}(H_1,\dots,H_n) = \mathbb{T}^n$.
2. *Let $\sigma, \tau \in \mathrm{G}_n$. Then $\sigma(H_1) = \tau(H_1),\dots,\sigma(H_n) = \tau(H_n)$ iff $\sigma = \tau t_\lambda$ for some element $t_\lambda \in \mathbb{T}^n$.*
3. $\mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{I}_n) = \mathrm{St}_{\mathrm{G}_n}(H_1,\dots,H_n) \subset \mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{A}_n)$.

Proof. 1. Since $\mathrm{G}_n \subseteq \mathbb{G}_n$ (Theorem 5.4),

$$\mathrm{St}_{\mathrm{G}_n}(H_1,\dots,H_n) = \mathrm{G}_n \cap \mathrm{St}_{\mathbb{G}_n}(H_1,\dots,H_n) = \mathrm{G}_n \cap \mathbb{T}^n \times \mathbb{U}_n = \mathbb{T}^n \times (\mathrm{G}_n \cap \mathbb{U}_n) = \mathbb{T}^n$$

since $\mathrm{G}_n \cap \mathbb{U}_n = \{e\}$, by the very definition of the group $\mathbb{U}_n$ (see (20)).

2. Statement 2 follows from statement 1.

3. Since $\mathbb{A}_n = S^{-1}\mathbb{I}_n$, $S \subseteq \mathbb{I}_{n,0}$ and $\mathrm{G}_n \subseteq \mathbb{G}_n$, we have the inclusion $\mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{I}_n) \subseteq \mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{A}_n)$. Now,

$$\mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{I}_n) = \mathrm{G}_n \cap \mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{A}_n) = \mathrm{G}_n \cap \mathbb{T}^n \times \mathbb{U}_n = \mathbb{T}^n = \mathrm{St}_{\mathrm{G}_n}(H_1,\dots,H_n),$$

by statement 1. By statement 1, the inclusion $\mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{I}_n) \subseteq \mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{A}_n)$ is a strict inclusion since $\mathrm{Aut}_{\mathbb{Z}^n\text{-gr}}(\mathbb{A}_n) = \mathbb{T}^n \times \mathbb{U}_n$ [9, Corollary 7.10]. □

5.5. The canonical form of $\sigma \in \mathrm{G}_n$

By Theorem 5.5, each automorphism σ of the algebra $\mathbb{I}_n$ is a *unique* product $st_\lambda\omega_\varphi$ where $s \in S_n$, $t_\lambda \in \mathbb{T}^n$, and ω_φ is an inner automorphism of the algebra $\mathbb{I}_n$ with $\varphi \in (1 + \mathfrak{a}_n)^*$, and the element φ is unique (Corollary 5.6).

Definition. The unique product $\sigma = st_\lambda\omega_\varphi$ is called the *canonical form* of the automorphism σ of the algebra $\mathbb{I}_n$.

Corollary 5.10. *Let $\sigma \in G_n$ and $\sigma = st_\lambda \omega_\varphi$ be its canonical form. Then the automorphisms s, t_λ and ω_φ can be effectively (in finitely many steps) found from the action of the automorphism σ on the elements $\{H_i, \partial_i, \int_i \mid i = 1, \dots, n\}$:*

$$\sigma(H_i) \equiv H_{s(i)} \mod \mathfrak{a}_n, \qquad \sigma(\partial_i) \equiv \lambda_i^{-1} \partial_{s(i)} \mod \mathfrak{a}_n, \qquad \sigma\left(\int_i\right) \equiv \lambda_i \int_{s(i)} \mod \mathfrak{a}_n,$$

and the elements φ and φ^{-1} are given by the formulae (26) and (27) respectively for the automorphism $(st_\lambda)^{-1}\sigma \in \mathrm{Inn}(\mathbb{I}_n) = \ker(\xi)$.

The next corollary is a criterion for an automorphism of the algebra $\mathbb{I}_n$ to be an inner automorphism.

Corollary 5.11. *Let $\sigma \in G_n$. The following statements are equivalent.*

1. $\sigma \in \mathrm{Inn}(\mathbb{I}_n)$.
2. $\sigma(\partial_i) \equiv \partial_i \mod \mathfrak{a}_n$ *for $i = 1, \dots, n$.*
3. $\sigma(\int_i) \equiv \int_i \mod \mathfrak{a}_n$ *for $i = 1, \dots, n$.*

Proof. The result follows from Theorem 5.5.(4) and Corollary 5.10. □

Corollary 5.12. *Let $\sigma \in G_n$. Then $\sigma \in \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ iff $\sigma(H_i) \equiv H_i \mod \mathfrak{a}_n$ for $i = 1, \dots, n$.*

Proof. This follows from Theorem 5.5.(4) and Corollary 5.10. □

5.6. A formula for the inverse φ^{-1} where $\sigma = \sigma_\varphi \in \mathrm{Inn}(\mathbb{I}_n)$ via $\sigma(\partial_i)$ and $\sigma(\int_j)$

By Corollary 5.6, for each inner automorphism $\sigma \in \mathrm{Inn}(\mathbb{I}_n)$ there exists a unique element $\varphi \in (1 + \mathfrak{a}_n)^*$ such that $\sigma = \sigma_\varphi : a \mapsto \varphi a \varphi^{-1}$ for all $a \in \mathbb{I}_n$. The next theorem presents a formula for the inverse φ^{-1} via the elements $\{\sigma(\partial_i), \sigma(\int_i) \mid i = 1, \dots, n\}$.

Theorem 5.13. *Let $\sigma = \sigma_\varphi \in \mathrm{Inn}(\mathbb{I}_n)$ where $\varphi \in (1 + \mathfrak{a}_n)^*$ ($\sigma_\varphi(a) = \varphi a \varphi^{-1}$ for all $a \in \mathbb{I}_n$, see Corollary 5.6). Then $\sigma_\varphi^{-1} = \sigma_{\varphi^{-1}}$ and*

$$\varphi^{-1} = q'\big(\{1, \dots, n\}, d\big) + \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} \left(\sum_{\alpha \in C_d(I)} \prod_{j \in I} \partial_j^{d - \alpha_j} \cdot \prod_{i \in I} \sigma\left(\int_i\right)^{d - \alpha_i} \cdot e'_{\alpha\alpha}(I) \right) p'(I, d) q'(CI, d) \qquad (27)$$

where d is as in (24) for $\sigma = \sigma_\varphi$, $e'_{\alpha\alpha}(I) := \prod_{i \in I} e'_{\alpha_i \alpha_i}(i)$ and $e'_{jj}(i) := \sigma(e_{jj}(i)) = \sigma(\int_i)^j \sigma(\partial_i)^j - \sigma(\int_i)^{j+1} \sigma(\partial_i)^{j+1}$; $p'(I, d) := \prod_{i \in I} p'(i, d)$ and $p'(i, d) := \sigma(p(i, d)) = \sum_{j=0}^{d-1} e'_{jj}(i)$; $q'(CI, d) := \sigma(q(CI, d)) = \prod_{i \in CI} (1 - p'(d, i))$.

Proof. We keep the notation of the proof of statement 4 of Theorem 5.5. In particular,

$$\varphi : P_n = \bigoplus_{\alpha \in \mathbb{N}^n} K x^{[\alpha]} \to P_n = \bigoplus_{\alpha \in \mathbb{N}^n} K x'^{[\alpha]}, \quad x^{[\alpha]} \mapsto x'^{[\alpha]}.$$

For each $\alpha \in \mathbb{N}^n$, the projection onto the summand $Kx'^{[\alpha]}$ of the polynomial algebra P_n is equal to $\varphi e_{\alpha\alpha}\varphi^{-1} = \sigma(e_{\alpha\alpha})$. For each subset I of the set $\{1, \dots, n\}$, let

$$P'_n(I, d) := \varphi\big(P_n(I, d)\big) = \bigoplus\big\{Kx'^{[\alpha]} \bigm| \alpha_i < d \text{ if } i \in I;\ \alpha_j \geqslant d \text{ if } j \in CI\big\},$$

see (22). Since $\varphi : P_n = \bigoplus_{I\subseteq\{1,\dots,n\}} P_n(I, d) \simeq P_n = \bigoplus_{I\subseteq\{1,\dots,n\}} P'_n(I, d)$, the projections onto the summand $P'_n(I, d)$ of P_n is equal to

$$\varphi p(I, d)q(I, d)\varphi^{-1} = \sigma\big(p(I, d)q(I, d)\big) = \sigma\big(p(I, d)\big)\sigma\big(q(I, d)\big) = p'(I, d)q'(I, d)$$

where $p'(I, d) = \sigma(p(I, d))$ and $q'(I, d) = \sigma(q(I, d))$. Then the inverse map φ^{-1} of the map φ in (26) is given by (27). To prove this let ψ be the RHS of (27). We have to show that $\psi : x'^{[\alpha]} \mapsto x^{[\alpha]}$ for all $\alpha \in \mathbb{N}^n$. Fix α, and let $I := \{i \mid \alpha_i < d\}$. Then $x'^{[\alpha]} = \prod_{i\in I} x_i'^{[\alpha_i]} \cdot \prod_{k\in CI} x_k'^{[\alpha_k]}$. If $I \neq \emptyset$ then, by Lemma 5.3,

$$\begin{aligned}\psi * x'^{[\alpha]} &= \prod_{j\in I} \partial_j^{d-\alpha_j} \cdot \prod_{i\in I} \sigma\Big(\int_i\Big)^{d-\alpha_i} * \prod_{i\in I} x_i'^{[\alpha_i]} \cdot \prod_{k\in CI} x_k'^{[\alpha_k]} = \prod_{j\in I} \partial_j^{d-\alpha_j} * \prod_{i\in I} x_i'^{[d]} \cdot \prod_{k\in CI} x_k'^{[\alpha_k]} \\ &= \prod_{j\in I} \partial_j^{d-\alpha_j} * \prod_{i\in I} x_i^{[d]} \cdot \prod_{k\in CI} x_k^{[\alpha_k]} \quad \big(\text{by (25)}\big) \\ &= x^{[\alpha]}.\end{aligned}$$

If $I \neq \emptyset$ then $\psi * x'^{[\alpha]} = q'(\{1, \dots, n\}, d) * x'^{[\alpha]} = x'^{[\alpha]} = x^{[\alpha]}$, by (25). This finishes the proof of the theorem. □

5.7. An inversion formula for $\sigma \in \mathrm{G}_n$

The next theorem gives an inversion formula for σ via the elements $\{\sigma(\partial_i), \sigma(\int_i) \mid i = 1, \dots, n\}$.

Theorem 5.14. *Let $\sigma \in \mathrm{G}_n$ and $\sigma = st_\lambda\omega_\varphi$ be its canonical form where $s \in S_n$, $t_\lambda \in \mathbb{T}^n$ and $\omega_\varphi \in \mathrm{Inn}(\mathbb{I}_n)$ for a unique element $\varphi \in (1 + \mathfrak{a}_n)^*$. Then*

$$\sigma^{-1} = s^{-1}t_{s(\lambda^{-1})}\omega_{st_\lambda(\varphi^{-1})} \tag{28}$$

is the canonical form of the automorphism σ^{-1} where the elements φ^{-1} and φ are given by the formulae (27) *and* (26) *respectively for the automorphism $(st_\lambda)^{-1}\sigma \in \mathrm{Inn}(\mathbb{I}_n)$.*

Proof. $\sigma^{-1} = s^{-1} \cdot st_\lambda^{-1}s^{-1} \cdot st_\lambda\omega_\varphi^{-1}(st_\lambda)^{-1} = s^{-1}t_{s(\lambda^{-1})}\omega_{st_\lambda(\varphi^{-1})}$. □

Theorem 5.15. *The centre of the group G_n is $\{e\}$.*

Proof. Let σ be an element of the centre of the group G_n. For all elements $\alpha, \beta \in \mathbb{N}^n$, $1 + e_{\alpha\beta} \in (1 + \mathfrak{a}_n)^*$, and so $\omega_{1+e_{\alpha\beta}} \in \mathrm{Inn}(\mathbb{I}_n)$. Then $\omega_{1+e_{\alpha\beta}} = \sigma\omega_{1+e_{\alpha\beta}}\sigma^{-1} = \omega_{1+\sigma(e_{\alpha\beta})}$, and so $1 + e_{\alpha\beta} = 1 + \sigma(e_{\alpha\beta})$ (Theorem 3.1.(2)), i.e. $e_{\alpha\beta} = \sigma(e_{\alpha\beta})$. By Corollary 3.7, $\sigma = e$. □

Let H be a subgroup of a group G. The *centralizer* $\mathrm{Cen}_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}$ of H in G is a subgroup of G. In the proof of Theorem 5.15, we have used only inner derivations of the algebra $\mathbb{A}_n$. So, in fact, we have proved there the next corollary.

Corollary 5.16. $\mathrm{Cen}_{\mathrm{G}_n}(\mathrm{Inn}(\mathbb{I}_n)) = \{e\}$.

For an algebra A and a subgroup G of its group of algebra automorphisms, the set $A^G := \{a \in A \mid \sigma(a) = a$ for all $\sigma \in G\}$ is called the *fixed algebra* or the *algebra of invariants* for the group G.

Theorem 5.17. $\mathbb{I}_n^{\mathrm{G}_n} = \mathbb{I}_n^{\mathrm{Inn}(\mathbb{I}_n)} = K$.

Proof. Since $K \subseteq \mathbb{I}_n^{\mathrm{G}_n} \subseteq \mathbb{I}_n^{\mathrm{Inn}(\mathbb{I}_n)}$, it suffices to show that $\mathbb{I}_n^{\mathrm{Inn}(\mathbb{I}_n)} = K$. For all $\alpha, \beta \in \mathbb{N}^n$, $1 + e_{\alpha\beta} \in (1 + \mathfrak{a}_n)^*$. Then $\omega_{1+e_{\alpha\beta}} \in \mathrm{Inn}(\mathbb{I}_n)$. If $a \in \mathbb{I}_n^{\mathrm{Inn}(\mathbb{I}_n)}$ then $a = \omega_{1+e_{\alpha\beta}}(a)$, and so $ae_{\alpha\beta} = e_{\alpha\beta}a$. By Lemma 5.18.(1), $a \in K$. Therefore, $\mathbb{I}_n^{\mathrm{Inn}(\mathbb{I}_n)} = K$. $\square$

Lemma 5.18.

1. $\mathrm{Cen}_{\mathbb{I}_n}(\{e_{\alpha\beta} \mid \alpha, \beta \in \mathbb{N}^n\}) = K$.
2. $\mathrm{Cen}_{\mathbb{I}_n}(\mathfrak{a}) = K$ *for all nonzero ideals* $\mathfrak{a}$ *of the algebra* $\mathbb{I}_n$.

Proof. 1. Recall that $\mathbb{I}_n \simeq \mathbb{I}_n^{op}$ and $F_n = \bigoplus_{\alpha,\beta\in\mathbb{N}^n} Ke_{\alpha\beta}$ is the least nonzero ideal of the algebra $\mathbb{I}_n$. Hence, F_n is a simple $\mathbb{I}_n$-bimodule. Since

$$_{\mathbb{I}_n}F_{n\mathbb{I}_n} \simeq {}_{\mathbb{I}_n\otimes\mathbb{I}_n^{op}}F_n \simeq {}_{\mathbb{I}_n\otimes\mathbb{I}_n}F_n \simeq {}_{\mathbb{I}_{2n}}F_n \simeq {}_{\mathbb{I}_{2n}}P_{2n},$$

we have $K \simeq \mathrm{End}_{\mathbb{I}_{2n}}(P_{2n}) \simeq \mathrm{Cen}_{\mathbb{I}_n}(F_n) = \mathrm{Cen}_{\mathbb{I}_n}(\{e_{\alpha\beta} \mid \alpha, \beta \in \mathbb{N}^n\})$.

2. Since $F_n \subseteq \mathfrak{a}$, we have $K \subseteq \mathrm{Cen}_{\mathbb{I}_n}(\mathfrak{a}) \subseteq \mathrm{Cen}_{\mathbb{I}_n}(F_n) = K$, by statement 1. Therefore, $\mathrm{Cen}_{\mathbb{I}_n}(\mathfrak{a}) = K$. $\square$

Theorem 5.19. *No proper prime factor algebra of* $\mathbb{I}_n$ *can be embedded into* $\mathbb{I}_n$ (*that is, for each nonzero prime ideal* $\mathfrak{p}$ *of the algebra* $\mathbb{I}_n$*, there is no algebra monomorphism from* $\mathbb{I}_n/\mathfrak{p}$ *into* $\mathbb{I}_n$).

Proof. By Corollary 2.3.(7), $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_s}$. Without loss of generality, we may assume that $\mathfrak{p} = \mathfrak{p}_1 + \cdots + \mathfrak{p}_s$. Suppose that there is a monomorphism $f : \mathbb{I}_n/\mathfrak{p} \to \mathbb{I}_n$, we seek a contradiction. For each element $a \in \mathbb{I}_n$, let $\bar{a} := a + \mathfrak{p}$. Notice that, for $i = 1, \ldots, s$, $\bar{\partial}_i \bar{\int}_i = 1$ and $\bar{\int}_i \bar{\partial}_i = \overline{1 - e_{00}(i)} = 1$ since $e_{00}(i) \in \mathfrak{p}_i \subseteq \mathfrak{p}$. The elements $\{\bar{\partial}_i, \bar{\int}_i \mid i = 1, \ldots, s\}$ are units of the algebra $\mathbb{I}_n/\mathfrak{p}$, hence their images under the map f are units of the algebra $\mathbb{I}_n$, i.e. $f(\int_i), f(\partial_i) \in \mathbb{I}_n^* = K^* \times (1 + \mathfrak{a}_n)^*$ (Theorem 3.1.(1)). We see that the image of the *simple non-commutative* algebra $B_s := \mathbb{I}_s/\mathfrak{a}_s$ under the compositions of homomorphisms $B_s := \mathbb{I}_s/\mathfrak{a}_s \to \mathbb{I}_n/\mathfrak{p} \xrightarrow{f} \mathbb{I}_n \to B_n = \mathbb{I}_n/\mathfrak{a}_n$ is the subalgebra of B_n generated by the images of the *commutative* elements $H_1, \ldots, H_n$, a contradiction. $\square$

The next lemma shows that in the algebra $\mathbb{I}_n$ there are non-invertible elements that are invertible as elements of the algebra $\mathrm{End}_K(P_n)$.

Lemma 5.20. $\mathbb{I}_n^* \subsetneqq \mathbb{I}_n \cap \mathrm{Aut}_K(P_n)$.

Proof. The element $1 - \partial_i \in \mathrm{End}_K(K[x_i])$ is an invertible linear map since ∂_i is a locally nilpotent derivation of the polynomial algebra $K[x_i]$, hence $u := \prod_{i=1}^n (1 - \partial_i) \in \mathbb{I}_n \cap \mathrm{Aut}_K(P_n)$. But $u \notin \mathbb{I}_n^*$ since the element $u + \mathfrak{a}_n$ is not a unit of the factor algebra $B_n = \mathbb{I}_n/\mathfrak{a}_n$ as $B_n^* = \bigcup_{\alpha\in\mathbb{Z}^n} K^*\partial^\alpha$. Therefore, $\mathbb{I}_n^* \subsetneqq \mathbb{I}_n \cap \mathrm{Aut}_K(P_n)$. $\square$

5.8. The group $\mathcal{G}_n := \mathrm{Aut}_{K\text{-alg}}(\mathcal{I}_n)$

Definition. (See [5].) The *algebra* $\mathbb{S}_n$ *of one-sided inverses* of P_n is an algebra generated over a field K of arbitrary characteristic by $2n$ elements $x_1, \ldots, x_n, y_n, \ldots, y_n$ that satisfy the defining relations:

$$y_1x_1 = \dots = y_nx_n = 1, \qquad [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \quad \text{for all } i \neq j,$$

where $[a, b] := ab - ba$ is the algebra commutator of elements a and b. Let $G_n := \mathrm{Aut}_{K\text{-alg}}(\mathbb{S}_n)$.

By the very definition, the algebra $\mathbb{S}_n \simeq \mathbb{S}_1^{\otimes n}$ is obtained from the polynomial algebra P_n by adding commuting, left (but not two-sided) inverses of its canonical generators. The algebra $\mathbb{S}_1$ is a well-known primitive algebra [14, Example 2, p. 35]. Over the field $\mathbb{C}$ of complex numbers, the completion of the algebra $\mathbb{S}_1$ is the *Toeplitz algebra* which is the C^*-algebra generated by a unilateral shift on the Hilbert space $l^2(\mathbb{N})$ (note that $y_1 = x_1^*$). The Toeplitz algebra is the universal C^*-algebra generated by a proper isometry.

The algebra $\mathcal{I}_n := K\langle \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \rangle$ of scalar integro-differential operators is isomorphic to the algebra $\mathbb{S}_n$:

$$\mathbb{S}_n \to \mathcal{I}_n, \quad x_i \mapsto \int_i, \quad y_i \mapsto \partial_i, \quad i = 1, \dots, n. \tag{29}$$

Since $\mathcal{I}_n = \bigotimes_{i=1}^n \mathcal{I}_1(i)$ where $\mathcal{I}_1(i) := K\langle \partial_i, \int_i \rangle$ and $\mathbb{S}_n = \bigotimes_{i=1}^n \mathbb{S}_1(i)$ where $\mathbb{S}_1(i) := K\langle x_i, y_i \rangle$, it suffices to prove the statement for $n = 1$. For $n = 1$, the algebra epimorphism $\mathbb{S}_1 \to \mathbb{I}_1$ is an isomorphism since any proper epimorphic image of the algebra $\mathbb{S}_1$ is commutative (see [5]) but the algebra $\mathcal{I}_1$ is non-commutative. The algebra $\mathbb{S}_n$ was studied in detail in [5], its group of automorphism and explicit generators were found in the papers [6–8].

Theorem 5.21.

1. $\mathcal{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathcal{I}_n)$ *and* $\mathrm{Inn}(\mathcal{I}_n) = \{\omega_u \mid u \in (1 + \mathfrak{a}_n')^*\} \simeq (1 + \mathfrak{a}_n')^*$, $\omega_u \mapsto u$, *where* $\mathfrak{a}_n' := \sum_{i=1}^n \mathcal{I}_n F(i)$ *is the only maximal ideal of the algebra* $\mathcal{I}_n$.
2. $\mathcal{G}_n = \{\sigma \in \mathrm{G}_n \mid \sigma(\mathcal{I}_n) = \mathcal{I}_n\} = \{\sigma \in \mathbb{G}_n \mid \sigma(\mathcal{I}_n) = \mathcal{I}_n\}$ *and* $\mathcal{G}_n$ *is a subgroup of the groups* G_n *and* $\mathbb{G}_n$.
3. *Each automorphism of the algebra* $\mathcal{I}_n$ *has a unique extension to an automorphism of the algebra* $\mathbb{I}_n$ *and* $\mathbb{A}_n$.
4. $\mathrm{G}_n \supseteq \mathcal{G}_n \supseteq S_n \ltimes \mathbb{T}^n \ltimes \underbrace{\mathrm{GL}_\infty(K) \ltimes \dots \ltimes \mathrm{GL}_\infty(K)}_{2^n - 1 \text{ times}}$.

Proof. 1. $\mathcal{I}_n \simeq \mathbb{S}_n$, $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$ and $\mathrm{Inn}(\mathbb{S}_n) = \{\omega_u \mid u \in (1 + \mathfrak{a}_n'')^*\} \simeq (1 + \mathfrak{a}_n'')^*$, $\omega_u \mapsto u$, where $\mathfrak{a}_n''$ is the only maximal ideal of the algebra $\mathbb{S}_n$ [6].

2. Statement 2 follows from statement 1, Theorem 5.5, Theorem 5.4 and Corollary 3.3.

3. Statement 3 follows from Theorem 3.6.

4. $G_n \supseteq S_n \ltimes \mathbb{T}^n \ltimes \underbrace{\mathrm{GL}_\infty(K) \ltimes \dots \ltimes \mathrm{GL}_\infty(K)}_{2^n - 1 \text{ times}}$ [6]. □

6. Stabilizers of the ideals of $\mathbb{I}_n$ in G_n

In this section, for each nonzero ideal $\mathfrak{a}$ of the algebra $\mathbb{I}_n$ its stabilizer $\mathrm{St}_{G_n}(\mathfrak{a}) := \{\sigma \in \mathrm{G}_n \mid \sigma(\mathfrak{a}) = \mathfrak{a}\}$ is found (Theorem 6.2) and it is shown that the stabilizer $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{a})$ has finite index in the group G_n (Corollary 6.3). When the ideal $\mathfrak{a}$ is either prime or generic, this result can be refined even further (Corollary 6.4, Corollary 6.5). In particular, when $n > 1$ the stabilizer of each height 1 prime of $\mathbb{I}_n$ is a maximal subgroup of G_n of index n (Corollary 6.4.(1)). It is shown that the ideal $\mathfrak{a}_n$ is the only nonzero, prime, G_n-invariant ideal of the algebra $\mathbb{I}_n$ (Corollary 6.4.(3)).

An ideal $\mathfrak{a}$ of $\mathbb{I}_n$ is called a *proper* ideal if $\mathfrak{a} \neq 0, \mathbb{I}_n$. For an ideal $\mathfrak{a}$ of the algebra $\mathbb{I}_n$, $\mathrm{Min}(\mathfrak{a})$ denotes the set of all the minimal primes over $\mathfrak{a}$. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are called *incomparable* if neither $\mathfrak{a} \subseteq \mathfrak{b}$ nor $\mathfrak{b} \subseteq \mathfrak{a}$. The ideals of the algebra $\mathbb{I}_n$ are classified in [10]. The next theorem shows that each ideal of the algebra $\mathbb{I}_n$ is completely determined by its minimal primes. We use this theorem in the proof of Theorem 6.2.

Theorem 6.1. *(See [10, Corollary 3.4].) Let $\mathfrak{a}$ be a proper ideal of the algebra $\mathbb{I}_n$. Then $\mathrm{Min}(\mathfrak{a})$ is a finite non-empty set, and the ideal $\mathfrak{a}$ is a unique product and a unique intersection of incomparable prime ideals of $\mathbb{I}_n$ (uniqueness is up to permutation). Moreover,*

$$a = \prod_{\mathfrak{p}\in\mathrm{Min}(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p}\in\mathrm{Min}(\mathfrak{a})} \mathfrak{p}.$$

Let Sub_n be the set of all the subsets of the set $\{1,\dots,n\}$. Sub_n is a partially ordered set with respect to '$\subseteq$'. Let SSub_n be the set of all the subsets of Sub_n. An element $\{X_1,\dots,X_s\}$ of SSub_n is called an *antichain* if for all $i \neq j$ such that $1 \leqslant i, j \leqslant s$ neither $X_i \subseteq X_j$ nor $X_i \supseteq X_j$. An empty set and one element set are antichains by definition. Let Inc_n be the subset of SSub_n that contains all the antichains of SSub_n. The number $\mathfrak{d}_n := |\mathrm{Inc}_n|$ is called the *Dedekind* number. The symmetric group S_n acts in the obvious way on the sets SSub_n and Inc_n ($\sigma \cdot \{X_1,\dots,X_s\} = \{\sigma(X_1),\dots,\sigma(X_s)\}$).

Theorem 6.2. *Let $\mathfrak{a}$ be a proper ideal of the algebra $\mathbb{I}_n$. Then*

$$\mathrm{St}_{\mathrm{G}_n}(\mathfrak{a}) = \mathrm{St}_{S_n}\big(\mathrm{Min}(\mathfrak{a})\big) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$$

where $\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a})) := \{\sigma \in S_n \mid \sigma(\mathfrak{q}) \in \mathrm{Min}(\mathfrak{a})$ for all $\mathfrak{q} \in \mathrm{Min}(\mathfrak{a})\}$. Moreover, if $\mathrm{Min}(\mathfrak{a}) = \{\mathfrak{q}_1,\dots,\mathfrak{q}_s\}$ and, for each number $t = 1,\dots,s$, $\mathfrak{q}_t = \sum_{i\in I_t} \mathfrak{p}_i$ for some subset I_t of $\{1,\dots,n\}$ then the group $\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))$ is the stabilizer in the group S_n of the element $\{I_1,\dots,I_s\}$ of SSub_n.

Remark. Note that the group

$$\mathrm{St}_{\mathrm{G}_n}\big(\mathrm{Min}(\mathfrak{a})\big) = \mathrm{St}_{S_n}\big(\{I_1,\dots,I_s\}\big) := \big\{\sigma \in S_n \bigm| \big\{\sigma(I_1),\dots,\sigma(I_s)\big\} = \{I_1,\dots,I_s\}\big\}$$

(and also the group $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{a})$) can be effectively computed in finitely many steps.

Proof of Theorem 6.2. Recall that each nonzero prime ideal of the algebra $\mathbb{I}_n$ is a unique sum of height one prime ideals of the algebra $\mathbb{I}_n$. By Theorem 6.1 and Corollary 5.8, $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{a}) \supseteq \mathrm{St}_{\mathrm{G}_n}(\mathcal{H}_1) = \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$. Since $\mathrm{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ (Theorem 5.5.(1)),

$$\mathrm{St}_{\mathrm{G}_n}(\mathfrak{a}) = \big(\mathrm{St}_{\mathrm{G}_n}(\mathfrak{a}) \cap S_n\big) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n) = \mathrm{St}_{S_n}(\mathfrak{a}) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n).$$

By Theorem 6.1, $\mathrm{St}_{S_n}(\mathfrak{a}) = \mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a})) = \mathrm{St}_{S_n}(\{I_1,\dots,I_s\})$, and the statement follows. □

The *index* of a subgroup H in a group G is denoted by $[G : H]$.

Corollary 6.3. *Let $\mathfrak{a}$ be a proper ideal of $\mathbb{I}_n$. Then $[\mathrm{G}_n : \mathrm{St}_{\mathrm{G}_n}(\mathfrak{a})] = |S_n : \mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))| < \infty$.*

Proof. This follows from Theorem 5.5.(1) and Theorem 6.2. □

Corollary 6.4.

1. $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}_i) \simeq S_{n-1} \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$, *for $i = 1,\dots,n$. Moreover, if $n > 1$ then the groups $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}_i)$ are maximal subgroups of G_n with $[\mathrm{G}_n : \mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}_i)] = n$ (if $n = 1$ then $\mathrm{St}_{\mathrm{G}_1}(\mathfrak{p}_1) = \mathrm{G}_1$, see statement 3).*
2. *Let $\mathfrak{p}$ be a nonzero prime ideal of the algebra $\mathbb{I}_n$ and $h = \mathrm{ht}(\mathfrak{p})$ be its height. Then $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}) \simeq (S_h \times S_{n-h}) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$.*
3. *The ideal $\mathfrak{a}_n$ is the only nonzero, prime, G_n-invariant ideal of the algebra $\mathbb{I}_n$.*
4. *Suppose that $n > 1$. Let $\mathfrak{p}$ be a nonzero prime ideal of the algebra $\mathbb{I}_n$. Then its stabilizer $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{p})$ is a maximal subgroup of G_n iff the ideal $\mathfrak{p}$ is of height one.*

Proof. 1. Clearly, $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}_i) \cap S_n = \{\tau \in S_n \mid \tau(\mathfrak{p}_i) = \mathfrak{p}_i\} \simeq S_{n-1}$. By Theorem 6.2, $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}_i) = S_{n-1} \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$. When $n > 1$, the group $\mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}_i)$ is a maximal subgroup of G_n since

$$S_{n-1} \simeq \mathrm{St}_{G_n}(\mathfrak{p}_i)/\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n) \subseteq G_n/\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n) \simeq S_n$$

and $S_{n-1} = \{\sigma \in S_n \mid \sigma(i) = i\}$ is a maximal subgroup of S_n. Clearly, $[\mathrm{G}_n : \mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}_i)] = [S_n : S_{n-1}] = n$.

2. By Corollary 2.3.(9), $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_h}$ for some distinct indices $i_1, \ldots, i_h \in \{1, \ldots, n\}$. Let $I = \{i_1, \ldots, i_h\}$ and CI be its complement in the set $\{1, \ldots, n\}$. Statement 2 follows from Theorem 6.2 and the fact that

$$\mathrm{St}_{\mathrm{G}_n}(\mathfrak{p}) \cap S_n = \{\sigma \in S_n \mid \sigma(I) = I,\ \sigma(CI) = CI\} \simeq S_h \times S_{n-h}.$$

3. Since $\mathfrak{a}_n = \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$, statement 3 follows from statement 2.

4. Statement 4 follows from statements 1 and 2. □

Next, we find the stabilizers of the generic ideals (see Corollary 6.5). First, we recall the definition of the *wreath product* $A \wr B$ of finite groups A and B. The set $\mathrm{Fun}(B, A)$ of all functions $f : B \to A$ is a group: $(fg)(b) := f(b)g(b)$ for all $b \in B$ where $g \in \mathrm{Fun}(B, A)$. There is a group homomorphism

$$B \to \mathrm{Aut}(\mathrm{Fun}(B, A)), \quad b_1 \mapsto (f \mapsto b_1(f) : b \mapsto f(b_1^{-1}b)).$$

Then the semidirect product $\mathrm{Fun}(B, A) \rtimes B$ is called the *wreath product* of the groups A and B denoted by $A \wr B$, and so the product in $A \wr B$ is given by the rule:

$$f_1 b_1 \cdot f_2 b_2 = f_1 b_1(f_2) b_1 b_2, \quad \text{where } f_1, f_2 \in \mathrm{Fun}(B, A),\ b_1, b_2 \in B.$$

Recall that each nonzero prime ideal $\mathfrak{p}$ of the algebra $\mathbb{I}_n$ is a unique sum $\mathfrak{p} = \sum_{i \in I} \mathfrak{p}_i$ of height one prime ideals. The set $\mathrm{Supp}(\mathfrak{p}) := \{\mathfrak{p}_i \mid i \in I\}$ is called the *support* of $\mathfrak{p}$.

Definition. We say that a proper ideal $\mathfrak{a}$ of $\mathbb{I}_n$ is *generic* if $\mathrm{Supp}(\mathfrak{p}) \cap \mathrm{Supp}(\mathfrak{q}) = \emptyset$ for all $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}(\mathfrak{a})$ such that $\mathfrak{p} \neq \mathfrak{q}$.

Corollary 6.5. *Let $\mathfrak{a}$ be a generic ideal of the algebra $\mathbb{I}_n$. The set $\mathrm{Min}(\mathfrak{a})$ of minimal primes over $\mathfrak{a}$ is the disjoint union of its non-empty subsets, $\mathrm{Min}_{h_1}(\mathfrak{a}) \cup \cdots \cup \mathrm{Min}_{h_t}(\mathfrak{a})$, where $1 \leqslant h_1 < \cdots < h_t \leqslant n$ and the set $\mathrm{Min}_{h_i}(\mathfrak{a})$ contains all the minimal primes over $\mathfrak{a}$ of height h_i. Let $n_i := |\mathrm{Min}_{h_i}(\mathfrak{a})|$. Then $\mathrm{St}_{G_n}(\mathfrak{a}) = (S_m \times \prod_{i=1}^t (S_{h_i} \wr S_{n_i})) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ where $m = n - \sum_{i=1}^t n_i h_i$.*

Proof. Suppose that $\mathrm{Min}(\mathfrak{a}) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$ and the sets $I_1, \ldots, I_s$ are defined in Theorem 6.2. Since the ideal $\mathfrak{a}$ is generic, the sets $I_1, \ldots, I_s$ are disjoint. By Theorem 6.2, we have to show that

$$\mathrm{St}_{S_m}(\{I_1, \ldots, I_s\}) \simeq S_m \times \prod_{i=1}^t (S_{h_i} \wr S_{n_i}). \tag{30}$$

The ideal $\mathfrak{a}$ is generic, and so the set $\{1, \ldots, n\}$ is the disjoint union $\bigcup_{i=0}^t M_i$ of its subsets where $M_i := \bigcup_{|I_j| = h_i} I_j$, $i = 1, \ldots, t$, and M_0 is the complement of the set $\bigcup_{i=1}^t M_i$. Let $S(M_i)$ be the symmetric group corresponding to the set M_i (i.e. the set of all bijections $M_i \to M_i$). Then each element $\sigma \in \mathrm{St}_{G_n}(\{I_1, \ldots, I_s\})$ is a unique product $\sigma = \sigma_0 \sigma_1 \cdots \sigma_t$ where $\sigma_i \in S(M_i)$. Moreover, σ_0 can be an arbitrary element of $S(M_0) \simeq S_m$, and, for $i \neq 0$, the element σ_i permutes the sets $\{I_j \mid |I_j| = h_i\}$ and simultaneously permutes the elements inside each of the sets I_j, i.e. $\sigma_i \in S_{h_i} \wr S_{n_i}$. Now, (30) is obvious. □

Corollary 6.6. *For each number $s = 1, \dots, n$, let $\mathfrak{b}_s := \prod_{|I|=s}(\sum_{i\in I} \mathfrak{p}_i)$ where I runs through all the subsets of the set $\{1, \dots, n\}$ that contain exactly s elements. The ideals $\mathfrak{b}_s$ are the only proper, G_n-invariant ideals of the algebra $\mathbb{I}_n$, and so there are precisely $n+2$ G_n-invariant ideals of the algebra $\mathbb{I}_n$.*

Proof. By Theorem 6.2, the ideals $\mathfrak{b}_s$ are G_n-invariant, and they are proper. The converse follows at once from the classification of ideals for the algebra $\mathbb{I}_n$ (Theorem 6.1) and Theorem 6.2. The ideals $\mathfrak{b}_s$ are distinct, by Theorem 6.1, and so there are precisely $n+2$ G_n-invariant ideals of the algebra $\mathbb{I}_n$. □

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