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Exponential Dichotomy and Trichotomy for Difference Equations

A. I. ALONSO†
Departamento de Matemática Aplicada a la Ingeniería
Universidad de Valladolid, 47011 Valladolid, Spain

JIALIN HONG‡
Institute of Computational Mathematics and Scientific/Engineering Computing
Chinese Academy of Sciences, Beijing 100080, P. R. China

R. OBAYA†
Departamento de Matemática Aplicada a la Ingeniería
Universidad de Valladolid, 47011 Valladolid, Spain

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Abstract—In this paper, a roughness theorem of exponential dichotomy and trichotomy of linear difference equations is proved. It is also shown that if an almost periodic difference equation has an exponential dichotomy on a sufficiently long finite interval, then it has one on $(-\infty, +\infty)$.

Keywords—Exponential dichotomy, Exponential trichotomy, Roughness, Almost periodic sequences.

1. INTRODUCTION

The theory of difference equations has received much attention because of its importance in various fields, such as numerical methods of differential equations and dynamical systems, finite elements techniques, control theory, and computer sciences (see [1–8]). The almost periodic type difference equations have been discussed in [5,9]. In this paper, we focus our attention on some properties of exponential dichotomy and trichotomy of linear difference equations. In the rest of this section, some fundamental concepts of this paper are given. The invariance of the exponential dichotomy and trichotomy under some perturbations, which is called roughness, is discussed in Section 2. The equivalence between the exponential dichotomy for linear difference equations with almost periodic coefficients in an infinite integer's interval and in a finite sufficiently long integer's interval is proved in Section 3. In what follows, we denote by $|\cdot|$ the Euclidean norm when the argument is a vector and the corresponding operator norm when the argument is a

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matrix. Let $l^\infty(J) := \{x : J \to R^d : x \text{ is bounded on } J \in \{Z, Z^+, Z^-\}\}$, where Z denotes the set of integer numbers, $Z^+ = \{n \in Z : n \geq 0\}$ and $Z^- = \{n \in Z : n \leq 0\}$. The set $l^\infty(J)$ is a Banach space endowed with the norm $\|x\| := \sup_{n\in J} |x(n)|$. Note that an element $x \in l^\infty(J)$ can be identified with the sequence $\{x(n)\}_{n\in J}$. Given a sequence $\{x(n)\}_{n\in J}$ and $p \in Z^+$, we define $\overline{x(n+1), x(n+p)} := \{x(n+1), \dots, x(n+p)\}$, called sequence interval with length p, and integer's interval with length p if $x(m) = m \in J$ for $n+1 \leq m \leq n+p$.

DEFINITION 1.1. *(See [5,9].) A sequence $x \in l^\infty(Z)$ is called almost periodic if for each $\varepsilon > 0$, the set*

$$T(x,\varepsilon) := \{\tau \in Z : |x(n+\tau) - x(n)| < \varepsilon \text{ for every } n \in Z\}$$

is relatively dense in Z, that is, there exists a positive integer $l(\varepsilon)$, which is called the length of contain interval, such that there is a $\tau \in T(x,\varepsilon)$ in every integer's interval with length $l(\varepsilon)$. A number $\tau \in T(x,\varepsilon)$ is called an ε-translation number of x. We denote by $\mathcal{AP}(Z)$ the set of almost periodic sequences. A matrix sequence $\{A(n)\}_{n\in Z}$ is called almost periodic if each one of its entries forms an almost periodic sequence.

REMARK 1.1. *The sequence $\{x(n)\}_{n\in Z}$ is almost periodic if and only if any sequence $\{k_i'\}_{i\in Z^+}$ of integers admits a subsequence $\{k_i\}_{i\in Z^+}$ such that $x(n+k_i)$ converges uniformly on $n \in Z$ as $i \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence (see Proposition 6 in [9, p. 1442]). Recall that almost periodic sequences and functions are closely related. In fact, a sequence $\{x(n)\}_{n\in Z}$ is almost periodic if and only if $x(n) = f(n)$ for every $n \in Z$, where f is an almost periodic function on R. If $x, y \in \mathcal{AP}(Z)$, then for each $\varepsilon > 0$, $T(x,\varepsilon) \cap T(y,\varepsilon) \neq \emptyset$ (see [5,9]).*

Now we consider the difference equation

$$x(n+1) = A(n)x(n), \qquad n \in Z, \tag{1.1}$$

where $A(n)$ is a $d \times d$ invertible matrix for each $n \in Z$. Let $X(t)$ be a fundamental matrix of (1.1) with $X(0) = I$. Then the transition matrix $\Phi(n,m) = X(n)X^{-1}(m)$ has the cocycle property [6, p. 267].

DEFINITION 1.2.

(1) *Equation (1.1) is said to have an exponential dichotomy on Z (respectively, Z^+, Z^-) if there exists a projection P ($P^2 = P$) and constants $K > 0$, $\alpha > 0$ such that for $m, n \in Z$ (respectively, Z^+, Z^-),*

$$\begin{aligned} \left|X(n)PX^{-1}(m)\right| &\leq Ke^{-\alpha(n-m)}, && n \geq m, \quad \text{and} \\ \left|X(n)(I-P)X^{-1}(m)\right| &\leq Ke^{-\alpha(m-n)}, && m \geq n, \end{aligned} \tag{1.2}$$

(see [1–3,5–12]).

(2) *Equation (1.1) is said to have an exponential trichotomy on Z if there exist projections P_1, P_2, P_3 with $P_iP_j = 0$ if $i \neq j$ and $P_1 + P_2 + P_3 = I$ (identity matrix), and constants $K > 0$, $\alpha > 0$ such that, for $m, n \in Z$,*

$$\begin{aligned} \left|X(n)P_1X^{-1}(m)\right| &\leq Ke^{-\alpha(n-m)}, && n \geq m, \\ \left|X(n)P_2X^{-1}(m)\right| &\leq Ke^{-\alpha(m-n)}, && m \geq n, \quad \text{and} \\ \left|X(n)P_3X^{-1}(m)\right| &\leq \begin{cases} Ke^{-\alpha(n-m)}, & 0 \leq m \leq n, \\ Ke^{-\alpha(m-n)}, & n \leq m \leq 0, \end{cases} \end{aligned} \tag{1.3}$$

(see [7]).

Obviously, equation (1.1) has an exponential trichotomy with $P_3 = 0$ if and only if it has an exponential dichotomy on Z. And in Section 3, it will be pointed out that the exponential trichotomy on Z is equivalent to the exponential dichotomy on Z if $\{A(n)\}_{n\in Z}$ is almost periodic (also see [5]).

To end this section, we point out that (1) in Definition 1.2 is equivalent to the definition of exponential dichotomy given in [6, p. 267].

REMARK 1.2.

(1) Equation (1.1) has an exponential dichotomy on $J \in \{Z, Z^+, Z^-\}$ with projection P and constants $K > 0$, $\alpha > 0$ if and only if there are positive constants K', α', and a family of projections $P(n)$, $n \in J$, such that

(i)
$$P(n+1)A(n) = A(n)P(n), \qquad \text{for all } n(n < 0 \text{ if } J = Z^-),$$

(ii)
$$\begin{aligned} \left|X(n)X^{-1}(m)P(m)\right| &\le K'e^{-\alpha'(n-m)}, && \text{for } n \ge m, \quad n,m \in J, \\ \left|X(n)X^{1}(m)(I - P(m))\right| &\le K'e^{-\alpha'(m-n)}, && \text{for } m \ge n, \quad n,m \in J. \end{aligned}$$

In fact, let $J = Z^+$ and equation (1.1) have an exponential dichotomy on J with projection P and constants $K > 0$, $\alpha > 0$. Taking $K' = K$, $\alpha' = \alpha$, $P(n) = X(n)PX^{-1}(n)$, then it follows immediately that (i) and (ii) are satisfied. Now let there be positive constants K', α', and a family of projections $P(n)$, $n \in J$ such that (i) and (ii) are satisfied. We take $K = K'$, $\alpha = \alpha'$, $P = P(0)$, then it follows from (i) that

$$\begin{aligned} P(m) &= A(m-1)P(m-1)A^{-1}(m-1) \\ &= A(m-1)A(m-2)\dots A(0)P(0)A^{-1}(0)\dots A^{-1}(m-2)A^{-1}(m-1) \\ &= X(m)P(0)X^{-1}(m) = X(m)PX^{-1}(m). \end{aligned}$$

Therefore,

$$\begin{aligned} \left|X(n)PX^{-1}(m)\right| &= \left|X(n)X^{-1}(m)P(m)\right| \le Ke^{-\alpha(n-m)}, && \text{for } n \ge m, \quad n,m \in J, \\ \left|X(n)(I-P)X^{-1}(m)\right| &= \left|X(n)X^{-1}(m)(I - P(m))\right| \le Ke^{-\alpha(m-n)}, && \text{for } m \ge n, \quad n,m \in J. \end{aligned}$$

The case of $J = Z$ or $J = Z^-$ can be verified in the same manner.

(2) Equation (1.1) has an exponential trichotomy on Z with projections P_i $(i = 1, 2, 3)$ and constants K, α if and only if there exist positive constants K', α', and three families of projections $P_1(n)$, $P_2(n)$, $P_3(n)$, $n \in Z$, such that

(i)
$$\text{for each } n \in Z,\ P_i(n)P_j(n) = 0, \quad \text{if } i \ne j \qquad \text{and} \qquad P_1(n) + P_2(n) + P_3(n) = I;$$

(ii)
$$P_i(n+1)A(n) = A(n)P_i(n), \quad \text{for all } n \in Z, \quad i = 1,2,3;$$

(iii)
$$\begin{aligned} |X(n)X^{-1}(m)P_1(m)| &\le K'e^{-\alpha'(n-m)}, && n \ge m, \\ |X(n)X^{-1}(m)P_2(m)| &\le K'e^{-\alpha'(m-n)}, && m \ge n, \\ |X(n)X^{-1}(m)P_3(m)| &\le \begin{cases} K'e^{-\alpha'(n-m)}, & 0 \le m \le n, \\ K'e^{-\alpha'(m-n)}, & n \le m \le 0. \end{cases} \end{aligned}$$

2. ROUGHNESS

In this section, we give a result on the roughness of exponential dichotomy and trichotomy of linear difference equations. We improve some known results on the invariance of exponential dichotomy and trichotomy under some perturbations. First of all, we summarize some fundamental results on this topic.

PROPOSITION 2.1. *Suppose that equation (1.1) has an exponential trichotomy on Z with projections P_i $(i = 1,2,3)$ and constants K, α. Let $B(n)$ be a $d \times d$ matrix function defined on Z such that $A(n) + B(n)$ is an invertible matrix function for every $n \in Z$. Then if*

$$\|B\| = \sup\{|B(n)|,\, n \in Z\} = \delta < \frac{e^{\alpha} - 1}{9K^2 e^{\alpha}},$$

we have that the perturbed equation

$$y(n+1) = (A(n) + B(n))y(n) \tag{2.1}$$

has an exponential trichotomy on Z with projections $\bar{P}_i(B)$ $(i = 1,2,3)$. Moreover if δ is sufficiently small we have $\operatorname{rank} \bar{P}_i(B) = \operatorname{rank} P_i$ $(i = 1,2,3)$ (see [7, p. 99]).

PROPOSITION 2.2. *Let $A(n)$ be a $d \times d$ invertible matrix defined for $n \in J \in \{Z, Z^+, Z^-\}$ such that for all n, $|A^{-1}(n)| \le M$ and such that equation (1.1) has an exponential dichotomy on J with constants K, α and projections $P(n)$ (see [6]). Suppose $0 < \delta < \alpha$ and $B(n)$ is a $d \times d$ matrix function defined for $n \in J$ and satisfying*

$$\begin{aligned} |B(n)| &\le M^{-1}, \\ 2K\left(1 + e^{-\alpha}\right)\left(1 - e^{-\alpha}\right)^{-1} |B(n)| &\le 1, \\ 2Ke^{\alpha}\left(e^{-\delta} + 1\right)\left(e^{\delta} - 1\right)^{-1} |B(n)| &\le 1. \end{aligned}$$

Then $A(n) + B(n)$ is invertible for all $n \in J$ and equation (2.1) has an exponential dichotomy on J with constants $2K\left(1 + e^{\delta}\right)\left(1 - e^{-\delta}\right)^{-1}$, $\alpha - \delta$ and projections of the same rank as for equation (1.1) (See [6, p. 276]).

PROPOSITION 2.3. *Equation (1.1) has an exponential trichotomy (respectively, dichotomy) on Z with projections P_i $(i = 1,2,3)$ (respectively, with projection P) if and only if the inhomogeneous equation*

$$y(n+1) = A(n)y(n) + f(n) \tag{2.2}$$

has at least one (respectively, has a unique) bounded solution $y(n)$ on Z for every bounded function $f(n)$, $n \in Z$. In fact, the bounded solution can be given by

$$y(n) = \sum_{m \in Z} G(n, m+1) f(m), \qquad n \in Z,$$

where

$$G(n,m) = \begin{cases} X(n)P_{-}X^{-1}(m), & \text{if } m \le 0 \le n, \quad m \le n \le 0, \\ -X(n)(I - P_{-})X^{-1}(m), & \text{if } n < m \le 0, \\ X(n)P_{+}X^{-1}(m), & \text{if } 0 < m \le n, \\ -X(n)(I - P_{+})X^{-1}(m), & \text{if } 0 \le n < m, \quad n \le 0 < m, \end{cases} \tag{2.3}$$

$$\text{respectively,} \quad G(n,m) = \begin{cases} X(n)PX^{-1}(m), & \text{if } m \le n, \\ -X(n)(I - P)X^{-1}(m), & \text{if } m > n, \end{cases}$$

with $P_+ = I - P_2$ and $P_- = P_1$, and

$$\|y\| \le 2K \frac{e^{\alpha} + 1}{e^{\alpha} - 1} \|f\| \left(\text{respectively, } \|y\| \le \frac{K\left(e^{\alpha} + 1\right)}{e^{\alpha} - 1} \|f\|\right). \tag{2.4}$$

REMARK 2.1. *The proof of Proposition 2.3 can be found in [6, pp. 272–273; 7, pp. 97–99; 11]. We only need to note that, in the case of the trichotomy,*

$$|G(n,m)| \le 2Ke^{-\alpha|n-m|}.$$

In fact,

$$\begin{aligned}
&\text{if } m \le 0 \le n \text{ or } m \le n \le 0,\\
&\text{then } |G(n,m)| = \left|X(n)P_- X^{-1}(m)\right| = \left|X(n)P_1 X^{-1}(m)\right| \le Ke^{-\alpha|n-m|};\\[1ex]
&\text{if } n < m \le 0,\\
&\text{then } |G(n,m)| = \left|X(n)(I-P_-)X^{-1}(m)\right| = \left|X(n)(P_2+P_3)X^{-1}(m)\right|\\
&\qquad \le \left|X(n)P_2X^{-1}(m)\right| + \left|X(n)P_3X^{-1}(m)\right| \le 2Ke^{-\alpha|n-m|};\\[1ex]
&\text{if } 0 < m \le n,\\
&\text{then } |G(n,m)| = \left|X(n)P_+X^{-1}(m)\right| = \left|X(n)\left(P_1+P_3\right)X^{-1}(m)\right|\\
&\qquad \le \left|X(n)P_1X^{-1}(m)\right| + \left|X(n)P_3X^{-1}(m)\right| \le 2Ke^{-\alpha|n-m|};\\[1ex]
&\text{if } 0 \le n < m \text{ or } n \le 0 < m,\\
&\text{then } |G(n,m)| = \left|X(n)(I-P_+)X^{-1}(m)\right| = \left|X(n)P_2X^{-1}(m)\right| \le Ke^{-\alpha|n-m|}.
\end{aligned}$$

THEOREM 2.1. *Let $A(n)$ be a $d \times d$ invertible matrix function defined for $n \in Z$ such that the linear difference equation (1.1) has an exponential trichotomy (respectively, dichotomy) on Z with constants $K > 1, \alpha > 0$ and projections P_i $(i = 1, 2, 3)$ (respectively, P). Suppose $B(n)$ is a $d \times d$ matrix function defined for $n \in Z$ and satisfying*

(1) *$A(n) + B(n)$ is invertible for all n;*
(2) *$L_B = \sup_{n\in Z}\sum_{m\in Z} e^{-\alpha|n-m-1|}|B(m)| < 1/2K$ (respectively, $< 1/K$).*

Then the perturbed difference equation (2.1) has an exponential trichotomy (respectively, dichotomy) on Z with projections $\tilde{P}_i$ $(i = 1, 2, 3)$ (respectively, $\tilde{P}$). Moreover, the projections in the exponential trichotomy (respectively, dichotomy) of the difference equation (2.1) have the same rank as for equation (1.1) if conditions (1) and (2) are replaced by

$$\begin{aligned}
&(1)' \quad \|A^{-1}\| \le M, \text{ for some constant } M > 0,\\
&(2)' \quad \|B\| \le \min\left\{M^{-1}, \frac{e^\alpha - 1}{2K\left(e^\alpha+1\right)} - \varepsilon\right\}\\
&\qquad \left(\text{respectively, } \|B\| \le \min\left\{M^{-1}, \frac{e^\alpha-1}{K(e^\alpha+1)} - \varepsilon\right\}\right),
\end{aligned} \tag{2.5}$$

for every sufficiently small $\varepsilon > 0$.

PROOF. In the case of the trichotomy, it follows from Proposition 2.3 that for every $f \in l^\infty(Z)$, the difference equation

$$x_0(n+1) = A(n)x_0(n) + f(n) \tag{2.6}$$

has a bounded solution $x_0(n)$ with $\|x_0\| \le \left(2K\left(e^\alpha+1\right)\right)\left(e^\alpha-1\right)\|f\|$. Consider the difference equation

$$x_1(n+1) = A(n)x_1(n) + B(n)x_0(n). \tag{2.7}$$

By Proposition 2.3, equation (2.7) has a solution

$$x_1(n) = \sum_{m\in Z} G(n, m+1)B(m)x_0(m), \qquad n \in Z, \tag{2.8}$$

with

$$\begin{aligned}|x_1(n)| &\leq \sum_{m\in Z} |G(n,m+1)||B(m)||x_0(m)| \\ &\leq 2K \sum_{m\in Z} e^{-\alpha|n-m-1|}|B(m)| \frac{2K\left(e^\alpha+1\right)}{e^\alpha-1}\|f\| \\ &\leq \frac{2K\left(e^\alpha+1\right)}{e^\alpha-1}\|f\|(2KL_B), \qquad n\in Z,\end{aligned}$$

where $G(n,m)$ is the same as in (2.3). We define inductively difference equations as

$$x_i(n+1) = A(n)x_i(n) + B(n)x_{i-1}(n), \qquad i=1,2,\dots. \tag{2.9}$$

Assume that $x_{i-1}(n)$ satisfies

$$|x_{i-1}(n)| \leq \frac{2K\left(e^\alpha+1\right)}{e^\alpha-1}\|f\|\left(2KL_B\right)^{i-1}, \qquad n\in Z, \tag{2.10}$$

then equation (2.9) has a solution

$$x_i(n) = \sum_{m\in Z} G(n,m+1)B(m)x_{i-1}(m), \qquad n\in Z, \tag{2.11}$$

with

$$\begin{aligned}|x_i(n)| &\leq \sum_{m\in Z} |G(n,m+1)||B(m)||x_{i-1}(m)| \\ &\leq 2K \sum_{m\in Z} e^{-\alpha|n-m-1|}|B(m)|\,\|x_{i-1}\| \\ &\leq \frac{2K\left(e^\alpha+1\right)}{e^\alpha-1}\|f\|\left(2KL_B\right)^i.\end{aligned}$$

Now let

$$x(n) = x_0(n) + \sum_{i=1}^{+\infty} x_i(n), \qquad n\in Z. \tag{2.12}$$

Then

$$|x(n)| \leq \frac{2K\left(e^\alpha+1\right)}{e^\alpha-1}\|f\|\sum_{i=0}^{+\infty}\left(2KL_B\right)^i = \frac{2K\left(e^\alpha+1\right)\|f\|}{\left(e^\alpha-1\right)\left(1-2KL_B\right)}, \qquad n\in Z. \tag{2.13}$$

And

$$\begin{aligned}x(n+1) &= x_0(n+1) + \sum_{i=1}^{+\infty} x_i(n+1) \\ &= A(n)x_0(n) + f(n) + A(n)\sum_{i=1}^{+\infty} x_i(n) + B(n)\sum_{i=1}^{+\infty} x_{i-1}(n) \\ &= A(n)\left(x_0(n) + \sum_{i=1}^{+\infty} x_i(n)\right) + B(n)\left(x_0(n) + \sum_{i=1}^{+\infty} x_i(n)\right) + f(n) \\ &= \left(A(n)+B(n)\right)x(n) + f(n),\end{aligned}$$

that is, $x(n)$ is a bounded solution of the difference equation

$$x(n+1) = \left(A(n)+B(n)\right)x(n) + f(n) \tag{2.14}$$

on Z. Therefore, equation (2.1) has an exponential trichotomy on Z.

Now assume that (1)′ and (2)′ hold. We prove that

$$\operatorname{rank}\bar{P}_i = \operatorname{rank}P_i, \qquad i = 1,2,3. \tag{2.15}$$

Let $M_0 = \{B : Z \to R^{d\times d}, B \text{ is bounded on } Z\}$ with the norm $\|B\| = \sup_{n\in Z}|B(n)|$. Then M_0 is a Banach space. And let

$$M_1 = \left\{B : B \in M_0 \text{ and } \|B\| \le \min\left\{M^{-1}, \frac{e^\alpha - 1}{2K(e^\alpha+1)} - \varepsilon\right\}\right\},$$
$$M_2 = \{B : B \in M_1 \text{ and (2.15) holds for the perturbed equation (2.1)}\}.$$

Then for any $B \in M_1$, equation (2.1) has an exponential trichotomy on Z with projections $P_i(B)$ $(i = 1,2,3)$, and M_1 is a connected subset of M_0, and the fact which we want to prove is equivalent to $M_1 = M_2$. Obviously, $O_{d\times d} \in M_2$, that is, $M_2 \ne \emptyset$, where $O_{d\times d}$ is the zero matrix. For any $\{B_m\}_{m\in Z^+} \subset M_2$ with $B_m \to B \in M_1$ (as $m \to +\infty$), it follows from Proposition 2.1 that there exists $N > 0$ such that the linear difference equation $x(n+1) = (A(n) + B_m(n))x(n)$ $(m > N)$ has an exponential trichotomy on Z with projections $P_i(B_m)$ $(i = 1,2,3)$ satisfying $\operatorname{rank}P_i(B) = \operatorname{rank}P_i(B_m)$, $(i = 1,2,3)$. This implies $B \in M_2$, that is, M_2 is a closed set. On the other hand, for any $B \in M_2$, the perturbed equation (2.1) has an exponential trichotomy on Z with projections $P_i(B)$ $(i = 1,2,3)$ and (2.15) holds. By Proposition 2.1, we know that there exists a $\delta > 0$ such that the difference equation $x(n+1) = (A(n) + C(n))x(n)$ has an exponential trichotomy on Z with projections $P_i(C)$ $(i = 1,2,3)$ satisfying $\operatorname{rank}P_i(C) = \operatorname{rank}P_i(B)$ $(i = 1,2,3)$ for every $C \in U = \{C : C \in M_1 \text{ and } \|C - B\|_{M_0} < \delta\}$. We thus have

$$\operatorname{rank}P_i(C) = \operatorname{rank}P_i(B) = \operatorname{rank}P_i, \qquad i = 1,2,3.$$

Consequently, $B \in U \subset M_2$, that is, M_2 is an open set in M_1. Hence, it follows from the connection property of M_1 that $M_1 = M_2$.

In the case of the exponential dichotomy, note that the bounded solution of the inhomogeneous equation is unique. The proof is similar to the previous case. This completes the proof.

REMARK 2.2.

(1) *Comparing with the known results on roughness (Proposition* 2.1, *Proposition* 2.2, *also see [6,7]), the radius of the perturbation for the invariance of exponential trichotomy and dichotomy in Theorem* 2.1 *are larger than those known. Speaking precisely, under the condition that $A(n) + B(n)$ is invertible for every $n \in Z$, the radius of the perturbation in the case of the trichotomy (respectively, dichotomy) is extended from $\|B\| \le (e^\alpha - 1)/(9K^2e^\alpha)$ (respectively, $\|B\| \le (e^\alpha - 1)/(2K(e^\alpha + 1)t)$ to at least $\|B\| < (e^\alpha - 1)/(2K(e^\alpha + 1)$ (respectively, $\|B\| < (e^\alpha - 1)/(K(e^\alpha + 1))$.*

(2) *In [13–15], the exact bound for exponential dichotomy roughness including the cases of strong and semistrong dichotomy are discussed for differential equations. The corresponding results for difference equations are not available in the literature. Work in this topic is in progress.*

(3) *The similar results of random difference equations [16] to Theorem* 2.1 *can be concluded by the methods used in the proofs of the previous theorems.*

3. EXPONENTIAL DICHOTOMY FOR ALMOST PERIODIC DIFFERENCE EQUATIONS

In [10,11], it is known that for an almost periodic linear differential equation

$$\frac{dx}{dt} = A(t)x, \tag{3.1}$$

where $A(t)$ is almost periodic on R, the following statements are equivalent:

(1) equation (3.1) has an exponential dichotomy on R;
(2) equation (3.1) has an exponential dichotomy on $[a, +\infty)$ for every $a \in R$;
(3) equation (3.1) has an exponential dichotomy on some sufficiently long finite interval.

In this section, we give the discrete version of the above equivalence. To do this, the following propositions are needed.

PROPOSITION 3.1. *Equation (1.1) has an exponential dichotomy on Z^+ if and only if there exist constants $0 < \theta < 1$, $T \geq 1$, $T \in Z^+$ such that*

$$|x(n)| \leq \theta \sup\{|x(u)| : |u-n| \leq T, u \in Z^+\}, \quad n \geq T \tag{3.2}$$

(see [3, pp. 296–297])

Combining Proposition 10 in [9, p. 1446] and the method of the proof for Lemma 1 in [11, p. 70], we have the following.

PROPOSITION 3.2. *Let $\{A(n)\}_{n\in Z}$ be almost periodic on Z. Then equation (1.1) has an exponential dichotomy on Z if and only if it has one on Z^+ (or Z^-).*

This result implies that the exponential trichotomy of equation (1.1) on Z is equivalent to the exponential dichotomy on Z if $\{A(n)\}_{n\in Z}$ is almost periodic.

By using induction, it can be deduced.

PROPOSITION 3.3. *Let $\{A(n)\}_{n\in Z}, \{B(n)\}_{n\in Z}$ be two matrix sequences whose elements are invertible matrices. If $\{A(n)\}_{n\in Z}$ and $\{B(n)\}_{n\in Z}$ satisfy:*

(1)
$$\max\left\{\sup_{n\in Z}|A(n)|, \sup_{n\in Z}|B(n)|, \sup_{n\in Z}\left|A^{-1}(n)\right|, \sup_{n\in Z}\left|B^{-1}(n)\right|\right\} \leq M < +\infty; \tag{3.3}$$

(2) *there exists a positive number $\varepsilon > 0$ such that*
$$\max\left\{\sup_{n\in Z}|A(n)-B(n)|, \sup_{n\in Z}\left|A^{-1}(n)-B^{-1}(n)\right|\right\} < \varepsilon. \tag{3.4}$$

Then
$$\left|Y(n)Y^{-1}(m) - X(n)X^{-1}(m)\right| \leq |n-m| M^{|n-m|-1}\varepsilon \tag{3.5}$$

holds for $n, m \in Z$, where $X(n)$ is a fundamental matrix for equation (1.1) and $Y(n)$ is one for equation $y(n+1) = B(n)y(n)$, with $X(0) = Y(0) = I$.

THEOREM 3.1. *Let $\{A(n)\}_{n\in Z}$ be a $d\times d$ invertible matrix sequence, defined and almost periodic on Z, such that*

(1) *$\left\{A^{-1}(n)\right\}_{n\in Z}$ is almost periodic;*
(2) *equation (1.1) has an exponential dichotomy on an integer's interval $\overline{0,T} = \{0,1,2,\dots,T\}$ ($T \in Z^+$) with constants $K \geq 1$ and $\alpha > 0$ such that $\alpha^{-1}\ln K > 1$.*

Consider $M = \max\left\{\|A\|+1, \|A^{-1}\|\right\}$, $f(t) = e^t - e^{-t}$ for $t \in (0, +\infty)$, $h = \alpha^{-1}\left(f^{-1}(8) + \ln K\right)$ and δ the largest positive number such that $hM^{h-1}\delta \leq 1$. Then if $T > 0$ is so large that $T \geq 4h$ and every integer's interval of length $[T/2]$ contains a common δ-translation number for $A(n)$ and $A^{-1}(n)$, equation (1.1) has an exponential dichotomy on Z with constants L and $\beta = h^{-1}\ln 3$, where L depends only on M, K and α, $[\cdot]$ is the greatest-integer function.

PROOF. Let $m \in Z$, $m \geq h$. Then it follows from $[a+b] \geq [a] + [b]$ for all $a, b \in R$ that there is a common δ-translation number τ for $A(n)$ and $A^{-1}(n)$ in the integer's interval $\overline{m - [3/4T], m - [1/4T]}$. Therefore, for all n,

$$|A(n) - A(n-\tau)| < \delta, \qquad \left|A^{-1}(n) - A^{-1}(n-\tau)\right| < \delta. \tag{3.6}$$

Since

$$\overline{m-\tau-\left[\frac{T}{4}\right], m-\tau+\left[\frac{T}{4}\right]} \subset \overline{0,T},$$

the difference equation

$$y(n+1) = A(n-\tau)y(n) \tag{3.7}$$

has an exponential dichotomy on $[m-T/4, m+T/4] \cap Z$ with constants K, α. Then since $T \geq 4h$ and

$$K^{-1}e^{\alpha h} - Ke^{-\alpha h} = 8,$$

it follows from [3, pp. 296–297] that if $y(n)$ is any solution of equation (3.7),

$$|y(m)| \leq \frac{1}{4}\sup\{|y(n)| : |n-m| \leq h\}.$$

Now let $x(n)$ be any solution of equation (1.1) and let $y(n)$ be the solution of equation (3.7) with $y(m) = x(m)$. From (3.6), Proposition 3.3, and the definition of δ, we have that for $0 \leq |n-m| \leq h$,

$$|y(n) - x(n)| = \left|(X(n-\tau)X^{-1}(m-\tau) - X(n)X^{-1}(m))x(m)\right| \leq |x(m)|.$$

Hence,

$$\begin{aligned} |x(m)| = |y(m)| &\leq \frac{1}{4}\sup\{|y(n)| : |n-m| \leq h\} \\ &\leq \frac{1}{4}\sup\{|x(n)| : |n-m| \leq h\} + \frac{1}{4}|x(m)|, \end{aligned}$$

that is,

$$|x(m)| \leq \frac{1}{3}\sup\{|x(n)| : |n-m| \leq h\}.$$

From Proposition 2 in [3, p. 297] and its proof and similar arguments to those in the proof of Theorem 1 in [10, p. 295], this proof is completed.

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