COMMUNICATION

EQUI-ASSIGNMENTS AND EXPLICIT SOLUTIONS OF A PARTICULAR DISCRETE OPTIMIZATION PROBLEM*

A. VOLPENTESTA

Dept. of Mathematics, University of Pisa, Piazza dei Cavalieri 2, Pisa, Italy

Communicated by F. Giannessi
Received March 1980

A concept of equi-assignment is introduced and characterized by means of a property based on the Euclidean algorithm for finding the greatest common divisor. Then a set of optimal solutions of a particular discrete optimization problem is found by determining a suitable equi-assignment.

1. Definitions and a characterization theorem

Throughout this paper \( \mathbb{Z}_+ \) and \( \mathbb{B} \) will denote the set of all nonnegative integers and the set \{0, 1\}, respectively. Define

\[
\mathbb{B}^n = \bigcup_{n=1}^{\infty} \mathbb{B}, \quad \text{where} \quad \mathbb{B}^n = \mathbb{B} \times \mathbb{B} \times \cdots \times \mathbb{B}.
\]

Consider the graph (cycle) \( G_m = (V_m, \Omega_m) \) where \( V_m = \{0, 1, \ldots, m - 1\} \) is the set of the vertices and \( \Omega_m = \{(0, 1), (1, 2), \ldots, (m - 2, m - 1), (m - 1, 0)\} \) is the set of the edges. We will say that the elements of a set \( J \subseteq V_m \) are \( m \)-consecutive iff the subgraph of \( G_m \) whose vertices are the elements of \( J \) and whose edges are those of \( \Omega_m \) joining two elements of \( J \), is connected. Let \( J_{r,m} = \{J \subseteq V_m : |J| = r \text{ and the elements of } J \text{ are } m \text{-consecutive}\} \).

Every element of \( \mathbb{B}^m \) can be regarded as an assignment of \( m \) 0–1 weights to the vertices of \( G_m \). We will say that \( x = (x_0, \ldots, x_{m-1}) \in \mathbb{B}^m \) is an equi-assignment iff the inequality

\[
\left| \sum_{j \in J_1} x_j - \sum_{j \in J_2} x_j \right| \leq 1
\]

holds for every \( (J_1, J_2) \in J_{2,m} \) and whatever the integer \( r \,(0 < r \leq m) \) may be. Consider now the following problem:

\[
\text{Let } t \text{ and } m \text{ be integers such that } 0 \leq t \leq m; \\
\text{find all the elements of } \mathbb{B}^m \text{ which are equi-assignments with exactly } t \text{ positive components.}
\]

*The results of this paper were presented at the Tenth Symposium on Mathematical Programming (MONTREAL—1979).
Denote by $E(t, m)$ the set of solutions of problem (1). In the next part of this section $E(t, m)$ will be determined by means of a procedure based on the Euclidean algorithm for finding the greatest common divisor between $t$ and $m$. Besides it will be shown that in $E(t, m)$ every element is a cyclic transformation of any other.

Denote by $u^{(n)}$ the null $n$-vector and by $e^{(n)}$ the $n$-vector with only the first component equal to one and all the others equal to zero. For every $d \in \mathbb{Z}_+$ the function $\phi^d$ from $B$ to $\tilde{B}$ is defined by

$$
\phi^d(x_0, \ldots, x_{m-1}) = (e^{(d+x_0)}, e^{(d+x_1)}, \ldots, e^{(d+x_{m-1})})
$$

$$
= (1, 0, \ldots, 0, 1, 0, \ldots, 0, \ldots, 1, 0, \ldots, 0).
$$

Let $v = (v_0, v_1, \ldots, v_n)$ be a positive integer vector. We define the function $\phi^v$ as the compositions of the functions $\phi^{v_i}$, $i = 0, \ldots, n$, i.e. $\phi^v = \phi^{v_n} \phi^{v_{n-1}} \ldots \phi^{v_0}$.

Given two integers $m, t$, ($0 < t \leq m$) let $q_0, \ldots, q_k$ and $r_1, \ldots, r_k$ be integers such that:

$$
m = q_0 t + r_1, \quad 0 \leq r_1 < t,
$$

$$
t = q_1 r_1 + r_2, \quad 0 \leq r_2 < r_1,
$$

$$
\ldots
$$

$$
r_{k-2} = q_{k-1} r_{k-1} + r_k, \quad 0 \leq r_k < r_{k-1},
$$

$$
r_{k-1} = q_k r_k.
$$

Let $q = (q_0, q_1, \ldots, q_k)$. Now we are able to state the main result of this section:

**Theorem 1.** A vector $x \in \mathbb{B}^m$ is an equi-assignment with exactly $t$ positive components if and only if it is a cyclic transformation of $\Phi^q(u^{(k)})$.

2. Explicit solutions of a particular discrete optimization problem

Consider the following discrete optimization problem:

$$
\min \sum_{j=0}^{m-1} x_j \tag{2a}
$$

$$
\sum_{j \in J} x_j \geq b \quad \forall J \in J_{k,m}, \tag{2b}
$$

$$
x_j \in \mathbb{Z}_+ \quad j = 0, \ldots, m-1 \tag{2c}
$$

where $m, k, b$ are positive integers and $k \leq m$. Problem (2abc) is defined by the 3-tuple $(m, k, b)$ and hence it will be denoted by $P(m, k, b)$. Such a problem was

---

A sketch of the proofs of this Theorem and of Theorem 2 which appears in Section 2 are given in the appendix.
A discrete optimization problem

originated in [2] as the formulation of a particular scheduling problem and in [1] a set of its optimal solutions has been characterized and explicitly determined by means of continued fractions. The next theorem shows how to find a set of optimal solutions of $P(m, k, b)$ by determining an element of $E(t, m)$.

**Theorem 2.** Let $m, k, b$ be nonnegative integers ($k \leq m$). Let

$$t = \left\lfloor \frac{mb}{k} \right\rfloor - \left\lfloor \frac{b}{k} \right\rfloor m$$

and

$$x = (x_0, \ldots, x_{m-1}) \in E(t, m).$$

Then

$$y = (x_0 + \left\lfloor \frac{b}{k} \right\rfloor, x_1 + \left\lceil \frac{b}{k} \right\rceil, \ldots, x_{m-1} + \left\lceil \frac{b}{k} \right\rceil)$$

is an optimal solution of $P(m, k, b)$.

It follows from Theorem 2 that the optimal value of $P(m, k, b)$ is $\left\lfloor \frac{mb}{k} \right\rfloor$ for any nonnegative integers $m, k, b$. We remark that if $x \in E(t, m)$ then every cyclic transformation of $x$ belongs to $E(t, m)$; hence, by using Theorem 1 and Theorem 2 we are able to determine explicitly a set of optimal solutions of $P(m, k, b)$. It is possible to show that such a set is the same as that one which is determined in [1].

**References**


**Appendix**

We will give a sketch of the proofs of Theorems 1 and 2. More details and other results related to the problems (1) and (2abc) can be found in [3, 4].

**Proof of Theorem 1.** Let $r_{-1} = m$, $r_0 = t$, $r_{k+1} = 0$. Denote by $\Theta_n$ the ordinary difference between integers mod $n$. Let $h \in \{0, 1, \ldots k\}$ and consider $x \in$

$\lfloor z \rfloor$ and $\lceil z \rceil$ denote the greatest integer less than or equal to $z$ and the smallest integer greater than or equal to $z$, respectively.
\( E(r_h, r_{h-1}) \). Set

\[
I(x) = \{i : x_i = 1\},
\]

\[
d_{ij} = j \Theta_{r_{h-1}}, i, \ \forall i, j \in I(x);
\]

\[
d_i(x) = \min_{j \in I(x)-\{i\}} d_{ij}(x), \ \forall i \in I(x);
\]

\[
d(x) = \min_{i \in I(x)} d_i(x);
\]

\[
I_1(x) = \{i \in I(x) : d_i(x) = d(x)\};
\]

\[
I_2(x) = \{i \in I(x) : d_i(x) = d(x) + 1\}.
\]

It is possible to prove the following assertions:

(A.1) The inequalities \( d(x) \leq d_i(x) \leq d(x) + 1 \) hold \( \forall i \in I(x) \).

(A.2) \( d(x) = q_h \), cardinality of \( I_1(x) = r_h - r_{h+1} \), cardinality of \( I_2(x) = r_{h+1} \).

(A.3) If \( h = k \), then \( x \in E(r_k, r_{k-1}) \) iff it is a cyclic transformation of \( \Phi^q(u^{(r_k)}) \).

(A.3) follows from (A.2) and it proves Theorem 1 in the special case \( k = 0 \). Let us consider now the general case \( k > 0 \) and assume \( h < k \). Let \( I(x) = \{i_0, \ldots, i_{n-1}\} \) with \( i_0 < i_1 < \cdots < i_{n-1} \) and set

\[
\Pi_0(x) = d_{i_0}(x) \quad \text{for} \quad j = 0, \ldots, r_{h-1},
\]

\[
\Pi(x) = (\Pi_0(x), \Pi_1(x), \ldots, \Pi_{r_{h-1}}(x)).
\]

Because of (A.1) and (A.2) we have that \( \Pi(x) \) is an element of \( B^k \) with exactly \( r_{h+1} \) positive components. Furthermore it can be proved that

\[
x \in E(r_h, r_{h-1}) \iff \Pi(x) \in E(r_{h+1}, r_h).
\]

The next assertion follows from (A.4) and from the definition of the function \( \Phi^q \):

\[
y \in E(r_{h+1}, r_h) \iff \Phi^q(y) \in E(r_h, r_{h-1}).
\]

Since (A.3) implies \( \Phi^q(u^{(r_h)}) \in E(r_h, r_{h-1}) \) by using more times (A.5) we have \( \Phi^q(u^{(r_h)}) \in E(t, m) \). Thus sufficiency is proved by remarking that every cyclic transformation of any element of \( E(t, m) \) is still an element of \( E(t, m) \). Let us prove necessity. Let \( x \in E(t, m) \) and define \( \Pi^{(2)}(x) = \Pi(\Pi(x)), \Pi^{(3)}(x) = \Pi(\Pi^{(2)}(x)) \) and so iteratively \( \Pi^{(k)}(x) = \Pi(\Pi^{(k-1)}(x)) \). It is not difficult to see that \( x \) is a cyclic transformation of \( \Phi^q(\Phi^q(\cdots \Phi^q(\Pi(x)) \cdots)) \). On the other hand, it follows from (A.3) and (A.4) that \( \Pi^{(k)}(x) \) is a cyclic transformation of \( \Phi^q(u^{(r_k)}) \). This implies that \( \Phi^q(\Phi^q(\cdots \Phi^q(\Pi^{(k)}(x)) \cdots)) \) is a cyclic transformation of \( \Phi^q(u^{(r_k)}) \) and thus we have necessity.

Proof of theorem 2. Set \( \beta = b - [b/k] k \). It is possible to prove the following
statement:

(A.6) if $x$ belongs to $E(t, m)$, then $\sum_{j \in J} x_j \geq \beta \ \forall J \in J(k, m)$. It follows from (A.6) that the inequality

$$\sum_{j \in J} y_j \geq k \left\lfloor \frac{b}{k} \right\rfloor + \beta = b$$

holds $\forall J \in J_{k,m}$, hence $y$ satisfies (2bc). Moreover we have

$$\sum_{j=0}^{m-1} y_j = m \left\lfloor \frac{b}{k} \right\rfloor + t = \left\lfloor \frac{mb}{k} \right\rfloor.$$ 

Since the optimal value of the linear program (2.ab) is $mb/k$, we have proved the optimality of $y$. 