ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT
Applied Mathematics Letters 17 (2004) 393–399

Applied Mathematics Letters
www.elsevier.com/locate/aml

The Long Wavelength Instability of High Prandtl Number Convective Flow

A. A. AL-ALI
Department of Mathematics & Computer Science
Faculty of Science, UAE University
P.O. Box 19214, Dubai, United Arab Emirates
aalali@uaeu.ac.ae

(Received October 2002; revised and accepted June 2003)

Abstract—This paper considers the stationary instability of convective flow between heated vertical planes in the limit of high Rayleigh number, $A \to \infty$. This limit is of practical importance in applications. The planes are held at temperatures that increase linearly with height and differ horizontally by a constant amount. This allows an exact solution of the Boussinesq equations whose nature depends on the value of a convective parameter γ. The lower branch of the neutral stability curve is obtained for general values of γ and large Prandtl numbers, σ, revealing a dramatic change in the stability properties of the system in the limit as $\sigma \to \infty$.

Keywords—Instability, Convective flow, Convective parameter γ, Rayleigh number A, Prandtl number σ.

1. INTRODUCTION

In a previous investigation of the stability of the high Rayleigh number ($A \gg 1$) buoyancy-driven flow between heated vertical planes [1–3], the lower branch of the neutral curve for stationary disturbances was obtained for an infinite Prandtl number fluid. The present note extends this analysis to consider how the lower branch is modified for large but finite values of the Prandtl number. This leads to the identification of a significant shift in the range of the convective parameter

$$\gamma = \left(\frac{\beta A}{4}\right)^{1/4} \tag{1.1}$$

for which stationary instability occurs. Here β is a measure of the positive vertical temperature gradient maintained over the two vertical planes and A is the Rayleigh number based on the gap width and the constant horizontal temperature difference between the planes. The stability of the buoyancy-driven flow which consists of fluid rising near the hot plane and descending near the cold plane has been studied numerically [4] and is related to the onset of multiple transverse rolls observed in experimental investigations of vertical slot flows [5–7]. It has been established that for infinite Prandtl number and high Rayleigh number ($A \gg 1$), stationary instability is limited to values of γ greater than a critical value $\gamma_0 \approx 6.30$ and the critical wavelength of the

0893-9659/04/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi: 10.1016/S0893-9659(04)00044-8

instability at γ_0 has been found [8]. In the present paper, the lower branch is traced at finite values of the Prandtl number where it emerges that instability occurs for general values of $\gamma \geq 0$. The main results are presented in Section 4 where it is shown that for $0 \leq \gamma \leq \gamma_0$, a thermal critical layer is formed midway between the planes. The results are summarized in Section 5.

2. FORMULATION

The nondimensional system governing stationary transverse disturbances is derived [6], and has the form

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = \theta' + i\alpha\sigma^{-1}A\left(\Psi'''\phi - \Psi'\left(\phi'' - \alpha^2\phi\right)\right), \tag{2.1}$$

$$\theta'' - \alpha^2\theta = i\alpha A(\Theta'\phi - \Psi'\theta) - 4\gamma^4\phi', \tag{2.2}$$

$$\phi = \phi' = \theta = 0, \qquad x = \pm\frac{1}{2}, \tag{2.3}$$

where primes denote differentiation with respect to x, (x, z) are horizontal and vertical coordinates nondimensionalised with respect to the gap width l^*,

$$\sigma = \frac{v^*}{k^*} \tag{2.4}$$

is the Prandtl number where v^* is the kinematic viscosity and k^* is the thermal diffusivity, and

$$A = \frac{\beta^* g \Delta T^* l^{*3}}{k^* v^*} \tag{2.5}$$

is the Rayleigh number where β^* is the coefficient of thermal expansion, g is the acceleration due to gravity, and ΔT^* is the horizontal temperature difference between the planes. The overall stream function and temperature, nondimensionalised with respect to k^* and ΔT^*, respectively, are

$$\psi = A\left(\Psi(x) + \epsilon\phi(x)e^{i\alpha z}\right), \tag{2.6}$$

$$T = \beta z + \Theta(x) + \epsilon\theta(x)e^{i\alpha z}, \tag{2.7}$$

where $\epsilon \ll 1$ denotes the size of the stationary perturbation and Ψ and Θ are the base flow and horizontal temperature fields which depend only on γ and are given in detail [8]. The functions Ψ and Θ are even and odd about $x = 0$, respectively.

3. LOWER BRANCH FOR FINITE PRANDTL NUMBERS

In the high Rayleigh number limit, the lower branch of the neutral stability curve is associated with small wavenumber disturbances such that

$$\alpha \sim \frac{\tilde{\alpha}}{A}, \qquad A \to \infty, \tag{3.1}$$

and it follows from (2.1)–(2.3) that $\tilde{\alpha}$ and the leading approximations to the perturbation functions ϕ and θ satisfy the reduced system

$$\phi'''' + i\tilde{\alpha}\sigma^{-1}\left(\Psi'\phi'' - \Psi'''\phi\right) = \theta', \tag{3.2}$$

$$\theta'' = i\tilde{\alpha}A\left(\Theta'\phi - \Psi'\theta\right) - 4\gamma^4\phi', \tag{3.3}$$

$$\phi = \phi' = \theta = 0, \qquad x = \pm\frac{1}{2}. \tag{3.4}$$

It has been shown previously [1] that this system has a solution for infinite Prandtl number in the region $\gamma > \gamma_0 \approx 6.30$ such that $\tilde{\alpha} \to \infty$ as $\gamma \to \gamma_0+$. This is shown in Figure 1. Here solutions

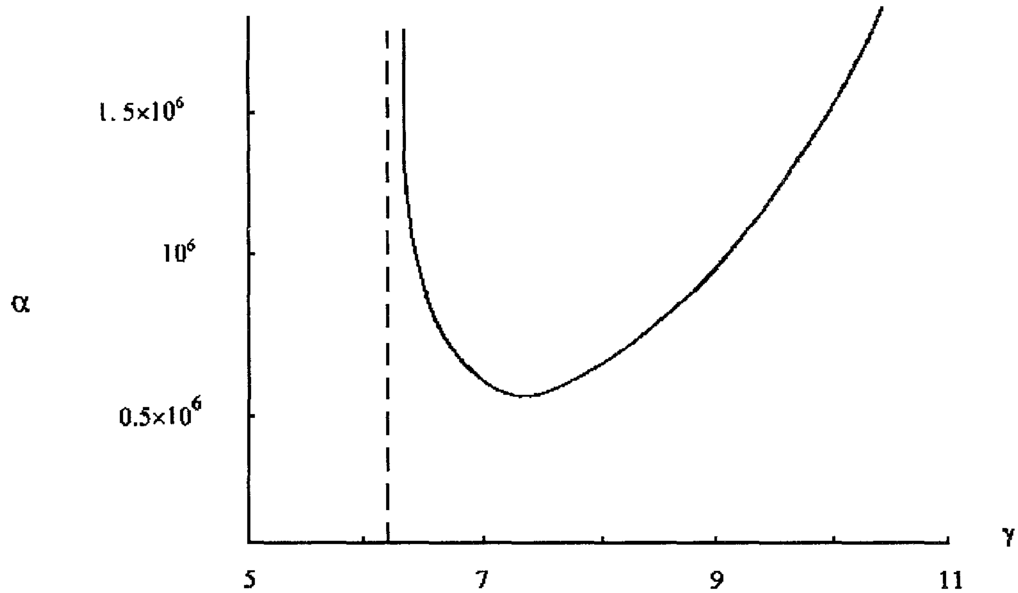

Figure 1.

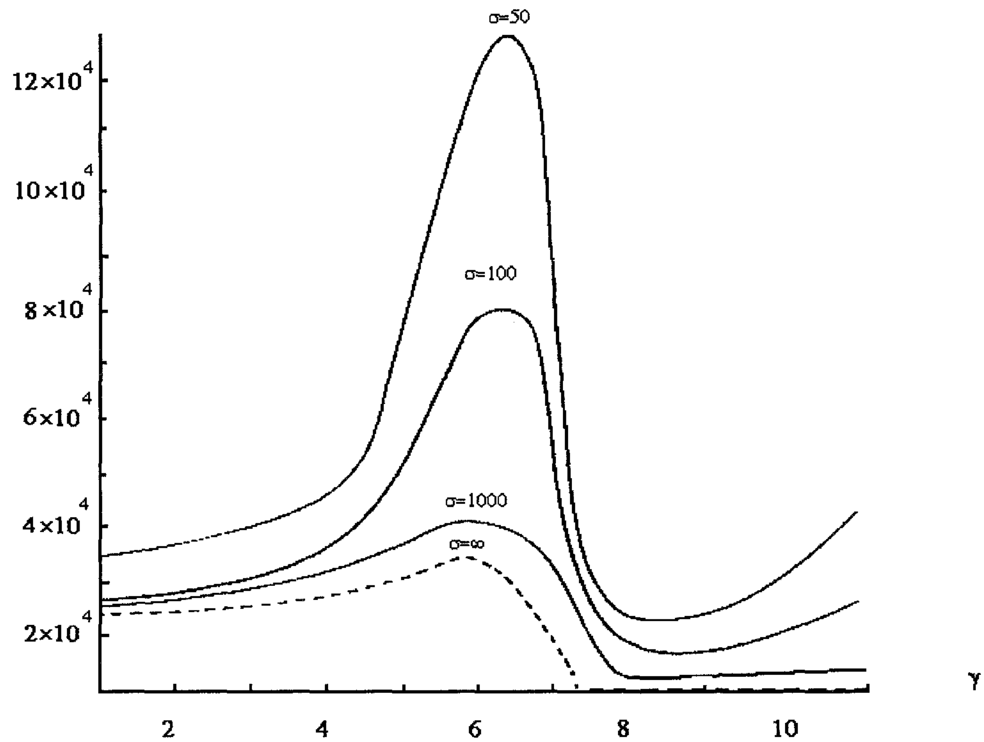

Figure 2.

were obtained for finite Prandtl numbers using a fourth-order Runge Kutta scheme. The system was converted into six first-order equations and three linearly independent complex solutions satisfying the boundary conditions at $x = -1/2$ computed across to $x = 1/2$. Applications of the boundary conditions there then led to a 3×3 complex determinant whose zeros were located to obtain $\tilde{\alpha}$ for given values of σ and γ. Results for $\sigma = 20$, 100, and 1000 are shown in Figure 2 where $\lambda = \tilde{\alpha}/\sigma$ is plotted in the range $0 \leq \gamma \leq 10$. Also shown is a dashed line corresponding to the limiting case $\sigma = \infty$ which is discussed in the next section.

4. LARGE PRANDTL NUMBER LIMIT, $\sigma \to \infty$

In order to determine the limiting form of the solution as $\sigma \to \infty$ in the region $\gamma < \gamma_0 \approx 6.30$, it is necessary to assume that $\lambda = \tilde{\alpha}/\sigma$ remains finite as $\sigma \to \infty$ so that (3.2) becomes

$$\phi'''' + i\lambda(\Psi'\phi'' - \Psi'''\phi) = \theta', \tag{4.1}$$

while to a first approximation, equation (3.3) reduces to

$$\theta = \frac{\Theta'}{\Psi'}\phi. \tag{4.2}$$

Conduction represented by the highest derivative term θ'' in (3.3) has been neglected in (4.2) but remains important in wall layers near $x = \pm 1/2$ and a thermal critical layer surrounding $x = 0$. Thus, (4.1) and (4.2) must be viewed as leading approximations in two outer zones $0 < |x| < 1/2$ either side of the critical layer. The solution of (4.1) and (4.2) for ϕ in $x < 0$ can be written in the form

$$\phi = \nu_1^- f_1(x) + \nu_2^- f_2(x), \tag{4.3}$$

where $\nu_{1,2}^-$ are complex constants and $f_{1,2}$ are complex functions uniquely defined as the solutions of the third-order systems

$$f_1''' - \frac{\Theta'}{\Psi'} f_1 = i\lambda\left(\Psi'' f_1 - \Psi' f_1'\right); \qquad (f_1, f_1', f_1'') = (0,0,1), \qquad \text{at } x = -\frac{1}{2}, \tag{4.4}$$

$$f_2''' - \frac{\Theta'}{\Psi'} f_2 = i\lambda\left(\Psi'' f_2 - \Psi' f_2'\right) + 1; \quad (f_2, f_2', f_2'') = (0,0,0), \qquad \text{at } x = -\frac{1}{2}. \tag{4.5}$$

Here it is assumed that $f_{1,2}$ must satisfy the full boundary conditions on ϕ at $x = -1/2$, in which case the behavior of θ obtained from (4.2) is also consistent with its boundary condition in (3.4). Detailed consideration of the wall layer of thickness order $\sigma^{-1/3}$ confirms this to be a consistent procedure.

The appropriate solution in $x > 0$ follows from the fact that Θ' and Ψ' are odd and even functions of x, respectively, and can be written as

$$\phi = \nu_1^+ f_1^*(-x) + \nu_2^+ f_2^*(-x), \tag{4.6}$$

where $*$ denotes the complex conjugate and $\nu_{1,2}^+$ are complex constants.

The general form of $f_j(x)$ as $x \to 0-$ is given by

$$f_j = a_j + b_j x + c_{j0} x^2 \ln|x| + c_j x^2 + d_j x^3 + o\left(x^4 \ln|x|\right), \tag{4.7}$$

where the constants a_j, b_j, and c_j $(j = 1, 2)$ are determined from a numerical solution of (4.4) and (4.5). The remaining coefficients are known in terms of these, so that, for example,

$$c_{j0} = \frac{\mu_0 a_j}{2\omega_1}, \qquad d_j = \frac{(\mu_0 b_j/\omega_1 + i\lambda\omega_1 a_j + j - 1)}{6}, \tag{4.8}$$

where $\mu_0 = \Theta'(0)$ and $\omega_1 = \Psi''(0)$ depend upon γ. This was carried out using series expansions in the neighborhood of $x = -1/2$ and then a fourth-order Runge-Kutta scheme to integrate the equations across to the neighborhood of $x = 0$. Here the solution was terminated at $x = x_s$ where x_s is small and negative, and the numerical solution for f_j, f_j', and f_j' equated to the corresponding values given by extended versions of the series expansion (4.7). This allowed the complex coefficients a_j, b_j, and c_j to be determined, and checks on accuracy were carried out using different step sizes in the Runge-Kutta scheme and by changing the value of x_s.

The eigenvalue λ can now be determined as a function of γ by considering continuity of the solution across the thermal critical layer centered on $x = 0$. This layer is of thickness $x \sim \sigma^{-1/3}$ and consideration of the solution there leads to four bridging conditions,

$$v_1^+ d_1^* + v_2^+ a_2^* = v_1^- a_1 + v_2^- a_2, \tag{4.9}$$

$$v_1^+ b_1^* + v_2^+ b_2^* = -v_1^- b_1 - v_2^- b_2, \tag{4.10}$$

$$v_1^+ c_1^* + v_2^+ c_2^* = v_1^- c_1 + v_2^- c_2 + i\mu_0\pi \frac{\left(v_1^+ a_1^* + v_2^+ a_2^*\right)}{2\omega_1}, \tag{4.11}$$

$$v_1^+ d_1^* + v_2^+ d_2^* = -v_1^- d_1 - v_2^- d_2. \tag{4.12}$$

Details of the analysis leading to (4.9)–(4.12) are similar to those described by [8] in relation to the critical layer structure near γ_0 and are omitted here; further information is given by [9]. Using the result for d_j in (4.8), it follows from (4.9), (4.10), and (4.12) that $v_2^+ = -v_2^-$ and then following elimination of v_2^- from (4.9)–(4.11) the condition for a nontrivial solution is found to be

$$\begin{aligned}(a_1^* b_1 + a_1 b_1^*)(c_2^* + c_2) + (a_1^* c_1 - a_1 c_1^*)(b_2^* - b_2) - (b_1^* c_1 + b_1 c_1^*)(a_2^* + a_2) \\ -i\mu_0\pi \frac{\left[(a_1 a_2^* b_1^* + a_1^* a_2 b_1) - (b_2^* - b_2)\, a_1 a_1^*\right]}{2\omega_1} = 0.\end{aligned} \tag{4.13}$$

It may be seen that the left-hand side of equation (4.13) is real and since μ_0 and ω_1 depend on γ and the coefficients a_j, b_j, c_j depend on γ and λ, it yields the eigenvalue λ as a function of γ. This is plotted as a dashed line in Figure 2, in the range $0 < \gamma < \gamma_0$. It was found that $\lambda \to 0$ as $\gamma \to \gamma_0 -$, with a local behavior given by

$$\lambda \sim m(\gamma_0 - \gamma), \qquad \gamma \to \gamma_0 -, \tag{4.14}$$

where $m \approx 5.065 \times 10^4$.

5. SUMMARY

It is seen that as $\sigma \to \infty$, the lower branch of the neutral stability curve adopts two distinct forms in the regions $\gamma < \gamma_0$ and $\gamma > \gamma_0$. For $\gamma < \gamma_0$, the wavenumber is proportional to the Prandtl number $\tilde{\alpha} \sim \lambda\sigma$ with λ finite, whereas for $\gamma > \gamma_0$ the wavenumber $\tilde{\alpha}$ is finite as $\sigma \to \infty$ and the limiting curve is that given by Figure 1. This appears as the dashed line $\lambda = 0$ for $\gamma > \gamma_0$ in Figure 2.

The results demonstrate that the lower branch of the neutral stability curve experiences dramatic transition as the Prandtl number becomes large. For finite Prandtl number, the lower branch extends all the way to $\gamma = 0$, so that long wavelength convection can occur with $\alpha = O(A^{-1})$ for any value of the convective parameter. As the Prandtl number increases, the wavenumber of the section of the lower branch in the range $0 \le \gamma < 6.3$ increases in proportion to the size of the Prandtl number so that when $\sigma \sim A$, the wavenumber α will be of order one, comparable with that of the upper branch, with the approximation used to obtain the lower branch equation in Section 3 no longer valid. Thus, it is reasonable to conjecture that as the Prandtl number increases through values of order A, both the lower and upper branches of the neutral stability curve are associated with order one wavenumbers in the region $0 \le \gamma < \gamma_0$ such that there is a minimum value γ_c of γ for instability which increases from 0 to γ_0 as σ/A increases from zero to infinity. This is shown schematically in Figure 3 and at $\gamma = 0$ is consistent with the critical point for the conduction regime obtained by [5] as $\alpha_c = 2.65$ when $A_c \sim 7.9 \times 10^3\sigma$ and $\sigma \gg 1$. It is also consistent with the results obtained by [7] which describe the nature of the solution near γ_0 when $\sigma = O(A^{4/3})$. These show that when $\sigma \sim A^{4/3}\sigma_0$, the location of the critical point (γ_c, α_c) is determined by

$$\gamma_c = \gamma_0 + A^{-2/3}\gamma_1, \qquad \alpha_c = A^{-1/3}\alpha_1, \tag{5.1}$$

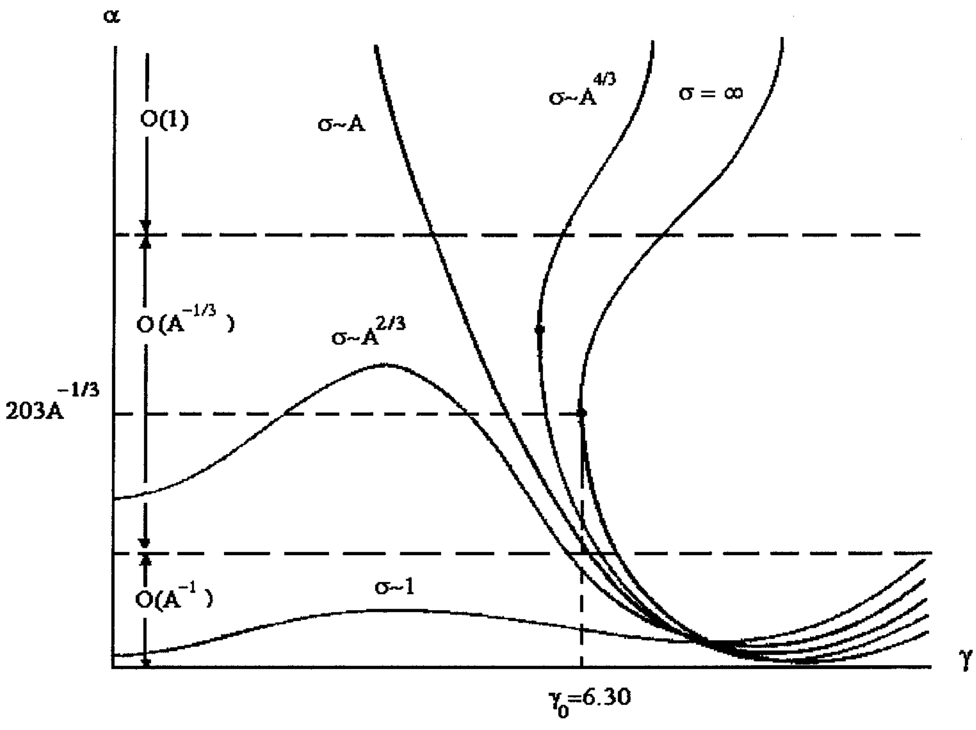

Figure 3.

where γ_1 and α_1 are the solutions of the cubic equation

$$\alpha_1^3 - c_0\sigma_0^{-1}\alpha_1^2 - c_1\gamma\alpha_1 + c_2 = 0 \tag{5.2}$$

for which γ_1 is a minimum. Here the coefficients of the equation are given by $c_0 = 0.00108$, $c_1 = 53.37$, and $c_2 = 1.677 \times 10^7$, respectively. As $\sigma_0 \to 0$, it follows that

$$\gamma_c \sim 6.30 - 5.46 \times 10^{-9}\left(\frac{A}{\sigma}\right)^2, \quad \alpha_c \sim 5.40 \times 10^{-4}\left(\frac{A}{\sigma}\right), \quad \left(A \ll \sigma \ll A^{4/3}\right), \tag{5.3}$$

consistent with finite changes to the values of γ_c, α_c when $\sigma = O(A)$. Note also that as $\sigma_0 \to 0$, the second and third terms dominate in the neutral stability curve (5.2) and the relation $\alpha_1 \sim c_1\sigma_0\gamma/c_0$ is consistent, to a reasonable degree of accuracy, with the linear behavior obtained in (4.14). As $\sigma_0 \to \infty$, $\gamma_1 \to 3(c_2/2)^{2/3}/c_1 = 2.32 \times 10^3$, and $\alpha_1 \to (c_2/2)^{1/3} = 2.03 \times 10^2$, and the infinite Prandtl number limit is finally achieved.

In a future investigation, it is hoped to confirm the above ideas by tracking the location of the critical point γ_c, α_c in the γ, α plane for Prandtl numbers of order A. This will allow the main adjustment in the region of stationary instability to be determined for large Rayleigh numbers and Prandtl numbers. It would also be of interest to consider how the travelling wave instability discussed by [4] develops at both high Rayleigh numbers and Prandtl numbers. Here the critical Rayleigh number A in the conduction regime ($\gamma = 0$) is of order $\sigma^{1/2}$ [10] and it seems likely that, in contrast to the stationary case, the instability is completely suppressed for all values of γ when the Prandtl number is infinite.

REFERENCES

1. P.G. Daniels, Convection in a vertical slot, *J. Fluid Mech.* **176**, 419–428, (1987).
2. P.G. Daniels, Stationary instability of the buoyancy-layer flow between heated vertical planes, *Proc. Roy. Soc. London* **A401**, 145, (1985).

3. P.G. Daniels, Stationary instability of the convective flow between differentially heated vertical planes, *J. Fluid Mech.* **203**, 525, (1989).
4. R.F. Bergholz, Instability in steady natural convection in a vertical fluid layer, *J. Fluid Mech.* **84**, 743–758, (1978).
5. J.W. Elder, Laminar free convection in a vertical slot, *J. Fluid Mech.* **23**, 77–85, (1965).
6. C.M. Vest and V.S. Arpaci, Stability of natural convection in a vertical slot, *J. Fluid Mech.* **36**, 1–15, (1969).
7. N. Seki, S. Fukusako and H.J. Inaba, Visual observation of natural convective flow in a narrow vertical cavity, *J. Fluid Mech.* **84**, 695–706, (1978).
8. A.A. Al-Ali and P.G. Daniels, On the wavelength of stationary instability in the convective flow between heated vertical planes, *Eur. J. Mech. B.* **15**, 413–428, (1996).
9. A.A. Al-Ali, Stability of convective flow between vertical planes, Ph.D. Thesis, City University, London, (1993).
10. S.A. Korpela, D. Gozum and C.B. Baxi, On the stability of the conduction regime of natural convection in a vertical slot, *Int. J. Heat Mass Transfer* **16**, 1683–1696, (1973).