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Enumeration of bilaterally symmetric 3-noncrossing partitions

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ABSTRACT

Schützenberger's theorem for the ordinary RSK correspondence naturally extends to Chen et al.'s correspondence for matchings and partitions. Thus the counting of bilaterally symmetric k-noncrossing partitions naturally arises as an analogue for involutions. In obtaining the analogous result for 3-noncrossing partitions, we use a different technique to develop a MAPLE package for 2-dimensional vacillating lattice walk enumeration problems. The package also applies to the hesitating case. As applications, we find several interesting relations for some special bilaterally symmetric partitions.

1. Introduction

A partition P of $[n] := \{1, 2, \ldots, n\}$ is a collection of nonempty subsets $\{B_1, B_2, \ldots, B_k\}$, whose disjoint union is $[n]$. The elements B_i are called *blocks* of P. An important special class of partitions are (complete) *matchings* of $[2n]$, which are partitions of $[2n]$ into n two-element blocks. Every partition P of $[n]$ has a graph representation, called *partition graph*, obtained by identifying vertex i with $(i, 0)$ in the plane for $i = 1, \ldots, n$, and drawing an arc connecting i and j above the horizontal axis whenever i and j are (numerically) consecutive in a block of P. Such an arc with $i < j$ is called an edge (i, j) of P, starting from i and ending at j. The vertices i and j are called the *left-hand endpoint* and the *right-hand endpoint* of the arc, respectively. A *singleton* is the element of a one-element block, and hence corresponds to an isolated vertex in the graph. Conversely, a graph on the vertex set $[n]$ is a partition graph if and only if each vertex is the left-hand (resp., right-hand) endpoint of at most one edge. For a partition P of $[n]$, let P^{refl} denote the partition obtained from P by reflecting in the vertical line $x = (n+1)/2$. Equivalently, (i, j) is an arc of P if and only if $(n+1-j, n+1-i)$ is an arc of P^{refl}.

A sequence $\emptyset = \nu^0, \nu^1, \ldots, \nu^{2n} = \lambda$ of Young diagrams is called a *vacillating tableau* of shape λ and length $2n$ if (i) ν^{2i+1} is obtained from ν^{2i} by doing nothing (i.e., $\nu^{2i+1} = \nu^{2i}$) or deleting a square, and (ii) ν^{2i} is obtained from ν^{2i-1} by doing nothing or adding a square.

In what follows, vacillating tableaux are always of shape $\emptyset$ unless specified otherwise. Recently, Chen et al. [5] established a bijection ϕ from partitions to vacillating tableaux. Using their bijection, crossings and nestings of a partition are characterized by its corresponding vacillating tableau. When restricting to matchings, the image of ϕ becomes the set of oscillating tableaux. (see Appendix A for definition).

For a vacillating tableau V, reading V backward still gives a vacillating tableau, denoted by V^{rev}. Schützenberger's theorem (see, e.g., [15] or [17, Chapter 7.11]) for the ordinary RSK correspondence (see, e.g., [10] or [17, Chapter 7.13]) naturally extends to the bijection ϕ. The result for partitions is stated as follows.

Theorem 1. *For any given partition P and vacillating tableau V, $\phi(P^{\mathrm{refl}}) = V^{\mathrm{rev}}$ if and only if $\phi(P) = V$.*

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This result and its analogy for matchings follows trivially from Fomin's growth diagram language. See [8,9]. The matching case is due to Roby [14] and the partition case is due to Krattenthaler [11].

A vacillating tableau V is said to be *palindromic* if $V = V^{\text{rev}}$. A partition P of $[n]$ is said to be *bilaterally symmetric* (bi-symmetric for short) if $P = P^{\text{refl}}$. Theorem 1 implies that P is bi-symmetric if and only if $V(P)$ is palindromic. The enumeration of bi-symmetric partitions and matchings are not hard, but turns out to be very difficult if we also consider the statistic of *crossing number* or *nesting number*. A k-subset $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ of the edge set of a partition P is said to be a k-crossing if $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. A k-noncrossing partition is a partition with no k-crossings. Some nice properties on crossings and nestings of partitions and matchings have been explored in [5]. Here we are interested in the enumeration of these objects.

The number of k-noncrossing matchings was enumerated in [5], and the number of bi-symmetric k-noncrossing matchings was enumerated in [21]. The number of partitions is well-known to be the Bell number, but a formula for the number of k-noncrossing partitions is only known for $k = 2$ and $k = 3$. See [3]. The number of bi-symmetric partitions was enumerated as the sequence A080107 in [16]. In this paper we enumerate bi-symmetric k-noncrossing partitions for $k = 2$ (in Appendix A) and $k = 3$, which are the same as palindromic vacillating tableaux of height bounded by k for $k = 1$ and $k = 2$.

Let $\widetilde{C}_3(n)$ be the number of bi-symmetric 3-noncrossing partitions of $[n]$. Then our main result is the following.

Proposition 2. *The numbers $\widetilde{C}_3(2n)$ satisfy $\widetilde{C}_3(0) = 1$, $\widetilde{C}_3(2) = 2$, $\widetilde{C}_3(4) = 7$, and*

$$27n(n+2)\widetilde{C}_3(2n) - 3(7n^2+26n+27)\widetilde{C}_3(2n+2) - (7n^2+50n+84)\widetilde{C}_3(2n+4) + (n+5)^2\widetilde{C}_3(2n+6) = 0. \quad (1)$$

The numbers $\widetilde{C}_3(2n+1)$ satisfy $\widetilde{C}_3(1) = 1$, $\widetilde{C}_3(3) = 3$, and

$$9(n^2+3n+2)\widetilde{C}_3(2n+1) - 2(5n^2+30n+43)\widetilde{C}_3(2n+3) + (n+4)(n+5)\widetilde{C}_3(2n+5) = 0. \quad (2)$$

Equivalently, their associated generating functions $\mathcal{G}_e(t) = \sum_{n\geq 0} \widetilde{C}_3(2n)t^n$ and $\mathcal{G}_o(t) = \sum_{n\geq 0} \widetilde{C}_3(2n+1)t^n$ satisfy

$$-4 - 6t - 6t^2 + (4 - 12t - 24t^2)\mathcal{G}_e(t) + (5t - 29t^2 - 57t^3 + 81t^4)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_e(t)$$
$$+ t^2(t-1)(3t+1)(9t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{G}_e(t) = 0, \quad (3)$$

$$6 + (-6 + 36t - 18t^2)\mathcal{G}_o(t) + (-6t + 50t^2 - 36t^3)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_o(t) - t^2(t-1)(9t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{G}_o(t) = 0. \quad (4)$$

The above result is analogous to that for $C_3(n)$, the number of 3-noncrossing partitions of $[n]$, in [3]. By a similar way we represent the generating functions as certain constant terms in Section 2. But the techniques differs thereafter. In proving our result, we develop a Maple package in Section 3 that applies to a class of two dimensional vacillating lattice walk enumeration problems. The package is also extended to the hesitating case in Section 4. As applications, we find several interesting results for some special bi-symmetric partitions. An extended abstract of this work appeared in the special volume of DMTCS 2008.

2. Lattice path interpretations and constant term expressions

In order to prove Proposition 2, we need to introduce the lattice path interpretations. Let S be a subset of $\mathbb{Z}^k$. An S-*vacillating lattice walk* of length n is a sequence of lattice points $p_0, p_1, \ldots, p_n$ in S such that (i) $p_{2i+1} = p_{2i}$ or $p_{2i+1} = p_{2i} - e_j$ for some unit coordinate vector e_j; (ii) $p_{2i} = p_{2i-1}$ or $p_{2i} = p_{2i-1} + e_j$ for some unit coordinate vector e_j. We are interested in two subsets of $\mathbb{Z}^k$: $Q_k = \mathbb{N}^k$ of nonnegative integer lattice points and $W_k = \{(a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k : a_1 > a_2 > \cdots > a_k \geq 0\}$ of Weyl lattice points. For two lattice points a and b in W_k (or Q_k), denote by $w_k(a, b, n)$ (or $q_k(a, b, n)$) the number of W_k (or Q_k)-vacillating lattice walks of length n starting at a and ending at b. Let $\delta = (k-1, k-2, \ldots, 0)$.

Let $C_k(n)$ be the number of k-noncrossing partitions of $[n]$. The following consequence of Chen et al.'s correspondence ϕ is the starting point of the enumeration for 3-noncrossing partitions, as well as for bi-symmetric 3-noncrossing partitions.

Theorem 3 (*Chen et al., [5]*). *The number $C_{k+1}(n)$ equals $w_k(\delta, \delta, 2n)$, i.e., the number of closed W_k-vacillating lattice walks of length $2n$ from δ to itself.*

By the correspondence ϕ, $\widetilde{C}_3(n)$ is the same as the number of palindromic vacillating tableaux of height bounded by 2 and length $2n$, and is the same as the number of palindromic W_2-vacillating lattice walks of length $2n$ that start and end at $(1, 0)$. Since such walks are palindromic, it is sufficient to consider only the first n steps of the lattice walks. We have

$$\widetilde{C}_3(n) = \sum_{b\in W_2} w_2((1,0), b, n). \quad (5)$$

Let us introduce the basic idea for solving the problem of determining $C_3(n)$, where the Q_2-vacillating lattice walks starting and ending at $(1,0)$ are considered. The same idea applies to determining $\widetilde{C}_3(n)$.

It was shown in [3] by using the reflection principle that

$$w_k(a,b,n)=\sum_{\pi\in\mathfrak{S}_k}(-1)^{\pi}q_k(\pi(a),b,n),$$

where $(-1)^{\pi}$ is the sign of π and $\pi(a_1,a_2,\ldots,a_k)=(a_{\pi(1)},a_{\pi(2)},\ldots,a_{\pi(k)})$. Thus the enumeration of $w_k(\delta,\delta,2n)$ reduces to that of $q_k(a,\delta,2n)$. Denote by $a_{i,j}(n)=q_2((1,0),(i,j),n)$. Let

$$F_e(x,y;t)=\sum_{i,j,n\geq 0}a_{i,j}(2n)x^iy^jt^{2n}$$

and

$$F_o(x,y;t)=\sum_{i,j,n\geq 0}a_{i,j}(2n+1)x^iy^jt^{2n+1}$$

be respectively the generating functions of lattice walks of even and odd length. By a step by step construction, one can set up functional equations for $F_e(x,y;t)$ and $F_o(x,y;t)$ and reduce the problem to solving the following functional equation:

$$K(x,y;t^2)F_o(x,y;t)/t=x(1+x^{-1}+y^{-1})-x^{-1}V_e(y;t^2)-y^{-1}H_e(x;t^2),$$

where $V_e(y;t^2)$ and $H_e(x;t^2)$ are respectively the generating functions for lattice walks of even length that start at $(1,0)$ and end on the vertical and horizontal axis, and the *kernel* of the equation $K(x,y;t)$ is given by

$$K(x,y;t)=1-t(1+x+y)(1+x^{-1}+y^{-1}).$$

By the *obstinate kernel method* of [1,2], one can finally obtain the generating function $\mathcal{C}(t)$ of $C_3(n)$ as

$$\mathcal{C}(t)=\mathrm{CT}_x\left((x^{-2}-x^2)(x^2+(x^{-2}+x+x^2)Y+(x^{-3}-x^{-1})Y^2-x^{-2}Y^3)\right),\tag{6}$$

where the operator CT_x extracts the constant term in x of series in $\mathbb{Q}[x,x^{-1}][[t]]$ and $Y=Y(x;t)$ is the unique power series in t satisfying $Y=t(1+x+Y)(1+(1+x^{-1})Y)$ given by

$$Y=\frac{1-(x^{-1}+3+x)t-\sqrt{\left(1-(1+x+x^{-1})t\right)^2-4t}}{2\left(1+x^{-1}\right)t}=(1+x)t+\cdots.\tag{7}$$

We shall mention that all this is done in the ring $Q[x,x^{-1},y,y^{-1}][[t]]$ of formal power series in t with coefficients Laurent polynomial in x and y.

This idea works in a similar way for lattice walks starting from a set of points and ending at $(1,0)$. For a set A of points, we denote by $A(x,y)=\sum_{(i,j)\in A}x^iy^j$ its generating function. Let $C_3^A(n)$ be the number of W_2-lattice walks of length $2n$ starting from points in A and ending at $(1,0)$, and let $\mathcal{C}^A(t)$ be the generating function of $C_3^A(n)$. For instance, $A_1(x,y)=x$ corresponds to the point $(1,0)$ and hence $C_3^{A_1}(n)=C_3(n)$ and $\mathcal{C}^{A_1}(t)=\mathcal{C}(t)$. For general A, with Y as in (7) the result of [3, Section 2.7] for $\mathcal{C}^A(t)$ can be summarized as follows.

Proposition 4. *For any set A of lattice points in W_2, we have*

$$\begin{aligned}\mathcal{C}^A(t)=\mathrm{CT}_x\big((x^{-2}-x^2)((x+Y+xY)A(x,Y)-(x^{-1}Y+Y+x^{-1}Y^2)A(x^{-1}Y,Y)\\+(x^{-1}Y+x^{-1}+x^{-2}Y)A(x^{-1}Y,x^{-1}))\big).\end{aligned}\tag{8}$$

Now it is natural to let $A_2(x,y)=\frac{x}{(1-x)(1-xy)}$, which corresponds to the set of all points in W_2. We shall also consider the following two closely related cases: $A_3(x,y)=x/(1-x)$ corresponds to the x-axis in W_2; $A_4(x,y)=x/(1-xy)$ corresponds to the diagonal in W_2. Define $e(n)=w_2((1,0),A_3,n)$ and $h(n)=w_2((1,0),A_4,n)$. Then at the same $e(n)$ (resp., $h(n)$) is the number of bi-symmetric 3-noncrossing partitions on $[n]$ whose central Young diagrams consist of at most one row (resp., two rows of squares of equal length including $\emptyset$).

Although our lattice walks for $\widetilde{C}_3(n)$ always start from $(1,0)$, which is different from that in Proposition 4, we will still use the formulas for $\mathcal{C}^A(t)$ by means of the following two observations:

(1) $w_2(a,b,2n)=w_2(b,a,2n)$, since W_2-vacillating lattice walks of even length are still W_2-vacillating if read backward. Thus by (5) $\widetilde{C}_3(2n)=C_3^{A_2}(2n)$, and similarly $e(2n)=C^{A_3}(2n)$ and $h(2n)=C^{A_4}(2n)$.

Table 1
Reducing the length from $2n+1$ to $2n$ by the step by step construction

The ending set (of odd length)	Generating function set (of even length)
A_2 (the set of all points in W_2)	$A_1(x,y)+2A_2'(x,y)+3A_2''(x,y)$
A_3 (the x-axis in W_2)	$(1+x+xy)A_3(x,y)$
A_4 (the diagonal in W_2)	$(1+x)A_4(x,y)$

Table 2
The first several numbers of vacillating lattice walks and their asymptotic estimate, where $\kappa_1 \approx 1691.643$, $\kappa_2 \approx 3.719$, $\kappa_3 \approx 11.156$, and $\kappa_4 = 2\kappa_5 \approx 16.732$

n	0	1	2	3	4	5	6	$\to$	∞
$w_2((1,0),A_1,2n)$	1	1	2	5	15	52	202	$\sim$	$\kappa_1 \cdot 9^n/n^7$
$w_2((1,0),A_2,2n)$	1	2	7	30	148	806	4716	$\sim$	$\kappa_2 \cdot 9^n/n^3$
$w_2((1,0),A_2,2n+1)$	1	3	12	57	303	1743	10629	$\sim$	$\kappa_3 \cdot 9^n/n^3$
$w_2((1,0),A_3,2_n)$	1	2	6	22	94	450	2346	$\sim$	$\kappa_4 \cdot 9^n/n^4$
$w_2((1,0),A_4,2n)$	1	1	3	11	47	225	1173	$\sim$	$\kappa_5 \cdot 9^n/n^4$

(2) By the step by step construction we have

$$w_2(a,b,2n+1) = w_2(a,b,2n) + w_2(a,b+(1,0),2n) + w_2(a,b+(0,1),2n).$$

However, we must take care of the boundary cases. A careful study yields

$$\sum_{b\in W_2} w_2(a,b,2n+1) = w_2(a,(1,0),2n) + 2\cdot \sum_{b\in A_2'} w_2(a,b,2n) + 3\cdot \sum_{b\in A_2''} w_2(a,b,2n),$$

where $A_2'(x,y) = x^2/(1-x) + x^2y/(1-xy)$ and $A_2''(x,y) = x^3y/((1-x)(1-xy))$.

A summary of the above is given in Table 1. With $\mathcal{G}_e(t)$ and $\mathcal{G}_o(t)$ as stated in Proposition 2 we have

$$\mathcal{G}_e(t) = \mathcal{C}^{A_2(x,y)}(t),$$

and

$$\mathcal{G}_o(t) = \mathcal{C}^{A_1(x,y)+2A_2'(x,y)+3A_2''(x,y)}(t).$$

Then by Proposition 4, $\mathcal{G}_e(t)$ and $\mathcal{G}_o(t)$ can be represented as certain constant terms. The cases for the other A's are similar. Such constant terms will be systematically dealt with by the Maple package developed in Section 3.

Several interesting results can be obtained for the A_3 and A_4 cases similarly.

Proposition 5. *For $n \geq 1$, we have $e(2n) = 2\cdot h(2n)$. Moreover $h(2)=1$, $h(4)=3$ and*

$$9n(n+3)h(2n) - 2(5n^2+26n+30)h(2n+2) + (n+4)(n+5)h(2n+4) = 0. \quad (9)$$

The proposition can be established by the following two differential equations, which can be easily shown by our package.

$$-6-6t+(6-18t)\mathcal{C}^{A_3}(t) + (6t-42t^2+36t^3)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}^{A_3}(t) + t^2(t-1)(9t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{C}^{A_3}(t) = 0, \quad (10)$$

$$-6+6t+(6-18t)\mathcal{C}^{A_4}(t) + (6t-42t^2+36t^3)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}^{A_4}(t) + t^2(t-1)(9t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{C}^{A_4}(t) = 0. \quad (11)$$

Proposition 6. *For $n \geq 0$, we have $e(2n+1) = e(2n+2)/2$, and $h(1)=1$, $h(3)=2$ and*

$$9(n+2)^2h(2n+1) - (10n^2+62n+93)h(2n+3) + (n+5)^2h(2n+5) = 0. \quad (12)$$

The sequence $(h(2n+1))_{n\geq 0}$ appears as A005802 in [16]. This suggests that $h(2n+1) = u_{n+1}$, the number of 1234-avoiding permutations of length $n+1$. It is easy to check that Eq. (12) coincides with the formula given by Mihailovs in the comments of A005802. We conclude this subsection by some asymptotic estimates in Table 2 using the method of [18].

3. Determine the constant terms by a maple package

In this section we will develop a MAPLE package to deal with constant term expressions for $\mathcal{C}^A(t)$. Our proof is based on the idea of Lipshitz [12], but for our particular problem we find a much smaller bound for the degree of the D-finiteness. Moreover, this bound is for a large class of power series and can be carried out by MAPLE. We find it better to work in the field $\mathbb{Q}((x))((t))$ of iterated Laurent series, which is also the field of Laurent series in t with coefficients Laurent series in x. See [19,20] for other applications of this field.

Many objects are easy to describe using

$$u = (x^{-1} + 2 + x) = x^{-1}(1 + x)^2.$$

Let

$$\Delta \equiv \Delta(x, t) = \sqrt{\left(1 - (1 + x + x^{-1})t\right)^2 - 4t} = \sqrt{(1 - (u - 1)t)^2 - 4t}.$$

Then it is easy to see that $\mathbb{Q}(x, t, \Delta) = \mathbb{Q}(x, t) \oplus \mathbb{Q}(x, t)\Delta$. Since

$$Y = \frac{1}{2}\frac{x - \left(1 + x^2 + 3\,x\right)t - x\Delta}{t\,(1 + x)}, \tag{13}$$

$\mathcal{C}^A(t)$ can be written as

$$\mathcal{C}^A(t) = \mathrm{CT}_x T_0 + \mathrm{CT}_x T_1 \Delta$$

for some $T_0, T_1 \in \mathbb{Q}(x, t)$. In our study, the series $A(x, y)$ is always in the form of $P(x, y)/\left((1 - x)(1 - xy)\right)$ for some polynomial $P(x, y)$. Consequently the rational functions T_0 and T_1 may have x, $1 + x$, D_1, D_2, and D_3 (but no more) as denominators, where

$$\begin{aligned}
D_1 &= -2t + (1 - 5t)x - 2tx^2 = x(1 - (2u + 1)t),\\
D_2 &= -t - 2tx - (3t - 1)x^2 - 2tx^3 - tx^4 = x^2(1 - (u - 1)^2 t),\\
D_3 &= t^2 + (2t^2 - 2t)x + (3t^2 - 6t + 1)x^2 + (2t^2 - 2t)x^3 + t^2x^4 = x^2((1 - (u - 1)t)^2 - 4t).
\end{aligned}$$

Of course one can write everything in terms of Y, but using Δ may significantly simplify the proof because the derivatives of Δ have simple expressions. Notice that $D_3 = \Delta^2 x^2$, and we have

$$\frac{\partial}{\partial x}\Delta(x, t) = \frac{t^2x^4 + (t^2 - t)x^3 - (t^2 - t)x - t^2}{xD_3}\Delta, \tag{14}$$

$$\frac{\partial}{\partial t}\Delta(x, t) = \frac{tx^4 + (2t - 1)x^3 + (3t - 3)x^2 + (2t - 1)x + t}{D_3}\Delta. \tag{15}$$

Let $\mathcal{L}$ be the (finite) $\mathbb{Q}(t)$-linear span of

$$\left\{\mathrm{CT}_x \frac{L(x, t)\Delta}{(1 + x)^p D_1^q D_2^r D_3^s} \mid (p, q, r, s) \in \mathbb{Z}^4, L(x, t) \text{ is a Laurent polynomial in } x\right\}.$$

We shall devote ourselves to prove the following result.

Proposition 7. *The linear span $\mathcal{L}$ is of dimension at most 3. More precisely, for any given $L(x, t)$, p, q, r and s, there exists a procedure to find rational functions $R(t), P(t), Q(t) \in \mathbb{Q}(t)$ such that*

$$\mathrm{CT}_x \frac{L(x, t)\Delta}{(1 + x)^p D_1^q D_2^r D_3^s} = R(t) + \mathrm{CT}_x (P(t) + Q(t)x)\Delta. \tag{16}$$

It is clear that $\mathcal{L}$ is closed under taking derivatives with respect to t. Thus we have

Corollary 8. *Every element in $\mathcal{L}$ is D-finite of order at most 2.*

The basic idea for proving Proposition 7 is to use the well-known formula

$$\mathrm{CT}_x x\frac{\partial}{\partial x}F(x, t) = 0, \quad \text{for all } F(x, t) \in \mathbb{Q}((x))((t)) \tag{P1}$$

to reduce elements of $\mathcal{L}$ into simple form. We need the following lemma.

Lemma 9. (a) *For all $k \in \mathbb{Z}$, we have*

$$\mathrm{CT}_x (x^k - x^{-k})\Delta = 0, \tag{P2}$$

$$\mathrm{CT}_x (x^k - x^{-k})\frac{x\Delta}{D_1} = 0, \tag{P3}$$

$$\mathrm{CT}_x (x^k - x^{-k})\frac{x^2\Delta}{D_2} = 0. \tag{P4}$$

(b)

$$\mathrm{CT}_x \frac{1-x}{1+x}\Delta = 1 - t. \tag{P5}$$

$$\mathrm{CT}_x \frac{1-x}{1+x}\frac{x\Delta}{D_1} = 1. \tag{P6}$$

$$\mathrm{CT}_x \frac{1-x}{1+x}\frac{x^2\Delta}{D_2} = 1. \tag{P7}$$

(c)

$$\mathrm{CT}_x (1-x)(1-3xt)\Delta = (1-t)^2. \tag{P8}$$

Proof. For brevity and similarity, we only prove (P4), (P7) and (P8). Using the easy fact

$$\mathrm{CT}_x F(x,t) = \mathrm{CT}_x F(x^{-1},t), \quad \text{if } F(x,t) \in \mathbb{Q}[x, 1/x][[t]],$$

we can prove (P4) by letting $F(x,t) = \Delta(x,t)/(D_2/x^2)$ and observing $F(x,t) = F(x^{-1},t)$.

For part (b), we use Jacobi's change of variable formula [20] in the one variable case:

Theorem 10 (*Jacobi's Residue Formula*). *Let $y = f(x) \in \mathbb{C}((x))$ be a Laurent series and let b be the integer such that $f(x)/x^b$ is a formal power series with nonzero constant term. Then for any formal series $G(y)$ such that the composition $G(f(x))$ is a Laurent series, we have*

$$\mathrm{CT}_x G(f(x))\frac{x}{f}\frac{\partial f}{\partial x} = b\,\mathrm{CT}_y G(y). \tag{17}$$

We make the change of variable by $f(x) = u = x^{-1} + 2 + x = x^{-1}(1+x)^2$ with $b = -1$. It is worth mentioning that the y on the right-hand side of (17) is understood to be the same as x^{-1} (or very large). For instance, $G(y) = 1/(1-y)$ should be expanded as $1/(-y(1-1/y)) = \sum_{n\ge 0} -y^{-1-n}$. See [20] for detailed explanation. Though this understanding is not used in our calculation since $G(y)$ will be taken as Laurent polynomials, it is crucial if we make a more natural change of variable by $f(x) = x^{-1} + 1 + x$.

Direct calculation shows that

$$\frac{x}{u}\frac{\partial u}{\partial x} = \frac{x^2}{(1+x)^2}(1 - x^{-2}) = -\frac{1-x}{1+x}.$$

Thus Jacobi's Residue Formula gives us the following equality

$$\mathrm{CT}_x G(u(x))\frac{1-x}{1+x} = \mathrm{CT}_u G(u).$$

Noticing that $G(u) = \frac{\sqrt{(1-(u-1)t)^2-4t}}{1-(u-1)^2t}$ is a power series in both u and t, we have

$$\mathrm{CT}_x \frac{1-x}{1+x}\frac{\Delta}{D_2/x^2} = \mathrm{CT}_u \frac{\sqrt{(1-(u-1)t)^2 - 4t}}{1-(u-1)^2 t} = \frac{\sqrt{(1+t)^2-4t}}{1-t} = 1.$$

(c) By (P1) and (P2), the following easily verified equation (from later calculation)

$$\frac{\partial}{\partial t}\frac{(1-x)(1-3tx)}{(1-t)^2}\Delta = \frac{1+3t}{(t-1)^3}(x - x^{-1})\Delta - x\frac{\partial}{\partial x}\frac{4tx^3 + (3t^2-7t)x^2 + (3t^2-4t-3)x + 3t^2 + t}{2(t-1)^3 tx}\Delta$$

shows that $\mathrm{CT}_x (1-x)(1-3xt)(1-t)^{-2}\Delta$ is a constant. Eq. (P8) thus follows by checking the $t = 0$ case. □

Remark 11. In Lemma 9, part (a) can also be regarded as applications of Jacobi's residue formula by letting $y = x^{-1}$. We suspect that Jacobi's residue formula can also be used to prove part (c), which arises naturally when proving the differential equation for $\mathcal{C}(t)$. See [3, Proposition 1].

Proof of Proposition 7. Let $\deg_x D$ be the degree of D in x. By classical results for partial fraction decompositions, we have the unique decomposition

$$\frac{L(x,t)}{(1+x)^p D_1^q D_2^r D_3^s} = l(x,t) + \sum_{i=1}^{p} \frac{P_i}{D_0^i} + \sum_{i=1}^{q} \frac{Q_i(x,t)}{D_1^i} + \sum_{i=1}^{r} \frac{R_i(x,t)}{D_2^i} + \sum_{i=1}^{s} \frac{S_i(x,t)}{D_3^i},$$

where $l(x,t)$ is a Laurent polynomial, $D_0 = 1+x$, $P_i \in \mathbb{R}$, $\deg_x Q_i(x,t) < \deg_x D_1$, $\deg_x R_i(x,t) < \deg_x D_2$ and $\deg_x S_i(x,t) < \deg_x D_3$ for all i.

We shall often use the the above decomposition when multiplied through by Δ, so we are actually dealing with a $\mathbb{Q}(t)$-linear combination of $x^k\Delta/D_i^j$, where $0 \le k \le \deg_x D_i$ if $j \ge 1$ and $k \in \mathbb{Z}$ if otherwise. Let us call $x^k\Delta/D_i^j$ together with its coefficient the *$x^k\Delta/D_i^j$-term*, and the collection of $x^k\Delta/D_i^j$-terms for $0 \le k \le \deg_x D_i$ the *$x^*\Delta/D_i^j$-term*. We will subtract by known constant terms to reduce our original constant term to simpler forms. One of our task is to reduce p, q, r, s to 0. If some of the p, q, r, s are already 0, the corresponding steps will be skipped.

Step 1: Reduce p, q, r to 1 by the following procedure. Successively eliminate the $x^*\Delta/D_2^r$-term, and then the $x^*\Delta/D_2^{r-1}, \ldots, x^*\Delta/D_2^2$-terms, and similarly for D_1 and D_0. The process works for any irreducible polynomial $D = D(x,t)$ that is coprime to x and D_3. Denote by $N\Delta/D^r$ the $x^*\Delta/D^r$-term where $N = N(x,t)$ and $\deg_x N < \deg_x D$. Noticing

$$D_3 = x^2\Delta^2 \Rightarrow \frac{\partial \Delta}{\partial x} = -\frac{\Delta}{x} + \frac{1}{2}\frac{\Delta}{D_3}\frac{\partial D_3}{\partial x},$$

we can eliminate $N\Delta/D^r$ for $r \ge 2$ by subtracting the partial fraction decomposition of the following constant term.

$$0 = \mathrm{CT}_x x \frac{\partial}{\partial x}\frac{S\Delta}{D^{r-1}} = \mathrm{CT}_x \left(\frac{x\Delta}{D^{r-1}}\frac{\partial S}{\partial x} - \frac{S\Delta}{D^{r-1}} + \frac{1}{2}\frac{xS\Delta}{D^{r-1}D_3}\frac{\partial D_3}{\partial x} + \frac{(1-r)xS\Delta}{D^r}\frac{\partial D}{\partial x} \right). \tag{18}$$

Here S is an appropriately chosen polynomial in x such that D divides $(1-r)\frac{\partial D}{\partial x}xS - N$. Since D is irreducible and coprime to x, it is coprime to $x\frac{\partial D}{\partial x}$. Therefore we can find polynomials α and β in x (by the Euclidean algorithm) such that

$$\alpha D + \beta x \frac{\partial D}{\partial x} = 1. \tag{19}$$

Now choose

$$S = N\beta/(1-r) \Rightarrow (1-r)\frac{\partial D}{\partial x}xS - N = -\alpha N D.$$

Step 2: Reduce p and q to 0. First eliminate the $x^*\Delta/D_0$-term by using (P5), which can be rewritten as

$$\mathrm{CT}_x(1+x)^{-1}\Delta = (1-t)/2 + 1/2\mathrm{CT}_x\Delta.$$

Next eliminate the $x^*\Delta/D_1$-term by subtracting a linear combination of the following two constant terms.

$$\mathrm{CT}_x \frac{1-x}{1+x}\frac{x\Delta}{D_1} - 1 = \mathrm{CT}_x\left(-\frac{\Delta}{t-1} - \frac{4t+(5t-1)x}{(t-1)D_1}\Delta \right) = 0,$$

$$\mathrm{CT}_x x\frac{\partial}{\partial x}\left(\ln(1-Y/x) - \frac{1}{2}\ln(D_1/x) \right)$$
$$= \mathrm{CT}_x\left(-\frac{1}{4} + \frac{\Delta}{4(t-1)} + \frac{x\Delta}{D_1} - \frac{t^2+(t^2-2t)x+(t^2-t-1)x^2+tx^3}{2(t-1)D_3}\Delta \right) = 0.$$

Step 3: Eliminate all the $x^k\Delta/D_2$-terms for $k = 1, 2, 3$ by using the following three constant terms.

$$\mathrm{CT}_x \frac{(x^3-x)\Delta}{D_2} = 0 \quad (\text{by (P4) with } k = 1)$$

$$\mathrm{CT}_x \frac{1-x}{1+x}\frac{x^2\Delta}{D_2} - 1 = \mathrm{CT}_x\left(-\frac{\Delta}{t-1} - \frac{2t+4tx+(3t-1)x^2+2tx^3}{(t-1)D_2}\Delta \right) = 0$$

$$\mathrm{CT}_x x\frac{\partial}{\partial x}\left(\ln(1-xY) - \frac{1}{2}\ln(D_2/x^2) \right)$$
$$= \mathrm{CT}_x\left(\frac{3}{4} + \frac{\Delta}{4(t-1)} - \frac{\Delta x^2}{D_2} - \frac{t^2+(t^2-2t)x+(3t^2-5t+1)x^2+tx^3}{2(t-1)D_3}\Delta \right) = 0.$$

Step 4: Reduce the current s to 0. Eliminate one by one the $x^*\Delta/D_3^\ell$-terms for $\ell = s, s-1, \ldots, 1$ similarly as in Step 1. By collecting terms in (18) (with $D = D_3$), with α and β in (19), we can eliminate $N\Delta/D_3^r$ by choosing

$$S = N\beta/(3/2 - r) \Rightarrow \left(\frac{3}{2} - r\right)\frac{\partial D_3}{\partial x} xS - N = -\alpha N D_3.$$

Step 5: Remove all of the $x^k\Delta$-terms for $k \le 0$ or $k \ge 2$. First eliminate all $x^k\Delta$-terms for $k < 0$ by (P2). Then eliminate all $x^k\Delta$-terms for $k = \ell, \ell - 1, \ldots, 3$, where $\ell = \max\{\deg_x l(x,t), \deg_x l(x^{-1},t)\}$, one by one by the formulas

$$x\frac{\partial}{\partial x} x^{1+i}\Delta^3 = \left((4+i)t^2x^{i+3} + bx^{i+2} + b'x^{i+1} + b''x^i\right)\cdot\Delta, \quad \text{for } i \ge 1,$$
$$x\frac{\partial}{\partial x} x\Delta^3 = \left(4t^2x^3 + (5t^2 - 5t)x^2 + (3t^2 + 1 - 6t)x - t^2 + t - 2\frac{t^2}{x}\right)\cdot\Delta,$$

in which the b's are independent of x. Finally eliminate $x^k\Delta$-terms by (P8) for $k = 2$, and by (P2) again for $k = -1$.

Step 6: Reduce r to 0. By Step 3, it is sufficient to eliminate the $x^0\Delta/D_2$-term. This is done by showing the following equality:

$$\begin{aligned}\mathrm{CT}_x\Bigg(&\frac{9}{32}\frac{1}{t^2(9t-1)}(-1 + 8t + 55t^2 - 440t^3 + 861t^4 - 528t^5 + 45t^6)\\ &+\Bigg(\frac{9}{32}\frac{1}{t^2(9t-1)}(1 - 5t - 74t^2 + 210t^3 - 87t^4 - 45t^5)\\ &+\frac{9}{32}\frac{1}{t^2(9t-1)}(-4 + 8t + 240t^2 - 552t^3 + 180t^4)tx\Bigg)\Delta - \frac{9}{8}\frac{\Delta(1 + 5t - 21t^2 + 15t^3)}{tD_2}\Bigg) = 0.\end{aligned} \tag{20}$$

Denote by $E(t)$ the left-hand side of the above equation. To show that $E(t) = 0$, we first show that $E(t)$ satisfies a D-finite equation. The method is typical.

Using Steps 1–5, we can rewrite

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}E(t) = \widetilde{R}_i(t) + \mathrm{CT}_x(\widetilde{P}_i(t) + \widetilde{Q}_i(t)x)\Delta + \mathrm{CT}_x\widetilde{S}_i(t)\Delta/D_2, \quad i = 0, 1, 2, 3, 4.$$

By solving the system of equations

$$\begin{cases} a\widetilde{R}_0(t) + b\widetilde{R}_1(t) + c\widetilde{R}_2(t) + d\widetilde{R}_3(t) + e\widetilde{R}_4(t) = 0\\ a\widetilde{P}_0(t) + b\widetilde{P}_1(t) + c\widetilde{P}_2(t) + d\widetilde{P}_3(t) + e\widetilde{P}_4(t) = 0\\ a\widetilde{Q}_0(t) + b\widetilde{Q}_1(t) + c\widetilde{Q}_2(t) + d\widetilde{Q}_3(t) + e\widetilde{Q}_4(t) = 0\\ a\widetilde{S}_0(t) + b\widetilde{S}_1(t) + c\widetilde{S}_2(t) + d\widetilde{S}_3(t) + e\widetilde{S}_4(t) = 0\end{cases}$$

for a, b, c, d, e independent of x, we get the nontrivial solution

$$a = 1, \qquad b = \frac{2t + 10t^2 - 42t^3 + 30t^4}{3 + 5t + 21t^2 - 45t^3}, \qquad c = d = e = 0.$$

This implies that $E(t) + b\cdot\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0$. Solving this differential equation gives

$$E(t) = C_0\left(15t^{\frac{3}{2}} - 21t^{\frac{1}{2}} + 5t^{-\frac{1}{2}} + t^{-\frac{3}{2}}\right)$$

for some constant C_0.

On the other hand, by using Maple to expand $E(t)$ as a series in t and then take constant term in x, we see that $E(t)$ is actually a power series in t with $E(0) = 0$. It then follows that C_0 must be 0 and hence $E(t) = 0$ as desired. □

Once Proposition 7 is established, the differential equations (e.g., (4)) can be proved by Maple. The package can be downloaded at http://www.combinatorics.net.cn/homepage/xin/maple/bs3np.txt.

4. Analogous results for bi-symmetric enhanced 3-noncrossing partitions

Chen et al. [5] also considered a variation of k-crossings (nestings), called enhanced k-crossings (nestings). Given a partition P of $[n]$, its enhanced graph representation is obtained by adding a loop to each isolated point in the graph representation of P. Then an enhanced k-crossing of P is a set of k edges $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ of the enhanced representation of P such that $i_1 < i_2 < \cdots < i_k \le j_1 < j_2 < \cdots < j_k$. Our approach for counting bi-symmetric 3-noncrossing partitions can be easily adapted to obtain analogous enumeration results for bi-symmetric partitions avoiding enhanced 3-crossings.

Let $\widetilde{E}_3(n)$ be the number of bi-symmetric partitions of $[n]$ avoiding enhanced 3-crossings. We obtain the following result.

Proposition 12. *The numbers $\widetilde{E}_3(2n)$ satisfy $\widetilde{E}_3(0)=1$, $\widetilde{E}_3(2)=2$, and*

$$8(n+3)(n+1)\widetilde{E}_3(2n)+(7n^2+41n+58)\widetilde{E}_3(2n+2)-(n+4)(n+5)\widetilde{E}_3(2n+4)=0.$$

The numbers $\widetilde{E}_3(2n+1)$ satisfy $\widetilde{E}_3(1)=1$, $\widetilde{E}_3(3)=3$, $\widetilde{E}_3(5)=11$, and

$$\begin{aligned}&32(n+2)^2\widetilde{E}_3(2n+1)+(36n^2+220n+328)\widetilde{E}_3(2n+3)\\&\quad+(3n^2+26n+56)\widetilde{E}_3(2n+5)-(n+6)^2\widetilde{E}_3(2n+7)=0.\end{aligned}$$

Equivalently, their associated generating functions $\mathcal{H}_e(t)=\sum_{n\geq 0}\widetilde{E}_3(2n)t^n$ and $\mathcal{H}_o(t)=\sum_{n\geq 0}\widetilde{E}_3(2n+1)t^n$ satisfy

$$6-(6-24t-24t^2)\mathcal{H}_e(t)-(6t-34t^2-40t^3)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_e(t)+t^2(t+1)(8t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}_e(t)=0,$$

$$\begin{aligned}&(9+32t+32t^2)+(-9+16t+144t^2+128t^3)\mathcal{H}_o(t)+t(-7+17t+184t^2+160t^3)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_o(t)\\&\quad+(4t+1)(8t-1)(t+1)t^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}_o(t)=0.\end{aligned}$$

We need the lattice walk interpretations. A hesitating lattice walk satisfies the following walking rules: when pairing every two steps from the beginning, each pair of steps has one of the following three types: (i) a stay step followed by an e_i step, (ii) a $-e_i$ step followed by a stay step, (iii) an e_i step followed by a $-e_j$ step. It was pointed out that partitions of $[n]$ avoiding enhanced $k+1$-crossings are in bijection with hesitating tableaux of height bounded by k under a map $\bar{\phi}$ in [5]. In turn, these hesitating tableaux are in one-to-one correspondence with certain W_k-hesitating lattice walks. For the $k=2$ case, this reduces to a bijection between partitions of $[n]$ avoiding enhanced 3-crossings and W_2-hesitating lattice walks of length $2n$ starting and ending at the point $(1,0)$.

Given a set A of points, let $E^A(n)$ be the number of W_2-hesitating lattice walks of length $2n$ starting from points in A and ending at $(1,0)$, and let $\mathcal{E}^A(t)$ be the generating function of $E^A(n)$. Similar approach as for the vacillating case can give us the following analogous result.

Proposition 13. *For any set A of lattice points in W_2, we have*

$$\mathcal{E}^A(t)=\mathrm{CT}_x\frac{(x^{-2}-x^3)\left(x\widetilde{Y}A(x,\widetilde{Y})-x^{-1}\widetilde{Y}^2A(x^{-1}\widetilde{Y},\widetilde{Y})+x^{-2}\widetilde{Y}A(x^{-1}\widetilde{Y},x^{-1})\right)}{t(1+x)},$$

where $\widetilde{Y}=\widetilde{Y}(x;t)$ is the unique power series in t satisfying $\widetilde{Y}=t(1+x^{-1})(1+\widetilde{Y})(x+\widetilde{Y})$ given by

$$\widetilde{Y}=\frac{1-tx^{-1}(1+x)^2-\sqrt{((1-tx^{-1}(1+x)^2)^2-4t^2x^{-1}(1+x)^2}}{2(1+x^{-1})t}.$$

Again, by the correspondence $\bar{\phi}$, a partition P of $[n]$ is bi-symmetric if and only if the corresponding hesitating lattice walk is palindromic. By a parallel argument as for the vacillating case, and observing that the $n+1$st pair of steps for each palindromic hesitating lattice walk of length $4n+2$ must be an e_i step followed by a $-e_i$ step for some i, we can obtain formulas for $\mathcal{H}_e(t)$ and $\mathcal{H}_o(t)$:

$$\begin{aligned}&\mathcal{H}_e(t)=\mathcal{E}^{A_2(x,y)}(t),\\&\mathcal{H}_o(t)=\mathcal{E}^{A_4(x,y)+2(A_2(x,y)-A_4(x,y))}(t)=\mathcal{E}^{2A_2(x,y)-A_4(x,y)}(t).\end{aligned}$$

The above observations and Proposition 13 enable us to develop a similar Maple package for 2-dimensional hesitating lattice walks enumerating problems. Actually we can use our package for the vacillating case by redefining some initial variables. See Appendix B. With our package, we can prove Proposition 12 in a second. Moreover, we find the following result.

Proposition 14. *The numbers $E^{A_3}(n)$ satisfy $E^{A_3}(0)=1$, $E^{A_3}(1)=2$, and*

$$8(n+1)(n+2)E^{A_3}(n)+(7n^2+49n+82)E^{A_3}(n+1)-(n+5)(n+6)E^{A_3}(n+2)=0.$$

Equivalently, its associated generating function satisfies that

$$12+4(-3+10t+4t^2)\mathcal{E}^{A_3}(t)+2t(-4+21t+16t^2)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^{A_3}(t)+t^2(t+1)(8t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{E}^{A_3}(t)=0.$$

Table 3
The first several numbers of hesitating lattice walks and their asymptotic estimate, where $\lambda_1 \approx 6670.312$, $\lambda_2 \approx 7.835$, $\lambda_3 \approx 15.669$, and $\lambda_4 \approx 46.988$

n	0	1	2	3	4	5	6	$\rightarrow$	∞
$\tilde{w}_2((1,0), A_1, 2n)$	1	1	2	5	15	51	191	$\sim$	$\lambda_1 \cdot 8^n/n^7$
$\tilde{w}_2((1,0), A_2, 2n)$	1	2	7	29	136	692	3739	$\sim$	$\lambda_2 \cdot 8^n/n^3$
$\tilde{w}_2((1,0), A_2, 2n+1)$	1	3	11	48	232	1207	6631	$\sim$	$\lambda_3 \cdot 8^n/n^3$
$\tilde{w}_2((1,0), A_3, 2n)$	1	2	6	22	92	422	2074	$\sim$	$\lambda_4 \cdot 8^n/n^4$

By searching through [16, A001181], we discover that the number of hesitating lattice walks of length $2n$ starting from $(1, 0)$ and ending in A_3 is equal to the number b_{n+1} of Baxter permutations of length $n + 1$. See [6]. To prove it, we use the formula

$$b_n = \frac{2}{n(n+1)^2} \sum_{k=0}^{n-1} \binom{n+1}{k} \binom{n+1}{k+1} \binom{n+1}{k+2},$$

and apply the creative telescoping of [13,22]. It is worth mentioning that b_n also counts the number of watermelons consisting of three vicious walkers. See [4,7]. Note that there are 8 possible pair of steps for W_2-hesitating lattice walks, and 8 possible 1-steps for watermelons consisting of three vicious walkers. Then a natural question arises: Can we find a bijection between them?

Let $\tilde{w}_2((1, 0), A, n)$ be the number of W_2-hesitating lattice walks of length n, starting at $(1, 0)$ and ending in A. We conclude this subsection by Table 3 of some asymptotic estimates.

5. Discussion

Since our discussion for the vacillating case and that for the hesitating case are similar to each other, we focus on the vacillating case.

The very general theory in [12] asserts that $\mathcal{C}^A(t)$ is *D-finite* if $A(x, y)$ is rational. That is, it satisfies a linear differential equation with polynomial coefficients, or equivalently, $C^A(n)$ satisfies a P-recurrence. However the degree of the equations suggested in [12] is usually too large for proving simple P-recurrences as we consider. Note that these recurrences can be easily guessed, using the Maple package Gfun. The recurrence for $C_3(n)$ was proved by using the Lagrange inversion formula to give a single sum formula and then applying the creative telescoping of [13]. However, the same route is difficult to apply to our case. The Lagrange inversion formula will give us a complicated double sum.

Actually our Maple package can produce the differential equation for $\mathcal{C}^A(t)$ for any $A = P(x, y)/((1 - x)(1 - xy))$ with $P(x, y)$ a polynomial. The whole process will be completed within seconds if $P(x, y)$ is simple. Two curious observations are worth mentioning. We have described how to write $\mathcal{C}^A(t)$ as $\mathrm{CT}_x T_0 + \mathrm{CT}_x T_1 \Delta$ for rational T_0 and T_1, and Proposition 7 deals with $\mathrm{CT}_x T_1 \Delta$. In practice, we find that (i) T_0 does not contain D_2 and D_3 as denominators; (ii) using the constant term identity

$$\mathrm{CT}_x \frac{4t + (5t-1)x}{2t + (5t-1)x + 2tx^2} = 1,$$

obtained by considering the constant term of $x\frac{\partial}{\partial x} \ln D_1$, one sees that $\mathrm{CT}_x T_0$ is always a rational function in t. We do not know why $\mathrm{CT}_x T_0$ is always rational, since this is not true if we take, e.g., $A(x, y) = x/(1 - x)^2$ or if we pick out a term from the sum in (8). It is not a problem even if T_0 has D_1, D_2, D_3 as denominators. We can suitably enlarge $\mathcal{L}$ and increase the dimension bound to fit in our package.

In the proof of Proposition 7, only using Steps 1, 4, 5, one can already give an upper bound for the dimension of $\mathcal{L}$. Theoretically one can prove differential equations like (3) similarly as in Step 6. Such equations, once proved, will reduce the upper bound of the dimension. Eq. (P8) is actually obtained when proving the differential equation satisfied by the generating function $\mathcal{C}(t)$ (See [3, Proposition 1]); The equation $E(t) = 0$ is obtained when proving (11). Finding small upper bounds for this type of problem may help in discovering and proving new formulas, and possibly reducing the upper bound again. This idea may well apply to other situations.

Our contribution is to reduce the upper bound to only 3. This results in a fast algorithm for 2-dimensional vacillating lattice walk enumeration problems. The number 3 should be the actual dimension of $\mathcal{L}$, since otherwise $\mathcal{C}(t)$ must satisfy a lower degree differential equation, which is not suggested by the Maple package Gfun. However it seems hard to prove the equality.

Moreover, it would be interesting to find some combinatorial proofs for the interesting relations stated in Propositions 5, 6 and 14.

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Appendix A. Enumeration of bi-symmetric noncrossing partitions

In this section we consider the enumeration of bi-symmetric noncrossing partitions. To state our result, we need the following definition: An *oscillating tableau* (or *up-down tableau*) of shape μ and length n is a sequence $(\emptyset = \mu^0, \mu^1, \dots, \mu^n = \mu)$ of partitions such that for all $1 \le i \le n-1$, the diagram of μ^i is obtained from μ^{i-1} by either adding or removing one square.

Proposition 15. *There is a bijection between the set of palindromic oscillating tableaux of length* $2n$ *and height bounded by* 1 *and the set of palindromic vacillating tableaux of length* $2n$ *and height bounded by* 1*. Moreover, both of them are enumerated by* $\binom{n}{\lfloor n/2 \rfloor}$.

To construct the bijection, it is convenient to introduce an intermediate set $\mathcal{W}(n)$ of all 01 words of length n with no initial segments containing more 0's than 1's. It is well-known that $|\mathcal{W}(n)| = \binom{n}{\lfloor n/2 \rfloor}$. For any 01 word w of length n and $s = 0$ or 1, define

$$\mathrm{odd}(w, s) = |\{i \text{ is odd} \mid w_i = s,\ i \in [n]\}|,$$
$$\mathrm{even}(w, s) = |\{i \text{ is even} \mid w_i = s,\ i \in [n]\}|.$$

Then we have the following characterization.

Lemma 16. $w \in \mathcal{W}(n) \Leftrightarrow$ *for any initial segment* w' of w, *even*$(w', 1) \ge$ *odd*$(w', 0)$.

Proof. Let w' be the initial segment of w of length m. Then we have the natural equality

$$\mathrm{even}(w', 1) + \mathrm{even}(w', 0) + \chi(m \text{ is odd}) = \mathrm{odd}(w', 0) + \mathrm{odd}(w', 1), \tag{A.1}$$

where $\chi(S)$ is 1 if the statement S is true and 0 otherwise. On the other hand, by definition $w \in \mathcal{W}(n)$ if and only if for every initial segment w' of w we have

$$\mathrm{even}(w', 1) + \mathrm{odd}(w', 1) \ge \mathrm{even}(w', 0) + \mathrm{odd}(w', 0). \tag{A.2}$$

Obviously (A.2) can be replaced with (A.1)+ (A.2), which is equivalent to *even*$(w', 1) \ge$ *odd*$(w', 0)$. □

Proof of Proposition 15. Given a palindromic oscillating tableau $O = (O_0, O_1, \dots, O_{2n})$ of length $2n$ and height bounded by 1, we have a natural encoding $\theta(O) = w = w_1 w_2 \cdots w_n \in \mathcal{W}(n)$ defined by $w_i = 1$ if O_i is obtained from O_{i-1} by adding a square, and $w_i = 0$ otherwise. Note that palindromic means that $(O_0, O_1, \dots, O_n)$ already carries all information of O.

Next we conclude the proposition by constructing a bijection η from the set of palindromic vacillating tableaux of length n and height bounded by 1 to $\mathcal{W}(n)$. Given such a tableau $V = (V_0, V_1, \dots, V_{2n})$, we define $\eta(V) = w = w_1 w_2 \cdots w_n$ according to the four cases: (i) if i is odd and $V_i = V_{i-1}$, then $w_i = 1$; (ii) if i is even and V_i is obtained from V_{i-1} by adding a square, then $w_i = 1$; (iii) if i is even and $V_i = V_{i-1}$, then $w_i = 0$; (iv) if i is odd and V_i is obtained from V_{i-1} by deleting a square, then $w_i = 0$. Clearly V is a vacillating tableaux if and only if the number of type (ii) moves is no less than the number of type (iv) moves in any initial segment of V. This is the same as that in any initial segment w' of w, $\mathrm{even}(w', 1) \ge \mathrm{odd}(w', 0)$, which is equivalent to $w \in \mathcal{W}(n)$ by Lemma 16. Thus η is the desired bijection. □

Example. Let $O = (0, 1, 2, 1, 2, 1, 0, \dot{1}, 0, 1, 2, 1, 2, 1, 0)$ be the palindromic oscillating tableau, where the integers stand for one row partitions and we put a $\cdot$ over the cental diagram. Then $\theta(O) = 1101001$, and the corresponding palindromic vacillating tableau is $(0, 0, 1, 0, 1, 1, 2, \dot{1}, 2, 1, 1, 0, 1, 0, 0)$.

Remark 17. A word $w \in \mathcal{W}(2n)$ consisting of n 1's and n 0's is called a *Dyck word*. Denote the set of such words by $\mathcal{D}(n)$. By the proof of Lemma 16, we observe that

$$w \in \mathcal{D}(n) \Leftrightarrow \mathrm{even}(w, 1) = \mathrm{odd}(w, 0), \quad w \in \mathcal{W}(2n).$$

When the bijections θ and η are restricted to $\mathcal{D}(n)$, we can obtain a bijection between noncrossing matchings of $[2n]$ and noncrossing partitions of $[n]$.

Appendix B. Initial variables for hesitating lattice walks

To apply the vacillating case package to the hesitating case, we reset the initial variables as follows:

$$u = (x^{-1} + 2 + x) = x^{-1}(1+x)^2,$$
$$\Delta \equiv \Delta(x, t) = \sqrt{((1 - tx^{-1}(1+x)^2)^2 - 4t^2 x^{-1}(1+x)^2} = \sqrt{(1-ut)^2 - 4ut^2},$$
$$D_1 = x - 2t(1+x)^2 = x(1 - 2tu),$$
$$D_2 = -t - 2tx - (2t-1)x^2 - 2tx^3 - tx^4 = x^2(1 + t - t(1-u)^2),$$
$$D_3 = t^2 - 2tx - (2t^2 + 4t - 1)x^2 - 2tx^3 + t^2x^4 = x^2((1-ut)^2 - 4ut^2).$$

Similarly, $D_3 = x^2\Delta^2$ and

$$\frac{\partial}{\partial x}\Delta(x,t) = \frac{t^2x^4 - tx^3 + tx - t^2}{xD_3}\Delta,$$
$$\frac{\partial}{\partial t}\Delta(x,t) = \frac{tx^4 - x^3 - 2tx^2 - 2x^2 - x + t}{D_3}\Delta.$$

The following is a replacement of Lemma 9.

Lemma 18. (a) *For all $k \in \mathbb{Z}$, we have*

$$\mathrm{CT}_x(x^k - x^{-k})\Delta = 0,$$
$$\mathrm{CT}_x(x^k - x^{-k})\frac{x\Delta}{D_1} = 0,$$
$$\mathrm{CT}_x(x^k - x^{-k})\frac{x^2\Delta}{D_2} = 0.$$

(b)

$$\mathrm{CT}_x\frac{1-x}{1+x}\Delta = 1.$$
$$\mathrm{CT}_x\frac{1-x}{1+x}\frac{x\Delta}{D_1} = 1.$$
$$\mathrm{CT}_x\frac{1-x}{1+x}\frac{x^2\Delta}{D_2} = 1.$$

(c)

$$\mathrm{CT}_x(1-x)(1+t-3xt)\Delta = 1.$$

The proofs of part (a) and (b) are similar. Part (c) follows from the following equality

$$\frac{\partial}{\partial t}(1-x)(1-3tx+t)\Delta = -\frac{1+4t}{1+t}(x-x^{-1})\Delta$$
$$-x\frac{\partial}{\partial x}\frac{-4t(t+1)x^3 + t(4t+7)x^2 + (4t^2+10t+3)x - (1+4t)t}{2t(1+t)x}\Delta.$$

The following equality is an analogy of (20).

$$\mathrm{CT}_x\left(\frac{3}{32}\frac{1-6t-32t^2+48t^3+64t^4}{t^2(t+1)^2(8t-1)} + \left(\frac{3}{32}\frac{1-4t-44t^2-48t^3}{t^2(1+t)^2(1-8t)} - \frac{3}{8}\frac{1-2t-28t^2-16t^3}{t(1+t)^2(1-8t)}x\right)\Delta\right.$$
$$\left.+\frac{3}{8}\frac{(1+4t)\Delta}{t(1+t)D_2}\right) = 0.$$

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